Morita equivalence for operator systems

Evgenios Kakariadis

Newcastle University

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Joint work with George Eleftherakis and Ivan Todorov

Why Morita equivalence?

Representations

- The main idea is to examine an object via its action(s) on its modules, rather than in itself:

Group G	Homomorphisms $G \to \mathrm{GL}(\mathrm{V})$, for V vector space.
Ring R	Left modules ${}_RM$.
Algebra A	Homomorphisms $A \to \mathrm{End}(V)$, for V vector space.
C*-algebra A	$*$-representations $A \to \mathscr{B}(H)$, for H Hilbert space.

- In this sense "Morita equivalence" means equivalent representation theories.
- To compare objects up to matricial representations, i.e., for rings we have that R is Morita equivalent to $M_n(R)$.
- To relate R and S via matricial approximate identities:

$$\begin{array}{ccc} R & \xrightarrow{\ \mathrm{id}\ } & R \\ & {}_{\phi_{k,n}}\searrow \quad \nearrow_{\psi_{k,n}} & \\ & M_{k,n}(S) & \end{array}$$

such that $\psi_{k,n} \circ \phi_{k,n} \to \mathrm{id}_R$.

Morita equivalence for rings

Equivalent views of Morita equivalence for associative rings R and S

- There are functors $\mathscr{F}: R\text{−Mod} \to S\text{−Mod}$ and $\mathscr{G}: S\text{−Mod} \to R\text{−Mod}$ such that

$$\mathscr{F} \circ \mathscr{G} \simeq \text{id and } \mathscr{G} \circ \mathscr{F} \simeq \text{id}.$$

- There are ${}_RM_S$ and ${}_SN_R$ such that

$$R \simeq M \otimes_S N \text{ and } S \simeq N \otimes_R M,$$

as bimodules.

- There are ${}_RM_S$ and ${}_SN_R$ and balanced module maps

$$(\cdot,\cdot): M \times N \to R \text{ and } [\cdot,\cdot]: N \times M \to S$$

that are compatible (wrt associativity).

- $\text{End}(R^{(\mathbb{N})}) \simeq \text{End}(S^{(\mathbb{N})})$ (stable isomorphism, Camillo 1984).

- Morita equivalent rings have isomorphic centers (and thus Morita equivalence for commutative rings is isomorphism).

Morita equivalence for C*-algebras

Equivalent views of Morita equivalence for C*-algebras A and B

- There is an imprititivity bimodule ${}_AM_B$.
- There is a C*-algebra C such that

$$C = \begin{bmatrix} A & M \\ M^* & B \end{bmatrix} \text{ and } A \text{ and } B \text{ are full.}$$

- The categories of left operator modules are equivalent.
- There are ${}_AM_B$ and ${}_BN_A$ C*-correspondences such that

$$A \simeq M \otimes_B N \text{ and } B \simeq N \otimes_A M, \text{ as bimodules (and we can choose } N = M^*).$$

- There are ${}_AM_B$ and ${}_BN_A$ C*-correspondences and balanced module maps

$$(\cdot,\cdot)\colon M \times N \to A \text{ and } [\cdot,\cdot]\colon N \times M \to B$$

that are compatible (wrt associativity).
- $A \otimes \mathbb{K} \simeq B \otimes \mathbb{K}$, when A and B are σ-unital.
- Morita equivalent C*-algebras have isomorphic centers.

Ternary rings of operators

- A ternary ring of operators (TRO) is a closed subspace $M \subseteq \mathscr{B}(H,K)$ such that $MM^*M \subseteq M$.
- TRO's = imprimitivity bimodules ($A = [MM^*]$ and $B = [M^*M]$).

Operator spaces

Definition (Eleftherakis-K. 2016)

- Two operator spaces $X \subseteq \mathscr{B}(H_1, H_2)$ and $Y \subseteq \mathscr{B}(K_1, K_2)$ are called (strongly) TRO equivalent if there are TRO's $M_1 \subseteq \mathscr{B}(K_1, H_1)$ and $M_2 \subseteq \mathscr{B}(K_2, H_2)$ such that

$$X = [M_2 Y M_1^*] \text{ and } Y = [M_2^* X M_1].$$

Note that X and Y are operator bimodules by $[M_2 M_2^*]$-$[M_1 M_1^*]$ and $[M_2^* M_2]$-$[M_1^* M_1]$.

- Two operator spaces X and Y are called (strongly) Δ-equivalent if they admit completely isometric maps with TRO equivalent ranges.

Main tools

- Suppose that (π, ϕ, σ) is a non-degenerate representation of the $[M_2 M_2^*]$-$[M_1 M_1^*]$-bimodule X in $\mathscr{B}(H_1, H_2)$. Then by using the functors $M_1^* \otimes_\bullet -$ and $M_2^* \otimes_\bullet -$ we get a non-degenerate representation (ρ, ψ, τ) of the $[M_2^* M_2]$-$[M_1^* M_1]$-bimodule Y in $\mathscr{B}(K_1, K_2)$ for

$$K_1 = M_1^* \otimes_\sigma H_1 \text{ and } K_2 = M_2^* \otimes_\pi H_2.$$

- Suppose that $[M_2 M_2^*]$ and $[M_1 M_1^*]$ admit cai's given by sequences $(\sum_{i=1}^k m_i m_i^*)_k$ and $(\sum_{j=1}^k n_j n_j^*)_k$. Then we can define the maps

$$\phi_k : X \to M_k(Y); x \mapsto [m_i^* x n_j] \text{ and } \psi_k : M_k(Y) \to X; [y_{ij}] \mapsto \sum_{i,j} m_i y_{ij} n_j^*$$

with $\psi_k \circ \phi_k(x) = \sum_{i,j} m_i m_i^* x n_j^* n_j \to x$ in norm.

Operator spaces

Main results for operator spaces (Eleftherakis-K. 2016)

1. TRO-equivalent spaces have equivalent bimodule representation theory.

2. TRO equivalence is an equivalence relation.

3. Strong Δ-equivalence is an equivalence relation.

4. Stable isomorphism ($X \otimes \mathcal{K} \simeq Y \otimes \mathcal{K}$) implies strong Δ-equivalence.

5. Strong Δ-equivalence is stable isomorphism in the presence of separability conditions (σ-unitality, or if the spaces are separable, or if the spaces are unital).

6. Strong Δ-equivalent operator spaces have strong Δ-equivalent TRO-envelopes.

7. Strong Δ-equivalent unital operator spaces have stable isomorphic C*-envelopes.

8. If two operator algebras with c.a.i.'s are strong Δ-equivalent as operator spaces then they are Morita equivalent in the sense of Blecher-Muhly-Paulsen, and thus Δ-equivalent in the sense of Eleftherakis.

9. Strong Δ-equivalence is Morita equivalence for C*-algebras.

10. Strong Δ-equivalent operator spaces admit Δ-equivalent second duals in the sense of Eleftherakis-Paulsen-Todorov.

Operator systems

Definition

- A (concrete) operator system $\mathscr{S}$ is a (closed) selfadjoint subspace of some $\mathscr{B}(H)$ that contains the unit I_H.
- The morphisms in this category are the unital completely positive maps (they are automatically completely contractive). The isomorphisms are the complete order embeddings (unital completely positive maps with an inverse that is unital completely positive).

Definition

- Two concrete operator systems $\mathscr{S} \subseteq \mathscr{B}(H)$ and $\mathscr{T} \subseteq \mathscr{B}(K)$ are called *TRO-equivalent* (denoted $\mathscr{S} \sim_{\text{TRO}} \mathscr{T}$), if there exists a non-degenerate TRO $M \subseteq \mathscr{B}(H,K)$ such that

$$\mathscr{S} = [M^* \mathscr{T} M] \text{ and } \mathscr{T} = [M \mathscr{S} M^*].$$

- Two concrete operator systems $\mathscr{S} \subseteq \mathscr{B}(H)$ and $\mathscr{T} \subseteq \mathscr{B}(K)$ are called *concretely bihomomorphically equivalent* if there exists an operator space $X \subseteq \mathscr{B}(H,K)$ such that X and X^* are non-degenerate (i.e. $I_H \in [X^*X]$ and $I_K \in [XX^*]$), and

$$\mathscr{S} = [X^* \mathscr{T} X] \text{ and } \mathscr{T} = [X \mathscr{S} X^*].$$

Proposition

If $\mathscr{S}$ and $\mathscr{T}$ are bihomomorphically equivalent by X, then they are TRO-equivalent by $M := [X\mathrm{C}^*(X^*X)]$.

Morita equivalence for operator systems

Definition

- Two operator systems $\mathscr{S}$ and $\mathscr{T}$ are called Δ-*equivalent* (denoted $\mathscr{S} \sim_{\Delta} \mathscr{T}$) if there exist Hilbert spaces H and K and unital complete order embeddings $\phi\colon \mathscr{S} \to \mathscr{B}(H)$ and $\psi\colon \mathscr{T} \to \mathscr{B}(K)$ such that $\phi(\mathscr{S}) \sim_{\text{TRO}} \psi(\mathscr{T})$.

- Two abstract operator systems $\mathscr{S}$ and $\mathscr{T}$ will be called *bihomomorphically equivalent* if there exist Hilbert spaces H and K, and unital complete order embeddings $\phi\colon \mathscr{S} \to \mathscr{B}(H)$ and $\psi\colon \mathscr{T} \to \mathscr{B}(K)$ such that the concrete operator systems $\phi(\mathscr{S})$ and $\psi(\mathscr{T})$ are concretely bihomomorphically equivalent. We write $\mathscr{S} \leftrightarrows \mathscr{T}$ to denote that $\mathscr{S}$ and $\mathscr{T}$ are bihomomorphically equivalent.

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. The following are equivalent:

1. *$\mathscr{T} \sim_{\Delta} \mathscr{S}$ as operator systems;*
2. *$\mathscr{T} \sim_{\Delta} \mathscr{S}$ as operator spaces;*
3. *$\mathscr{S} \otimes \mathbb{K} \simeq \mathscr{T} \otimes \mathbb{K}$ via a completely isometric isomorphism;*
4. *$\mathscr{S} \otimes \mathbb{K} \simeq \mathscr{T} \otimes \mathbb{K}$ via a completely positive completely isometric isomorphism;*
5. *$\mathscr{S} \otimes \mathbb{K} \simeq \mathscr{T} \otimes \mathbb{K}$ via a hermitian completely isometric isomorphism.*

Morita equivalence for operator systems

Definition

Let $\mathscr{I}(\mathscr{S})$ be the injective envelope of $\mathscr{S}$ and write

$$\mathscr{A}_{\mathscr{S}} := \{a \in \mathscr{I}(\mathscr{S}) \mid a\mathscr{S} \subseteq \mathscr{S} \text{ and } a^*\mathscr{S} \subseteq \mathscr{S}\}.$$

Letting $\iota_{\mathrm{env}} : \mathscr{S} \to \mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) \subseteq \mathscr{I}(\mathscr{S})$ be the canonical embedding, we note that the unitality condition yields that $\mathscr{A}_{\mathscr{S}} \subseteq \iota_{\mathrm{env}}(\mathscr{S})$; we can thus consider $\mathscr{A}_{\mathscr{S}}$ as being contained in $\mathscr{S}$.

Proposition

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. If $\mathscr{S} \sim_{\Delta} \mathscr{T}$ then there exist complete order embeddings $\phi : \mathscr{S} \to \mathscr{B}(H)$ and $\psi : \mathscr{T} \to \mathscr{B}(K)$, and a non-degenerate TRO $M \subseteq \mathscr{B}(H,K)$, such that $\phi(\mathscr{S}) \sim_{\mathrm{TRO}} \psi(\mathscr{T})$ via M, and in addition

$$\phi(\mathscr{A}_{\mathscr{S}}) = [M^*M] \quad \textit{and} \quad \psi(\mathscr{A}_{\mathscr{T}}) = [MM^*].$$

Furthermore,

$$\mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) \simeq \mathrm{C}^*(\phi(\mathscr{S})) \quad \textit{and} \quad \mathrm{C}^*_{\mathrm{env}}(\mathscr{T}) \simeq \mathrm{C}^*(\psi(\mathscr{T})),$$

and consequently $\mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) \sim_{\Delta} \mathrm{C}^*_{\mathrm{env}}(\mathscr{T})$.

Morita contexts for operator systems

Definition

Let $\mathscr{S}$ and $\mathscr{T}$ be abstract operator systems and M be a TRO. We say that the quintuple $(\mathscr{S},\mathscr{T},M,[\cdot,\cdot,\cdot],(\cdot,\cdot,\cdot))$ is a Δ-*pre-context* if:

(i) the C*-algebras $[M^*M]$ and $[MM^*]$ are unital;

(ii) $\mathscr{S}$ is a C*-bimodule over $[M^*M]$ and $\mathscr{T}$ is a C*-bimodule over $[MM^*]$;

(iii) $[\cdot,\cdot,\cdot]\colon M^*\times\mathscr{T}\times M\longrightarrow\mathscr{S}$ and $(\cdot,\cdot,\cdot)\colon M\times\mathscr{S}\times M^*\longrightarrow\mathscr{T}$ are completely bounded completely positive maps, modular over $[M^*M]$ and $[MM^*]$ on the outer variables (with unital module actions), and

(iv) the associativity relations

$$(m_1,[m_2^*,t,m_3],m_4^*) = (m_1m_2^*)\cdot t\cdot(m_3m_4^*)$$

and

$$[m_1^*,(m_2,s,m_3^*),m_4] = (m_1^*m_2)\cdot s\cdot(m_3^*m_4)$$

hold for all $s\in\mathscr{S}$, $t\in\mathscr{T}$ and all $m_1,m_2,m_3,m_4\in M$.

A Δ-pre-context is called a Δ-*context* if the trilinear maps $[\cdot,\cdot,\cdot]$ and $(\cdot,\cdot,\cdot)$ are completely contractive and the relations

$$(m_1,1_{\mathscr{S}},m_2^*) = (m_1m_2^*)\cdot 1_{\mathscr{T}} \quad\text{and}\quad [m_1^*,1_{\mathscr{S}},m_2] = (m_1^*m_2)\cdot 1_{\mathscr{S}} \tag{1}$$

hold for all $m_1,m_2\in M$.

Morita contexts for operator systems

Definition

Let $\mathscr{S}$ and $\mathscr{T}$ be abstract operator systems and X be an abstract operator space. We say that the quintuple $(\mathscr{S}, \mathscr{T}, X, [\cdot,\cdot,\cdot], (\cdot,\cdot,\cdot))$ is a *bihomomorphism pre-context* if:

(i) X is non-degenerate;

(ii) $[\cdot,\cdot,\cdot]\colon X^* \times \mathscr{T} \times X \longrightarrow \mathscr{S}$ and $(\cdot,\cdot,\cdot)\colon X \times \mathscr{S} \times X^* \longrightarrow \mathscr{T}$ are completely bounded completely positive maps such that

$$[X^*, 1_{\mathscr{T}}, X] \subseteq \mathscr{A}_{\mathscr{S}} \quad \text{and} \quad (X, 1_{\mathscr{S}}, X^*) \subseteq \mathscr{A}_{\mathscr{T}};$$

(iii) the associativity relations

$$[x_1^*, (x_2, s, x_3^*), x_4] = [x_1^*, 1_{\mathscr{T}}, x_2] \cdot s \cdot [x_3^*, 1_{\mathscr{T}}, x_4]$$

and

$$(x_1, [x_2^*, t, x_3], x_4^*) = (x_1, 1_{\mathscr{S}}, x_2^*) \cdot t \cdot (x_3, 1_{\mathscr{S}}, x_4^*)$$

hold for all $s \in \mathscr{S}, t \in \mathscr{T}$ and all $x_1, x_2, x_3, x_4 \in X$.

A bihomomorphism pre-context is called a *bihomomorphism context* if the trilinear maps $[\cdot,\cdot,\cdot]$ and $(\cdot,\cdot,\cdot)$ are completely contractive and there exist semi-units $((\underline{x}_i)_i, (\underline{y}_i)_i)$ and $((\underline{z}_i)_i, (\underline{w}_i)_i)$ over X and X^*, respectively, such that

$$\lim_i [\underline{x}_i^*, 1_{\mathscr{T}} \otimes I, \underline{y}_i] = 1_{\mathscr{S}} \quad \text{and} \quad \lim_i (\underline{z}_i, 1_{\mathscr{S}} \otimes I, \underline{w}_i^*) = 1_{\mathscr{T}}. \tag{2}$$

Morita contexts for operator systems

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be (abstract) operator systems. The following are equivalent:

1. $\mathscr{S} \sim_\Delta \mathscr{T}$;
2. $\mathscr{S} \leftrightarrows \mathscr{T}$;
3. *there exists a Δ-context for $\mathscr{S}$ and $\mathscr{T}$;*
4. *there exists a bihomomorphism context for $\mathscr{S}$ and $\mathscr{T}$.*

Proof

(2) implies (1) concretely. A realization is a context, and so (1) implies (3) and (2) implies (4). A familiar "trick" by using $M \otimes -$ gives that (3) implies (1). An extension of this trick is required for showing that (4) implies (2):

- Suppose we have a bihomomorphism context and start with $\phi \colon \mathscr{S} \to \mathscr{I}(\mathscr{S}) \subseteq \mathscr{B}(H)$.

- Let K be the Hausdorff completion of $X \odot H$ wrt

$$\langle x_1 \otimes h_1, x_2 \otimes h_2 \rangle := \langle \phi([x_2^*, 1_{\mathscr{T}}, x_1]) h_1, h_2 \rangle_H .$$

- Let $\theta \colon X \to \mathscr{B}(H,K)$ be given by $\theta(x)h = x \otimes h$.

- Let $\psi \colon \mathscr{T} \to \mathscr{B}(K)$ such that $\langle \psi(t)(x_1 \otimes h_1), x_2 \otimes h_2 \rangle_K = \langle \phi([x_2^*, t, x_1]) h_1, h_2 \rangle_H$. Existence of semi-units plus that ϕ is c.is. gives that ψ is completely isometric.

- Then it follows that $\phi(\mathscr{S})$ and $\psi(\mathscr{T})$ are bihomomorphically equivalent by $\theta(X)$, since $\theta(x_2)^* \psi(t) \theta(x_1) = \phi([x_2^*, t, x_1])$.

Morita invariants and applications

Definition

- We write $\mathrm{Rep}_{C^*}(\mathscr{S})$ for the class of ucp maps of $\mathscr{S}$ that restrict to a $*$-homomorphism on $\mathscr{A}_{\mathscr{S}}$.
- We write $\mathrm{Ker}_{\mathscr{A}_{\mathscr{S}}}(\mathscr{S})$ for the set of all subspaces $\mathscr{J}$ for which there exists an element (H_ϕ, ϕ) of $\mathrm{Rep}_{C^*}(\mathscr{S})$ such that $\mathscr{J} = \ker(\phi)$.

Theorem

Let $\mathscr{S}, \mathscr{T}$ be operator systems with $\mathscr{S} \sim_\Delta \mathscr{T}$. Then

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{Id}_{\mathrm{Rep}_{C^*}(\mathscr{S})} \quad \textit{and} \quad \mathscr{F} \circ \mathscr{G} \cong \mathrm{Id}_{\mathrm{Rep}_{C^*}(\mathscr{T})},$$

up to natural equivalence. In particular, the categories $\mathrm{Rep}_{C^*}(\mathscr{S})$ and $\mathrm{Rep}_{C^*}(\mathscr{T})$ are equivalent via completely contractive functors.
Moreover the lattices $\mathrm{Ker}_{\mathscr{A}_{\mathscr{S}}}(\mathscr{S})$ and $\mathrm{Ker}_{\mathscr{A}_{\mathscr{T}}}(\mathscr{T})$ are isomorphic.

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems with $\mathscr{S} \sim_\Delta \mathscr{T}$. The functors $\mathscr{F}$ and $\mathscr{G}$ preserve the maximal representations and the Choquet representations. In addition, $\mathscr{S}$ is hyperrigid if and only if $\mathscr{T}$ is hyperrigid.

Morita invariants and applications

Definition

Let $\mathrm{Hmod}(\mathscr{S})$ have the same objects as $\mathrm{Rep}_{C^*}(\mathscr{S})$, but given objects $\Gamma_1 = (H_1, \phi_1)$ and $\Gamma_2 = (H_2, \phi_2)$, the set $\mathrm{Int}_{\mathscr{S}}(\Gamma_1, \Gamma_2)$ of morphisms from Γ_1 to Γ_2 is defined by letting

$$\mathrm{Int}_{\mathscr{S}}(\Gamma_1, \Gamma_2) = \{T \in \mathscr{B}(H_1, H_2) \mid T\phi_1(s) = \phi_2(s)T \text{ for all } s \in \mathscr{S}\};$$

we call the elements $T \in \mathrm{Int}_{\mathscr{S}}(\Gamma_1, \Gamma_2)$ *intertwiners* of the pair (Γ_1, Γ_2).

Theorem

Let $\mathscr{S}, \mathscr{T}$ be operator systems with $\mathscr{S} \sim_\Delta \mathscr{T}$. We have that

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{Id}_{\mathrm{Hmod}(\mathscr{S})} \quad \textit{and} \quad \mathscr{F} \circ \mathscr{G} \cong \mathrm{Id}_{\mathrm{Hmod}(\mathscr{T})},$$

up to natural equivalence. In particular, the categories $\mathrm{Hmod}(\mathscr{S})$ *and* $\mathrm{Hmod}(\mathscr{T})$ *are equivalent through completely contractive functors.*

Tensor products of operator systems

Remarks

- Introduced by Kavruk–Paulsen–Todorov–Tomforde (2010).

- Given two operator systems $(\mathscr{S}, \{P_n\}_n, e_{\mathscr{S}})$ and $(\mathscr{T}, \{Q_n\}_n, e_{\mathscr{T}})$, by an *operator system structure* on $\mathscr{S} \odot \mathscr{T}$ we mean a family of cones $\tau := \{C_n\}_n$ such that:

1. $C_n \subseteq M_n(\mathscr{S} \odot \mathscr{T})$;
2. $(\mathscr{S} \odot \mathscr{T}, \{C_n\}_n, e_{\mathscr{S}} \otimes e_{\mathscr{T}})$ defines an operator system denoted by $\mathscr{S} \otimes_\tau \mathscr{T}$;
3. $P_n \odot Q_m \subseteq C_{nm}$ for all $n, m \in \mathbb{N}$;
4. if $\phi\colon \mathscr{S} \to M_n$ and $\psi\colon \mathscr{T} \to M_m$ are unital completely positive maps, then $\phi \otimes \psi\colon \mathscr{S} \otimes_\tau \mathscr{T} \to M_{nm}$ is a unital completely positive map.

- Given two operator structures τ_1 and τ_2 on $\mathscr{S} \odot \mathscr{T}$, we say that $\tau_2 \leq \tau_1$ if the identity map on $\mathscr{S} \odot \mathscr{T}$ extends to a unital completely positive map $\mathscr{S} \otimes_{\tau_1} \mathscr{T} \to \mathscr{S} \otimes_{\tau_2} \mathscr{T}$; equivalently that

$$M_n(\mathscr{S} \otimes_{\tau_1} \mathscr{T})^+ \subseteq M_n(\mathscr{S} \otimes_{\tau_2} \mathscr{T})^+ \text{ for every } n \in \mathbb{N}.$$

- We say that τ is *functorial* if for every unital completely positive maps $\phi\colon \mathscr{S}_1 \to \mathscr{S}_2$ and $\psi\colon \mathscr{T}_1 \to \mathscr{T}_2$ of operator systems we get that the linear map $\phi \otimes \psi$ defines a unital completely positive map from $\mathscr{S}_1 \otimes_\tau \mathscr{S}_2$ to $\mathscr{S}_2 \otimes_\tau \mathscr{T}_2$.

- An operator system tensor product τ is *associative* if the natural isomorphism $(\mathscr{R} \odot \mathscr{S}) \odot \mathscr{T} \simeq \mathscr{R} \odot (\mathscr{S} \odot \mathscr{T})$ extends to a complete order isomorphism from $(\mathscr{R} \otimes_\tau \mathscr{S}) \otimes_\tau \mathscr{T}$ onto $\mathscr{R} \otimes_\tau (\mathscr{S} \otimes_\tau \mathscr{T})$.

Tensor products of operator systems

Tensor products on $\mathscr{S} \odot \mathscr{T}$

- The *maximal tensor product* $\mathscr{S} \otimes_{\max} \mathscr{R}$ is characterised by the property that it linearises every jointly completely positive bilinear map from $\mathscr{S} \times \mathscr{R}$ into an operator system, to a completely positive map.

- The *minimal* (or *spatial*) *tensor product* $\mathscr{S} \otimes \mathscr{R}$ is given as the (concrete) closed operator subsystem of $\mathscr{B}(H \otimes K)$ arising from (any) unital complete order embeddings $\mathscr{S} \hookrightarrow \mathscr{B}(H)$ and $\mathscr{R} \hookrightarrow \mathscr{B}(K)$.

- In the *commuting tensor product* $\mathscr{S} \otimes_{\mathrm{c}} \mathscr{R}$, an element $u \in M_n(\mathscr{S} \odot \mathscr{R})$ is positive if it has positive images under the maps $(\phi \cdot \psi)^{(n)}$ (where $(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$), for all unital completely positive maps ϕ and ψ with commuting ranges, defined on $\mathscr{S}$ and $\mathscr{R}$, respectively.

- The *essential left tensor product* $\mathscr{S} \otimes_{\mathrm{el}} \mathscr{R}$ is defined by the requirement that the inclusion map $\mathscr{S} \odot \mathscr{R} \subseteq \mathscr{I}(\mathscr{S}) \otimes_{\max} \mathscr{R}$ lifts to a complete order embedding on $\mathscr{S} \otimes_{\mathrm{el}} \mathscr{R}$.

- The *essential right tensor product* $\mathscr{S} \otimes_{\mathrm{er}} \mathscr{R}$ is defined by the requirement that the inclusion map $\mathscr{S} \odot \mathscr{R} \subseteq \mathscr{S} \otimes_{\max} \mathscr{I}(\mathscr{R})$ lifts to a complete order embedding on $\mathscr{S} \otimes_{\mathrm{er}} \mathscr{R}$.

Tensor products of operator systems

Properties

- For C*-algebras $\mathscr{A}$ and $\mathscr{B}$ the maximal operator system t.p. $\mathscr{A} \otimes_{\max} \mathscr{B}$ coincides with the operator system arising from the inclusion $\mathscr{A} \odot \mathscr{B} \hookrightarrow \mathscr{A} \otimes_{C^*\text{-}\max} \mathscr{B}$.
- $\mathscr{S} \otimes_c \mathscr{A} = \mathscr{S} \otimes_{\max} \mathscr{A}$ for every C*-algebra $\mathscr{A}$.
- $\mathscr{S} \otimes_c \mathscr{T}$ coincides with the operator system arising from the inclusion $\mathscr{S} \odot \mathscr{T} \hookrightarrow C^*_{\max}(\mathscr{S}) \otimes_{\max} C^*_{\max}(\mathscr{T})$.
- $\mathscr{S} \otimes_c \mathscr{T}$ coincides with the operator system arising from the inclusion $\mathscr{S} \odot \mathscr{T} \hookrightarrow \mathscr{S} \otimes_c C^*_{\max}(\mathscr{T})$.
- Those tensor product structures satisfy:

$$\min \quad \leq \quad \mathrm{er} \quad , \quad \mathrm{el} \quad \leq \quad \mathrm{c} \quad \leq \quad \max .$$

Definition

We say that $\mathscr{S}$ is (τ', τ)-nuclear if for every operator system $\mathscr{R}$ the identity map on $\mathscr{S} \odot \mathscr{R}$ lifts to a complete order isomorphism $\mathscr{S} \otimes_\tau \mathscr{R} \to \mathscr{S} \otimes_{\tau'} \mathscr{R}$. We say that $\mathscr{S}$ is *nuclear* if it is $(\min, \max)$-nuclear.

Tensor products of operator systems

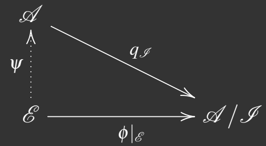
Nuclearity relations on $\mathscr{S}$ (KPTT 2010)

1. $\mathscr{S}$ is $(\min, \mathrm{el})$-nuclear iff it is exact; i.e., if for every C*-algebra $\mathscr{A}$ with an ideal $\mathscr{I}$ we have that the well-defined map

$$\mathscr{A} \otimes \mathscr{S} \Big/ \mathscr{I} \otimes \mathscr{S} \to (\mathscr{A} / \mathscr{I}) \otimes \mathscr{S}$$

is a completely isometric isomorphism.

2. $\mathscr{S}$ is $(\min, \mathrm{er})$-nuclear iff it has the operator system local lifting property (OSLLP); i.e., if whenever $\mathscr{A}$ is a unital C*-algebra with a closed ideal $\mathscr{I} \lhd \mathscr{A}$ and canonical quotient map $q_{\mathscr{I}} \colon \mathscr{A} \to \mathscr{A} / \mathscr{I}$, and $\phi \colon \mathscr{S} \to \mathscr{A} / \mathscr{I}$ is a unital completely positive map, then for every finite-dimensional operator system $\mathscr{E} \subseteq \mathscr{S}$ there is a unital completely positive map $\psi \colon \mathscr{E} \to \mathscr{A}$ such that the diagram



is commutative.

Tensor products of operator systems

Nuclearity relations on $\mathscr{S}$ (KPTT 2010)

3. $\mathscr{S}$ is $(\mathrm{el}, \max)$-nuclear iff it has the weak expectation property (WEP); i.e., if the canonical inclusion $\iota\colon \mathscr{S} \to \mathscr{S}^{dd}$ extends to a unital completely positive map $\phi\colon \mathscr{I}(\mathscr{S}) \to \mathscr{S}^{dd}$. Here $\mathscr{S}^{dd}$ denotes the double dual operator system of $\mathscr{S}$.

4. $\mathscr{S}$ is $(\mathrm{el}, \mathrm{c})$-nuclear iff it has the double commutant expectation property (DCEP); i.e., if for any unital complete order embedding $\iota\colon \mathscr{S} \to \mathscr{B}(H)$ there exists a unital completely positive extension $\phi\colon \mathscr{I}(\mathscr{S}) \to \iota(\mathscr{S})'' \subseteq \mathscr{B}(H)$.

Corollary (EKT 2021)

Nuclearity, exactness, the (OSLLP), the (WEP), and the (DCEP) are invariants of Δ-equivalence of operator systems.

Tensor products of operator systems

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems such that $\mathscr{S} \sim_\Delta \mathscr{T}$. Let (τ', τ) be a pair of functorial tensor products with $\tau' \leq \tau$ such that:

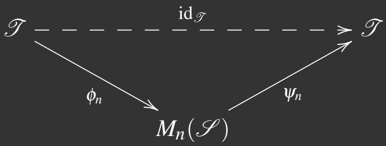
1. τ' *and* τ *are associative tensor products; or*
2. τ' *is associative and* $\tau = \mathrm{el}$*; or*
3. τ' *is associative and* $\tau = \mathrm{er}$*; or*
4. $\tau' = \mathrm{el}$ *and* τ *is associative; or*
5. $\tau' = \mathrm{er}$ *and* τ *is associative; or*
6. $(\tau', \tau) = (\mathrm{el}, \mathrm{c})$.

Then $\mathscr{S}$ *is* (τ', τ)*-nuclear if and only if* $\mathscr{T}$ *is* (τ', τ)*-nuclear.*

Tensor products of operator systems

Proof.

Due to Δ-equivalence we have an approximately commutative diagram:



of completely positive completely contractive maps.

Let $\mathscr{R}$ be an operator system. If the c.o.i. $\mathscr{S} \otimes_\tau \mathscr{R} \simeq \mathscr{S} \otimes_{\tau'} \mathscr{R}$ gives a c.o.i.

$$M_n(\mathscr{S}) \otimes_\tau \mathscr{R} \simeq M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R}, \text{ for every } n \in \mathbb{N},$$

then we are done, as we obtain a sequence of ucp maps

$$\mathscr{T} \otimes_{\tau'} \mathscr{R} \to M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R} \to M_n(\mathscr{S}) \otimes_\tau \mathscr{R} \to \mathscr{T} \otimes_\tau \mathscr{R},$$

with limit the identity map $\mathscr{T} \otimes_{\tau'} \mathscr{R} \to \mathscr{T} \otimes_\tau \mathscr{R}$. □

Tensor products of operator systems

Lemma

Let $\mathscr{S}$ be an operator system that is (τ', τ)-nuclear for $\tau' \leq \tau$ such that:

1. *τ' and τ are associative tensor products; or*
2. *τ' is associative and $\tau = \mathrm{el}$; or*
3. *τ' is associative and $\tau = \mathrm{er}$; or*
4. *$\tau' = \mathrm{el}$ and τ is associative; or*
5. *$\tau' = \mathrm{er}$ and τ is associative; or*
6. *$(\tau', \tau) = (\mathrm{el}, \mathrm{c})$.*

Then the operator system $M_n(\mathscr{S})$ is (τ', τ)-nuclear.

Proof

(i) Suppose that both τ and τ' are associative. Since M_n is nuclear, we have the complete order isomorphisms

$$\begin{aligned} M_n(\mathscr{S}) \otimes_\tau \mathscr{R} &\simeq (M_n \otimes \mathscr{S}) \otimes_\tau \mathscr{R} \simeq M_n \otimes_\tau \mathscr{S} \otimes_\tau \mathscr{R} \simeq M_n \otimes (\mathscr{S} \otimes_\tau \mathscr{R}) \\ &\simeq M_n \otimes (\mathscr{S} \otimes_{\tau'} \mathscr{R}) \simeq M_n \otimes_{\tau'} \mathscr{S} \otimes_{\tau'} \mathscr{R} \simeq (M_n \otimes \mathscr{S}) \otimes_{\tau'} \mathscr{R} \simeq M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R}. \end{aligned}$$

Proof

(ii) Suppose τ' is associative and $\tau = \mathrm{el}$. Using that $\mathscr{I}(M_n(\mathscr{S})) = M_n(\mathscr{I}(\mathscr{S}))$, the nuclearity of M_n (and thus also preservation of inclusions by $M_n \otimes_{\max} -$), the associativity of τ' and the (τ', el)-nuclearity of $\mathscr{S}$, we have the complete order isomorphisms

$$\begin{aligned} M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R} &\simeq M_n \otimes_{\tau'} (\mathscr{S} \otimes_{\tau'} \mathscr{R}) \simeq M_n \otimes_{\max} (\mathscr{S} \otimes_{\mathrm{el}} \mathscr{R}) \\ &\hookrightarrow M_n \otimes_{\max} (\mathscr{I}(\mathscr{S}) \otimes_{\max} \mathscr{R}) \simeq M_n(\mathscr{I}(S)) \otimes_{\max} \mathscr{R} \simeq \mathscr{I}(M_n(\mathscr{S})) \otimes_{\max} \mathscr{R}. \end{aligned}$$

Therefore we have the following diagram that fixes $M_n(\mathscr{S}) \odot \mathscr{R}$:

$$\begin{array}{ccc} M_n(\mathscr{S}) \otimes_{\mathrm{el}} \mathscr{R} & \longrightarrow & M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R} \\ & \searrow & \downarrow \\ & & \mathscr{I}(M_n(\mathscr{S})) \otimes_{\max} \mathscr{R}, \end{array}$$

where the horizontal arrow is given by $\tau' \leq \mathrm{el}$, the diagonal arrow is a complete order embedding by the definition of $\otimes_{\mathrm{el}}$, and the vertical arrow was shown to be a complete order embedding. This shows that the horizontal map is a complete order isomorphism.

Tensor products of operator systems

Proof

(iii) Suppose that τ' is associative and $\tau = \mathrm{er}$. Using the nuclearity of M_n (and thus also preservation of inclusions by $M_n \otimes_{\max} -$), the associativity of τ' and the (τ', er)-nuclearity of $\mathscr{S}$, we have the complete order isomorphisms

$$\begin{aligned} M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R} &\simeq M_n \otimes_{\tau'} (\mathscr{S} \otimes_{\tau'} \mathscr{R}) \simeq M_n \otimes_{\max} (\mathscr{S} \otimes_{\mathrm{er}} \mathscr{R}) \\ &\hookrightarrow M_n \otimes_{\max} (\mathscr{S} \otimes_{\max} \mathscr{I}(\mathscr{R})) \simeq M_n(\mathscr{S}) \otimes_{\max} \mathscr{I}(\mathscr{R}). \end{aligned}$$

Therefore we have the following diagram that fixes $M_n(\mathscr{S}) \odot \mathscr{R}$:

$$\begin{array}{ccc} M_n(\mathscr{S}) \otimes_{\mathrm{er}} \mathscr{R} & \longrightarrow & M_n(\mathscr{S}) \otimes_{\tau'} \mathscr{R} \\ & \searrow & \downarrow \\ & & M_n(\mathscr{S}) \otimes_{\max} \mathscr{I}(\mathscr{R}), \end{array}$$

where the horizontal arrow is given by $\tau' \leq \mathrm{er}$, the diagonal arrow is a complete order embedding by the definition of $\otimes_{\mathrm{er}}$, and the vertical arrow was shown to be a complete order embedding. This shows that the horizontal map is a complete order isomorphism.

Tensor products of operator systems

Proof

(iv) Suppose that $\tau' = \mathrm{el}$ and τ is associative. Using that $\mathscr{I}(M_n(\mathscr{S})) = M_n(\mathscr{I}(\mathscr{S}))$, the nuclearity of M_n (and thus also preservation of inclusions by $M_n \otimes_{\max} -$), the associativity of τ and the (el, τ)-nuclearity of $\mathscr{S}$, we have the complete order isomorphisms

$$\begin{aligned} M_n(\mathscr{S}) \otimes_\tau \mathscr{R} &\simeq M_n \otimes_\tau (\mathscr{S} \otimes_\tau \mathscr{R}) \simeq M_n \otimes_{\max} (\mathscr{S} \otimes_{\mathrm{el}} \mathscr{R}) \\ &\hookrightarrow M_n \otimes_{\max} (\mathscr{I}(\mathscr{S}) \otimes_{\max} \mathscr{R}) \simeq M_n(\mathscr{I}(S)) \otimes_{\max} \mathscr{R} \simeq \mathscr{I}(M_n(\mathscr{S})) \otimes_{\max} \mathscr{R}. \end{aligned}$$

Therefore we have the following diagram that fixes $M_n(\mathscr{S}) \odot \mathscr{R}$:

$$\begin{array}{ccc} M_n(\mathscr{S}) \otimes_\tau \mathscr{R} & \longrightarrow & M_n(\mathscr{S}) \otimes_{\mathrm{el}} \mathscr{R} \\ & \searrow & \downarrow \\ & & \mathscr{I}(M_n(\mathscr{S})) \otimes_{\max} \mathscr{R}, \end{array}$$

where the horizontal arrow is given by $\mathrm{el} \leq \tau$, the vertical arrow is a complete order embedding by the definition of $\otimes_{\mathrm{el}}$, and the diagonal arrow was shown to be a complete order embedding. This shows that the horizontal map is a complete order isomorphism.

Tensor products of operator systems

Proof

(v) Suppose that $\tau' = \mathrm{er}$ and τ is associative. Using the nuclearity of M_n (and thus also preservation of inclusions by $M_n \otimes_{\max} -$), the associativity of τ and the (er,τ)-nuclearity of $\mathscr{S}$, we have the complete order isomorphisms

$$\begin{aligned} M_n(\mathscr{S}) \otimes_\tau \mathscr{R} &\simeq M_n \otimes_\tau (\mathscr{S} \otimes_\tau \mathscr{R}) \simeq M_n \otimes_{\max} (\mathscr{S} \otimes_{\mathrm{er}} \mathscr{R}) \\ &\hookrightarrow M_n \otimes_{\max} (\mathscr{S} \otimes_{\max} \mathscr{I}(\mathscr{R})) \simeq M_n(\mathscr{S}) \otimes_{\max} \mathscr{I}(\mathscr{R}). \end{aligned}$$

Therefore we have the following diagram that fixes $M_n(\mathscr{S}) \odot \mathscr{R}$:

$$\begin{array}{ccc} M_n(\mathscr{S}) \otimes_\tau \mathscr{R} & \longrightarrow & M_n(\mathscr{S}) \otimes_{\mathrm{er}} \mathscr{R} \\ & \searrow & \downarrow \\ & & M_n(\mathscr{S}) \otimes_{\max} \mathscr{I}(\mathscr{R}), \end{array}$$

where the horizontal arrow is given by $\mathrm{er} \leq \tau$, the vertical arrow is a complete order embedding by the definition of $\otimes_{\mathrm{er}}$, and the diagonal arrow was shown to be a complete order embedding. This shows that the horizontal map is a complete order isomorphism.

Tensor products of operator systems

Proof

(vi) Suppose that $(\tau', \tau) = (\mathrm{el}, \mathrm{c})$. First we show we show that there is a complete order isomorphism

$$M_n(\mathscr{S}) \otimes_{\mathrm{c}} \mathscr{R} \simeq (M_n \otimes \mathscr{S}) \otimes_{\mathrm{c}} \mathscr{R} \simeq M_n \otimes (\mathscr{S} \otimes_{\mathrm{c}} \mathscr{R}) \tag{3}$$

that extends the canonical inclusions of the algebraic tensor product. Recall that

$$\mathscr{S} \otimes_{\mathrm{c}} \mathscr{R} \hookrightarrow \mathrm{C}^*_{\max}(\mathscr{S}) \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{R}).$$

Using the nuclearity of M_n and the preservation of inclusions by $M_n \otimes -$, we have that

$$\begin{aligned} M_n \otimes (\mathscr{S} \otimes_{\mathrm{c}} \mathscr{R}) &\hookrightarrow M_n \otimes (\mathrm{C}^*_{\max}(\mathscr{S}) \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{R})) \\ &\simeq M_n \otimes_{\max} (\mathrm{C}^*_{\max}(\mathscr{S}) \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{R})) \simeq (M_n \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{S})) \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{R}), \end{aligned}$$

where we used the associativity of $\otimes_{\max}$. This map takes values in

$$[(M_n \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{S})) \odot \iota_{\max}(\mathscr{R})]^{-(M_n \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{S})) \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{R})} \simeq (M_n \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{S})) \otimes_{\mathrm{c}} \mathscr{R},$$

as $M_n \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{S})$ is a C*-algebra (see equation (**??**)). In particular, we get a complete order embedding that takes values in

$$[M_n \odot \iota_{\max}(\mathscr{S})]^{-M_n \otimes_{\max} \mathrm{C}^*_{\max}(\mathscr{S})} \otimes_{\mathrm{c}} \mathscr{R} \simeq (M_n \otimes \mathscr{S}) \otimes_{\mathrm{c}} \mathscr{R},$$

where we used the nuclearity of M_n.

Tensor products of operator systems

Proof

We thus obtain a complete order embedding that preserves the copies of $M_n \odot (\mathscr{S} \odot \mathscr{R}) = (M_n \odot \mathscr{S}) \odot \mathscr{R}$, and thus it is surjective. The required identification (3) now follows.
Now the proof follows as in item (iv). That is, use that $\mathscr{I}(M_n(\mathscr{S})) = M_n(\mathscr{I}(\mathscr{S}))$, the nuclearity of M_n (and thus also preservation of inclusions by $M_n \otimes_{\max} -$), the associativity result, and the (el, τ)-nuclearity of $\mathscr{S}$, to get

$$\begin{aligned} M_n(\mathscr{S}) \otimes_{\mathrm{c}} \mathscr{R} &\simeq M_n \otimes (\mathscr{S} \otimes_{\mathrm{c}} \mathscr{R}) \simeq M_n \otimes_{\max} (\mathscr{S} \otimes_{\mathrm{el}} \mathscr{R}) \\ &\hookrightarrow M_n \otimes_{\max} (\mathscr{I}(\mathscr{S}) \otimes_{\max} \mathscr{R}) \simeq M_n(\mathscr{I}(S)) \otimes_{\max} \mathscr{R} \simeq \mathscr{I}(M_n(\mathscr{S})) \otimes_{\max} \mathscr{R}. \end{aligned}$$

Therefore we have the following diagram that fixes $M_n(\mathscr{S}) \odot \mathscr{R}$:

$$\begin{array}{ccc} M_n(\mathscr{S}) \otimes_{\mathrm{c}} \mathscr{R} & \longrightarrow & M_n(\mathscr{S}) \otimes_{\mathrm{el}} \mathscr{R} \\ & \searrow & \downarrow \\ & & \mathscr{I}(M_n(\mathscr{S})) \otimes_{\max} \mathscr{R}, \end{array}$$

where the horizontal arrow is given by $\mathrm{el} \leq \mathrm{c}$, the vertical arrow is a complete order embedding by the definition of $\otimes_{\mathrm{el}}$, and the diagonal arrow was shown to be a complete order embedding. This shows that the horizontal map is a complete order isomorphism.

Function systems

Definition

A *function system* is an operator subsystem of the (commutative) C*-algebra $\mathrm{C}(X)$.
(Prototype: the space $A(K)$ of continuous affine functions on a compact convex subset K of a locally convex space.)

Definition

The *centre of $\mathscr{S}$* is defined by

$$\mathscr{Z}(\mathscr{S}) := \mathscr{Z}(\mathrm{C}^*_{\mathrm{env}}(\mathscr{S})) \cap \mathscr{S} = \{x \in \mathscr{S} \mid \iota_{\mathrm{env}}(x)y = y\iota_{\mathrm{env}}(x) \text{ for all } y \in \mathrm{C}^*_{\mathrm{env}}(\mathscr{S})\}.$$

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. If $\mathscr{S} \sim_\Delta \mathscr{T}$ then $\mathscr{Z}(\mathscr{S}) \simeq_{\text{c.o.i.}} \mathscr{Z}(\mathscr{T})$. Consequently an operator system $\mathscr{S}$ is Δ-equivalent to a function system if and only if $\mathscr{S} \sim_\Delta \mathscr{Z}(\mathscr{S})$.
Hence Δ-equivalence coincides with c.o.i. for function systems.

Rigid systems

Definition

An operator system $\mathscr{S}$ is called *rigid* if $\mathscr{A}_{\mathscr{S}} = \mathbb{C}$.

Example

The operator system $C(S^1)^{(n)} \subseteq M_n$ of Toeplitz matrices, i.e.,

$$T := \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{bmatrix}.$$

Corollary

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. If $\mathscr{S}$ is rigid, then $\mathscr{S} \sim_\Delta \mathscr{T}$ if and only if there exists $k \in \mathbb{N}$ such that $\mathscr{T} \simeq_{\text{c.o.i.}} M_k(\mathscr{S})$. Moreover, if $\mathscr{S} \sim_\Delta \mathscr{T}$ then $\mathscr{A}_{\mathscr{T}} \simeq M_k$.

Proof.

Choose a TRO M such that $[M^*M] \simeq \mathscr{A}_{\mathscr{S}} = \mathbb{C}$ and $[MM^*] \simeq \mathscr{A}_{\mathscr{T}}$ (which is unital). Hence M is a finite dimensional Hilbert space (and set $k = \dim M$). □

Non-commutative graphs

Definition (Duan–Severini–Winter 2013)

A *non-commutative (NC) graph* is an operator subsystem of M_d, for some $d \in \mathbb{N}$.

Example

Let G be an undirected graph on a vertex set $[d] := \{1, \ldots, d\}$. We write $i \sim j$ if $\{i, j\}$ is an edge of G, and $i \simeq j$ if $i \sim j$ or $i = j$. Following Duan–Severini–Winter, let

$$\mathscr{S}_G = \operatorname{span}\{E_{ij} \mid i \simeq j\} \subseteq M_d,$$

for the canonical matrix unit system $(E_{ij})_{i,j=1}^{d}$ of M_d. As G is undirected, $\mathscr{S}_G$ is an operator system; operator systems of this form are called *graph operator systems*. We can see that $\mathrm{C}^*(\mathscr{S}_G) = \oplus_{j=1}^{n} M_{d_j}$.

Non-commutative graphs

Definition

We say that $\mathscr{S} \subseteq M_d$ *acts reducibly* if there exists a non-trivial subspace $L \subseteq \mathbb{C}^d$ such that

1. L is reducing for $\mathscr{S}$;
2. the restriction map of $\mathscr{S}$ to L is completely isometric.

We say that $\mathscr{S} \subseteq M_d$ *acts irreducibly* if there exists no such non-trivial subspace $L \subseteq \mathbb{C}^d$.

Proposition (Arveson 2011)

Let $\mathscr{S} \subseteq M_d$ be an non-commutative graph. If $\mathscr{S}$ acts irreducibly then

$$\mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) = \oplus_{j=1}^{n} M_{d_j},\ d_j, n \in \mathbb{N}, d_1 + \cdots + d_n = d.$$

for some $d_j, n \in \mathbb{N}$. Consequently,

$$\mathscr{A}_{\mathscr{S}} = \oplus_{j=1}^{k} M_{r_j},\ r_j, k \in \mathbb{N}, n \leq k, r_1 + \cdots + r_k = d.$$

Non-commutative graphs

Definition (equivalent to Stahlke 2016)

Let H and K be finite dimensional Hilbert spaces and $\mathscr{S} \subseteq \mathscr{B}(H)$ and $\mathscr{T} \subseteq \mathscr{B}(K)$ be non-commutative graphs. A *cohomomorphism* from $\mathscr{T}$ to $\mathscr{S}$ is a unital completely positive map $\Phi\colon \mathscr{B}(K) \to \mathscr{B}(H)$, which admits a Kraus representation

$$\Phi(T) = \sum_{i=1}^{r} A_i^* T A_i \quad \text{such that} \quad A_i^* \mathscr{T} A_j \subseteq \mathscr{S} \text{ for all } i, j \in [r];$$

if such a map exists, we write $\mathscr{T} \to \mathscr{S}$.

Proposition

Let $\mathscr{S}$ and $\mathscr{T}$ be non-commutative graphs. There is a cohomomorphism from $\mathscr{T}$ to $\mathscr{S}$ if and only if there exists an operator space X such that $I \in [X^*X]$ and $X^* \mathscr{T} X \subseteq \mathscr{S}$, if and only if there exists a TRO M such that $M^* \mathscr{T} M \subseteq \mathscr{S}$.

Remark

Hence two non-commutative graphs are TRO-equivalent if and only if there are finitely many A_i, B_j in a non-generate operator space X such that

$$\operatorname{span}\{A_i \mid i \in [r]\} = X = \operatorname{span}\{B_j \mid j \in [r']\},$$

with

$$A_{i_1}^* \mathscr{T} A_{i_2} \subseteq \mathscr{S}, \text{ for } i_1, i_2 \in [r] \quad \text{and} \quad B_{j_1} \mathscr{S} B_{j_2}^* \subseteq \mathscr{T} \text{ for } j_1, j_2 \in [r'].$$

Non-commutative graphs

Theorem

Let $\mathscr{S} \subseteq M_d$ and $\mathscr{T} \subseteq M_{d'}$ be irreducibly acting non-commutative graphs. Then $\mathscr{S} \sim_\Delta \mathscr{T}$ if and only if $\mathscr{S} \sim_{\mathrm{TRO}} \mathscr{T}$.

Lemma (Katavolos, Paulsen, Todorov)

Suppose that M is a TRO such that

$$[M^*M] = \oplus_{j=1}^{k} M_{r_j} \quad \text{and} \quad [MM^*] = \oplus_{i=1}^{m} M_{\ell_i},$$

and set $r := \sum_{j=1}^{k} r_j$ and $\ell := \sum_{i=1}^{m} \ell_i$. Then there exist $N \in \mathbb{N}$ and surjective maps $g\colon [r] \to [N]$ and $f\colon [\ell] \to [N]$ such that

$$M = \{(a_{i,j}) \mid a_{i,j} = 0 \text{ if } f(i) \neq g(j)\} \subseteq M_{\ell,r}.$$

Non-commutative graphs

Theorem

Let $\mathscr{S} \subseteq M_d$ and $\mathscr{T} \subseteq M_{d'}$ be irreducibly acting non-commutative graphs. Then $\mathscr{S} \sim_\Delta \mathscr{T}$ if and only if $\mathscr{S} \sim_{\text{TRO}} \mathscr{T}$.

Proof

Assume $\mathscr{S} \subseteq M_d$ and take $\psi\colon \mathscr{T} \to \mathscr{B}(M \otimes_{[M^*M]} \mathbb{C}^d)$. Then ψ is irreducible and remains to show that it is unitarily equivalent to the identity representation.

- M attains a specific form due to the Lemma, inducing a unitary map $U\colon M \otimes_{[M^*M]} \mathbb{C}^d \to \mathbb{C}^{d'}; x \otimes \xi \mapsto x\xi$. Hence we can assume that $\psi\colon \mathscr{T} \to \mathscr{B}(\mathbb{C}^{d'})$.

- Now ψ exends to a $*$-automorphism of the C*-envelope, which is finite dimensional, and so ψ is unitarily equivalent to the identity representation by some V.

- The TRO equivalence is given by VUM. □

Non-commutative graphs

Definition

Let G and H be graphs with finite vertex sets $[k]$ and $[m]$, respectively. We say that G is a *pullback of H*, if there exists a map $f\colon [k] \to [m]$ such that

$$x \simeq x' \text{ in } G \quad \text{if and only if} \quad f(x) \simeq f(x') \text{ in } H.$$

Example

The pullback of a single vertex is a complete graph.

Corollary

Let G and H be graphs. The following are equivalent:

1. $\mathscr{S}_G \sim_{\mathrm{TRO}} \mathscr{S}_H$;
2. $\mathscr{S}_G \sim_{\Delta} \mathscr{S}_H$;
3. *G and H are pullbacks of isomorphic graphs.*

Proof

The Lemma gives the connection between TRO's and pullbacks.

Δ-embeddings

Definition

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems.

1. We say that $\mathscr{S}$ Δ-*embeds* in $\mathscr{T}$ (denoted $\mathscr{S} \subset_\Delta \mathscr{T}$) if $\mathscr{S} \sim_\Delta p[\psi(\mathscr{T})]$ for a complete order embedding ψ of $\mathscr{T}$ and a projection $p \in \psi(\mathscr{T})'$.
2. We say that $\mathscr{S}$ Δ_{env}-*embeds* in $\mathscr{T}$ (denoted $\mathscr{S} \subset_{\Delta_{\text{env}}} \mathscr{T}$) if $\mathscr{S} \sim_\Delta p[\iota_{\text{env}}^{**}(\mathscr{T})]$ for the complete order embedding $\iota_{\text{env}}^{**} \colon \mathscr{T} \to \mathrm{C}^*_{\text{env}}(\mathscr{T})^{**}$ and a projection $p \in \iota_{\text{env}}^{**}(\mathscr{T})'$.

Proposition

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. Then

(i) $\mathscr{S} \subset_\Delta \mathscr{T}$ *if and only if there is a $*$-epimorphism* $\mathrm{C}^*_{\max}(\mathscr{T}) \otimes \mathbb{K} \to \mathrm{C}^*_{\text{env}}(\mathscr{S}) \otimes \mathbb{K}$ *that maps* $\iota_{\max}(\mathscr{T}) \otimes \mathbb{K}$ *onto* $\iota_{\text{env}}(\mathscr{S}) \otimes \mathbb{K}$*;*

(ii) $\mathscr{S} \subset_{\Delta_{\text{env}}} \mathscr{T}$ *if and only if there is a $*$-epimorphism* $\mathrm{C}^*_{\text{env}}(\mathscr{T}) \otimes \mathbb{K} \to \mathrm{C}^*_{\text{env}}(\mathscr{S}) \otimes \mathbb{K}$ *that maps* $\iota_{\text{env}}(\mathscr{T}) \otimes \mathbb{K}$ *onto* $\iota_{\text{env}}(\mathscr{S}) \otimes \mathbb{K}$.

Corollary

Δ_{env}-*embedding of operator systems is a transitive relation.*

Δ-embeddings

Remark

In the case of commutative C*-algebras we have that the relations Δ-embeding and Δ_{env}-embedding both coincide with the existence of surjective $*$-homomorphisms. We also note that Δ_{env}-embedding is not anti-symmetric modulo Δ-equivalence.

Remark

In contrast with the case of commutative C*-algebras, Δ_{env}-embedding is rigid for operator systems of graphs. Indeed, the C*-envelope of an operator system of a graph is a finite-dimensional C*-algebra, with the summands corresponding to the disjoint components of the graph. Therefore $\mathscr{S}_G \subset_{\Delta_{\mathrm{env}}} \mathscr{S}_H$ if and only if $\mathscr{S}_G \sim_\Delta \mathscr{S}_{H'}$ for H' a disjoint union of connected components of H; equivalently, if and only if G and H' have isomorphic pullbacks for H' a disjoint union of connected components of H.

Δ-embeddings

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. If $\mathscr{S} \subset_{\Delta_{\text{env}}} \mathscr{T}$ and $\mathscr{T} \subset_{\Delta_{\text{env}}} \mathscr{S}$ then there exist projections p in the commutant of $\mathrm{C}^*_{\text{env}}(\mathscr{T})^{**}$ and q in the commutant of $\mathrm{C}^*_{\text{env}}(\mathscr{S})^{**}$ such that

$$p[\iota^{**}_{\text{env}}(\mathscr{T})] \sim_\Delta q[\iota^{**}_{\text{env}}(\mathscr{S})] \quad \textit{and} \quad (1-p)[\iota^{**}_{\text{env}}(\mathscr{T})] \sim_\Delta (1-q)[\iota^{**}_{\text{env}}(\mathscr{S})].$$

Example

The converse of the Theorem does not hold. For a counterexample let

$$\mathscr{A} := \{(x_n) \mid \lim_{n\to\infty} x_n \text{ exists}\}$$

as a C*-subalgebra of ℓ^∞. By setting $p = 1 \oplus 0$ we see that

$$p(\mathscr{A} \oplus \mathscr{A}) \simeq \mathscr{A}_{2\mathbb{N}} \quad \text{and} \quad (1-p)(\mathscr{A} \oplus \mathscr{A}) \simeq \mathscr{A}_{2\mathbb{N}+1}.$$

However, although we have a $*$-epimorphism $\mathscr{A} \oplus \mathscr{A} \to \mathscr{A}$, we cannot have a $*$-epimorphism $\mathscr{A} \to \mathscr{A} \oplus \mathscr{A}$.

Thank you for your attention!

Stay safe, and physically and mentally healthy.