A (discrete) **C*-dynamical system** is a triple \((A, \alpha, G)\) where \(\alpha : G \to \text{Aut}(A)\) is a group morphism into the group of \(*\)-automorphisms of \(A\).

**Definition**

A **covariant representation** of a C*-dynamical system \((\mathcal{A}, \alpha, G)\) on a Hilbert space \(H\) is a pair \((\pi, U : H)\) where \(\pi\) is a \(*\)-representation of \(\mathcal{A}\) on \(H\), \(U\) is a unitary representation of \(G\) on the same \(H\) and \(\pi\) and \(U\) are connected by the **covariance condition**:

\[
\pi(\alpha_g(a)) = U_g \pi(a) U^*_g \quad (a \in \mathcal{A}, g \in G).
\]
Example

Let $\Omega$ be a locally compact Hausdorff space, $G$ a group of homeomorphisms of $\Omega$, $\mu$ a $G$-invariant Borel measure on $\Omega$ (thus $\mu(tS) = \mu(S)$ for all $t \in G$ and $S \subseteq \Omega$ Borel).

Let $A = C_0(\Omega)$ and $\alpha_t(a) = a \circ t^{-1}$.

Represent $A$ on $H = L^2(\Omega, \mu)$ as multiplication operators:

$$\pi(a)f = af \quad (a \in A, f \in H).$$

Represent $G$ on $H$ by composition:

$$U_t f = f \circ t^{-1}$$

(the fact that each $U_t$ is unitary follows from the fact that $\mu$ is $G$-invariant).

The pair $(\pi, U)$ is covariant.
The twisted convolution algebra

\[ A \otimes c_{00}(G) = c_{00}(G; A) = \{ f : G \to A : \text{supp} f \text{ finite} \} \]

This is the linear span of the functions \( a \otimes f, a \in A, f \in c_{00}(G) \) where

\[ (a \otimes f)(t) = af(t) \in A \]

It is also the linear span of the functions \( a \otimes \delta_s, a \in A, s \in G \) where

\[ (a \otimes \delta_s)(t) = \begin{cases} a, & t = s \\ 0, & t \neq s \end{cases} \]

So \( f = \sum_t f(t) \otimes \delta_t \).

Given covariant pair \( (\pi, U : H) \), define \( (\pi \times U)(a \otimes \delta_s) = \pi(a) U_s \), i.e.

\[ (\pi \times U) \left( \sum_t f(t) \otimes \delta_t \right) := \sum_t \pi(f(t)) U_t \in B(H) \]
The twisted convolution algebra

Want to define $\ast$-algebra structure on $A \otimes c_{00}(G)$ making $\pi \times U$ a $\ast$-representation: covariance requires

$$\pi(a)U_s\pi(b)U_r = \pi(a)\pi(\alpha_s(b))U_sU_r,$$ so

$$(a \otimes \delta_s) \ast (b \otimes \delta_r) = (a\alpha_s(b)) \otimes \delta_{sr}$$

i.e. $$(\phi \ast \psi)(t) = \sum_{sr=t} \phi(s)\alpha_s(\psi(r)) = \sum_{s \in G} \phi(s)\alpha_s(\psi(s^{-1}t)).$$

and

$$(\pi(a)U_s)^\ast = U_{s^{-1}}\pi(a^\ast) = \pi(\alpha_{s^{-1}}(a^\ast))U_{s^{-1}},$$ so

$$(a \otimes \delta_s)^\ast = \alpha_{s^{-1}}(a^\ast) \otimes \delta_{s^{-1}}$$

i.e. $$\phi^\ast(t) = \alpha_t(\phi^\ast(t^{-1})).$$
The (full) crossed product

**Definition**

*The completion of the twisted convolution algebra $(A \otimes c_{00}(G), \ast)$ with respect to*

$$\| f \| := \sup \{ \| (\pi \times U)(f) \| : (\pi, U : H) \text{ covariant rep.} \} \quad (*)$$

*is called* the (full) crossed product $A \rtimes_\alpha G$.

It is a $C^*$-seminorm, but why a norm?
Existence of covariant representations

For each *-rep $\pi : A \to \mathcal{B}(H_0)$ define

$$H = H_0 \otimes \ell^2(G) \cong \ell^2(G, H_0).$$
Define a representation $\tilde{\pi}$ of $A$ on $H$ by

$$\tilde{\pi}(a)(x \otimes \delta_s) = \pi(\alpha_{s^{-1}}a)x \otimes \delta_s, \quad \text{i.e.} \quad \tilde{\pi}(a) = \text{diag}(\pi(\alpha_{s^{-1}}a))$$

$$(\tilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))(\xi(t)) \quad (a \in A, \xi \in \ell^2(G, H_0)). \quad (1)$$

Define a unitary representation $\Lambda$ of $G$ on $H$ by

$$\Lambda_s(x \otimes \delta_t) = x \otimes \delta_{st}, \quad \text{i.e.}$$

$$(\Lambda_s\xi)(t) = \xi(s^{-1}t) \quad (s \in G, \xi \in \ell^2(G, H_0)). \quad (2)$$

This is covariant; and if $\pi$ is faithful on $A$, then $\tilde{\pi} \times \Lambda$ is faithful on the convolution algebra $A \otimes c_{00}(G)$. 
\[ \tilde{\pi} \times \Lambda \text{ is faithful on } A \otimes c_{00}(G) \]

Indeed, if \( f = \sum_t f(t) \otimes \delta_t \in A \otimes c_{00}(G) \), then, for each \( x \in H_0 \),

\[
(\tilde{\pi} \times \Lambda)(f)(x \otimes \delta_1) = \left( \sum_t \tilde{\pi}(f(t))\Lambda_t \right) (x \otimes \delta_1)
\]

\[
= \sum_t \tilde{\pi}(f(t))(x \otimes \delta_t)
\]

\[
= \sum_t \pi(\alpha_{t-1}(f(t)))x \otimes \delta_t
\]

\[
\Rightarrow \| (\tilde{\pi} \times \Lambda)(f)(x \otimes \delta_1) \|^2 = \sum_t \| \pi(\alpha_{t-1}(f(t)))x \|^2 \quad (3)
\]

hence if \( (\tilde{\pi} \times \Lambda)(f) = 0 \) then for each \( x \in H_0 \) and \( t \in G \) we have \( \pi(\alpha_{t-1}(f(t)))x = 0 \) and so each \( f(t) \) vanishes since \( \pi \circ \alpha_{t-1} \) is injective.
Thus (*) defines a norm $\| \cdot \|$ on $A \otimes c_{00}(G)$. The completion of $A \otimes c_{00}(G)$ with respect to the (a priori smaller) norm

$$\| f \|_r := \| (\tilde{\pi} \times \Lambda)(f) \|$$

is called the reduced crossed product $A \rtimes_r G$. It coincides with $A \rtimes G$ when $G$ is abelian, or compact, but not necessarily when $G = \mathbb{F}_2$. 

Existence of covariant representations
If \( G = \mathbb{Z} \), \( A = \mathbb{C} \) and \( \alpha \) is the trivial action, then the unitary \( V := \Lambda_1 \) is just the bilateral shift on \( \ell^2(\mathbb{Z}) \), which is unitarily equivalent to multiplication by \( z \) on \( L^2(\mathbb{T}) \). If \( \pi \) is the identity representation of \( \mathbb{C} \) as operators on \( \mathbb{C} \), then the representation \( \tilde{\pi} \times \Lambda \) extends to a faithful representation of \( A \rtimes_{id} \mathbb{Z} \) on \( L^2(\mathbb{T}) \). If \( \phi = \sum \phi_k \otimes \delta_k \) is in \( c_{oo}(\mathbb{Z}) \), then \( (\tilde{\pi} \times \Lambda)(\phi) = \sum \phi_k V^k \) is the operator of multiplication by the function \( \sum \phi_k z^k \), whose norm is precisely the supremum norm of the function.

Since such functions are dense in \( C(\mathbb{T}) \), it follows that \( \mathbb{C} \times_{id} \mathbb{Z} \) is isometrically isomorphic to \( C(\mathbb{T}) \).

The dense subalgebra \( \ell^1(\mathbb{Z}) \) of \( \mathbb{C} \times_{id} \mathbb{Z} \) is mapped by \( \tilde{\pi} \times \Lambda \) to the Wiener algebra, that is the algebra of all \( f \in C(\mathbb{T}) \) whose Fourier series is absolutely convergent.
For $\phi = \sum_t \phi_t \otimes \delta_t \in A \otimes c_{00}(G)$ call $\phi_t$ the $t$-th Fourier coefficient of $\phi$.

Fix a faithful rep. $\pi_0$ of $A$. Note that by (3),

$$\|\pi_0(\alpha_t^{-1}(\phi_t))\| \leq \|(\tilde{\pi}_0 \times \Lambda)(\phi)\|$$

for each $t \in G$. Now

$$\|\phi_s\|_A = \|\pi_0(\alpha_s^{-1}(\phi_s))\| \leq \sup\{\|(\pi \times U)(\phi)\| : \pi \times U \text{ covariant pair}\} = \|\phi\|.$$

Hence the map

$$E_s : A \otimes c_{00}(G) \to A : \phi \to \phi_s$$

is contractive, so extends to a contraction

$$E_s : A \rtimes_\alpha G \to A.$$

Clearly if $a \in A \rtimes_\alpha G$ has $(\tilde{\pi}_0 \times \Lambda)(a) = 0$ then each $E_s(a) = 0$. Hence if the “Fourier transform” is injective, the reduced crossed product coincides with the full crossed product.
Abelian groups

For $G$ abelian, let $\hat{G} = \{ \gamma : G \to \mathbb{T} : \text{cts homom.} \}$ be the dual group. For $\gamma \in \hat{G}$, let

$$\theta_\gamma \left( \sum_g \phi_g \otimes \delta_g \right) = \sum_g \phi_g \otimes \gamma(g) \delta_g.$$ 

Each $\theta_\gamma$ extends to an isometric *-automorphism of $\mathcal{A} \times_\alpha G$ and

$$E_t(B) \otimes \delta_t = \int_{\hat{G}} \theta_\gamma(B) \gamma(t^{-1}) d\gamma \quad \forall B \in \mathcal{A} \times_\alpha G, \forall t \in G. \quad (*)$$

Let $\xi$ be a continuous linear form on $\mathcal{A} \times_\alpha G$. Let $f(\gamma) = \xi(\theta_\gamma(B))$; its Fourier transform is

$$\hat{f}(t) = \int_{\hat{G}} f(\gamma) \gamma(t^{-1}) d\gamma = \xi(E_t(B) \otimes \delta_t).$$

So if each $E_t(B)$ is zero, then $\hat{f} = 0$ and so $f = 0$; therefore $B = 0$. 
Abelian groups

Note that $\theta_{\gamma}(a \otimes \delta_e) = a$ when $a \in \mathcal{A}$ hence $E_e(a \otimes \delta_e) = a$ by (*).

Identify $\mathcal{A}$ with its image $\{a \otimes \delta_e : a \in \mathcal{A}\}$ in $\mathcal{A} \times \alpha G$. The map

$$E_e : \mathcal{A} \times \alpha G \to \mathcal{A}$$

is a contractive projection, and

$$E_e(aBc) = E_e(aBc) \otimes \delta_e = \int_{\Gamma} \theta_{\gamma}(aBc)\gamma(e^{-1})d\gamma = \int_{\Gamma} a\theta_{\gamma}(B)cd\gamma$$

$$= a(E_e(B))c$$

*conditional expectation*. Also, faithful:

$$0 = E_e(B^*B) = \int_{\Gamma} \theta_{\gamma}(B^*B)d\gamma \Rightarrow B^*B = 0 \Rightarrow B = 0$$

because $\gamma \to \theta_{\gamma}(B^*B)$ is nonneg. and continuous.
There exists a C*-algebra $\mathcal{B}$ satisfying

(a) There exist embeddings $i_A : \mathcal{A} \rightarrow \mathcal{B}$ (a $^*$-representation, necessarily 1-1) and $i_G : G \rightarrow U(\mathcal{B})$ (a - necessarily injective-group homomorphism into the unitary group $U(\mathcal{B})$ of $\mathcal{B}$) satisfying

$$i_A(\alpha_s(x)) = i_G(s)i_A(x)i_G(s)^* \text{ for all } x \in \mathcal{A}, s \in G;$$

(b) for every covariant representation $(\pi, U; H)$ of $(\mathcal{A}, G, \alpha)$, there is a non-degenerate representation $\pi \times U$ of $\mathcal{B}$ with $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_G$;

(c) the linear span of $\{i_A(x)i_G(s) : x \in \mathcal{A}, s \in G\}$ is dense in $\mathcal{B}$.

This C*-algebra $\mathcal{B}$ is unique (up to $^*$-isomorphism) and is the crossed product $\mathcal{A} \rtimes_\alpha G$. 
Fix $\theta \in \mathbb{R}$ s.t. $\frac{\theta}{2\pi}$ is irrational and write $\lambda = e^{i\theta}$. Let $\mathcal{A} = C(\mathbb{T})$, $G = \mathbb{Z}$ and

$$(\alpha_n f)(z) = f(\lambda^n z) \quad (f \in \mathcal{A}, n \in \mathbb{Z}, z \in \mathbb{T}).$$

The crossed product $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z} := \mathcal{A}_\theta$ is called the irrational rotation algebra.

The reduced representation on $H = L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$:

$$(\tilde{\pi} \times \Lambda) \left( \sum_{|k| \leq n} f_k \otimes \delta_k \right) = \left( \sum_{|k| \leq n} \tilde{\pi}(f_k) \Lambda_k \right)$$

where $\pi : C(\mathbb{T}) \to B(L^2(\mathbb{T})) : \pi(f)g = fg \quad (g \in L^2(\mathbb{T}))$

(see (1) and (2)) is faithful on $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$ since $\mathbb{Z}$ is abelian.
But the representation on $L^2(\mathbb{T})$ given by

$$(\pi \times \lambda) \left( \sum_{|k| \leq n} f_k \otimes \delta_k \right) = \left( \sum_{|k| \leq n} \pi(f_k) \lambda_k \right)$$

(where $\lambda_k = U^k$ with $U(\delta_k) = \delta_{k+1}$ the bilateral shift) is also faithful because Lebesgue measure is ergodic for the irrational rotation.

So we have two isometric representations of the same C*-algebra, $\mathcal{A}_\theta$. 
The irrational rotation algebra

But if we take $w^*$ closures:

$$\overline{((\tilde{\pi} \times \Lambda)(\mathcal{A}_\theta))}^{w^*} = L^\infty(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$$

the weak-* crossed product, which we have seen is a type II$_1$ factor.

On the other hand

$$\overline{((\pi \times \lambda)(\mathcal{A}_\theta))}^{w^*} = \mathcal{B}(L^2(\mathbb{T}))$$

(because $\pi \times \lambda$ is irreducible - ergodicity) so we get a type I$_1$ factor.

These two von Neumann algebras cannot be isomorphic (not even algebraically) for example because in $\mathcal{B}(L^2(\mathbb{T}))$ the unilateral shift $S$ satisfies $S^*S = I \neq SS^*$ whereas in $L^\infty(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$ the relation $s^*s = I$ implies $ss^* = I$. 
Generalisations:
• \( \mathcal{A} \) is now an operator algebra (preferably unital), i.e. a norm closed subalgebra of a C*-algebra, not necessarily selfadjoint (for example, the upper triangular matrices on \( \ell^2 \) or the disk algebra \( A(\mathbb{D}) \)).
• \( G \) is replaced by a unital sub-semigroup \( G^+ \) of a group \( G \) (preferably abelian)
• the action \( \alpha \) is now a homomorphism \( \alpha : G^+ \to \text{End}(\mathcal{A}) \) where \( \text{End}(\mathcal{A}) \) consists of all homomorphisms \( \mathcal{A} \to \mathcal{A} \) which are *completely contractive*.
(On a C*-algebra, every \(*\)-homomorphism is completely contractive)
The triple \((\mathcal{A}, \alpha, G^+)\) is called a semigroup dynamical system.
A covariant representation \((\pi, T; H)\) of \((A, \alpha, G^+)\) is:

\[
\pi : A \to \mathcal{B}(H) \quad \text{compl. contractive representation}
\]

\[
T : G^+ \to \mathcal{B}(H) \quad \text{contactions s.t. } T_{s+t} = T_s T_t.
\]

\[
\pi(f) T_s = T_s \pi(\alpha_s(f)), \quad f \in A, \ s \in G^+ \quad \text{(covariance)}.
\]

The covariance algebra \(c_{00}(G^+, \alpha, A)\) is \(c_{00}(G^+) \otimes A\) as a linear space with

\[
(\delta_t \otimes f) \ast (\delta_s \otimes g) = \delta_{t+s} \otimes \alpha_s(f)g.
\]

To define a norm\(^1\), fix a family \(\mathcal{F}\) of covariant pairs and put

\[
\left\| \sum_{k} \delta_{t_k} \otimes f_k \right\|_{\mathcal{F}} := \sup \left\{ \left\| \sum_{k} T_{t_k} \pi(f_k) \right\|_{\mathcal{B}(H)} : (\pi, T : H) \in \mathcal{F} \right\}
\]

\(^1\)on the quotient by ker\(\| \cdot \|_{\mathcal{F}}\), if necessary
To get an operator algebra structure need norms on $n \times n$ matrices for all $n \in \mathbb{N}$: Given $F_k = [f^{(k)}_{i,j}] \in M_n(\mathcal{A})$, for each covariant rep. $(\pi, T : H)$ get operator $[T_{t_k} \pi(f^{(k)}_{i,j})]$ on $H^n$. Define

$$\left\| \sum_k \delta_{t_k} \otimes F_k \right\|_{n, \mathcal{F}} := \sup \left\{ \left\| \sum_k [T_{t_k} \pi(f^{(k)}_{i,j})] \right\|_{\mathcal{B}(H^n)} : (\pi, T : H) \in \mathcal{F} \right\}$$

**Definition**

*The semicrossed product* $\mathcal{A} \rtimes_\alpha \mathcal{G}^+$ *is the Hausdorff* $^2$ *completion of* $c_{00}(\mathcal{G}^+, \alpha, \mathcal{A})$ *with respect to* $\| \cdot \|_{\mathcal{F}_c}$ *where* $\mathcal{F}_c$ *denotes the family of all contractive covariant pairs.*

When one restricts to the family $\mathcal{F}_{is}$ of all isometric covariant pairs, one obtains the isometric semicrossed product $\mathcal{A} \rtimes_{is} \mathcal{G}^+$.  

$^2$i.e. the completion of the quotient modulo the ideal $\ker \| \cdot \|_{\mathcal{F}_c}$
As before, fix $\theta \in \mathbb{R}$ s.t. $\frac{\theta}{2\pi}$ is irrational. Let $\mathcal{A} = C(\mathbb{T})$, $G = \mathbb{Z}$ and

$$(\alpha_n f)(z) = f(e^{in\theta} z) \quad (f \in \mathcal{A}, n \in \mathbb{Z}, z \in \mathbb{T}).$$

The semicrossed product $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}_+$ is a closed subalgebra of the irrational rotation algebra $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$ (why?). Thus the representation $\pi \times \lambda : C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z} \to \mathcal{B}(L^2(\mathbb{T}))$ restricts to an isometric representation of $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}_+$ given by (flip)

$$(\pi \times \lambda)(\sum_{k=0}^{n} \delta_k \otimes f_k) = \sum_{k=0}^{n} V^k \pi(f_k)$$

where $V$ is the generator $\lambda_1$ of $\{\lambda_n : n \in \mathbb{Z}_+\}$ given by $(Vg)(z) = g(e^{i\theta} z)$, $g \in L^2(\mathbb{T})$. 
The C*-algebra $C(\mathbb{T})$ is the closed algebra generated by $\zeta$ and $\bar{\zeta}$, where $\zeta(z) = z$; hence $\pi(C(\mathbb{T}) \subseteq B(L^2(\mathbb{T})))$ is generated by $U := \pi(\zeta)$ and $U^*$. Therefore $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}_+$ is generated by \{U, U^*, V\} and $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z} = \mathcal{A}_\theta$ is generated by \{U, U^*, V, V^*\}.

$$UV = e^{i\theta} VU$$ the Weyl relation.

**Proposition**

The $w^*$-closed subalgebra of $B(L^2(\mathbb{T}))$ generated by \{U, V\} is the nest algebra $\text{Alg}\mathcal{N}$ of all operators $T \in B(L^2(\mathbb{T}))$ leaving all elements of $\mathcal{N} = \{N_n : n \in \mathbb{Z}\}$ invariant, where $N_n = \{f \in L^2(\mathbb{T}) : \hat{f}(k) = 0, k < n\}$. 
Example: the irrational rotation

After Fourier transform $L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z})$:

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 1 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 1 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 1 & 0 & 0 \\
\vdots & 0 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \lambda^2 & 0 & 0 & 0 & 0 \\
\vdots & 0 & \lambda & 0 & 0 & 0 \\
\vdots & 0 & 0 & 1 & 0 & 0 \\
\vdots & 0 & 0 & 0 & \lambda & 0 \\
\vdots & 0 & 0 & 0 & 0 & \lambda^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

Write $A_\theta^+$ and $A_\theta^{++}$ for the norm-closed subalgebras of $A_\theta$ generated by $\{U, V, V^*\}$ and $\{U, V, I\}$ respectively.
Example: the irrational rotation

Note that $U(N_m) = N_{m+1} \subset N_m$ and $V(N_m) = N_m$. It follows that $U, V$ and $V^*$ lie in the nest algebra $Alg\mathcal{N}$ and so

$$A^{++}_\theta \subset A^+_\theta \subseteq A_\theta \cap Alg\mathcal{N}.$$ 

We have shown that the weak-* closure of $A^{++}_\theta$ is the whole of $Alg\mathcal{N}$. Thus

$$W^*(A^{++}_\theta) = W^*(A^+_\theta) = Alg\mathcal{N}.$$ 

Since $A_\theta$ is an irreducible C*-algebra, its w* closure is $B(H)$.

**Proposition**

We have $A^+_\theta = A_\theta \cap Alg\mathcal{N}$. In other words $A^+_\theta$ is a nest subalgebra of a C*-algebra.