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Subject:
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To:

HK

Crossed and semicrossed products I

$\mathcal{A}$ $C^*$-algebra

$G$ discrete group

$\alpha : G \longrightarrow \mathrm{Aut}\,\mathcal{A} \equiv *$-automorphisms of $\mathcal{A}$

We want operator algebra(s) that capture the action of $G$ on $\mathcal{A}$.

A covariant representation $(\pi, U, \mathcal{H})$ of the dynamical system $(\mathcal{A}, G, \alpha)$ consists of

(i) $\mathcal{H}$ Hilbert space

(ii) $\pi : \mathcal{A} \longrightarrow B(\mathcal{H})$ $*$-representation

(iii) $U : G \longrightarrow B(\mathcal{H})$ unitary repn of $G$

so that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^* , \ \forall s \in G$$

Do such covariant representations exist?

Lets look closer at the case where $G = \mathbb{Z}$

In that case the action $\alpha : G \to \operatorname{Aut} \mathcal{Q}$ reduces to a single automorphism $\alpha \in \operatorname{Aut} \mathcal{Q}$ and its iterates $\alpha^{(n)} \equiv \alpha \circ \alpha \cdots \circ \alpha$

Similarly, a covariant representation $(\pi, \mathcal{U}, \mathcal{H})$ consists of a single unitary $\mathcal{U} \in \mathcal{B}(\mathcal{H})$ so that

$$\mathcal{U} \pi(a) \mathcal{U}^* = \pi(\alpha(a)), \quad \forall a \in \mathcal{Q}.$$

Starting with a $*$-repn $\pi : \mathcal{Q} \longrightarrow \mathcal{B}(\mathcal{H})$ we may produce a covariant repn of $(\mathcal{Q}, \alpha)$ as follows

$$\hat{\pi}(a) = \begin{bmatrix} \ddots & & & & & & \\ & \pi(\alpha^2(a)) & & & & \mathbf{0} & \\ & & \pi(\alpha(a)) & & & & \\ & & & \boxed{\pi(a)} & & & \\ & \mathbf{0} & & & \pi(\alpha^{-1}(a)) & & \\ & & & & & \pi(\alpha^{-2}(a)) & \\ & & & & & & \ddots \end{bmatrix}$$

$$\mathcal{U} = \begin{bmatrix} \ddots & & & & & \\ & 0 & & & \mathbf{O} & \\ & I & 0 & & & \\ & & I & \boxed{0} & & \\ & \mathbf{O} & & I & 0 & \\ & & & & I & 0 \\ & & & & & \ddots \end{bmatrix}$$

We leave it as an excercise to verify that $(\hat{\pi}, \mathcal{U})$ as above forms a covariant repn for $(\mathcal{Q}, \alpha)$

For more general dynamical systems $(\mathcal{Q}, G, \alpha$

the existence of covariant representations is verified in a similar way. Specifically.

If $\pi : \mathcal{Q} \to B(\mathcal{H})$ is a $*$-representation, then consider the Hilbert space

$$\hat{\mathcal{H}} = \bigoplus_{s \in G} \mathcal{H}_s, \quad \text{with } \mathcal{H}_s = \mathcal{H}, \ \forall s \in G$$

Let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$ and define orthonormal basis in $\mathcal{H}_s$ by

$$e_n^{(s)} = e_n \quad , \quad \forall\, n, s$$

Given $t \in G$, let $U_t \in B(\hat{\mathcal{H}})$ defined as

$$U_t(e_n^{(s)}) = e_n^{(ts)} \quad , \quad \forall\, n, s.$$

Also for each $a \in \mathcal{Q}$ we define $\hat{\pi}(a)$ to be the "diagonal" operator with

$$\hat{\pi}(a)\big|_{\mathcal{H}_s} = \pi(\alpha^{s^{-1}}(a)) \quad , \quad s \in G.$$

<u>Definition</u>. If $(\mathcal{Q}, G, \alpha)$ is a $C^*$-dynamical system then

$$\mathcal{Q} \rtimes_\alpha G$$

will denote the universal $C^*$-algebra for all covariant repns of $(\mathcal{Q}, G, \alpha)$, i.e., there exists a covariant repn $(\bar{\pi}, \bar{U}, \mathcal{Q} \rtimes_\alpha G)$ so that f any other covariant repn $(\pi, U, \mathcal{H})$ there exists a representation

$$\pi \times U : \mathcal{Q} \rtimes_\alpha G \longrightarrow B(\mathcal{H})$$

so that

$$(\pi \times U)(\pi(a)) = \pi(a), \quad \forall a \in \mathcal{Q}.$$
$$(\pi \times U)(\bar{U}_t) = U_t, \quad \forall t \in G$$

$\mathcal{Q} \rtimes_\alpha G$ is called the full crossed product

The reduced crossed product $\mathcal{Q} \rtimes_{\alpha,r} G$ is the $C^*$-algebra generated by the covariant representation $(\hat{\pi}, U)$ with $\pi$ being the universal representation of $\mathcal{Q}$, i.e.,

$$\mathcal{Q} \rtimes_{\alpha,r} G = (\hat{\pi} \rtimes U)(\mathcal{Q} \rtimes_\alpha G).$$

On certain occasions, we can characterize the faithful repns of $\mathcal{Q} \rtimes_\alpha G$.

<u>Definition</u>. A covariant repn $(\pi, U, \mathcal{H})$ of $(\mathcal{Q}, G, \cdot$ with $G$ abelian, is said to admit a gauge action if there exists group representation

$$\delta : \hat{G} \longrightarrow \mathrm{Aut}\Big((\pi \rtimes U)(\mathcal{Q} \rtimes_\alpha G)\Big)$$

so that

(i) $\delta_\gamma(\pi(a)) = \pi(a)$

(ii) $\delta_\gamma(U_s) = \gamma(s)\, U_s \qquad \forall \gamma \in \hat{G}, \ s \in G$

(ii) $\delta_\gamma(u_s) = \gamma(s)\, u_s$ , $\forall \gamma \in \hat{G}$, $s \in G$.

In the case where $G = \mathbb{Z}$, we are simply request

$$\delta : \mathbb{T} \longrightarrow \mathrm{Aut}\left(\pi \rtimes \mathcal{U}(\mathcal{C} \rtimes_\alpha \mathbb{Z})\right)$$

so that
$$\delta_z(\pi(a)) = \pi(a) \quad \text{and} \quad \delta_z(\mathcal{U}) = z\mathcal{U}, \quad z \in \mathbb{T}$$

The identity repn of $\mathcal{O} \rtimes_\alpha G$ admits a gauge action $\Delta$.

Reason: If $(\pi, \mathcal{U})$ is a covariant repn of $(\mathcal{O}, G, \alpha)$, then $(\pi, \gamma \cdot \mathcal{U})$, $\gamma \in \hat{G}$, is also a covariant repn, where

$$\gamma \mathcal{U} : t \longrightarrow \gamma(t)\, u_t \quad , \quad t \in G$$

<u>PROPOSITION 1</u> Let $(\mathcal{O}, G, \alpha)$ be an abelian $C^*$-dynamical system and $(\pi, \mathcal{U}, \mathcal{H})$ a faithful covariant repn. Then, $\pi \rtimes \mathcal{U}$ is a faithful if and only if $(\pi, \mathcal{U})$ admits a gauge action.

<u>Proof</u> Define a map

$$E_\Delta : \mathcal{O} \rtimes_\alpha G \longrightarrow \mathcal{O} \qquad \sum a_s u_s$$

by

$$E_\Delta(x) = \int_{\widehat{G}} \Delta_\gamma(x)\, d\gamma \quad \in \mathcal{O}$$

and verify on polynomials that it maps on $\mathcal{O} \subseteq \mathcal{O} \rtimes_\alpha G$
Furthermore, $E_\Delta$ is faithful, i.e.

$$E_\Delta(x^* x) = 0 \Rightarrow x^* x = 0 \ , \quad x \in \mathcal{O} \rtimes_\alpha G.$$

A similar map $E_\Gamma$ can be defined on $(\pi \rtimes U)(\mathcal{O} \rtimes_\alpha G)$
We have a commutative diagram

$$\begin{tikzcd}
x^*x \in \mathcal{O} \rtimes_\alpha G \arrow[r, "E_\Delta"] \arrow[d, "\pi \rtimes U"] & \mathcal{O} \arrow[d, "\pi"', "\pi \text{ faithful}"] \\
(\pi \rtimes U)(\mathcal{O} \rtimes_\alpha G) \arrow[r, "E_\Gamma"] & \pi(\mathcal{O})
\end{tikzcd}$$

Clearly, if $(\pi \rtimes U)(x^* x) = 0$, then

$$\pi \circ E_\Delta(x^* x) = 0$$

$\blacksquare$

<u>Corollary 2.</u> If $G$ is abelian, then

$$\mathcal{O} \rtimes_\alpha G \cong \mathcal{O} \rtimes_{\alpha, r} G$$

<u>Proof</u>

Notice that $\mathcal{O} \rtimes_{\alpha,r} G$ admits a gauge actio via conjugation by unitaries.

( In the $G = \mathbb{Z}$ case the action is

$$\partial_z : x \longrightarrow \mathcal{U}_z \times \mathcal{U}_z \quad , \; t \in \mathbb{T}$$

with

$$\mathcal{U}_z = \begin{bmatrix} \ddots & & & & & \\ & z^{-1} & & & 0 & \\ & & 1 & & & \\ & 0 & & z & & \\ & & & & z^2 & \\ & & & & & \ddots \end{bmatrix}, \quad t \in \mathbb{T}$$

)

<u>Remark</u> For any discrete group $G$, $\mathcal{Q} \rtimes_{\alpha, r}$ admits a faithful expectation on $\mathcal{Q}$.

We will now restrict our attention to abelian $C^*$-algebras and we will address the simplicity of $\mathcal{Q} \rtimes_\alpha^r G$

Let $X$ be a compact $T_2$ space and let $G$ be a discrete group acting on $X$ by homeomorphisms. We therefore have an action of $G$ on $C(X)$ by $*$-automorhisms given by

$$\alpha_s(f)(x) = f(s^{-1}x)$$

We say that $G$ acts (topologically) freely on $X$ if for every $s \in G$, the set

$$\{x \in X \mid sx = x\}$$

has empty interior

We say that the action of $G$ on $X$ is **minimal** iff for every $x \in X$, the set $Gx \subseteq X$ is dense.

<u>THEOREM 3.</u> Let $X$ be a compact $T_2$ space and assume that a group $G$ acts on $X$ freely and minimally. Then

$$C(X) \rtimes_r G$$

is a simple $C^*$-algebra

If $G$ is abelian, then the converse is also true

The proof will follow from several steps

<u>Lemma 4.</u> Let $G$ be a discrete group acting freely ~~on~~ a compact $T_2$ space $X$ and let $s_1, s_2, \ldots, s_n \in$ Given any open $U \subseteq X$, there exists open $V \subseteq U$ so that

$$s_i V \cap V = \emptyset$$

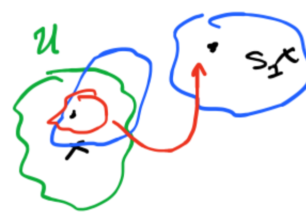
$s_i V$

**Proof.** Choose

$$x \in \bigcap_{i=1}^{n} \{ x \in X \mid s_i x \neq x \} \cap \mathcal{U} \neq \phi$$

and for each $i = 1, 2, \dots, n$ let $V_i \subseteq X$ open so tha

$$s_i V_i \cap V_i = \phi$$




Then $V = \left( \bigcap_{i=1}^{n} V_i \right) \cap \mathcal{U}$. ∎

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**Lemma 5.** Let $X$, $G$ and $s_1, s_2, \dots, s_n$ as above. $f \in C(X)$ and $n \in \mathbb{N}$, there exist $g \in C(X)$ so ·

(i) $0 \leq g(x) \leq 1 \quad \forall x \in X$

(ii) $\|fg\| \geq \|f\| - \frac{1}{n}$, and

(iii) $\alpha_{s_i^{-1}}(g) g = 0, \quad \forall\, i = 1, 2, \dots, n$

**Proof** Let

$$\mathcal{U} = \{ x \in X \mid \quad |f(x)| > \|f\| - \tfrac{1}{n} \}$$

and choose $V \subseteq \mathcal{U}$ as in Lemma 4. Now use Urysohn's Lemma to obtain $g \in C(X)$ satisfying (i) vanishing outside $V$. Then $\alpha_{s_i^{-1}}(g)$ vanishes ou- $s_i V$ and the conclusion follows ∎

We now prove a more general result that Theorem 3.

<u>THEOREM 6</u>. Let $G$ be a discrete group acting f a compact $T_2$ space $X$.
Assume that $J \subseteq C(X) \rtimes_r G$ is a closed ideal

$$J \cap C(X) = \{0\}$$

Then $J = \{0\}$. (Intersection property)

Proof. Assume that $J \neq \{0\}$ and consider $0 \leq c \in \operatorname{Ker}\pi$, where

$$\pi : C(X) \rtimes_r G \longrightarrow C(X) \rtimes_r G / J$$

By assumption, $\pi$ is an isometry on $C(X) \subseteq C(X$
Let $f_e \equiv E(c)$. Since $E$ is faithful, $f_e \neq 0$
Let $n \in \mathbb{N}$ and consider a polynomial

$$c_n = \sum f^{(n)}_{s_i} u_{s_i}$$

with $\| c - c_n \| \leq \frac{1}{n}$. For $f^{(n)}_e$ and $s_1, s_2, \dots, s$
as above, let $g$ be as in Lemma 5. Then

$$\| g f_e^{(n)} \| \geq \| f_0^{(n)} \| - \frac{1}{n} \qquad (*)$$

On the other hand

$$g^{\frac{1}{2}} f_{s_i} U_{s_i} g^{\frac{1}{2}} = f_{s_i} U_{s_i} \alpha_{s_i^{-1}}(g^{\frac{1}{2}}) g^{\frac{1}{2}} = 0$$

Therefore

$$\begin{aligned} \| \pi ( g^{\frac{1}{2}} f_e^{(n)} g^{\frac{1}{2}} ) \| &= \| \pi ( g^{\frac{1}{2}} c_n g^{\frac{1}{2}} ) \| \\ &= \| \pi ( g^{\frac{1}{2}} (c_n - c) g^{\frac{1}{2}} ) \| \\ &\leq \| c_n - c \| \leq \frac{1}{n} \end{aligned}$$

and since $\pi$ is faithful

$$\| g f_e^{(n)} \| \leq \frac{1}{n}$$

Combine with $(*)$

$$\| f_e^{(n)} \| - \frac{1}{n} \leq \| g f_e^{(n)} \| \leq \frac{1}{n}$$

and so $\| f_e^{(n)} \| \leq \frac{2}{n}$.

But the sequence $\{ f^{(n)} \}$ [illegible] that the sequence $\{f_e\}$ new approximates

and so $f_e = 0$, a contradiction ∎

We now finalize

<u>THEOREM 3.</u> Let $X$ be a compact $T_2$ space and assume that a group $G$ acts on $X$ freely and minimaly. Then

$$C(X) \rtimes_r G$$

is a simple $C^*$-algebra

If $G$ is abelian, then the converse is also true

<u>Proof</u> Assume that $G$ acts freely and minimaly

Let

$$J \subseteq C(X) \rtimes_r G$$

non-trivial ideal.

Hence

$$J \cap C(X) = Z(K) = \{ f \in C(X) \mid f(K) = \{0\} \}$$

for some compact $K \subseteq X$. By assumption $K \neq X$.

Now, if $K \neq \emptyset$, then

$$\mathcal{U}_s^*(J\cap C(X))\mathcal{U}_s = J\cap C(X)$$

$$\mathcal{U}_s^* Z(K)\mathcal{U}_s = Z(K)$$

$$Z(sK) = Z(K)$$

and so

$$sK = K \quad , \forall s\in G$$

i.e., $K = X$, a contradiction. Hence $K=\emptyset$ and $J = C(X)\rtimes_\alpha G$.

For the abelian case, assume that $C(X)\rtimes_\alpha G$ is simple. Then $G$ has to act minimaly on $X$, or otherwise, any $G$-invariant ideal $J\subseteq C(X)$ produces an ideal $J\rtimes_\alpha G\subseteq C(X)\rtimes_\alpha G$

To show that $G$ acts freely assume that there is $s\in G$ and open $\mathcal{U}\subseteq X$ consisting of fixed poi- for $s\in G$. Let $f\in C(X)$ with $f(X\setminus\mathcal{U}) = \{0\}$.

Pick any $x\in X$ and consider the representation $\pi_x$ of $C(X)\rtimes_\alpha G$ with

$$\pi_x(f)\,\delta_{tx} = f(tx)\delta_{tx} \quad , \quad f\in C(X)\ ,\ t\in G$$

$$\pi_x(\mathcal{U}_s)\,\delta_{tx} = \delta_{stx} \quad , \quad s,t\in G$$

where $\{\delta_{tx}\}_{t\in G}$ is an o.b. for $\mathcal{H}_x$.

Claim: $\pi_x(f - f u_s) = 0$

If $tx$ is a fixed point for $s$, then

$$(f - f u_s)\delta_{tx} = f(tx)\delta_{tx} - \pi(f)\delta_{\cancel{s}tx} = 0$$

If $tx$ not. fixed then $tx \notin \mathcal{U}$ and $stx \notin \mathcal{U}$ (or otherwise $stx$ fixed for $s^{-1}$ and so

$$\mathcal{U} \ni stx = s^{-1}stx = tx \;)$$

Again

$$(f - f u_s)\delta_{tx} = f(\cancel{tx})\delta_{tx} - f(\cancel{stx})\delta_{stx}$$

Since $C(X) \rtimes_\alpha G$ is simple, $\pi$ is faithful and so $f - f u_s = 0 \Rightarrow f = 0$ by the ~~faithfulness~~ of the expectation. $\square$

## Concluding Remarks

PROBLEM: Characterize the simplicity of $C(X) \rtimes_{\alpha,r} G$, $\mathcal{A} \rtimes_{\alpha,r} G$

when $G$ is not abelian (actually amenable)

If $X$ is a singleton

$$C(X) \rtimes_{\alpha,r} G = C^*_r(G)$$

The problem was solved in 2014 by Kalantar and Kennedy ( (Crelle's Journal)

Furstenberg boundary of $G$: a universal compact $T_2$ space $\partial_G$ on which $G$ acts minimaly and proximaly

<u>THEOREM</u> $C^*_r(G)$ is simple iff the action of $G$ on $\partial G$ is topol. free.