Morita equivalence for operator systems

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Why Morita equivalence?

Representations

- The main idea is to examine an object via its action(s) on its modules, rather than in itself:

Group G	Homomorphisms $G \to \mathrm{GL(V)}$, for V vector space.
Ring R	Left modules ${}_RM$.
Algebra A	Homomorphisms $A \to \mathrm{End}(V)$, for V vector space.
C*-algebra A	$*$-representations $A \to \mathscr{B}(H)$, for H Hilbert space.

- In this sense "Morita equivalence" means equivalent representation theories.

- To compare objects up to matricial representations, i.e., for rings we have that R is Morita equivalent to $M_n(R)$.

- To relate R and S via matricial approximate identities:

$$\begin{array}{ccc} R & \xrightarrow{\quad \mathrm{id} \quad} & R \\ & {\scriptstyle \phi_{k,n}} \searrow \qquad \nearrow {\scriptstyle \psi_{k,n}} & \\ & M_{k,n}(S) & \end{array} \qquad \text{such that } \psi_{k,n} \circ \phi_{k,n} \to \mathrm{id}_R.$$

Morita equivalence for rings

Equivalent views of Morita equivalence for associative rings R and S

- There are functors $\mathscr{F}: R-\mathrm{Mod} \to S-\mathrm{Mod}$ and $\mathscr{G}: S-\mathrm{Mod} \to R-\mathrm{Mod}$ such that

$$\mathscr{F} \circ \mathscr{G} \simeq \mathrm{id} \text{ and } \mathscr{G} \circ \mathscr{F} \simeq \mathrm{id}.$$

- There are ${}_R M_S$ and ${}_S N_R$ such that

$$R \simeq M \otimes_S N \text{ and } S \simeq N \otimes_R M,$$

as bimodules.

- There are ${}_R M_S$ and ${}_S N_R$ and balanced module maps

$$(\cdot,\cdot): M \times N \to R \text{ and } [\cdot,\cdot]: N \times M \to S$$

that are compatible (wrt associativity).

- $\mathrm{End}(R^{(\mathbb{N})}) \simeq \mathrm{End}(S^{(\mathbb{N})})$ (stable isomorphism, Camillo 1984).

- Morita equivalent rings have isomorphic centers (and thus Morita equivalence for commutative rings is isomorphism).

Morita equivalence for C*-algebras

Equivalent views of Morita equivalence for C*-algebras A and B

- There is an impritivity bimodule ${}_AM_B$, i.e., M is an A-B-bimodule such that

$$[M^*M] = B \text{ and } [MM^*] = A.$$

- There is a C*-algebra C such that

$$C = \begin{bmatrix} A & M \\ M^* & B \end{bmatrix} \text{ and } A \text{ and } B \text{ are full.}$$

- The categories of left operator modules are equivalent.
- There are ${}_AM_B$ and ${}_BN_A$ C*-correspondences such that

$$A \simeq M \otimes_B N \text{ and } B \simeq N \otimes_A M, \text{ as bimodules (and we can choose } N = M^*).$$

- There are ${}_AM_B$ and ${}_BN_A$ C*-correspondences and balanced module maps

$$(\cdot,\cdot)\colon M \times N \to A \text{ and } [\cdot,\cdot]\colon N \times M \to B$$

that are compatible (wrt associativity).
- $A \otimes \mathbb{K} \simeq B \otimes \mathbb{K}$, when A and B are σ-unital.
- Morita equivalent C*-algebras have isomorphic centers.

Morita equivalence beyond C*-algebras

Decompositions

- Consider the decomposition $A \simeq M \otimes_B N$ and $B \simeq N \otimes_A M$, as the starting point, and replace $\otimes$ with the Haagerup tensor product $\otimes_h$.
- Introduced by Blecher-Muhly-Paulsen, and studied by Blecher, Kashyap et al.
- It recovers most of the C*-Morita results, but it does not recover stable isomorphism.
- It is based on that an involution or a multiplication is not available.

"Unitary equivalence"

- Consider the concrete realization $M^*AM \subseteq B$ and $MBM^* \subseteq A$, as the starting point, for an imprimitivity bimodule ${}_AM_B$, and extend up to faithful representations.
- Introduced by Eleftherakis, and studied by Eleftherakis, K., Paulsen, Todorov et al.
- It recovers most of the C*-Morita results, *and* the stable isomorphism.
- Aims to use involution and multiplication to the maximum.

Ternary rings of operators

- A ternary ring of operators (TRO) is a closed subspace $M \subseteq \mathscr{B}(H,K)$ such that $MM^*M \subseteq M$.
- TRO's = imprimitivity bimodules ($A = [MM^*]$ and $B = [M^*M]$).

Operator spaces

Definitions

- A (concrete) operator space X is a norm-closed subspace of $\mathscr{B}(H_1, H_2)$. It is called unital if $H_1 = H_2$ and $I \in X$.

- The morphisms in this category are given by completely contractive maps. A map $\phi : X \to B(K_1, K_2)$ is non-degenerate if both $[\phi(X)K_1] = K_2$ and $[\phi(X)^* K_2] = K_1$. Every map has a non-degenerate compression.

- Consider the embedding of a unital operator space X inside its injective envelope $\mathscr{I}(X)$, and endow $\mathscr{I}(X)$ with the Choi-Effros structure. The C*-algebra generated by the copy of X is called the C*-envelope, and it is the *smallest* C*-algebra generated by a ucis map of X. In the non-unital case a similar construction gives the smallest TRO, i.e., the TRO-envelope.

Definition (Eleftherakis-K. 2016)

- Two operator spaces $X \subseteq \mathscr{B}(H_1, H_2)$ and $Y \subseteq \mathscr{B}(K_1, K_2)$ are called (strongly) TRO equivalent if there are TRO's $M_1 \subseteq \mathscr{B}(K_1, H_1)$ and $M_2 \subseteq \mathscr{B}(K_2, H_2)$ such that

$$X = [M_2 Y M_1^*] \text{ and } Y = [M_2^* X M_1].$$

Note that X and Y are operator bimodules by $[M_2 M_2^*]$-$[M_1 M_1^*]$ and $[M_2^* M_2]$-$[M_1^* M_1]$.

- Two operator spaces X and Y are called (strongly) Δ-equivalent if they admit completely isometric maps with TRO equivalent ranges.

Operator spaces

Main tool 1

- Suppose that X and Y are TRO equivalent by M_1 and M_2 so that

$$X = [M_2 Y M_1^*] \text{ and } Y = [M_2^* X M_1].$$

Suppose that (π, ϕ, σ) is a non-degenerate representation of the $[M_2 M_2^*]$-$[M_1 M_1^*]$-bimodule X in $\mathscr{B}(H_1, H_2)$. Then by using the functors $M_1^* \otimes_\bullet -$ and $M_2^* \otimes_\bullet -$ we get a non-degenerate representation (ρ, ψ, τ) of the $[M_2^* M_2]$-$[M_1^* M_1]$-bimodule Y in $\mathscr{B}(K_1, K_2)$ for

$$K_1 = M_1^* \otimes_\sigma H_1 \text{ and } K_2 = M_2^* \otimes_\pi H_2.$$

Applying twice gives the original one up to unitary equivalence. Moreover $[\phi(X)]$ and $[\psi(Y)]$ are TRO equivalent; and if ϕ is cis then so is ψ.

- Hence we can always assume the $[M_2 M_2^*]$-$[M_1 M_1^*]$-bimodule X sits inside $\mathscr{I}(X)$. This is helpful for showing that Δ-equivalence is an equivalence relation.

Main tool 2

- Suppose that $[M_2 M_2^*]$ and $[M_1 M_1^*]$ admit cai's given by sequences $(\sum_{i=1}^k m_i m_i^*)_k$ and $(\sum_{j=1}^k n_j n_j^*)_k$. Then we can define the maps

$$\phi_k \colon X \to M_k(Y); x \mapsto [m_i^* x n_j] \text{ and } \psi_k \colon M_k(Y) \to X; [y_{ij}] \mapsto \textstyle\sum_{i,j} m_i y_{ij} n_j^*$$

with $\psi_k \circ \phi_k(x) = \sum_{i,j} m_i m_i^* x n_j^* n_j \to x$ in norm.

- This is helpful for approximation arguments.

Operator spaces

Main results for operator spaces (Eleftherakis-K. 2016)

1. TRO-equivalent spaces have equivalent bimodule representation theory.

2. TRO equivalence is an equivalence relation.

3. Strong Δ-equivalence is an equivalence relation.

4. Stable isomorphism ($X \otimes \mathscr{K} \simeq Y \otimes \mathscr{K}$) implies strong Δ-equivalence.

5. Strong Δ-equivalence is stable isomorphism in the presence of separability conditions (σ-unitality, or if the spaces are separable, or if the spaces are unital).

6. Strong Δ-equivalent operator spaces have strong Δ-equivalent TRO-envelopes.

7. Strong Δ-equivalent unital operator spaces have stable isomorphic C*-envelopes.

8. If two operator algebras with c.a.i.'s are strong Δ-equivalent as operator spaces then they are Morita equivalent in the sense of Blecher-Muhly-Paulsen, and thus Δ-equivalent in the sense of Eleftherakis.

9. Strong Δ-equivalence is Morita equivalence for C*-algebras.

10. Strong Δ-equivalent operator spaces admit Δ-equivalent second duals in the sense of Eleftherakis-Paulsen-Todorov.

Operator spaces

Proposition

TRO-equivalent spaces have equivalent bimodule representation theory.

Proof.

Suppose that $X = [M_2 Y M_1^*]$ and $Y = [M_2^* X M_1]$. Then the (Main Tool 1) applies. □

Theorem

TRO equivalence is an equivalence relation.

Proof.

For transitivity, suppose that

$$X = [M_2 Y M_1^*]^{-\|\cdot\|}, Y = [M_2^* X M_1]^{-\|\cdot\|} = [N_2 Z N_1^*]^{-\|\cdot\|}, Z = [N_2^* Y N_1]^{-\|\cdot\|}.$$

Then $X = \left[L_2 Z L_1^*\right]^{-\|\cdot\|}$ and $Z = \left[L_2^* X L_1\right]^{-\|\cdot\|}$ for the TRO's

$$L_1 := [M_1 D_1 N_1]^{-\|\cdot\|} \quad \text{and} \quad L_2 := [M_2 D_2 N_2]^{-\|\cdot\|}$$

where $D_i := \mathrm{C}^*(\{M_i^* M_i \cup N_i N_i^*\})$. The proof uses that $[D_2 Y]^{-\|\cdot\|} = [Y D_1]^{-\|\cdot\|} = Y$. □

Operator spaces

Definition

We write $\mathscr{I}(X)$ for the injective envelope of an operator space X, i.e.:

1. for $Y \subseteq Z$, every cc map $\phi : Y \to \mathscr{I}(X)$ extends to a cc map $\phi' : Z \to \mathscr{I}(X)$;
2. $\iota : X \hookrightarrow \mathscr{I}(X)$;
3. if $\iota(X) \subseteq Y \subseteq \mathscr{I}(X)$ and Y is injective then $Y = \mathscr{I}(X)$.

Theorem (Hamana)

The injective envelope of a unital operator space exists.

Theorem

If X is not unital then consider its Paulsen system

$$\mathscr{S}(X) := \left\{ \begin{bmatrix} \lambda & x_1 \\ x_2 & \mu \end{bmatrix} \mid x_1, x_2 \in X, \lambda, \mu \in \mathbb{C} \right\}$$

and the scheme

$$X \hookrightarrow \mathscr{S}(X) \hookrightarrow \mathscr{I}(\mathscr{S}(X)) = \begin{bmatrix} \mathscr{I}_{11}(X) & \mathscr{I}_{12}(X) \\ \mathscr{I}_{21}(X) & \mathscr{I}_{22}(X) \end{bmatrix}.$$

Then $\mathscr{I}(X) = \mathscr{I}_{12}(X)$.

Operator spaces

Definition

We write $\mathscr{A}_l(X) := \{s \in \mathscr{I}_{11}(X) \mid sX \subseteq X \text{ and } s^*X \subseteq X\}$, for the C*-algebra of left multipliers, and likewise for $\mathscr{A}_r(X)$.

Proposition

If ${}_AX_B$ is an operator space non-degenerate bimodule then there is a bimodule representation $(\pi,\phi,\sigma)\colon (A,X,B) \to (\mathscr{A}_l(X), \iota(X), \mathscr{A}_r(X))$ such that ϕ is a complete isometry.

Remark

If $X = [M_2 Y M_1^*]$ and $Y = [M_2^* X M_1]$ then the bimodules are non-degenerate. By the (Main Tool 1), choosing $(\pi,\phi,\sigma)\colon ([M_2M_2^*], X, [M_1M_1^*]) \to (\mathscr{A}_l(X), \iota(X), \mathscr{A}_r(X))$ induces (ρ,ψ,τ) for $([M_2^*M_2], Y, [M_1^*M_1])$ such that $\iota(X)$ and $\psi(Y)$ are TRO equivalent.

Theorem

Strong Δ-equivalence is an equivalence relation.

Proof.

For transitivity, suppose that X and Y are strong Δ-equivalent and that Y and Z are strong Δ-equivalent. Then we can choose cis maps ϕ for X and θ for Z such that $\phi(X)$ and $\iota(Y)$ are TRO-equivalent, and $\iota(Y)$ and $\theta(Z)$ are TRO equivalent. Then transitivity of TRO equivalence imply that $\phi(X)$ and $\theta(Z)$ are TRO equivalent, and so X and Z are strong Δ-equivalent. □

Operator spaces

Definition

For a complete isometric map $i\colon X \to Y$ into a TRO Y, let $\mathscr{T}(i(X))$ be the TRO spanned by

$$i(x_1)i(x_2)^* i(x_3)i(x_4)^* \cdots i(x_{2n})^* i(x_{2n+1}) \text{ for } n \geq 0 \text{ and } x_1, \ldots, x_{2n+1} \in X,$$

and their limits. We say that $(\mathscr{T}(i(X)), i)$ is *a TRO extension of* X. The TRO extension of X generated in $\mathscr{I}(\mathscr{S}(X))$ will be denoted by $\mathscr{T}_{\text{env}}(X)$.

Definition

A triple morphism of a TRO satisfies $\theta(x_1 x_2^* x_3) = \theta(x_1)\theta(x_2)^*\theta(x_3)$ (and it is automatically cc). Moreover it induces $*$-homomorphisms $\pi(x_1 x_2^*) = \theta(x_1)\theta(x_2)^*$ and $\rho(x_2^* x_3) = \theta(x_2)^*\theta(x_3)$.

Theorem (Hamana)

Given any TRO extension (Z, j) of X there exists a necessarily unique and surjective triple morphism $\theta\colon Z \to \mathscr{T}_{\text{env}}(X)$ such that $\theta(j(x)) = x$. The TRO space $\mathscr{T}_{\text{env}}(X)$ is called the TRO envelope of X.
If X is unital then the TRO envelope is a C-algebra, called the C*-envelope of X.*

Operator spaces

Theorem

Strong Δ-equivalent (unital) operator spaces have strong Δ-equivalent TRO envelopes (C-envelopes).*

Proof

Wlog assume that $X \subseteq \mathscr{I}(\mathscr{S}(X))$ and $Y \subseteq \mathscr{B}(H_1, H_2)$ such that

$$X = [M_1 Y M_2^*]^{-\|\cdot\|} \quad \text{and} \quad Y = [M_2 X M_1^*]^{-\|\cdot\|}.$$

Then $\mathscr{T}(X) = \mathscr{T}_{\text{env}}(X)$ and we have to show that $\mathscr{T}(Y) \simeq \mathscr{T}_{\text{env}}(Y)$. Suffices to find a TRO morphism $\mathscr{T}_{\text{env}}(Y) \to \mathscr{T}(Y)$ fixing Y.
Note here that $\iota\colon Y \to \mathscr{T}_{\text{env}}(Y) \subseteq \mathscr{I}(\mathscr{S}(Y))$ induces cis maps ϕ, θ_1, θ_2 such that

$$\phi(X) = [\theta_1(M_1)\iota(Y)\theta_2(M_2)^*]^{-\|\cdot\|} \quad \text{and} \quad \iota(Y) = [\theta_2(M_2)\phi(X)\theta_1(M_1)^*]^{-\|\cdot\|}.$$

Let $\widetilde{\psi}\colon \mathscr{T}(\phi(X)) \to \mathscr{T}_{\text{env}}(X)$ be the induced TRO morphism.

Operator spaces

Proof cont'd.

By the (Main Tool 2) we have the scheme

$$\begin{array}{ccccc} \iota(Y) & & \dashrightarrow & & Y \\ & \searrow & & \nearrow & \\ & M_{m,n}(\phi(X)) & \longrightarrow & M_{m,n}(X) & \end{array}$$

which lifts to

$$\begin{array}{ccccc} \mathscr{T}_{\text{env}}(Y) & & \dashrightarrow & & \mathscr{T}(Y) \\ & \searrow & & \nearrow & \\ & M_{m,n}(\mathscr{T}(\phi(X))) & \xrightarrow{\widetilde{\psi}^{(m,n)}} & M_{m,n}(\mathscr{T}_{\text{env}}(X)) & \end{array}$$

and the horizontal arrow is a TRO morphism in the limit. □

Operator systems

Definition

- A (concrete) operator system $\mathscr{S}$ is a (closed) selfadjoint subspace of some $\mathscr{B}(H)$ that contains the unit I_H.

- The morphisms in this category are the unital completely positive maps (they are automatically completely contractive). The isomorphisms are the complete order embeddings (unital completely positive maps with an inverse that is unital completely positive).

Choi-Effros Theorem (1977)

Let $\mathscr{S}$ be a $*$-vector space such that:

1. for each n we are given a cone $\mathscr{C}_n$ in $M_n(\mathscr{S})_h$,
2. $\mathscr{C}_n \cap (-\mathscr{C}_n) = (0)$ for every n,
3. $M_{m,n} \cdot \mathscr{C}_n \cdot M_{n,m} \subseteq \mathscr{C}_m$,
4. there exists an $e \in \mathscr{S}_h$ such that: for every $x \in \mathscr{S}_h$ there exists an $r \geq 0$ so that $re + x \in \mathscr{C}_1$ (order unit); $re + x \in \mathscr{C}_1$ for all $r > 0$ implies that $x \in \mathscr{C}_1$ (Archimedean order unit); and $e \otimes 1_n$ is an Archimedean order unit at every level (Archimedean matrix order unit).

Then there exists a complete order embedding $\phi \colon \mathscr{S} \to \mathscr{S}'$ for a concrete operator system $\mathscr{S}' \subseteq \mathscr{B}(H)$ such that $\phi(e) = I_H$ and $\phi(\mathscr{S})$ is dense in $\mathscr{S}'$.

Operator systems

Why operator systems? Perhaps, for encoding missing information.

- (Stinespring Theorem) A map $\phi\colon \mathscr{A} \to \mathscr{B}(H)$ of a C*-algebra $\mathscr{A}$ is ucp iff $\phi = P_H \pi(\cdot)|_H$ for a $*$-homomorphism $\pi\colon \mathscr{A} \to \mathscr{B}(K)$.

- Completely positive maps are the analogue of positive measures of commutative C*-algebras.

- Operator systems arise through the range of unital completely positive maps of C*-algebras.

- Essential for injective envelopes, e.g., $X \subseteq \mathscr{B}(H,K)$ then the Paulsen system is

$$\mathscr{S}(X) := \begin{bmatrix} \mathbb{C} & X \\ X^* & \mathbb{C} \end{bmatrix}.$$

- They find applications in Quantum Information Theory through the positive operator valued measures.

- They appear in positive definite completion problems, i.e., for determining whether the unspecified positions of a partial (or incomplete) matrix can be completed in a desired subclass of positive definite matrices.

- Given an undirected graph $\mathscr{G}$ we define $\mathscr{S}_{\mathscr{G}} := \{E_{ij} \mid i \simeq j\}$ which is an operator system inside M_n (for the n vertices of $\mathscr{G}$).

- Operator systems related to tolerance relations, which are reflexive and symmetric but not transitive (Connes-van Suijlekom 2020).

- Fejer-Riesz operator systems $\mathscr{F}_n := \{f \in \mathrm{C}(\mathbb{T}) : \operatorname{supp}\widehat{f} \subseteq [-n,n]\}$ (Connes-van Suijlekom 2021).

Morita equivalence for operator systems

Definition

Two concrete operator systems $\mathscr{S} \subseteq \mathscr{B}(H)$ and $\mathscr{T} \subseteq \mathscr{B}(K)$ are called *TRO-equivalent* (denoted $\mathscr{S} \sim_{\text{TRO}} \mathscr{T}$), if there exists a non-degenerate TRO $M \subseteq \mathscr{B}(H,K)$ such that

$$\mathscr{S} = [M^* \mathscr{T} M] \text{ and } \mathscr{T} = [M \mathscr{S} M^*].$$

Remark

Note that in this setup we have $1_{\mathscr{S}} \in [M^*M]$ and $1_{\mathscr{T}} = [MM^*]$ and so

$$MM^* \mathscr{T} \cup \mathscr{T} MM^* \subseteq \mathscr{T} \text{ and } M^*M\mathscr{S} \cup \mathscr{S} M^*M \subseteq \mathscr{S}.$$

Definition

Two concrete operator systems $\mathscr{S} \subseteq \mathscr{B}(H)$ and $\mathscr{T} \subseteq \mathscr{B}(K)$ are called *concretely bihomomorphically equivalent* if there exists an operator space $X \subseteq \mathscr{B}(H,K)$ such that X and X^* are non-degenerate (i.e. $I_H \in [X^*X]$ and $I_K \in [XX^*]$), and

$$\mathscr{S} = [X^* \mathscr{T} X] \text{ and } \mathscr{T} = [X \mathscr{S} X^*].$$

Proposition

If $\mathscr{S}$ and $\mathscr{T}$ are bihomomorphically equivalent by X, then they are TRO-equivalent by $M := [X\mathrm{C}^*(X^*X)]$.

Morita equivalence for operator systems

Definition

- Two operator systems $\mathscr{S}$ and $\mathscr{T}$ are called Δ-*equivalent* (denoted $\mathscr{S} \sim_\Delta \mathscr{T}$) if there exist Hilbert spaces H and K and unital complete order embeddings $\phi\colon \mathscr{S} \to \mathscr{B}(H)$ and $\psi\colon \mathscr{T} \to \mathscr{B}(K)$ such that $\phi(\mathscr{S}) \sim_{\text{TRO}} \psi(\mathscr{T})$.

- Two abstract operator systems $\mathscr{S}$ and $\mathscr{T}$ will be called *bihomomorphically equivalent* if there exist Hilbert spaces H and K, and unital complete order embeddings $\phi\colon \mathscr{S} \to \mathscr{B}(H)$ and $\psi\colon \mathscr{T} \to \mathscr{B}(K)$ such that the concrete operator systems $\phi(\mathscr{S})$ and $\psi(\mathscr{T})$ are concretely bihomomorphically equivalent. We write $\mathscr{S} \leftrightarrows \mathscr{T}$ to denote that $\mathscr{S}$ and $\mathscr{T}$ are bihomomorphically equivalent.

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. The following are equivalent:

1. $\mathscr{T} \sim_\Delta \mathscr{S}$ *as operator systems;*
2. $\mathscr{T} \sim_\Delta \mathscr{S}$ *as operator spaces;*
3. $\mathscr{S} \otimes \mathbb{K} \simeq \mathscr{T} \otimes \mathbb{K}$ *via a completely isometric isomorphism;*
4. $\mathscr{S} \otimes \mathbb{K} \simeq \mathscr{T} \otimes \mathbb{K}$ *via a completely positive completely isometric isomorphism;*
5. $\mathscr{S} \otimes \mathbb{K} \simeq \mathscr{T} \otimes \mathbb{K}$ *via a hermitian completely isometric isomorphism.*

Morita equivalence for operator systems

Definition

Let $\mathscr{I}(\mathscr{S})$ be the injective envelope of $\mathscr{S}$ and write

$$\mathscr{A}_{\mathscr{S}} := \{a \in \mathscr{I}(\mathscr{S}) \mid a\mathscr{S} \subseteq \mathscr{S} \text{ and } a^*\mathscr{S} \subseteq \mathscr{S}\}.$$

Letting $\iota_{\mathrm{env}} \colon \mathscr{S} \to \mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) \subseteq \mathscr{I}(\mathscr{S})$ be the canonical embedding, we note that the unitality condition yields that $\mathscr{A}_{\mathscr{S}} \subseteq \iota_{\mathrm{env}}(\mathscr{S})$; we can thus consider $\mathscr{A}_{\mathscr{S}}$ as being contained in $\mathscr{S}$.

Proposition

Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. If $\mathscr{S} \sim_\Delta \mathscr{T}$ then there exist complete order embeddings $\phi \colon \mathscr{S} \to \mathscr{B}(H)$ and $\psi \colon \mathscr{T} \to \mathscr{B}(K)$, and a non-degenerate TRO $M \subseteq \mathscr{B}(H,K)$, such that $\phi(\mathscr{S}) \sim_{\mathrm{TRO}} \psi(\mathscr{T})$ via M, and in addition

$$\phi(\mathscr{A}_{\mathscr{S}}) = [M^*M] \quad \textit{and} \quad \psi(\mathscr{A}_{\mathscr{T}}) = [MM^*].$$

Furthermore,

$$\mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) \simeq \mathrm{C}^*(\phi(\mathscr{S})) \quad \textit{and} \quad \mathrm{C}^*_{\mathrm{env}}(\mathscr{T}) \simeq \mathrm{C}^*(\psi(\mathscr{T})),$$

and consequently $\mathrm{C}^*_{\mathrm{env}}(\mathscr{S}) \sim_\Delta \mathrm{C}^*_{\mathrm{env}}(\mathscr{T})$.

Morita contexts for operator systems

Definition

Let $\mathscr{S}$ and $\mathscr{T}$ be abstract operator systems and M be a TRO. We say that the quintuple $(\mathscr{S}, \mathscr{T}, M, [\cdot,\cdot,\cdot], (\cdot,\cdot,\cdot))$ is a Δ-*pre-context* if:

(i) the C*-algebras $[M^*M]$ and $[MM^*]$ are unital;

(ii) $\mathscr{S}$ is a C*-bimodule over $[M^*M]$ and $\mathscr{T}$ is a C*-bimodule over $[MM^*]$;

(iii) $[\cdot,\cdot,\cdot]\colon M^* \times \mathscr{T} \times M \longrightarrow \mathscr{S}$ and $(\cdot,\cdot,\cdot)\colon M \times \mathscr{S} \times M^* \longrightarrow \mathscr{T}$ are completely bounded completely positive maps, modular over $[M^*M]$ and $[MM^*]$ on the outer variables (with unital module actions), and

(iv) the associativity relations

$$(m_1, [m_2^*, t, m_3], m_4^*) = (m_1 m_2^*) \cdot t \cdot (m_3 m_4^*)$$

and

$$[m_1^*, (m_2, s, m_3^*), m_4] = (m_1^* m_2) \cdot s \cdot (m_3^* m_4)$$

hold for all $s \in \mathscr{S}$, $t \in \mathscr{T}$ and all $m_1, m_2, m_3, m_4 \in M$.

A Δ-pre-context is called a Δ-*context* if the trilinear maps $[\cdot,\cdot,\cdot]$ and $(\cdot,\cdot,\cdot)$ are completely contractive and the relations

$$(m_1, 1_{\mathscr{S}}, m_2^*) = (m_1 m_2^*) \cdot 1_{\mathscr{T}} \quad \text{and} \quad [m_1^*, 1_{\mathscr{S}}, m_2] = (m_1^* m_2) \cdot 1_{\mathscr{S}} \tag{1}$$

hold for all $m_1, m_2 \in M$.

Morita contexts for operator systems

Remark

Let $\big(\mathscr{S},\mathscr{T},M,[\cdot,\cdot,\cdot],(\cdot,\cdot,\cdot)\big)$ *be a* Δ*-context. Then the following hold for all* $s\in\mathscr{S}$, $t\in\mathscr{T}$ *and all* $m_i,n_i\in M$, $i=1,2,3$*:*

1. $[m_1^*,(m_2,[m_3^*,t,n_3],n_2^*),n_1]=[m_1^*m_2m_3^*,t,n_3n_2^*n_1]$;
2. $(m_1,[m_2^*,(m_3,s,n_3^*),n_2],n_1^*)=(m_1m_2^*m_3,s,n_3^*n_2n_1^*)$;
3. $(m_1,1_{\mathscr{S}},m_2^*)=1_{\mathscr{T}}\cdot m_1m_2^*$;
4. $[m_1^*,1_{\mathscr{T}},m_2]=1_{\mathscr{S}}\cdot m_1^*m_2$;
5. $[m_1^*,(m_2,1_{\mathscr{S}},n_2^*),n_1]=[m_1^*,1_{\mathscr{T}},m_2n_2^*n_1]=[m_1^*m_2n_2^*,1_{\mathscr{T}},n_1]$;
6. $(m_1,[m_2^*,1_{\mathscr{T}},n_2],n_1^*)=(m_1,1_{\mathscr{S}},m_2^*n_2n_1^*)=(m_1m_2^*n_2,1_{\mathscr{S}},n_1^*)$.

Morita contexts for operator systems

Definition

Let $\mathscr{S}$ and $\mathscr{T}$ be abstract operator systems and X be an abstract operator space. We say that the quintuple $\big(\mathscr{S}, \mathscr{T}, X, [\cdot,\cdot,\cdot], (\cdot,\cdot,\cdot)\big)$ is a *bihomomorphism pre-context* if:

(i) X is non-degenerate;

(ii) $[\cdot,\cdot,\cdot]\colon X^* \times \mathscr{T} \times X \longrightarrow \mathscr{S}$ and $(\cdot,\cdot,\cdot)\colon X \times \mathscr{S} \times X^* \longrightarrow \mathscr{T}$ are completely bounded completely positive maps such that

$$[X^*, 1_{\mathscr{T}}, X] \subseteq \mathscr{A}_{\mathscr{S}} \quad \text{and} \quad (X, 1_{\mathscr{S}}, X^*) \subseteq \mathscr{A}_{\mathscr{T}};$$

(iii) the associativity relations

$$[x_1^*, (x_2, s, x_3^*), x_4] = [x_1^*, 1_{\mathscr{T}}, x_2] \cdot s \cdot [x_3^*, 1_{\mathscr{T}}, x_4]$$

and

$$(x_1, [x_2^*, t, x_3], x_4^*) = (x_1, 1_{\mathscr{S}}, x_2^*) \cdot t \cdot (x_3, 1_{\mathscr{S}}, x_4^*)$$

hold for all $s \in \mathscr{S}, t \in \mathscr{T}$ and all $x_1, x_2, x_3, x_4 \in X$.

A bihomomorphism pre-context is called a *bihomomorphism context* if the trilinear maps $[\cdot,\cdot,\cdot]$ and $(\cdot,\cdot,\cdot)$ are completely contractive and there exist semi-units $((\underline{x}_i)_i, (\underline{y}_i)_i)$ and $((\underline{z}_i)_i, (\underline{w}_i)_i)$ over X and X^*, respectively, such that

$$\lim_i [\underline{x}_i^*, 1_{\mathscr{T}} \otimes I, \underline{y}_i] = 1_{\mathscr{S}} \quad \text{and} \quad \lim_i (\underline{z}_i, 1_{\mathscr{S}} \otimes I, \underline{w}_i^*) = 1_{\mathscr{T}}. \tag{2}$$

Morita contexts for operator systems

Theorem

Let $\mathscr{S}$ and $\mathscr{T}$ be (abstract) operator systems. The following are equivalent:

1. $\mathscr{S} \sim_\Delta \mathscr{T}$;
2. $\mathscr{S} \leftrightarrows \mathscr{T}$;
3. *there exists a Δ-context for $\mathscr{S}$ and $\mathscr{T}$;*
4. *there exists a bihomomorphism context for $\mathscr{S}$ and $\mathscr{T}$.*

Proof

We have seen that (2) implies (1).

A realization is a context, and so (1) implies (3) and (2) implies (4).

A familiar "trick" by using $M \otimes -$ gives that (3) implies (1).

An extension of this trick is required for showing that (4) implies (2).

Morita contexts for operator systems

Proof continued

- Suppose we have a bihomomorphism context and start with $\phi\colon \mathscr{S} \to \mathscr{I}(\mathscr{S}) \subseteq \mathscr{B}(H)$.
- Let K be the Hausdorff completion of $X \odot H$ wrt

$$\langle x_1 \otimes h_1, x_2 \otimes h_2 \rangle := \langle \phi([x_2^*, 1_{\mathscr{T}}, x_1])h_1, h_2 \rangle_H.$$

- Let $\theta\colon X \to \mathscr{B}(H,K)$ be given by $\theta(x)h = x \otimes h$, and note that

$$\theta(x)^*(x' \otimes h') = \phi([x^*, 1_{\mathscr{T}}, x'])h'$$

- Let $\psi\colon \mathscr{T} \to \mathscr{B}(K)$ such that

$$\langle \psi(t)(x_1 \otimes h_1), x_2 \otimes h_2 \rangle_K = \langle \phi([x_2^*, t, x_1])h_1, h_2 \rangle_H.$$

The existence of semi-units and the fact that ϕ is completely isometric give that ψ is completely isometric.

- Then it follows that $\phi(\mathscr{S})$ and $\psi(\mathscr{T})$ are bihomomorphically equivalent by $\theta(X)$, basically by using that

$$\theta(x_2)^* \psi(t) \theta(x_1) = \phi([x_2^*, t, x_1])$$

and the dual (by repeating for ψ and showing unitary equivalence with ϕ).