Notes on the irrational rotation nonselfadjoint algebras

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(partly based on joint work with M. Anoussis and I.G. Todorov)

As before, \(^1\) fix \(\theta \in \mathbb{R}\) s.t. \(\frac{\theta}{2\pi}\) is irrational. Let \(\mathcal{A} = C(\mathbb{T})\), \(G = \mathbb{Z}\) and

\[(\alpha_n f)(z) = f(e^{i\theta n} z) \quad (f \in \mathcal{A}, n \in \mathbb{Z}, z \in \mathbb{T}).\]

The semicrossed product \(C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}_+\) is a closed subalgebra of the irrational rotation algebra \(C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}\) (why?). \(^2\)

Thus the representation \(\pi \times \lambda : C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z} \to \mathcal{B}(L^2(\mathbb{T}))\) restricts to an isometric representation of \(C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}_+\) given by (flip) \(^3\)

\[\left(\pi \times \lambda\right) \left(\sum_{k=0}^{n} \delta_k \otimes f_k\right) = \sum_{k=0}^{n} V^k \pi(f_k)\]

where \(V\) is the generator \(\lambda_1\) of \(\{\lambda_n : n \in \mathbb{Z}_+\}\) given by

\[(Vg)(z) = g(e^{i\theta} z), \quad g \in L^2(\mathbb{T}).\]

The \(C^*\)-algebra \(C(\mathbb{T})\) is the closed algebra generated by \(\zeta\) and \(\bar{\zeta}\), where \(\zeta(z) = z\); hence \(\pi(C(\mathbb{T})) \subseteq \mathcal{B}(L^2(\mathbb{T}))\) is generated by \(U := \pi(\zeta)\) and \(U^* = \pi(\bar{\zeta})\). Therefore \(C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}_+\) is generated by \(\{U, U^*, V\}\) and the crossed product \(C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z} := \mathcal{A}_0\) is generated by \(\{U, U^*, V, V^*\}\). They satisfy:

\[UV = e^{i\theta} VU\quad \text{(the Weyl relation)}.

For \(\mu \in \mathbb{T}\), let \(V_{\mu} : L^2(\mathbb{T}) \to L^2(\mathbb{T})\) be given by \(V_{\mu} f = f_{\mu}\) where \(f_{\mu}(z) = f(\mu z)\). The map \(\mu \to V_{\mu}\) is a SOT-continuous group homomorphism into

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\(^2\)This is non-trivial; one shows (see e.g. [2]) that the norm of the semicrossed product (defined as a sup over all covariant pairs \((\pi, T)\) with \(T\) a contraction coincides with the (a priori smaller) norm of the crossed product, where the sup is taken over \((\pi, U)\) with \(U\) unitary.

\(^3\)This is a minor technicality.
the unitary group of $\mathcal{B}(L^2(\mathbb{T}))$; hence it is weak$^*$ continuous (because it takes values in a ball). Observe that $V_{e^{i\theta}} = V$. Note that $(V\mu f)(n) = \mu f(n).$ \footnote{f \to \hat{f} : L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ is the Fourier transform.} More generally, if $a = (a_n)_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$, let $D_a$ be given by $(D_a f)(n) = a_n f(n);$ thus $D_a$ is the image, under conjugation by the Fourier transform, of the diagonal operator on $l^2(\mathbb{Z})$ given by $(x_j) \to (a_j x_j).$ Let $\mathcal{D} = \{D_a : a \in \ell^\infty(\mathbb{Z})\}$. This is a masa on $L^2(\mathbb{T})$ (being unit. equivalent to the diagonal masa on $l^2(\mathbb{Z})$).

\begin{lemma}
The weak$^*$ closed operator algebra on $L^2(\mathbb{T})$ generated by $V$ coincides with $\mathcal{D}$. In particular, it contains $V^*$.
\end{lemma}

\begin{proof}
Since $\mu \to V_\mu$ is weak$^*$ continuous and $\{e^{in\theta} : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$, the weak$^*$ closed algebra generated by the set $\{V^n : n \in \mathbb{N}\}$ equals $\{V_\mu : \mu \in \mathbb{T}\}$ and hence is selfadjoint. Since it is weak$^*$ closed, it is equal to its bicommutant which is clearly $\mathcal{D}$. \hfill \Box
\end{proof}

\begin{remark}
By contrast, the weak$^*$ closed operator algebra on $L^2(\mathbb{T})$ generated by $U$ is non-selfadjoint; it is equal to $\{M_f : f \in H^\infty(\mathbb{T})\}$. In fact, the weak$^*$ closed operator algebra generated by $U$ and $V$ does not contain $U^*$. \footnote{This also follows from Proposition 3, since $U^*$ does not leave $N_m$ invariant.}
\end{remark}

\begin{proof}
Recall that $H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : \hat{f}(k) = 0 \text{ for } k < 0\}.$

It is well-known that the Cesaro means of the Fourier series of any $f \in L^\infty(\mathbb{T})$ converges to $f$ in the weak$^*$ topology on $L^\infty(\mathbb{T})$ induced by $L^1(\mathbb{T})$. Thus any $f \in H^\infty(\mathbb{T})$ is a weak$^*$ limit of polynomials in $\zeta$ (analytic polynomials) and hence $M_f$ is a weak$^*$ limit of polynomials in $U$.

On the other hand, the weak$^*$ continuous linear form $T \to \langle T \zeta^0, \zeta^{-1} \rangle$ annihilates all polynomials in $U, V$ (since $\langle U^k V l \zeta^0, \zeta^{-1} \rangle = \langle U^k \zeta^0, \zeta^{-1} \rangle = 0$ when $k \geq 0$) but does not annihilate $U^*$ (since $\langle U^* \zeta^0, \zeta^{-1} \rangle = 1\).$ \hfill \Box
\end{proof}

\begin{proposition}
The $w^*$-closed subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ generated by $\{U, V\}$ is the nest algebra $\text{Alg} N$ of all operators $T \in \mathcal{B}(L^2(\mathbb{T}))$ leaving all elements of $N = \{N_n : n \in \mathbb{Z}\}$ invariant, where $N_n = \{f \in L^2(\mathbb{T}) : \hat{f}(k) = 0, k < n\}$.
\end{proposition}

\begin{proof}
Clearly, $U$ and $V$ belong to $\text{Alg} N$, hence so does the weak$^*$ closed operator algebra that they generate.

By Lemma 1, the weak$^*$ closed algebra generated by $V$ is equal to $\mathcal{D}$. On the other hand, if $a \in c_{00}(\mathbb{Z})$, the matrix of $U^k D_a$ with respect to the basis $\{\zeta^k\}_{k \in \mathbb{Z}}$ has the sequence $a$ at the $l$-th diagonal below the main diagonal
and zeros elsewhere. It follows that all lower triangular matrix units belong to the weak* closed algebra generated by $U$ and $V$, and hence it equals $\text{Alg} N$.

**Remark 4.** Observe that $\text{Lat} \{U, V\} = \{\zeta^k H^2 : k \in \mathbb{Z}\}$. Thus, every invariant subspace of $\{U, V\}$ is actually reduced by the semigroup generated by $V$; hence it is invariant under the “larger” semigroup generated by $\{U, V, V^{-1}\}$.

After Fourier transform $L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$:

$$U \sim \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 \\ \vdots & 0 & 1 & 0 & 0 \\ \vdots & 0 & 0 & 1 & 0 \\ \end{bmatrix}, \quad V \sim \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \lambda^2 & 0 & 0 & 0 \\ \vdots & 0 & \lambda & 0 & 0 \\ \vdots & 0 & 0 & \lambda & 0 \\ \vdots & 0 & 0 & 0 & \lambda^2 \\ \end{bmatrix}$$

We write $A^+_{\theta}$ and $A^{++}_{\theta}$ for the norm-closed subalgebras of $A_{\theta}$ generated by $\{U, V, V^*\}$ and $\{U, V, I\}$ respectively.

Now $N$ becomes the $\mathbb{Z}$-ordered nest on $\ell^2(\mathbb{Z})$ with non-trivial elements $N_m, m \in \mathbb{Z}$ where $N_m = [e_k : k \geq m]$; thus $U(N_m) = N_{m+1} \subset N_m$ and $V(N_m) = N_m$. It follows that $U, V$ and $V^*$ lie in the nest algebra $\text{Alg} (N)$ and so

$$A^{++}_{\theta} \subset A^+_{\theta} \subset A_{\theta} \cap \text{Alg} (N).$$

We have shown in Proposition 3 that the weak-* closure of $A^{++}_{\theta}$ is the whole of $\text{Alg} (N)$, and so the same is true for the w* closure of $A^+_{\theta}$ (but this might be obvious anyway). Thus

$$W^*(A^{++}_{\theta}) = W^*(A^+_{\theta}) = \text{Alg} (N).$$

On the other hand, since $A_{\theta}$ is an irreducible C*-algebra, its w* closure is $B(H)$.

For the proof of the following Proposition, we shall need a conditional expectation $\Psi : A_{\theta} \to C^*(V)$. This is constructed as follows (see [1, Theorem VI.1.1] for more details):
There is a *-automorphism \( \rho_t \) of \( A_\theta \) given on the generators by \( \rho_t(U) = e^{it}U \) and \( \rho_t(V) = V \). Moreover for all \( a \in A_\theta \) the map \( t \rightarrow \rho_t(a) \) is (norm-) continuous. Thus the integral

\[
\Psi(a) = \frac{1}{2\pi} \int_0^{2\pi} \rho_t(a) dt
\]

exists. It is easy to see that \( \Psi \) is linear positive unital and \( \| \Psi \| = 1 \).

On easily verifies that if \( a = \sum_{k,l} c_{k,l} V^k U^l \) is a finite sum,

\[
\Psi(a) = \sum_{k,0} c_{k,0} V^k U^0.
\]

It follows by continuity of \( \Psi \) that it is an idempotent mapping \( A_\theta \) onto the \( C^* \)-subalgebra \( C^*(V) \) generated by \( V \); it is a conditional expectation.

The following formula holds for all \( a \in A_\theta \)

\[
\Psi(a) = \lim_n \frac{1}{2n+1} \sum_{|k| \leq n} V^k a V^{-k}.
\]

This can be verified when \( a \) is a finite sum as above; \(^6\) hence it is valid on the whole of \( A_\theta \) by continuity.

Any \( a \in A_\theta \) has a formal expansion:

\[
a \sim \sum_{n \in \mathbb{Z}} \Psi(U^n) a U^n.
\]

When \( a = \sum_{k,l} c_{k,l} V^k U^l \) is a finite sum, one verifies that the above formula is in fact an equality.

For general \( a \in A_\theta \), the means of the partial sums sum \( s_k(a) = \sum_{|n| \leq k} \Psi(U^{-n}) U^n \) converge in norm to \( a \). Indeed,

\[
\sigma_m(a) := \frac{1}{m+1} (s_0(a) + \cdots + s_m(a))
\]

\[
= \sum_{|n| \leq m} \left( 1 - \frac{|n|}{m+1} \right) \Psi(U^{-n}) U^n
\]

\[
= \sum_{|n| \leq m} \left( 1 - \frac{|n|}{m+1} \right) \int_0^{2\pi} \rho_t(U^{-n}) \frac{dt}{2\pi} U^n
\]

\[
= \sum_{|n| \leq m} \left( 1 - \frac{|n|}{m+1} \right) \int_0^{2\pi} \rho_t(U^{-n}) e^{-int} \frac{dt}{2\pi} = \int_0^{2\pi} \rho_t(a) e^{-int} \frac{dt}{2\pi}
\]

\(^6\) use the fact that \( \lim_n \frac{1}{n+1} \sum_{|k| \leq n} e^{iks} = 0 \) unless \( s = 0 \)
where $k_n$ is Féjer’s kernel. We all know that if $f$ is continuous (even Banach-space valued), then $\frac{2\pi}{t} f(t)K_n(t) \to f(0)$ and so $\sigma_n(a) \to \rho_0(a) = a$.

**Proposition 5.** We have $A_\theta^+ = A_\theta \cap \text{Alg}(\mathcal{N})$. In other words $A_\theta^+$ is a nest subalgebra of a $C^*$-algebra.

**Proof.** Suppose that $a \in A_\theta \cap \text{Alg}(\mathcal{N})$, so $a(N_m) \subseteq N_m$ for all $m \in \mathbb{Z}$.

In order to show that $a \in A_\theta^+$, it suffices by the discussion above to show that in the expansion $\sum_{n \in \mathbb{Z}} \psi(aU^{-n})U^n$ the terms $\psi(aU^{-n})U^n$ vanish when $n < 0$.

**Claim.** Since $a(N_m) \subseteq N_m$, the same holds for each monomial $\psi(aU^{-n})U^n$.

**Proof.** Recall that $\psi(aU^{-n})$ is the limit of convex sums of terms $V^kaU^{-n}V^{-k}$ for which we have

$$V^kaU^{-n}V^{-k}(N_m) = V^kaU^{-n}(N_m) \subseteq V^ka(N_{m-n}) \subseteq V^k(N_{m-n}) = N_{m-n}$$

and therefore $\psi(aU^{-n})(N_m) \subseteq N_{m-n}$ so that $\psi(aU^{-n})U^n(N_m) \subseteq \psi(aU^{-n})N_{m+n} \subseteq N_m$ as claimed.

Now $\psi(aU^{-n}) \in C^*(V)$; write $\psi(aU^{-n}) = f_n(V)$ for some continuous function $f_n$ on the spectrum of $V$ (which is actually the whole of $T$; why?).

By the Claim, for each $m \in \mathbb{Z}$, we have $f_n(V)U^n e_m \in N_m$ and so $\langle f_n(V)U^n e_m, e_{m+p} \rangle = 0$ when $p < 0$. But $f_n(V)U^n e_m = f_n(V)e_{n+m} = f_n(\beta^{n+m})e_{n+m}$ (where we write $\beta = e^{i\theta}$) and so we obtain

$$0 = \langle f_n(V)U^n e_m, e_{m+p} \rangle = \langle f_n(\beta^{m+n})e_{m+n}, e_{m+p} \rangle = f_p(\beta^{m+p}).$$

Since this holds for all $m \in \mathbb{Z}$ and $\theta$ is irrational and $f_p$ is continuous, it follows that $f_p$ must vanish identically. Thus $f_p(V) = \psi(aU^{-p}) = 0$ for all $p < 0$, and so $a \sim \sum_{n \geq 0} \psi(aU^{-n})U^n$ is in $A_\theta^+$. \hfill $\square$

**Proposition 6.** The inclusion $A_\theta^{++} \subset A_\theta^+$ is proper.

**Proof.** If $p(U,V) = \sum_{k,l \geq 0} c_{k,l} U^kV^l$ is a polynomial in $A_\theta^{++}$, then the diagonal of $V^* - p(U,V)$ is the operator $V^* - \sum_{l \geq 0} c_{0,l}V^n$ whose norm is at least 1. Thus $\|V^* - p(U,V)\| \geq 1$ for every polynomial in $A_\theta^{++}$, showing that $V^*$ is (in $A_\theta^+$ but not in $A_\theta^{++}$.
In more detail, for each \( n \in \mathbb{Z} \) (writing \( \beta = e^{i\theta} \) again)

\[
\langle (V^* - p(U,V))e_n, e_n \rangle = \langle V^*e_n, e_n \rangle - \sum_{k,l \geq 0} c_{k,l} \langle U^k V^l e_n, e_n \rangle
\]

\[
= \langle \bar{\beta}^n e_n, e_n \rangle - \sum_{k,l \geq 0} c_{k,l} \langle U^k \bar{\beta}^l e_n, e_n \rangle
\]

\[
= \bar{\beta}^n - \sum_{k,l \geq 0} c_{k,l} \bar{\beta}^l \langle e_{n+k}, e_n \rangle = \bar{\beta}^n - \sum_{l \geq 0} c_{0,l} \bar{\beta}^l
\]

and thus

\[
\|V^* - p(U,V)\| \geq \sup_n |\langle (V^* - p(U,V))e_n, e_n \rangle| = \sup_n |\bar{\beta}^n - \sum_{l \geq 0} c_{0,l} \bar{\beta}^l|.
\]

But since \( \{\beta^n : n \in \mathbb{Z}\} \) is dense in \( T \), the last supremum is the same as

\[
\sup\{ |\bar{z} - \sum_{l \geq 0} c_{0,l} \bar{z}^l| : z \in T \}. \]

But this is at least 1. \(^7\) Therefore finally

\[
\|V^* - p(U,V)\| \geq 1
\]

for any polynomial \( p(U,V) \), and so \( V^* \) is not in the closure \( \mathcal{A}_b^+ \) of such polynomials. \( \square \)

References


\(^7\) The functions \( \zeta_k(z) = z^k \) are orthonormal in \( L^2(T) \), and so

\[
\sup\{ |\bar{z} - \sum_{l \geq 0} c_{0,l} \bar{z}^l| : z \in T \}^2 = \left| \zeta_{-1} - \sum_{l \geq 0} c_{0,l} \zeta_l \right|^2
\]

\[
\geq \left| \zeta_{-1} - \sum_{l \geq 0} c_{0,l} \zeta_l \right|^2 = |\zeta_{-1}|^2 + \sum_{l \geq 0} c_{0,l} |\zeta_l|^2 \geq 1.
\]