Detecting ideals in reduced crossed product C^*-algebras of topological dynamical systems

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Γ group, $\Gamma \curvearrowright X$ loc cpt

$$\sum_g f_g u_g \in C_0(X) \rtimes_{\mathrm{alg}} \Gamma \subseteq C_0(X) \rtimes_{\mathrm{red}} \Gamma$$

Theorem Let Γ be a group from the class $\mathcal{U}$ and $\Gamma \curvearrowright X$ on a loc. cpt Hausdorff space. Then every non-zero ideal of $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ intersects $C_0(X) \rtimes_{\mathrm{alg}} \Gamma$ non-trivially.

Examples of groups in $\mathcal{U}$

- lattices in connected Lie groups
- groups that are linear over the integers in a number field.
- acylindrally hyperbolic groups
- virtually polycyclic groups

Comparison with other ways to describe ideals

- Ideals in very tame situations can be described completely (Mackey machine)
- Simplicity of $C_0(X) \rtimes \Gamma$ can be characterised by a certain condition on the field of stabilisers + minimality (Kawabe)

More generally, minimality can be removed at the cost of "only" obtaining the ideal intersection property for $C_0(X) \subseteq C_0(X) \rtimes \Gamma$.

Brown - Nagy - Reznikoff - Sims - Williams: $\mathcal{G} = \underline{\Gamma \ltimes X} \ni (g, x)$ transformation groupoid, $\mathcal{I}^{\mathcal{G}}$ interior of the isotropy bundle. Then $C^*_{red}(\mathcal{I}^{\mathcal{G}}) \subseteq C^*_{red}(\mathcal{G})$ has the ideal intersection property.

Comparing with other results on "C^*-uniqueness"

- $\ell^1(\Gamma)$ has a unique C^*-norm for Γ virtually nilpotent (Boidol 80's)
  This is equ. to $\ell^1(\Gamma) \subseteq C^*(\Gamma)$ having the ideal int. prop.

  $\rightsquigarrow$ here amenability is a minimum requirement

  <u>Question</u> Is $\ell^1(\Gamma)$ C^*-unique for all amenable groups?

- Variation of C^*-uniqueness for $\ell^1\Gamma$, using $\mathbb{C}\Gamma$ instead (Grigorchuk-Musat-Rørdam).
  Unclear picture which groups satisfy this condition:
  $(\mathbb{Z}/2\mathbb{Z})^{\oplus \mathbb{Z}} \rtimes \mathbb{Z}$ is C^*-unique (Ozawa).

The class $\mathcal{U}$

Kalantar - Kennedy: Γ is C^*-simple iff
Breuillard - KK - Ozawa $\Gamma \curvearrowright \partial_F \Gamma$ is (top.) free

Furstenberg boundary

$\rightsquigarrow$ Any point stabiliser of $\Gamma \curvearrowright \partial_F \Gamma$ is called a Furstenberg subgroup of Γ.

Definition We denote by $\mathcal{U}$ the class of all groups Γ such that for all f.g. subgroups $\Lambda \leq \Gamma$ the following three conditions are satisfied:

1) the Furstenberg subgroup(s) of Λ is its amenable radical.
2) Every amenable subgroup of Λ is virt. solvable.
3) There is $l \in \mathbb{N}$ such that every solvable subgroup of Λ is polycyclic of Hirsch length bounded by l.

Ideas of the proof

We first look at group algebras ($X = \{pt\}$).
Here, e.g. Γ finite-by-C^*-simple
can be checked to satisfy the ℓ^1-ideal int. prop
by results of Bryder-Kennedy:

$$1 \to \underset{\text{finite}}{F} \to \Gamma \to \underset{C^*\text{-simple}}{\Lambda} \to 1$$

gives a decomposition of $\ell^1(\Gamma) \leq C^*_{red}(\Gamma)$
into a twisted crossed product of $C^*(F)$ by Λ..
Some lines of work reduce the problem to twisted
crossed products of a Λ-simple C^*-algebra.

- Let $\Gamma \in \mathcal{U}$ and $R = R_{am}(\Gamma)$. Let
$P \leq R$ be a finite index characteristic poly-$\mathbb{Z}$
subgroup. We may assume that P is infinite.
Let $A \leq P$ be the last non-trivial term
in the derived series. It's some $\mathbb{Z}^n$.

Associated with $A \trianglelefteq \Gamma$ is a twisted groupoid
$(\mathcal{G}, \Sigma)$ with units $\hat{A}$.

Generalisations of Brown-Nagy-Reznikoff-Sims-Williams
to twisted groupoids by Armstrong allow to reduce
considerations to the interior of the isotropy bundle
of $(\mathcal{G}, \Sigma)$. The construction ensures that its
fibres can be treated by a suitable induction
hypothesis

Results of Austad-Ortega: ideal int. prop for
L^1-algebras of groupoids.