Isoperimetric constants of metric probability spaces

Seminar on Functional Analysis and Operator Algebras

December 4, 2020
Poincaré inequality and concentration

Let \((X, d, \mu)\) be a metric probability space. The concentration function of \(X\) is defined on \((0, \infty)\) by

\[
\alpha_\mu(t) := \sup\{1 - \mu(A_t) : \mu(A) \geq 1/2\}
\]

where the supremum runs over all sets \(A\) in the Borel \(\sigma\)-algebra \(B(X)\) with \(\mu(A) \geq 1/2\), and where \(A_t = \{x : d(x, A) < t\}\) is the \(t\)-extension of \(A\).

We say that \(\mu\) has exponential concentration on \((X, d)\) if there exist constants \(C, c > 0\) such that, for every \(t > 0\),

\[
\alpha_\mu(t) \leq Ce^{-ct}.
\]

Recall that a function \(f : (X, d) \rightarrow \mathbb{R}\) is called Lipschitz if there exists \(\sigma \geq 0\) such that \(|f(x) - f(y)| \leq \sigma d(x, y)\) for all \(x, y \in X\), and the smallest such constant \(\sigma\) is denoted by \(\|f\|_{\text{Lip}}\).

We say that \(f\) is locally Lipschitz if for every \(x \in X\) there exists a neighborhood \(U_x\) of \(x\) such that \(f \big|_{U_x}\) is Lipschitz. For every locally Lipschitz function \(f\) we define (in the continuous case)

\[
|\nabla f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.
\]
We say that \( \mu \) satisfies a Poincaré inequality with constant \( \theta \) if
\[
\text{Var}_\mu(f) \leq \theta^2 \int |\nabla f|^2 \, d\mu
\]
for every locally Lipschitz function \( f : X \to \mathbb{R} \), where
\[
\text{Var}_\mu(f) = \mathbb{E}_\mu (f - \mathbb{E}_\mu(f))^2 = \mathbb{E}_\mu(f^2) - (\mathbb{E}_\mu(f))^2.
\]

**Theorem (Gromov-Milman)**

Let \((X, d, \mu)\) be a metric probability space. If \( \mu \) satisfies a Poincaré inequality with constant \( \theta \), then \( \mu \) has exponential concentration. More precisely,
\[
\alpha_\mu(t) \leq \exp\left(-\frac{t^3}{3\theta}\right).
\]

We present an argument that uses the notion of the expansion coefficient of \( \mu \).

This is defined for every \( \varepsilon > 0 \) as follows:
\[
\text{Exp}_\mu(\varepsilon) = \sup\{s \geq 1 : \mu(B_{\varepsilon}) \geq s\mu(B) \text{ for all } B \in \mathcal{B}(X) \text{ with } \mu(B_{\varepsilon}) \leq 1/2\}.
\]
Poincaré inequality and concentration

**Theorem**

Assume that for some $\varepsilon > 0$ we have $\text{Exp}_\mu(\varepsilon) \geq s > 1$. Then, for every $t > 0$ we have $\alpha_\mu(t) \leq \frac{s}{2} s^{-t/\varepsilon}$.

- Let $A \subseteq X$ with $\mu(A) \geq \frac{1}{2}$ and let $t > 0$. There exists $k \geq 0$ such that $k\varepsilon \leq t < (k + 1)\varepsilon$. Setting $B_0 = X \setminus A$ and $B_j = X \setminus A_{j\varepsilon}$, for every $1 \leq j \leq k$ we check that $(B_j)_{\varepsilon} \subseteq B_{j-1} \subseteq X \setminus A$.
- Applying the definition of the expansion coefficient to $B_j$ (as $\mu(B_j) \leq 1/2$) and the assumption that $\text{Exp}_\mu(\varepsilon) \geq s$ we get

  \[ \mu(B_j) = \mu(X \setminus A_{j\varepsilon}) \leq \frac{1}{s} \mu(X \setminus A_{(j-1)\varepsilon}) = \frac{1}{s} \mu(B_{j-1}), \]

  for all $1 \leq j \leq k$.
- Then, we have

  \[
  \mu(X \setminus A_t) \leq \mu(X \setminus A_{k\varepsilon}) \leq \frac{1}{s} \mu(X \setminus A_{(k-1)\varepsilon}) \leq \frac{1}{s^2} \mu(X \setminus A_{(k-2)\varepsilon}) \\
  \leq \cdots \leq \frac{1}{s^k} \mu(X \setminus A) \leq \frac{1}{2} s^{-k} \leq \frac{1}{2} s^{-\left(\frac{t}{\varepsilon} - 1\right)}
  \]

  where the last inequality follows from $t < (k + 1)\varepsilon$. 

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Isoperimetric constants

December 4, 2020 4 / 27
Poincaré inequality and concentration

Theorem (Gromov-Milman)

Let \((X, d, \mu)\) be a metric probability space. If \(\mu\) satisfies a Poincaré inequality with constant \(\theta\), then \(\mu\) has exponential concentration. More precisely, 
\[
\alpha_\mu(t) \leq \exp\left(-t/(3\theta)\right).
\]

- For the proof of the Gromov-Milman theorem we shall show that if \(\mu\) satisfies a Poincaré inequality with constant \(\theta\) then \(\text{Exp}_\mu(\sqrt{2}\theta) \geq 2\).
- Let \(\sqrt{2}\theta = \varepsilon > 0\) and consider \(B \subseteq X\) such that \(A = X \setminus B_\varepsilon\) satisfies \(\mu(A) \geq 1/2\). We set \(a = \mu(A)\), \(b = \mu(B)\). Note that \(d(A, B) \geq \varepsilon\).
- Define \(f : X \to \mathbb{R}\) by \(f(x) = \frac{1}{a} - \frac{1}{\varepsilon} \left(\frac{1}{a} + \frac{1}{b}\right) \min\{\varepsilon, d(x, A)\}\).
- Then, \(f(x) = 1/a\) on \(A\), \(f(x) = -1/b\) on \(B\) and

\[
|\nabla f|(x) \leq \frac{1}{\varepsilon} \left(\frac{1}{a} + \frac{1}{b}\right)
\]

for all \(x \in X\), while \(|\nabla f|(x) = 0\) on a set of measure \(a + b\).
- Consequently,

\[
\int |\nabla f|^2 d\mu \leq \frac{1}{\varepsilon^2} \left(\frac{1}{a} + \frac{1}{b}\right)^2 (1 - a - b).
\]
On the other hand, if \( m = \mathbb{E}_{\mu}(f) \) we have

\[
\text{Var}_{\mu}(f) \geq \int_A (f - m)^2 d\mu + \int_B (f - m)^2 d\mu \geq a \left( \frac{1}{a} - m \right)^2 + b \left( -\frac{1}{b} - m \right)^2 \geq \frac{1}{a} + \frac{1}{b}.
\]

From the Poincaré inequality we get

\[
\left( \frac{1}{a} + \frac{1}{b} \right) \leq \frac{\theta^2}{\varepsilon^2} \left( \frac{1}{a} + \frac{1}{b} \right)^2 (1 - a - b),
\]

and hence \( \frac{\varepsilon^2}{\theta^2} \leq \frac{a + b}{ab} (1 - a - b) \leq \frac{1 - a - b}{ab} = \frac{1 - a}{ab} - \frac{1}{a} \).

Solving for \( b \) we have

\[
b \leq \frac{1 - a}{a} \cdot \frac{1}{\frac{1}{a} + \frac{\varepsilon^2}{\theta^2}} = \frac{1 - a}{1 + \frac{ae^2}{\theta^2}} \leq \frac{1 - a}{1 + \frac{\varepsilon^2}{2\theta^2}}.
\]

Having chosen \( \varepsilon = \sqrt{2\theta} \), this implies

\[
\mu(B) \leq \frac{1}{2} \mu(B_{\varepsilon}),
\]

as claimed. Since \( B \) was arbitrary, we conclude that \( \text{Exp}_{\mu}(\sqrt{2\theta}) \geq 2 \).

Then,

\[
\alpha_{\mu}(t) \leq \exp \left( -\frac{\ln 2}{\sqrt{2\theta}} t \right) \leq \exp \left( -\frac{t}{3\theta} \right).
\]
Let $\mu$ be a Borel probability measure on $\mathbb{R}^n$. For every Borel subset $A$ of $\mathbb{R}^n$, the Minkowski content of $A$ with respect to $\mu$ is defined as

$$\mu^+(A) = \liminf_{t \to 0^+} \frac{\mu(A_t) - \mu(A)}{t}.$$ 

The isoperimetric ratio of $A$ is defined as follows:

$$\chi_\mu(A) := \frac{\mu^+(A)}{\min\{\mu(A), 1 - \mu(A)\}}.$$ 

Then, we define the Cheeger constant $\chi_\mu$ of $\mu$ setting

$$\chi_\mu := \inf\{\chi_\mu(A) : A \text{ Borel } \subset \mathbb{R}^n\}.$$
Cheeger constant

Theorem (Rothaus, Cheeger, Maz’ya)

Let \( \mu \) be a Borel probability measure on \( \mathbb{R}^n \) with Cheeger constant \( \chi_\mu \). Let \( \alpha_1 \) be the largest constant with the following property: for every integrable, locally Lipschitz function \( f : \mathbb{R}^n \to \mathbb{R} \),

\[
\alpha_1 \int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)| \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x).
\]

Then, \( \alpha_1 \leq \chi_\mu \leq 2\alpha_1 \).

First we show that \( \chi_\mu \leq 2\alpha_1 \).

- Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an integrable, locally Lipschitz function. We may assume that \( f \) is bounded from below and hence, by adding a suitable constant, that \( f > 0 \).

- The co-area formula shows that

\[
\int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x) \geq \int_0^\infty \mu^+\left(\{x : f(x) > s\}\right) \, ds \\
\geq \chi_\mu \int_0^\infty \min\{\mu(A(s)), 1 - \mu(A(s))\} \, ds,
\]

where \( A(s) = \{f > s\} \).
• Setting $A(s) = \{ f > s \}$ we saw that
\[
\int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x) \geq \chi_{\mu} \int_0^\infty \min\{\mu(A(s)), 1 - \mu(A(s))\} \, ds.
\]

• Using the fact that $\|1_B - \mathbb{E}_\mu(1_B)\|_1 = 2\mu(B)(1 - \mu(B))$ for every Borel subset $B$ of $\mathbb{R}^n$, and the simple identity $\mathbb{E}_\mu(f(g - \mathbb{E}_\mu(g))) = \mathbb{E}_\mu(g(f - \mathbb{E}_\mu(f)))$, we may write
\[
\int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x) \geq \chi_{\mu} \int_0^\infty \mu(A(s))(1 - \mu(A(s))) \, ds
\]
\[
= \frac{\chi_{\mu}}{2} \int_0^\infty \|1_{A(s)} - \mathbb{E}_\mu(1_{A(s)})\|_1 \, ds
\]
\[
\geq \frac{\chi_{\mu}}{2} \sup \left\{ \int_0^\infty \int_{\mathbb{R}^n} (1_{A(s)} - \mathbb{E}_\mu(1_{A(s)}))g \, d\mu \, ds : \|g\|_\infty \leq 1 \right\}
\]
\[
= \frac{\chi_{\mu}}{2} \sup \left\{ \int_0^\infty \int_{\mathbb{R}^n} 1_{A(s)}(g - \mathbb{E}_\mu(g)) \, d\mu \, ds : \|g\|_\infty \leq 1 \right\}
\]
\[
= \frac{\chi_{\mu}}{2} \sup \left\{ \int_{\mathbb{R}^n} f(g - \mathbb{E}_\mu(g)) \, d\mu : \|g\|_\infty \leq 1 \right\}
\]
\[
= \frac{\chi_{\mu}}{2} \sup \left\{ \int_{\mathbb{R}^n} g(f - \mathbb{E}_\mu(f)) \, d\mu : \|g\|_\infty \leq 1 \right\} = \frac{\chi_{\mu}}{2} \|f - \mathbb{E}_\mu(f)\|_1.
\]
This shows that $\chi_{\mu} \leq 2\alpha_1$. 
Recall that $\alpha_1$ is the largest constant so that
\[\alpha_1 \int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)| \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x)\]
for locally Lipschitz functions. Now, we want to show that $\alpha_1 \leq \chi_\mu$.

Consider any closed subset $A$ of $\mathbb{R}^n$ and for small $\varepsilon > 0$ we define the function
\[f_\varepsilon(x) = \max \left\{ 0, 1 - \frac{d(x, A_\varepsilon)}{\varepsilon - \varepsilon^2} \right\}.
\]
Then, $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon \equiv 1$ on $A_\varepsilon \supseteq A$, $f \equiv 0$ on $\{x : d(x, A) > \varepsilon\}$, and $f_\varepsilon \rightarrow 1_A$ as $\varepsilon \rightarrow 0$.

Finally, $f_\varepsilon$ is Lipschitz: we have
\[|f_\varepsilon(x) - f_\varepsilon(y)| \leq \frac{1}{\varepsilon(1 - \varepsilon)} \left| d(x, A_\varepsilon) - d(y, A_\varepsilon) \right| \leq \frac{|x - y|}{\varepsilon(1 - \varepsilon)},
\]
therefore $|\nabla f_\varepsilon(x)| \leq (\varepsilon - \varepsilon^2)^{-1}$.

Since $\nabla f_\varepsilon(x) = 0$ on $C = \{x : d(x, A) > \varepsilon\} \cup \{x : d(x, A) < \varepsilon^2\}$, we get
\[
\int_{\mathbb{R}^n} |\nabla f_\varepsilon(x)| \, d\mu(x) \leq \int_{\mathbb{R}^n \setminus C} |\nabla f_\varepsilon(x)| \, d\mu(x)
\leq \frac{1}{1 - \varepsilon} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon} - \frac{\varepsilon}{1 - \varepsilon} \frac{\mu(A_\varepsilon^2) - \mu(A)}{\varepsilon^2}.
\]
We have assumed that
\[
\alpha_1 \int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)| \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x).
\]

Therefore,
\[
\alpha_1 \int_{\mathbb{R}^n} |f_\varepsilon(x) - \mathbb{E}_\mu(f_\varepsilon)| \, d\mu(x) \leq \frac{1}{1 - \varepsilon} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon} - \frac{\varepsilon}{1 - \varepsilon} \frac{\mu(A_\varepsilon^2) - \mu(A)}{\varepsilon^2}.
\]

Letting \( \varepsilon \to 0^+ \) we see that
\[
\mu^+(A) \geq \alpha_1 \|1_A - \mathbb{E}_\mu(1_A)\|_1 = 2\alpha_1 \mu(A)(1 - \mu(A)).
\]

This shows that \( \chi_\mu \geq \alpha_1 \).

**Definition**

\[
\psi_\mu = \frac{1}{\chi_\mu}, \text{ the reciprocal Cheeger constant.}
\]
Recall that a Borel probability measure $\mu$ on $\mathbb{R}^n$ satisfies the Poincaré inequality with constant $\vartheta > 0$ if

$$\text{Var}_\mu(f) \leq \vartheta^2 \int |\nabla f|^2 d\mu,$$

for all smooth functions $f$ on $\mathbb{R}^n$, where

$$\text{Var}_\mu(g) = \mathbb{E}_\mu(g^2) - (\mathbb{E}_\mu(g))^2$$

is the variance of $g$ with respect to $\mu$.

The Poincaré constant $\vartheta_\mu$ of $\mu$ is the smallest constant $\vartheta > 0$ for which the Poincaré inequality is satisfied for all $f$.

**Theorem (Maz’ya, Cheeger)**

Let $\mu$ be a Borel probability measure with reciprocal Cheeger constant $\psi_\mu$. Then its Poincaré constant $\vartheta_\mu$ satisfies

$$\vartheta_\mu \leq 2\psi_\mu.$$
Poincaré constant and Cheeger constant

- By the co-area formula and the definition of the Cheeger constant, for every positive integrable locally Lipschitz function $g$ we have

$$\chi_\mu \int_0^\infty \min\{\mu\{g \geq s\}, 1 - \mu\{g \geq s\}\} \, ds \leq \int_0^\infty \mu^+\{g \geq s\} \, ds$$

$$\leq \int_{\mathbb{R}^n} |\nabla g| \, d\mu.$$

- Let $f$ be an integrable locally Lipschitz function and set $m = \text{med}(f)$. Then, we have $\mu\{f \geq m\} \geq \frac{1}{2}$ and $\mu\{f \leq m\} \geq \frac{1}{2}$.

- We set $f^+ = \max\{f - m, 0\}$ and $f^- = -\min\{f - m, 0\}$. Then, $f - m = f^+ - f^-$ and by the definition of $m$ we have

$$\mu\{(f^+)^2 \geq s\} \leq \frac{1}{2} \quad \text{and} \quad \mu\{(f^-)^2 \geq s\} \leq \frac{1}{2}$$

for all $s > 0$. 
Poincaré constant and Cheeger constant

Using
\[ \chi \mu \int_0^{\infty} \min \{ \mu(g \geq s), 1 - \mu(g \geq s) \} \, ds \leq \int_{\mathbb{R}^n} |\nabla g| \, d\mu \]

with \( g = (f^+)^2 \) and \( g = (f^-)^2 \) and applying integration by parts we see that

\[ \chi \mu \int_{\mathbb{R}^n} |f - m|^2 \, d\mu = \chi \mu \int_{\mathbb{R}^n} (f^+)^2 \, d\mu + \chi \mu \int_{\mathbb{R}^n} (f^-)^2 \, d\mu \]

\[ = \chi \mu \int_0^{\infty} \mu(\{(f^+)^2 \geq s\}) \, ds + \chi \mu \int_0^{\infty} \mu(\{(f^-)^2 \geq s\}) \, ds \]

\[ \leq \int_{\mathbb{R}^n} |\nabla ((f^+)^2)| \, d\mu + \int_{\mathbb{R}^n} |\nabla ((f^-)^2)| \, d\mu \]

\[ = \int_{\mathbb{R}^n} (|\nabla ((f^+)^2)| + |\nabla ((f^-)^2)|) \, d\mu. \]

Note that
\[ |\nabla ((f^+)^2)| + |\nabla ((f^-)^2)| \leq 2 |f - m| |\nabla f|. \]

Therefore, applying the Cauchy-Schwarz inequality we see that

\[ \chi \mu \int_{\mathbb{R}^n} |f - m|^2 \, d\mu \leq 2 \left( \int_{\mathbb{R}^n} |f - m|^2 \, d\mu \right)^{1/2} \left( \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \right)^{1/2}. \]
We saw that
\[ \chi_{\mu} \int_{\mathbb{R}^n} |f - m|^2 d\mu \leq 2 \left( \int_{\mathbb{R}^n} |f - m|^2 d\mu \right)^{1/2} \left( \int_{\mathbb{R}^n} |\nabla f|^2 d\mu \right)^{1/2}. \]

This gives
\[ \frac{\chi_{\mu}^2}{4} \int_{\mathbb{R}^n} |f - m|^2 d\mu \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\mu. \]

Since
\[ \int_{\mathbb{R}^n} |f - \mathbb{E}_\mu(f)|^2 d\mu = \min_{\alpha \in \mathbb{R}} \int_{\mathbb{R}^n} |f - \alpha|^2 d\mu \leq \int_{\mathbb{R}^n} |f - m|^2 d\mu \]
and \( f \) was arbitrary, we get \( \varphi_{\mu}^2 \leq 4\chi_{\mu}^{-2} = 4\psi_{\mu}^2 \).
A Borel probability measure $\mu$ on $\mathbb{R}^n$ is called log-concave if for all compact subsets $A, B$ of $\mathbb{R}^n$ and all $0 < \lambda < 1$ we have

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda} \mu(B)^\lambda.$$

**Theorem (Buser, Ledoux)**

Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$ with reciprocal Cheeger constant $\psi_\mu$. Then its Poincaré constant $\vartheta_\mu$ satisfies

$$\psi_\mu \leq c \vartheta_\mu.$$
Isotropic measures

- We say that a Borel probability measure $\mu$ on $\mathbb{R}^n$ is isotropic if $\text{bar} (\mu) = \int_{\mathbb{R}^n} x d\mu (x) = 0$ and $\mu$ satisfies the isotropic condition
  $$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu (x) = 1, \quad \theta \in S^{n-1}. $$

- Similarly, we shall say that a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with barycenter $\text{bar} (f) = 0$ is isotropic if $\int f (x) dx = 1$ and the measure $d\mu (x) = f (x) dx$ is isotropic.

- A convex body $K$ of volume 1 in $\mathbb{R}^n$ with barycenter at the origin is called isotropic if
  $$\int_{K} \langle x, \theta \rangle^2 dx = L_K^2 $$
  for some constant $L_K > 0$ (the isotropic constant of $K$) and all $\theta \in S^{n-1}$.

- One can check that $K$ is isotropic if and only if the function $f_K := L_K^n \frac{1}{L_K} \mathbf{1}_K$ is an isotropic log-concave function.

- Every non-degenerate absolutely continuous probability measure $\mu$ has an isotropic image $\nu = \mu \circ S$, where $S : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map. Similarly, every log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < \infty$ has an isotropic image: there exist an affine isomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$ and a positive number $a$ such that $af \circ S$ is isotropic.
Let $f$ be a log-concave function with finite, positive integral. The covariance matrix $\text{Cov}(f)$ is the matrix with entries

$$\text{Cov}(f)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f(x) \, dx}{\int_{\mathbb{R}^n} f(x) \, dx} - \frac{\int_{\mathbb{R}^n} x_i f(x) \, dx}{\int_{\mathbb{R}^n} f(x) \, dx} \frac{\int_{\mathbb{R}^n} x_j f(x) \, dx}{\int_{\mathbb{R}^n} f(x) \, dx}.$$  

If $f$ is the density of a measure $\mu$, we denote this matrix also by $\text{Cov}(\mu)$. Note that if $f$ is isotropic then $\text{Cov}(f)$ is the identity matrix.

The isotropic constant of $f$ is defined by

$$L_f := \left( \frac{\sup_{x \in \mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x) \, dx} \right)^{\frac{1}{n}} \left[ \det \text{Cov}(f) \right]^{\frac{1}{2n}}.$$  

(and given a log-concave measure $\mu$ with density $f_\mu$, we let $L_\mu := L_{f_\mu}$).

It is easy to check that the isotropic constant $L_\mu$ is an affine invariant.
Conjecture 1: Isotropic constant

- One can also prove that if $f: \mathbb{R}^n \to [0, \infty)$ is a log-concave density, then
  \[
  nL_f^2 = \inf_{S \in SL_n} \left( \sup_{x \in \mathbb{R}^n} f(x) \right)^{2/n} \int_{\mathbb{R}^n} |S(x) + y|^2 f(x) \, dx.
  \]

- If $f: \mathbb{R}^n \to [0, \infty)$ is an isotropic log-concave function then
  \[
  L_f = \|f\|_\infty^{1/n} \geq c,
  \]
  where $c > 0$ is an absolute constant.

This would imply that a convex body of volume one, in any dimension, has at least one hyperplane section with volume bounded from below by an absolute constant (slicing problem).
Conjecture 1: Isotropic constant

- Define
  \[ L_n := \sup \{ L_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n \}. \]

- Then, Conjecture 1 states that \( L_n \leq C \) for an absolute constant \( C > 0 \).

- Around 1985-6 (published in 1991), Bourgain introduced this conjecture and obtained the upper bound \( L_n \leq c^{4/\sqrt{n}} \ln n \).

- In 2006 the estimate was improved by Klartag, who showed that the logarithmic factor can be omitted.

**Theorem (Bourgain/Klartag)**

*There exists an absolute constant \( c > 0 \) such that \( L_n \leq c^{4/\sqrt{n}} \) for all \( n \geq 1 \).*
KLS-conjecture

- Kannan, Lovász and Simonovits conjectured in 1994 that the isoperimetric ratio of any Borel set $A$ with respect to the uniform measure $\mu_K$ on a convex body $K$ in $\mathbb{R}^n$ (defined by $\mu_K(A) = \frac{\text{vol}_n(K \cap A)}{\text{vol}_n(K)}$) should be, up to an absolute constant, at least as large as the minimal isoperimetric ratio over all half-spaces.

\[
\chi(K) \geq c \cdot \inf_H \frac{\mu_K^+(H)}{\min\{\mu_K(H), \mu_K(\mathbb{R}^n \setminus H)\}}
\]

for some absolute constant $c > 0$, where the infimum is over all half-spaces $H$ in $\mathbb{R}^n$.

- Their interest in this parameter was related to the study of randomized volume algorithms.
- Since the isoperimetric ratio of a half-space is basically a one-dimensional quantity, one can obtain an explicit formula for this infimum. Then, one arrives at the following conjecture:
KLS-conjecture

Conjecture 2

\[ \chi(K) \approx 1/\sqrt{\lambda(K)} \]

where \( \lambda(K) \) is the largest eigenvalue of the matrix of inertia \( M_{ij} := \int_K x_i x_j dx \) of \( K \).

- They actually proved that one always has \( \chi(K) \leq 10/\sqrt{\lambda(K)} \), therefore the question is about the lower bound.

Theorem (Kannan-Lovász-Simonovits)

For every convex body \( K \) in \( \mathbb{R}^n \) one has

\[ \chi(K) \geq \frac{\ln 2}{l_1(K)}. \]

- Here,

\[ l_1(K) := \frac{1}{\text{vol}_n(K)} \int_K |x - \text{bar}(K)| \, dx. \]

- If \( K \) is isotropic this gives \( \chi(K) \geq c/(\sqrt{n}L_K) \).

- In fact, one may find literature on the subject before their work, and there were known lower bounds for \( \chi(K) \) of order \( 1/\text{diam}(K) \).
Another approach to the KLS-conjecture is due to Bobkov.

**Theorem (Bobkov)**

*Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$. Then we have*

$$\chi_\mu \geq \frac{c}{\|f\|_{L_2(\mu)}},$$

*where $f(x) = |x - \text{bar}(\mu)|$ and $c > 0$ is an absolute constant.*

- If $\mu$ is isotropic this gives $\chi_\mu \geq c/\sqrt{n}$.
Conjecture 2

\[ \chi(K) \approx \frac{1}{\sqrt{\lambda(K)}} \]

where \( \lambda(K) \) is the largest eigenvalue of the matrix of inertia \( M_{ij} := \int_K x_i x_j dx \) of \( K \).

- For an isotropic convex body \( K \) this becomes \( \chi(K) \approx \frac{1}{L_K} \).

KLS-Conjecture

For every isotropic log-concave probability measure \( \mu \) on \( \mathbb{R}^n \) one has \( \chi_\mu \geq c \), where \( c > 0 \) is an absolute constant.

Theorem (Eldan-Klartag)

\[ L_n \leq C \psi_n = C / \chi_n. \]

- In other words, the KLS-conjecture is stronger than Conjecture 1 about the isotropic constant.
The currently best known results are due to Lee and Vempala and are consequences of the following theorem:

**Theorem (Lee-Vempala)**

If $\mu$ is a log-concave probability measure on $\mathbb{R}^n$ with covariance matrix $A$ then

$$\psi_\mu \leq c \left( \text{tr}(A^2) \right)^{1/4}$$

where $c > 0$ is an absolute constant.

- If we make the additional assumption that $\mu$ is isotropic then we obtain the upper bound
  $$\psi_\mu \leq c^4 \sqrt{n}.$$  

- The approach of Lee and Vempala is based on Eldan’s stochastic localization.
An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture

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Abstract

We prove an almost constant lower bound of the isoperimetric coefficient in the KLS conjecture. The lower bound has dimension dependency $d^{-\omega(1)}$. When the dimension is large enough, our lower bound is tighter than the previous best bound which has dimension dependency $d^{-1/4}$. Improving the isoperimetric coefficient in the KLS conjecture has many implications, including improvements of the bounds in the thin-shell conjecture and in the slicing conjecture, better concentration inequalities for Lipschitz functions of log-concave measures and better mixing time bounds for MCMC sampling algorithms on log-concave measures.
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Quick Bio:

I am a postdoc fellow at ETH Foundations of Data Science (ETH-FDS) in ETH Zürich under the supervision of Prof. Peter Bühlmann. Previously, I obtained my PhD in the Department of Statistics at UC Berkeley in 2019. My PhD study was advised by Prof. Bin Yu. During my PhD, I am fortunate to also work with Prof. Martin Wainwright and Prof. Jack Gallant.

My main research interests lie on statistical machine learning, optimization and the applications in neuroscience. In particular, I am interested in domain adaptation, stability, MCMC sampling algorithms, convolutional neural networks and statistical problems that arise from computational neuroscience. Before my PhD study, I obtained my Diplome d'Ingénieur (Eng. Deg. in Applied Mathematics) at Ecole Polytechnique in France.