Operator systems

Plan

Lecture 1 : Classical period
Lecture 2 : Modern times
Lecture 3 : Tunes from the last decade

Lecture 1 : Some fundamental theorems

Motivation.

Hilbert space H $\longleftrightarrow$ quantum system

$\sigma \in \mathcal{T}(H)^+$, $\operatorname{tr}\sigma = 1$ $\longleftrightarrow$ quantum state

$\mathcal{T}(H)^* \cong \mathcal{B}(H)$

$A \in \mathcal{B}(H)_{sa}$ $\longleftrightarrow$ observable

Evolution of the state of a system:

$$\sigma \longrightarrow \Phi(\sigma)$$

Properties of Φ $\quad$ $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ linear.

$\sigma \geq 0 \Rightarrow \Phi(\sigma) \geq 0$, $\quad$ $\operatorname{tr}(\Phi(\sigma)) = \operatorname{tr}(\sigma)$

H, K $\qquad$ $H \otimes K$ $\qquad$ $\Phi \otimes \Psi : \mathcal{T}(H \otimes K) \to \mathcal{T}(H \otimes K)$

positive as well.

If $\dim K = n$ then

$\mathcal{B}(K) = \mathcal{T}(K) \cong M_n$

$\varphi = \mathrm{id}_n : M_n \to M_n$

Heisenberg picture: $\Phi^* : \mathcal{B}(H) \to \mathcal{B}(H)$
unital $\Phi^*(I) = I$.

Def $\mathcal{A}, \mathcal{B}$ C^*-algebras.
$\phi : \mathcal{A} \to \mathcal{B}$ completely positive iff

$T \in M_n(\mathcal{A})^+ \Rightarrow (\phi \otimes id_n)(T) \in M_n(\mathcal{B})^+$, $\forall n \in \mathbb{N}$.

$\{(a_{ij}) : a_{ij} \in \mathcal{A}\} \cong \mathcal{A} \otimes M_n$

$$T = (a_{ij})_{ij=1}^n \Rightarrow \underbrace{\phi \otimes id_n}_{\phi^{(n)}}(T) = (\phi(a_{ij}))_{ij=1}^n$$

Example 1 $*$-homomorphisms

Example 2 Conjugations

Ex. 1 $\pi : \mathcal{A} \to \mathcal{B}$ $*$-homom.
$\Rightarrow \pi$ c.p.

$T \in M_n(\mathcal{A})^+$

$T = A^* A$, $A \in M_n(\mathcal{A})$.

$\Rightarrow T = \Sigma$ matrices of the form
$$(a_i a_j^*)_{ij=1}^n$$

$a_1, a_2, \ldots, a_n \in \mathcal{A}$

$$\pi^{(n)}[(a_i a_j^*)] = \left(\pi(a_i)\pi(a_j)^*\right)_{i,j=1}^{n}$$

$$= \begin{bmatrix} \pi(a_1) \\ \pi(a_2) \\ \vdots \\ \pi(a_n) \end{bmatrix} \begin{bmatrix} \pi(a_1) \\ \vdots \\ \pi(a_n) \end{bmatrix}^* \geq 0.$$

<u>Ex. 2</u> $\phi(T) = A^* T A$, $A : K \to H$.

$\mathcal{A} \subseteq \mathcal{B}(H)$, $\mathcal{B} \subseteq \mathcal{B}(K)$.

$$\phi^{(n)}(\cdot) = A^{(n)*} \bullet A^{(n)}$$

$$= \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix}.$$

The Stinespring Theorem

$\phi : \mathcal{A} \to \mathcal{B}(H)$ is completely positive iff

$$\phi(a) = V^* \pi(a) V, \quad a \in \mathcal{A}.$$

$\pi : \mathcal{A} \to \mathcal{B}(K)$, $V : H \to K$ bdd.

$\quad$ $*$-homom.

Idea of proof

pre-inner product on $\mathcal{A} \otimes H$

$$\langle x \otimes \xi, y \otimes \eta \rangle := \langle \phi(y^* x) \xi, \eta \rangle .$$

$x, y \in \mathcal{A}$, $\xi, \eta \in H$

$\rightsquigarrow$ Hilbert space K

$$\pi(a)(x \otimes \xi) := ax \otimes \xi$$

$$V \xi := 1 \otimes \xi$$

Some consequences of the Stinespring thm

1) Normal c.p. maps on $\mathcal{B}(H)$
(channels in the Heisenberg picture)

2) POVM's, PVM's, and Naimark.

1) $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ c.p. weak*-cont.

$\phi_0 : \mathcal{K}(H) \to \mathcal{B}(H)$

$\mathcal{K}(H) \hookrightarrow$ compact op.

$$\phi_0(a) = V^* \pi(a) V$$

$\pi : \mathcal{K}(H) \to \mathcal{B}(K)$.

$$\pi(a) = a \otimes I \qquad a \xrightarrow{\pi} \begin{bmatrix} a & & & \\ & a & & 0 \\ & 0 & a & \\ & & & \ddots \end{bmatrix}.$$

$$K = H \otimes \ell^2$$

$$V : H \to H \otimes \ell^2 \rightsquigarrow V = (v_i)_{i \in \mathbb{N}}$$

$$\phi_0(a) = \sum_{i=1}^{\infty} v_i^* a v_i \ .$$

If $\dim H = n < \infty$, $\phi : M_n \to M_n$ c.p.

$$\Rightarrow \phi(a) = \sum_{i=1}^{k} v_i^* a v_i \ . \quad \text{Kraus rep.}$$

2) POVM : A_i $i = 1 \dots n$,

$A_i \geq 0$,

$\sum_{i=1}^{n} A_i = I$ $\quad A_i \in \mathcal{B}(H)$.

PVM : A_i : projections + POVM.

Assume $(E_i)_{i=1}^{n}$ PVM on H

$P : H \to K$, $K \subseteq H$.

$(PE_iP)_{i=1}^{n}$ POVM.

Naimark Dilation: these are all.

$\mathcal{D}_n$: all diagonal matrices in M_n.

$\phi : \mathcal{D}_n \to \mathcal{B}(H)$ u.c.p. $\Leftrightarrow$

POVM's : $A_i = \phi(e_i e_i^*)$.

$\pi : \mathcal{D}_n \to \mathcal{B}(K)$ $*$-homom:

PVM's: $E_i = \pi(e_i e_i^*)$.

Insufficiency of C^*-algebras

Example: Spectral truncation
(Connes – van Suijlekom, 2021)

For $n \in \mathbb{N}$,

$$S_n = \{ f \in C(\mathbb{T}) : \hat{f}(k) = 0 \text{ if } |k| \geq n \}.$$

Fejer-Riesz operator system

Def An operator system is a linear subspace $S \subseteq A$ (A a unital C^*-algebra) s.t.
$1 \in S$ and $x \in S \Rightarrow x^* \in S$.

Positive cones Assuming $S \subseteq B(H)$,

$$M_n(S)^+ = M_n(S) \cap B(H^n)^+.$$

Def $\phi : S \to T$ linear map between operator systems is

(i) positive if $x \in S^+ \Rightarrow \phi(x) \in T^+$

(ii) n-positive if $\phi^{(n)} : M_n(S) \to M_n(T)$ is positive,

$$\phi^{(n)}((x_{ij})) = (\phi(x_{ij}))$$

(iii) completely positive if n-positive $\forall n \in \mathbb{N}$.

Operator subsystems

1) Containment $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{B}(H)$

2) Complete order embedding $\mathcal{S} \subseteq_{coi} \mathcal{T}$.

$\mathcal{S}$ abstract object

$\phi: \mathcal{S} \to \mathcal{B}(H)$ representations

$\mathcal{S}, \mathcal{T}$ $\qquad \phi: \mathcal{S} \to \mathcal{T}$ ucp

+ injective, surjective

+ $\phi^{-1}: \mathcal{T} \to \mathcal{S}$: c.p.

$\mathcal{S}, \mathcal{T}$: c.o.i.

$\phi: \mathcal{S} \longrightarrow \mathcal{T}$ complete order embedding.
(not surjective).

$$\phi^{(n)}(M_n(\mathcal{S})) \cap M_n(\mathcal{T})^+$$
$$\parallel$$
$$\phi^{(n)}(M_n(\mathcal{S})^+).$$

$\mathcal{S} \subseteq \mathcal{B}(H)$ $\quad M_n(\mathcal{S})^+ \subseteq \mathcal{B}(H^n)$

Interaction between order and metric

Theorem $\phi : \mathcal{S} \to \mathcal{T}$ c.p.

Then $\|\phi\|_{cb} = \|\phi(1)\|$.

Recall $\phi^{(n)} : \underbrace{M_n(\mathcal{S})}_{\|\cdot\|_n} \to \underbrace{M_n(\mathcal{T})}$

$$\sup_{n \in \mathbb{N}} \|\phi^{(n)}\| =: \|\phi\|_{cb} < \infty$$

Idea $x \in \mathcal{S}$

$$\|x\| \leq 1 \iff \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \in M_2(\mathcal{S})^+.$$

$$\downarrow \phi^{(2)}$$

$$\|\phi(x)\| \leq 1. \iff \begin{bmatrix} 1 & \phi(x) \\ \phi(x)^* & 1 \end{bmatrix} \in M_2(\mathcal{T})^+$$

Q What can one say about c.p. maps defined on an operator system?

Special case:

$\phi : \mathcal{S} \to \mathbb{C}$ positive, $\mathcal{S} \subseteq \mathcal{A}$

$\Rightarrow \phi(x) = \langle \pi(x)\xi, \xi \rangle$, for some *-rep. $\pi : \mathcal{A} \to \mathcal{B}(K)$, $\xi \in K$.

Krein's Theorem: One can extend a positive functional:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "f"] \arrow[d, phantom, "\cap"] & \mathbb{C} \\ \mathcal{T} \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Arveson's Extension Theorem:

Let $\mathcal{S}, \mathcal{T}$ operator systems, $\mathcal{S} \subseteq \mathcal{T}$.
If $\phi : \mathcal{S} \to B(H)$ c.p. then $\exists\ \psi : \mathcal{T} \to B(H)$, s.t. $\psi|_{\mathcal{S}} = \phi$. cp.

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\phi"] \arrow[d, phantom, "\cap"] & B(H) \\ \mathcal{T} \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Sketch of proof.

Step 1 Claim true if $\dim H < \infty$.

Step 2 Claim true for any H.

Step 1 $\phi : S \to M_n$ $\qquad (n = \dim H)$.

$$\begin{tikzcd} \phi : S \to M_n & \phi_f \\ f_\phi : M_n(S) \to \mathbb{C} \arrow[u, leftrightarrow] & f \arrow[u] \end{tikzcd}$$

$\phi \to f_\phi$, $\quad f \to \phi_f$ mutual inverses

ϕ cp $\iff$ f_ϕ positive.

$$\begin{tikzcd} M_n(S) \arrow[r, "f_\phi"] \arrow[d, phantom, "\cap"] & \mathbb{C} \\ M_n(A) \arrow[ur, "g"'] & \end{tikzcd}$$

$g \rightsquigarrow \phi_g$

$\phi : S \to M_n$

$f_\phi : M_n(S) \to \mathbb{C}$

$$f_\phi\big((x_{ij})_{ij}\big) = \frac{1}{n}\sum_{i,j}\langle \phi(x_{ij})\, e_j, e_i\rangle .$$

$f : M_n(S) \to \mathbb{C}$

$\leadsto \quad \phi_f : S \to M_n.$

$$\phi_f(x) = n\sum_{i,j} f(x \otimes E_{ij})\, E_{ij}$$

<u>Step 2</u> fix H arbitrary.

$\mathcal{P} = \{P \text{ proj. on } H, \ \operatorname{rank} P < \infty\}.$

$\phi : S \to B(H).$

$\phi_P : S \to B(PH) \cong M_n.$

$$\phi_P(x) = P\,\phi(x)\,P$$

$\psi_P : T \to B(PH) \subseteq B(H) \qquad$ CP.

Let $\psi : T \to B(H)$ be a cluster point of the $(\psi_P)_{P \in \mathcal{P}}$.

Then ψ: extension of ϕ.

BW topology $\quad CP_c(\mathcal{T}, B(H))$

$\varphi_\alpha \to \varphi \quad$ if $\quad \varphi_\alpha(x) \xrightarrow{w^*} \varphi(x)$, $\forall x \in \mathcal{T}$.

Example (universality).

$$S_1 = \mathrm{span}\{1, \zeta, \bar{\zeta}\} \subseteq C(\mathbb{T})$$

$$\zeta(z) = z, \quad z \in \mathbb{T}.$$

Theorem The unital completely positive maps $\phi : S_1 \to \mathcal{B}(H)$ are entirely described by the contractions $T \in \mathcal{B}(H)$ via the assignment

$$\phi(\zeta) = T,$$

$$\phi(\bar{\zeta}) = T^*,$$

$$\phi(1) = I.$$

T contraction $\rightsquigarrow$

$$U = \begin{bmatrix} T & (I - TT^*)^{1/2} \\ (I - T^*T)^{1/2} & T^* \end{bmatrix}.$$

$P : H \oplus H \to H \qquad PUP = T$

$\phi : S_1 \to \mathcal{B}(H) \qquad \phi(\zeta) = T$ contr.

U unitary in $\mathcal{B}(H \oplus H)$.

$\pi : C(\mathbb{T}) \to \mathcal{B}(H \oplus H)$ $*$-rep.

$\rightsquigarrow \quad \phi(f) = P\pi(f)P$.

$$\begin{tikzcd}
\phi : \mathcal{S}_1 \arrow[r] & \mathcal{B}(H) \\
C(\mathbb{T}) \arrow[ur, "\psi"'] \arrow[u, phantom, "\cap" rotate=180] &
\end{tikzcd}$$

$\psi(f) = V^* \pi(f) V$. $\quad \pi(z)$ unitary.

Example ("Partial knowledge")

The positive completion problem:

Suppose G is a graph with vertex set $[n]=\{1,\dots,n\}$ and a partially defined matrix (λ_{ij}), for $i \simeq j$, when can we specify the remaining entries as to obtain a positive matrix?

$$\begin{bmatrix} * & * & ? & & * \\ * & * & & & \\ ? & & * & & \\ & & & \ddots & \\ * & & & & * \end{bmatrix}.$$

For a graph G with vertex set $[n]$, let

$$S_G = \operatorname{span}\{E_{ij} : i \sim j \text{ or } i = j\}.$$

(E_{ij}: matrix units in M_n)

$S \subseteq M_n$. $\quad S = S_G \Leftrightarrow D_n S D_n \subseteq S.$

Necessary condition: partial positivity

$\alpha \subseteq [n] \qquad \alpha \times \alpha \subseteq G$

T: partially defined matrix.

Schur product: $T * X = [t_{ij} x_{ij}]$

$(T = [t_{ij}],\ X = [x_{ij}])$

Consider the map $S_T : X \to T * X$

If T partially defined along the graph G, the map S_T is well-defined on $\mathcal{S}_G$:

$$S_T : \mathcal{S}_G \to \mathcal{S}_G$$

Theorem (Paulsen-Power-Smith). G fixed.

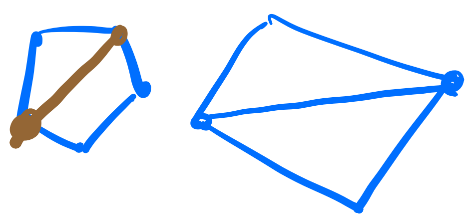
(i) T p.p. $\Leftrightarrow$ $S_T(\mathrm{Rank}_1^+) \subseteq \mathcal{S}_G^+$

(ii) T has a positive completion $\Leftrightarrow$ S_T c.p.

$\leadsto$ The positive completion problem can be always solved for a graph G

$\Leftrightarrow$ $\sum \mathrm{Rank}_1^+ = \mathcal{S}_G^+$ Paulsen-Power-Smith.

$\Leftrightarrow$ G chordal.



$v_1, v_2, v_3, \ldots v_n$

$v_k \sim v_i$, $v_k \sim v_j$ $\Rightarrow$ $v_i \sim v_j$ $\quad$ $i, j > k$.

perfect elimination

(iii) $X \to T * X$ positive map

$\Leftrightarrow$ T positive.

Q Which are the c.p. maps

$\phi : M_n \to \mathcal{B}(H)$?

Choi's Theorem: Precisely those ϕ

s.t. $\quad [\phi(E_{ij})]_{i,j} \in M_n(\mathcal{B}(H))^+$

Idea $\quad [E_{ij}]_{i,j} \in M_n(M_n)^+$

Conversely: $\quad [E_{ij}]_{i,j} \in M_n(\mathcal{B}(H))^+$ $\quad E_{ij} \in \mathcal{B}(H)$

$\phi(E_{ij}) = E_{ij} \quad \rightsquigarrow \phi : M_n \to \mathcal{B}(H).$

$[E_i^* E_j]_{i,j}$

Tensor products

$\mathcal{S} \otimes \mathcal{T}$ op. tensor product.

1) **Minimal:** $\mathcal{S} \subseteq \mathcal{B}(H)$, $\mathcal{T} \subseteq \mathcal{B}(K)$

$\rightsquigarrow \mathcal{S} \otimes_{min} \mathcal{T} \subseteq \mathcal{B}(H \otimes K)$

Injective: $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{T} \subseteq \mathcal{B}$

$\rightsquigarrow \mathcal{S} \otimes_{min} \mathcal{T} \subseteq_{c.o.i.} \mathcal{A} \otimes_{min} \mathcal{B}$

2) **Commuting:** $\mathcal{S} \otimes_c \mathcal{T}$.

$(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$

Define $u \in (\mathcal{S} \otimes_c \mathcal{T})^+$ if $(\phi \cdot \psi)(u) \in \mathcal{B}(H)^+$,

for all (ϕ, ψ).

$$\begin{tikzcd} \mathcal{S} \arrow[dr, "\phi"] & \\ & \mathcal{B}(H) \\ \mathcal{T} \arrow[ur, "\psi"'] & \end{tikzcd}$$

commuting ranges

The universal C^*-algebra of an op. system

$\mathcal{S}$ an op. sys. Representing

$\phi : \mathcal{S} \longrightarrow \mathcal{B}(H)$, we obtain

a C^*-cover of $\mathcal{S}$, $C^*(\phi(\mathcal{S}))$.

Thus: $\mathcal{S} \cong \phi(\mathcal{S}) \subseteq C^*(\phi(\mathcal{S}))$.

Kirchberg - Wasserman:

$\exists\, C^*_u(\mathcal{S})$ and $\iota : \mathcal{S} \longrightarrow C^*_u(\mathcal{S})$, C^*-cover,

such that

$$\begin{tikzcd} S \arrow[r, "\text{ucp } \phi"] \arrow[d, "\iota"'] & B(H) \\ C^*_u(S) \arrow[ur, dashed, "*\text{-hom. } \pi"'] & \end{tikzcd}$$

$\pi \circ \iota = \phi$

"Concrete" representation of $S \otimes_c T$:

$$S \otimes_c T \subseteq_{coi} C^*_u(S) \otimes_{max} C^*_u(T).$$

Sketch of proof:

Consider $u \in (S \otimes T) \cap \left(C^*_u(S) \otimes_{max} C^*_u(T) \right)^+$

Need to show: $u \in (S \otimes_c T)^+$.

$\phi \cdot \psi(u) \geq 0$?

$\|$

$(\pi \cdot \rho)(u) \geq 0$.

Take

$$\begin{tikzcd} C^*_u(S) \arrow[drr, bend right, "\pi"] & S \arrow[l, "\iota"'] \arrow[dr, "\phi"] & \\ & & B(H) \\ C^*_u(T) \arrow[urr, bend left, "\rho"] & T \arrow[l] \arrow[ur, "\psi"'] & \end{tikzcd}$$

3) Maximal: $S \otimes_{max} T$

Take $X \in M_k(S)^+$, $Y \in M_m(T)^+$

Generate a cone in $M_n(S \otimes T)$ by $\alpha^* \overbrace{(X \otimes Y)}^{\in M_{km}(S \otimes T)} \alpha$,

$\alpha \in M_{km,n}$

Concrete rep. of the dual:

$$(S \otimes_{max} T)^d \cong S^d \otimes_{min} T^d$$ finite dimensional

Stochastic matrices

quantum no-signalling correlations

X, A finite sets. M_X : matrices over X.
$(M_X \cong \mathcal{L}(\mathbb{C}^X))$.

Stochastic operator matrix over (X, A):

$$E = (E_{xx'aa'})_{xx'aa'} \in (M_X \otimes M_A \otimes \mathcal{B}(H))^+,$$

with $\quad \mathrm{Tr}_A E = I \quad$ i.e.

$$\sum_a E_{xx'aa} = \delta_{x,x'} I, \quad x, x' \in X.$$

$V : H^X \to K^A$

Factorisation: $\exists$ isometry $V = (V_{a,x})_{a,x}$,

$V_{a,x} : H \to K$, $a \in A$, $x \in X$, s.t.

$$E_{xx'aa'} = V_{ax}^* V_{a'x'}, \quad x, x' \in X, a, a' \in A$$

Idea: $\Phi : \varepsilon_{aa'} \to [E_{xx'aa'}]_{xx'}$, $M_A \to M_X \otimes \mathcal{B}(H)$ ucp
$\rightsquigarrow$ Stinespring.

Q Universal operator system?

Start with the universal TRO of an isometry $V = (V_{a,x})$, say $\mathcal{V}$.

Thus: $\mathcal{V} = \mathrm{TRO}(\{V_{a,x} : a \in A, x \in X,$
$(x, y, z) \to x y^* z.$ $\qquad (V_{a,x})_{a,x}$ an isometry $\})$.

! Ternary representations of $\mathcal{V}$ correspond to

Block operator isometries V.

Let $\mathcal{B} = [V^* V]$ (Morita theory)

Let $\mathcal{E}_{X,A} = C^*(\{ e_{xx'aa'} : x, x', a, a' \})$

where $e_{xx'aa'} = V^*_{ax} V_{a'x'}$.

Universal property:

$*$-rep. of $\mathcal{E}_{X,A}$

$\longleftrightarrow$ isometries $V = (V_{a,x})$.

$\longleftrightarrow$ stochastic matrices E

Operator system:

$\mathcal{T}_{X,A} = \operatorname{span}\{ e_{xx'aa'} : x, x', a, a' \}$.

universal for stochastic op. matrices

$e_{xx'aa'} \xrightarrow{\Phi} E_{xx'aa'}$ $\rightarrow$ concrete

ucp maps $\mathcal{T}_{X,A} \to \mathcal{B}(H)$

$\updownarrow$

stochastic matrices via

A family of POVM's,

$(E_{x,a})_{a\in A}$, $x \in X$ (no-signalling correlation)

$\downarrow$

stochastic matrix $E = \sum \varepsilon_{x,x} \otimes \varepsilon_{a,a} \otimes E_{x,a}$

$\in$

$(\mathcal{D}_X \otimes \mathcal{D}_A \otimes \mathcal{B}(H))^+$

Duan - Winter

Quantum no-signalling correlations

$\Gamma : M_X \otimes M_Y \longrightarrow M_A \otimes M_B$

quantum channels (c.p. + trace-pr.)

s.t. $\Gamma_{X \to A} : M_X \to M_A$,

$\Gamma_{Y \to B} : M_Y \to M_B$ well-defined.

$$\Gamma_{X \to A}(\omega) = Tr_B(\Gamma(\omega \otimes \omega')) \in M_A$$

$\omega \in M_X$, $\omega' \in M_Y$

$$\Gamma^* : M_A \otimes M_B \to M_X \otimes M_Y$$

no-signalling:

$$\Gamma^*(M_A \otimes 1) \subseteq M_X \otimes 1$$

$$\Gamma^*(1 \otimes M_B) \subseteq 1 \otimes M_Y$$

quantum input - quantum output non-local games

$\Gamma : \mathcal{D}_X \otimes \mathcal{D}_Y \to \mathcal{D}_A \otimes \mathcal{D}_B$ classical

Theorem The states on

(a) $\mathcal{T}_{X,A} \otimes_{max} \mathcal{T}_{Y,B}$.

(b) $\mathcal{T}_{X,A} \otimes_{c} \mathcal{T}_{Y,B}$

(c) $\mathcal{T}_{X,A} \otimes_{min} \mathcal{T}_{Y,B}$

correspond to correlation types

(i) ns
(ii) qc
(iii) qa

(last series of lectures at this seminar)

via the assignment

$$s : \mathcal{T}_{X,A} \otimes \mathcal{T}_{Y,B} \to \mathbb{C} \qquad M_X \otimes M_Y$$

$$\rightsquigarrow \quad \Gamma_s : M_{XY} \to M_{AB},$$

$$\Gamma_s(\varepsilon_{xx'} \otimes \varepsilon_{yy'}) = \sum_{aa'bb'} s(e_{xx'aa'} \otimes e_{yy'bb'}) \, \varepsilon_{aa'} \otimes \varepsilon_{bb'}$$

$$s \longrightarrow \Gamma_s$$

$$(\mathcal{T}_{X,A} \otimes \mathcal{T}_{Y,B})^* \longrightarrow \mathcal{L}(M_{XY}, M_{AB}).$$

1-1 map.,

onto all "no-signalling" maps.

$\Gamma \longrightarrow s_{\Gamma}$