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HK

Crossed and semicrossed products III

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Hilbert C^*-module: $(X, \mathfrak{A}, \langle \cdot | \cdot \rangle)$

(i) $\mathfrak{A}$: C^*-algebra, unital
(ii) $(X, \|\cdot\|)$ is a Banach space and a right $\mathfrak{A}$-modul
(iii) $\langle \cdot | \cdot \rangle$ is an $\mathfrak{A}$-valued inner product, i.e.,

$$\langle \xi | \eta a \rangle = \langle \xi | \eta \rangle a$$
$$\langle \xi | \eta \rangle^* = \langle \eta | \xi \rangle$$
$$\langle \xi | \zeta + \lambda \eta \rangle = \langle \xi | \zeta \rangle + \lambda \langle \xi | \eta \rangle, \quad \xi, \zeta, \eta \in X, \ a$$
$$\lambda \in \mathbb{C}$$

and
$$\|\xi\|^2 = \|\langle \xi | \xi \rangle\|, \quad \forall \xi \in X.$$

C^*-correspondence: $(X, \mathfrak{A}, \varphi)$

$(X, \mathfrak{A}, \langle \cdot | \cdot \rangle)$ Hilbert C^*-module

$\varphi : \mathfrak{A} \to \mathcal{L}(X)$ $*$-representation into the adjointable operators on X, unital

which allows us to view X as a left $\mathfrak{A}$-modul

<u>Example</u> (Concrete C^*-correspondences)

Consider $X \subseteq B(\mathcal{H})$ any closed $\mathcal{Q}$-bimo
where $\mathcal{Q} \subseteq B(\mathcal{H})$ is any $\underset{\text{unital}}{\wedge}$ C^*-algebra, satisfying

$$X^* X \subseteq \mathcal{Q}$$

(Here $\langle \xi, \eta \rangle := \xi^* \eta$ and

$\varphi : \mathcal{Q} \to \mathcal{L}(X) : a \to M_a =$ left multiplication by a

<u>Example</u> The C^*-correspondence $\mathcal{Q}_\alpha$.

Let $\mathcal{Q}$ be a $\overset{\text{unital}}{\vee}$ C^*-algebra and $\alpha : \mathcal{Q} \to \mathcal{Q}$ $*$-
Consider

$$\mathcal{Q}_\alpha = \mathcal{Q}$$

as a C^*-correspondence over $\mathcal{Q}$ with

$$\varphi(a)\xi b := \alpha(a)\xi b, \quad \text{and}$$
$$\langle \xi | \eta \rangle := \xi^* \eta, \qquad a, b, \xi, \eta \in \mathcal{Q}.$$

A Toeplitz representation for $(X, \mathcal{Q}, \varphi)$ is a triple $(\pi,$
where

(i) $\pi : \mathcal{A} \longrightarrow B$ $*$-representation (non-de

(ii) $t : X \longrightarrow B$ linear map satisfying

$$t(a\cdot\xi\cdot a') = \pi(a)\,t(\xi)\,\pi(a')$$

and

(iii) $\pi(\langle \xi,\eta\rangle) = t(\xi)^* t(\eta)$, $\quad a,a'\in\mathcal{A}$, $\xi,$

The Toeplitz C^*-algebra $\mathcal{T}(X,\mathcal{A},\varphi)$ or simp is the universal C^*-algebra for all Toeplitz re of $(X,\mathcal{A},\varphi)$.

$$\begin{tikzcd}
(X,\mathcal{A}) \arrow[r, "{(\bar\pi,\bar t)}", "\text{injective}"'] \arrow[dr, "{(\pi,t)}"'] & \mathcal{T}(X) = C^*(\bar\pi,\bar t) \arrow[d, dashed, "\exists\ \pi\times t"] \\
 & B
\end{tikzcd}$$

The tensor algebra $\mathcal{T}^+(X)$ is the norm-close of $\mathcal{T}(X)$ generated by X and $\mathcal{A}$.

If $(X,\mathcal{A})$ is a concrete C^*-correspondence

$$\mathcal{T}(X) \cong C^*(\mathcal{A}\otimes I,\ X\otimes S) \subseteq B(\mathcal{H}\otimes\ell^2$$

where S is the forward shift on $\ell^2(\mathbb{N})$ (Kat

Let (π, t) be a Toeplitz repn. of
Then

$$\pi(a)\,t(1) = t(a\cdot 1) = t(1\cdot\alpha(a))$$

$$= t(1)\,\pi(\alpha(a))$$

Also

$$t(1)^*t(1) = \pi(\langle 1, 1\rangle) = \pi(1) = I$$

and so the pair $(\pi, t(1))$ forms an is covariant representation of $(\mathcal{Q}, \alpha)$.

The "right" C^*-algebra

Let $(X, \mathcal{Q})$ be a represented C^*-correspond and let

$$K(X) = C^*(X X^*) = \overline{\operatorname{span}}\{ST^* \mid S, T \in X\}$$

View both $\mathcal{Q}$ and $K(X)$ as multiplication (from left) operators on X. Let $I \subseteq \mathcal{Q}$ be the kernel of this association

$$\mathcal{Q} \ni a \longrightarrow M_a \in B(X)$$

A representation $(\pi, t, \mathcal{H})$ of $(X, \mathcal{Q})$ is said Katsura covariant iff whenever

$$M_a = M_K \;, \quad a \in I^{\perp}, \; K \in k(X) \qquad I$$

we have

$$\pi(a) = t_*(K) \quad (\quad t_*(ST^*) = t(S)t$$

JFA (2004)

Translated into the context of semicrossed products

* The correspondence $\mathcal{Q}_\alpha$ can be viewed as a repr C^*-correspondence

$$X = V\mathcal{Q} \quad \text{over the } C^*\text{-algebra } \mathcal{Q}$$

where V is the universal isometry with $aV = V\alpha(a)$,

* If $M_a = 0$, $a \in \mathcal{Q}$, as a multiplication operator on

$$0 = aVb = V\alpha(a)b \quad \Rightarrow \quad \alpha(a) = 0$$

i.e.

$$I = \ker\alpha \quad \subseteq \mathcal{Q}$$

* $K(\mathcal{Q}_\alpha) = V\mathcal{Q}V^*$ $\quad (\; V a \, b^* V^*$

* Assume (π, t) is Katsura covariant

If $M_a = M_u = M_{Va'V^*}$, for some $a' \in \mathcal{Q}$

$$aVb = Va'V^*Vb$$

$\nabla_{\alpha(a)} b = \nabla_{a'} b$

$\alpha(a) = a'$

and so a and $\nabla_{\alpha(a)} \nabla^*$ when represented should be equal, provided $a \in (\ker \alpha)^\perp$

(π, V) Katsura covariant $\Longleftrightarrow$ $V_{\alpha(a)} V^* = a$, $\forall$

In general

$\mathcal{O}_X \equiv$ universal C^*-alg. for Katsura covaria[nt] representations.

THM (Katsoulis and Kribs, 2004)

$$\mathcal{O}_X \cong C^*_{env}(\mathcal{T}_X^+)$$

(For more info and the latest on this line of rese[arch] see the talks at Hilbert C^*-modules Online Weeken[d] at http://mech.math.msu.su/~manuilov/HCM.html)

The isomorphism problem for tensor algebras

$\mathcal{L}^+(X,\mathcal{Q}) \cong \mathcal{L}^+(Y,\mathcal{B})$ if and only if $(X,\mathcal{Q})$ a

$(Y,\mathcal{B})$ are unitarilly equivalent, i.e., there exist

$$\rho : \mathcal{Q} \to \mathcal{B} \quad *\text{-isomorphism}$$

and ρ-unitary $U : X \to Y$ with ρ^{-1}-adjoi

$U^* : Y \to X$ so that $UU^* = I_Y$ and $U^*U = I_X$

$$U(a \xi a') = \rho(a)\, U(\xi)\, \rho(a') \qquad \forall\, a, a' \in \mathcal{Q},\ \xi \in$$
$$U^*(b \eta b') = \rho^{-1}(b)\, U^*(\eta)\, \rho^{-1}(b'), \quad \forall\, b, b' \in \mathcal{B},\ \eta$$

and

$$\langle U\xi, \eta \rangle = \rho(\langle \xi, U^*\eta \rangle), \quad \xi \in X,\ \eta \in$$

PROPOSITION If $(X,\mathcal{Q})$ and $(Y,\mathcal{B})$ are unitarill equivalent then $\mathcal{L}^+(X)$ and $\mathcal{L}^+(Y)$ are isomorphi (and also $\mathcal{L}(X) \cong \mathcal{L}(Y)$).

Proof. Let (U,ρ) be the unitary pair implementin the equivalence between $(X,\mathcal{Q})$ and $(Y,\mathcal{B})$. If $\xi_1, \xi_2, \dots, \xi_n \in X$ and $a_1, a_2, \dots, a_m \in \mathcal{Q}$ then for any representation (π, t) of $(Y,\mathcal{B})$ we have

$$(\pi \times t)\big(p(U\xi_1, \dots, U\xi_n, \rho(a_1), \dots, \rho(a_m))\big)$$

$$= (\pi \circ \rho) \times (t \circ U)\big(p(\xi_1, \dots, \xi_n, a_1, \dots, a_m)\big)$$

and so

$$\| p(U\xi_1, \dots, U\xi_n, \rho(a_1), \dots, \rho(a_m))\| \leq$$

$$\leq \| p(\xi_1, \dots, \xi_n, a_1, \dots, a_m)\|$$

Hence the association

$$a \longrightarrow \rho(a)$$
$$\xi \longrightarrow U\xi$$

extends to a well-defined $*$-homomorphism $\rho \times U$ with inverse $\rho^{-1} \times U^*$ ∎

Towards the converse, we have the following

PROPOSITION. If $\varphi : Z^+(X) \longrightarrow Z^+(Y)$ is an isometric isomorphism then $\varphi|_{\mathcal{Q}} : \mathcal{Q} \longrightarrow \mathcal{B}$ is a $*$-isomorphism.

Proof. Note that $Z^+(X) \cap Z^+(X)^* = \mathcal{Q}$ (Fourier series)

If $w \in Z^+(X) \cap Z^+(\mathcal{Q})^*$ is a unitary, then $w^{-1} \in$ and $\|w\| = \|w^{-1}\| = 1$.

Then, $\varphi(w)^{-1} \in Z^+(Y)$ and $\|\varphi(w)\| = \|\varphi(w)^{-1}\| =$

But then $\varphi(w)$ has to be a unitary, i.e.,

$$\varphi(w) \in Z^+(Y)^* \cap Z^+(Y)^+ = \mathcal{B}$$

Also

$$\varphi(w^*) = \varphi(w^{-1}) = \varphi(w)^{-1} = \varphi(w)^*$$

, i.e., φ is $*$-isomorphism. ∎

We will examine the Isomorphism Problem for the C^*-correspondences $\mathcal{A}_\alpha$, for unital $\mathcal{A}$ and α.

PROPOSITION The C^*-correspondences $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$ are unitarilly equivalent iff the dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjug
i.e., there exists a $*$-isomorphism

$$\rho : \mathcal{A} \longrightarrow \mathcal{B}$$

and a unitary $u \in \mathcal{B}$ so that

$$\beta(b) = u\left(\rho \circ \alpha \circ \rho^{-1}(b)\right)u^*$$

<u>Proof</u> Assume that $\mathcal{A}_\alpha \cong \mathcal{B}_\beta$ via a unitary $(\rho, \mathcal{U})$ and let $u = \mathcal{U}1 \in \mathcal{B}$

Then

$$u^*u = \mathcal{U}1^*\mathcal{U}1 = \langle \mathcal{U}1, \mathcal{U}1 \rangle = \langle 1, 1 \rangle = 1$$

Also $uu^* = 1$. Indeed, first notice that

$$\left.\begin{aligned} 1 = \mathcal{U}(\mathcal{U}^*(1)) &= \mathcal{U}(1 \cdot \mathcal{U}^*(1)) \\ &= u(1)\rho(\mathcal{U}^*(1)) \end{aligned}\right\} \Rightarrow \begin{aligned} &\rho(\mathcal{U}^*(1)) = \mathcal{U}(1 \\ \text{or } &\rho^{-1}(u^*) = \mathcal{U}^*(1) \end{aligned}$$

and so

$$1 = \mathcal{U}(\mathcal{U}^*(1)) = \mathcal{U}(1 \cdot \rho^{-1}(u^*)) = \mathcal{U}(1)\rho(\rho^{-1}(u^*)) = uu^*$$

Finally :

$$\begin{aligned} u\rho(\alpha(a)) &= \mathcal{U}(1)\rho(\alpha(a)) = \mathcal{U}(\alpha(a)) \\ &= \mathcal{U}(a \cdot 1) \\ &= \rho(a) \cdot \mathcal{U}(1) \\ &= \beta(\rho(a))\mathcal{U}(1) \\ &= (\beta \circ \rho)(a)u, \quad a \in \mathcal{A} \end{aligned}$$

Replace a with $\rho^{-1}(b)$, $b \in \mathcal{B}$, and we get

$$u(\rho \circ \alpha \circ \rho^{-1}(b)) = \beta(b)u$$

or

$$u(\rho \circ \alpha \circ \rho^{-1}(b))u^* = \beta(b), \quad b \in \mathcal{B}$$

For the converse, set $\mathcal{U}(a) (= \mathcal{U}(1 \cdot a)) = u \cdot \rho(a)$ and note that $\mathcal{U}^*(b) = \rho^{-1}(u^* \cdot b)$, $b \in \mathcal{B}$

This problem has a long history ...

When $\mathcal{A}$ is abelian it was introduced by Arveson (1 and further studied by Peters, Hadwin and Hoover, Power.

It was finally solved by Davidson and Katsoulis in 2008 (Crelle's 261). The isomorphisms considered there were algebraic.

Beyond $\mathcal{A}$ abelian, the isomorphisms considered a isometric

Muhly and Solel (PLMS, 2000) for α autom. wi full Connes spectrum.

Davidson and Kats. (Math. Ann., 2008) for α automorph $\mathcal{A}$ simple, sep.

Davidson and Kakariadis (IMRN, 2014) for injecti surjective (unital) endomorphism and $\mathcal{A}$ arbitrary C^*-algebra

Many other cases considered as well, eg, $\mathcal{A}$ finite and α arbitrary (unital) endomorphism.

<u>Conjecture</u> (Davidson and Kakariadis) Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be unital C^*-dynamical systems. If $\mathcal{Z}^+(\mathcal{A}, \alpha)$ and $\mathcal{Z}^+(\mathcal{B}, \beta)$ are isometrically isomorphic then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate

<u>THEOREM</u> (Katsoulis and Ramsey, 2020) Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be unital C^*-dynamical systems. If $\mathcal{Z}^+(\mathcal{A}, \alpha)$ and $\mathcal{Z}^+(\mathcal{B}, \beta)$ are isometrically isomorphic then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ are outer conjugate.

Lets take a closer look at $\mathcal{Z}^+(\mathcal{A}, \alpha)$ ...

Let (π, t) be a Toeplitz repn. of $\mathcal{A}_\alpha$

Then

$$\pi(a)t(1) = t(a \cdot 1) = t(1\,\alpha(a))$$

$$= t(1)\pi(\alpha(a))$$

Also

$$t(1)^* t(1) = \pi(\langle 1, 1\rangle) = \pi(1) = I$$

and so the pair $(\pi, t(1))$ forms an isometric covariant representation of $(\mathcal{A}, \alpha)$.

Hence $\mathcal{Z}^+(\mathcal{A}, \alpha)$ has a crossed product like structure by being the univeral operator algebra for isometric covariant representations of $(\mathcal{A}, \alpha)$

The algebra $\mathcal{Z}^+(\mathcal{A}, \alpha)$ is called the semi-crossed product of $(\mathcal{A}, \alpha)$ and is usually denoted as

$$\mathcal{A} \rtimes_\alpha \mathbb{Z}^+ \subseteq \mathcal{A} \rtimes_\alpha \mathbb{Z}$$
if α autom.

Generalizations of main theorem.

A multivariable C^*-dynamical system consists of a C^*-algebra $\mathcal{A}$ and (unital) $*$-en $\alpha_1, \alpha_2, \ldots, \alpha_n$.

A row-isometric covariant representation of $(\mathcal{A}$ consists of

$$\pi : \mathcal{A} \longrightarrow B \quad \text{non-degen. } *\text{-hom}$$

and a row isometry $(V_1, V_2, \ldots, V_n) \in B^{(1,n)}$ that

$$\pi(a) V_i = V_i \pi(\alpha_i(a)), \quad \forall a \in \mathcal{A}$$

If $(\mathcal{Q}, \vec{\alpha}=(\alpha_1, \alpha_2, \dots, \alpha_n))$ is a multivariable syst
then the tensor algebra $\mathcal{T}^+(\mathcal{Q}, \vec{\alpha})$ is the univ
operator algebra for all $\underset{\text{row}}{\text{isometric}}$ covariant repr
of $(\mathcal{Q}, \vec{\alpha})$.

<u>THEOREM</u> (Kats. and Ramsey, 2020) Two tensor algebras $\mathcal{T}^+(\mathcal{Q}, \vec{\alpha})$ and $\mathcal{T}^+(\mathcal{B}, \vec{\beta})$ are compl. isometrically isomorphic iff the dynamical systems $(\mathcal{Q}, \vec{\alpha})$ $(\mathcal{B}, \vec{\beta})$ are unitarilly equivalent after a conjugation, i.e., there exists a unitary $u \in M_{m,n}$ (and a $*$-isomorphism $\rho: \mathcal{Q} \to \mathcal{B}$ so that

$$\begin{pmatrix} \beta_1(b) & & & 0 \\ & \beta_2(b) & & \\ & & \ddots & \\ 0 & & & \beta_n(b) \end{pmatrix} = u \begin{pmatrix} \rho\circ\alpha_1\circ\rho^{-1}(b) & & 0 \\ & \ddots & \\ 0 & & \rho\circ\alpha_n\circ\rho^{-1}(b) \end{pmatrix} u^*, \ \forall$$

This extends earlier work of Kakariadis and (J. Noncommutative Geom., 2014) who did the autom case and the case where $\mathcal{Q}, \mathcal{B}$ are finite with the assumption of an isometric isomorphism. (In cases also $m=n$ as a consequence.)

Two multivariable dynamical systems $(X, \sigma_1, \sigma_2, \dots, \sigma_n)$, $(Y, \tau_1, \tau_2, \dots, \tau_n)$ on compact

Spaces X, Y are said to be piecewise conjugate if there exists an open cover

$$\{ U_a \mid a \in S_n \}$$

of X and homeomorphism

$$\gamma : X \to Y$$

so that

$$\gamma^{-1} \tau_i \gamma \mid U_a = \sigma_{a(i)} \mid U_a$$

for each $a \in S_n$.

THEOREM (Davidson and Katsoulis, 2011)
If the tensor algebras $\mathcal{T}^+(X, \sigma)$ and $\mathcal{T}^+(Y, \tau)$ are algebraically isomorphic then (X, σ) and (Y, τ) are piecewise conjugate.

The converse true if $n = 2, 3$ or $X \subseteq \mathbb{R}$

Hence

$$C(X)_\sigma \cong C(Y)_\tau \Rightarrow (X, \sigma), (Y, \tau) \text{ piecewise conj}$$

$$\overset{???}{\Leftarrow}$$

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