Operator Theory Seminar

Brief notes by A.K., March 2012

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1 Positive definite functions and Hilbert spaces

We begin by describing a construction that appears in very many contexts; we will see more specific manifestations later on.

Let[1] X be a nonempty set. A function

$$u : X \times X \to \mathbb{C}$$

is said to be **positive (semi-) definite** if, for all $n \in \mathbb{N}$, all $x_1 \ldots, x_n \in X$, and all $\lambda_1 \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{k,j=1}^{n} u(x_k, x_j)\lambda_j \bar{\lambda}_k \geq 0. \qquad (*)$$

This is equivalent to requiring that, for any finite subset $X_f = \{x_1 \ldots, x_n\}$ of X the matrix $u(X_f) := [u(x_k, x_j)]$ induces a positive operator on the Hilbert space $\ell^2(X_f) = (\mathbb{C}^n, \|\cdot\|_2)$. Indeed relation $(*)$ can be written

$$\left\langle u(X_f) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \right\rangle \geq 0.$$

Example 1.1 *If we are given a Hilbert space H and a function $f : X \to H$, then*

$$u(x, y) := \langle f(x), f(y) \rangle_H \quad (x, y \in X)$$

is a positive definite function.

Conversely,

[1]sem1, April 11, 2012

Proposition 1.2 *Given (X, u) where X is a set and $u : X \times X \to \mathbb{C}$ is positive definite, there exists $(H(u), f)$ where $H(u)$ is a Hilbert space and a function $f : X \to H(u)$ such that*

$$u(x, y) := \langle f(x), f(y) \rangle \quad \textit{for all } \ x, y \in X.$$

Moreover, f is minimal *in the sense that the linear span $[f(X)]$ of $f(X)$ is dense in $H(u)$.*

Idea of the proof On the linear space

$$c_{00}(X) = \{\xi : X \to \mathbb{C} : \operatorname{supp} \xi \ \text{ is finite } \}$$

(here $\operatorname{supp} \xi = \{x \in X : \xi(x) \neq 0\}$) define the form

$$\langle \xi, \eta \rangle_0 = \sum_{x,y \in X} u(x, y) \xi(x) \overline{\xi(y)}$$

and prove that this sesquilinear form has all the properties of a scalar product, [2] except possibly that

$$N := \{\xi \in c_{00}(X) : \langle \xi, \xi \rangle_0 = 0\}$$

may contain non-zero vectors. The Cauchy-Schwarz inequality (!) for $\langle \cdot, \cdot \rangle_0$ shows that N is a subspace and so $\langle \cdot, \cdot \rangle_0$ induces a true scalar product on the quotient space $H_0 := c_{00}(X)/N$ given by

$$\langle \xi + N, \eta + N \rangle = \langle \xi, \eta \rangle_0 .$$

(of course $\xi + N$ is the coset $\{\xi + \zeta : \zeta \in N\}$).

The Hilbert space $H(u)$ is defined to be the completion of $(H_0, \langle \cdot, \cdot \rangle)$.

The function $f : X \to H(u)$ is defined by

$$f(x) = \delta_x + N$$

where $\delta \in c_{00}(X)$ is given by

$$\delta_x(y) := \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$$

It is immediate from the definition of the scalar product that

$$\langle f(x), f(y) \rangle = \langle \delta_x, \delta_y \rangle_0 = u(x, y) \quad \text{for all } \ x, y \in X.$$

Note that the family $\{\delta_x : x \in X\}$ is an algebraic basis of $c_{00}(X)$; hence the linear span $[f(X)] = H_0$ is dense in $H(u)$ as required. $\square$

Remark 1.3 *It is not hard to verify that, if X has a topology making u continuous, then the map $f : X \to H(u)$ constructed above is also continuous.*

How to construct isometries on $H(u)$ from u-preserving maps on X.
To be slightly more general, suppose we are given two pairs (X_1, u_1) and (X_2, u_2) and a map $\phi : X_1 \to X_2$ such that

$$u_2(\phi(x), \phi(y)) = u_1(x, y) \qquad \text{for all } \quad x, y \in X_1.$$

[2] $\langle \xi, \xi \rangle_0 \geq 0$ is immediate from (*), and $\langle \eta, \xi \rangle_0 = \overline{\langle \xi, \eta \rangle_0}$ follows from this by polarisation

If $f_k : X_k \to H(u_k)$ $(l = 1, 2)$ are the maps of the Proposition, then the previous equality can be rewritten

$$\langle f_2(\phi(x)), f_2(\phi(y))\rangle_{H(u_2)} = u_2(\phi(x), \phi(y)) = u_1(x, y) = \langle f_1(x), f_1(y)\rangle_{H(u_1)}, \quad x, y \in X_1.$$

It follows that if we define a map V_o on $f_1(X_1)$ by

$$V_o(f_1(x) = f_2(\phi(x)), \quad x \in X_1$$

then this map extends by linearity to a map $V_0 : [f_1(X_1)] \to H(u_2)$ satisfying

$$\|V_0(\xi)\|_{H(u_2)} = \|\xi\|_{H(u_1)}, \quad \xi \in [f_1(X_1)]$$

and hence extends by continuity to an isometry $V_\phi : H(u_1) \to H(u_2)$, since $[f_1(X_1)]$ is dense in $H(u_2)$. This isometry *implements* ϕ in the sense that, on the generators of $H(u_1)$,

$$V_\phi(f_1(x) = f_2(\phi(x)), \quad x \in X_1.$$

Note that, in case ϕ maps X_1 onto X_2, the range of V_ϕ contains the whole of $[f_2(X_2)]$, and hence V_ϕ is a bijection: a unitary operator.

Remark 1.4 *It is immediate from the definitions that $V_{\phi\circ\psi} = V_\phi V_\psi$; thus the correspondence $\phi \to V_\phi$ is* covariant.

A unitary representation of the symmetry group G_u of (X, u)
This is the group consisting of all bijections $\phi : X \to X$ that preserve u in the sense that $u(\phi(x), \phi(y)) = u(x, y)$ for all $x, y \in X$.

It follows from the above remark that for all $\phi \in G_u$ the isometry V_ϕ in fact belongs to the group $\mathcal{U}(H(u))$ of unitary operators on $H(u)$ and the map

$$G_u \to \mathcal{U}(H(u)) : \phi \to V_\phi$$

is a group homomorphism: it is a unitary representation of G_u on $H(u)$.

Example 1.5 *Let $X = \mathbb{D}$, the open unit disc in $\mathbb{C}$, and define u by*

$$u(z, w) = \frac{1}{1 - \bar{w}z}, \quad z, w \in \mathbb{D}$$

(this is known as the Szegö kernel).

A way to see that this is positive definite is to write it in the form

$$u(z, w) = \sum_{n=0}^{\infty} z^n \bar{w}^n = \langle f(z), f(w)\rangle_{\ell^2}$$

where

$$f : \mathbb{D} \to \ell^2 : z \to (1, z, z^2, \dots).$$

Then Example 1.1 shows that u must be positive definite.

There is a very fruitful connection with analytic function theory: consider, for each $w \in \mathbb{D}$, the function k_w given by

$$k_w(\lambda) = \sum_{n=1}^{\infty} \bar{w}^n \lambda^n = \langle f(\lambda), f(w)\rangle_{\ell^2}.$$

This converges for all $\lambda \in \mathbb{D}$, hence defines an analytic function on $\mathbb{D}$. It has the following remarkable property:

Consider the space $H^2(\mathbb{D})$ of all analytic functions h whose power series representation $h(z) = \sum_n a_n z^n$ has square-summable coefficients (i.e. $\sum |a_n|^2 < \infty$). This is isomorphic to ℓ^2 (via the map $h \to (a_n)$) and hence a Hilbert space for the inner product

$$\langle h, h_1 \rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n$$

if $h(z) = \sum_n a_n z^n$ and $h_1(z) = \sum_n b_n z^n$. Then for each $h \in H^2(\mathbb{D})$ we have

$$\langle h, k_w \rangle = \sum_n a_n w^n = h(w), \quad w \in \mathbb{D}.$$

Thus the actual *value* of the function h at w can be found from its scalar product with k_w.

Example 1.6 *Now let X be a Hilbert space (finite dimensional or not). Define u by*

$$u(x, y) = \exp \langle x, y \rangle_X, \quad x, y \in X.$$

The Hilbert space $H(u)$ obtained from (X, u) is called the symmetric Fock space over X. *Any unitary operator $\phi : X \to X$ preserves u, of course; the associated unitary operator V_ϕ on $H(u)$ is called* the second quantization *of the unitary ϕ.*

Exercise Show that the function u in the last example is indeed positive definite.

2 Positive linear maps and dilations

Definition 1 *A* **Banach *-algebra** *$\mathcal{A}$ is a complex algebra which is a Banach space such that*

$$\|ab\| \le \|a\| \, \|b\| \quad \textit{for all } \ a, b \in \mathcal{A}$$

(this makes multiplication continuous) which is equipped with a map $\mathcal{A} \to \mathcal{A} : a \to a^*$ satisfying $(a+\lambda b)^* = a^* + \bar{\lambda} b$, $(ab)^* = b^* a^*$ and $(a^*)^* = a$ (an involution*) which is isometric: $\|a^*\| = \|a\|$.*

An *example* is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$, or more generally a closed subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed under the map $T \to T^*$. These are C^*-algebras: they satisfy the much more rigid C^* property: $\|T^*T\| = \|T\|^2$.

Example Recall that the Banach space $\ell^1(\mathbb{Z})$ consists of all summable functions $a : \mathbb{Z} \to \mathbb{C}$ with norm $\|a\|_1 = \sum_n |a(n)|$. Each $a \in \ell^1(\mathbb{Z})$ is the absolutely convergent sum

$$a = \sum_{n \in \mathbb{Z}} a(n) \delta_n.$$

Defining

$$a * b = \sum_{n,m \in \mathbb{Z}} a(n) b(m) \delta_{n+m} = \sum_k \left(\sum_n a(n) b(k-n) \right) \delta_k$$

$$\text{and} \quad a^* = \sum_n \overline{a(n)} \delta_{-n} = \sum_n \overline{a(-k)} \delta_k.$$

we obtain an abelian Banach *-algebra with identity δ_0 of norm 1.

This is not a C*-algebra; however it can be embedded as a dense subalgebra of a C*-algebra. [3] One way of doing this is as follows:

If $a \in \ell^1(\mathbb{Z})$ the series

$$f_a(z) = \sum_n a(n) z^n, \qquad z = e^{it} \in \mathbb{T}$$

converges absolutely, hence defines a continuous function on the circle, the (inverse) 'Fourier transform' of a.

The map $a \to f_a$ is easily seen to be a *-homomorphism, and it is injective. Moreover its range contains all 'trigonometric polynomials' $\sum_{n=-N}^{N} a(n) z^n$ and is therefore dense in the C*-algebra $(C(\mathbb{T}), \|\cdot\|_\infty)$ of all continuous functions on the circle (by the Stone-Weierstrass theorem). Note also that $\|f_a\|_\infty \leq \|a\|_1$: the embedding is contractive. $\square$

Positvity and complete poisitivity If $\mathcal{A}$ is a Banach *-algebra and $\mathcal{H}$ a Hilbert space, a linear map

$$\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

is said to be **positive** provided that

$$\langle \phi(a^* a)\xi, \xi \rangle \geq 0 \quad \text{for all } a \in \mathcal{A} \text{ and } \xi \in \mathcal{H}$$

i.e. provided that for all $a \in \mathcal{A}$ the operator $\phi(a^* a)$ is a positive operator on $\mathcal{H}$.

The map ϕ is said to be **completely positive (CP)** if for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in \mathcal{A}$ the operator matrix $[\phi(a_i^* a_j)]$ defines a positive operator on the Hilbert space direct sum $\mathcal{H}^n$. Equivalently, if for all $n \in \mathbb{N}$, all $a_1, \ldots, a_n \in \mathcal{A}$ and all $\xi_1, \ldots, a_n \in \mathcal{A}$ we have

$$\left\langle [\phi(a_i^* a_j)] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle = \sum_{i,j=1}^{n} \langle \phi(a_i^* a_j)\xi_j, \xi_i \rangle \geq 0.$$

Examples 2.1 *(i) Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a *-representation. The map π is CP:*

$$\sum_{i,j=1}^{n} \langle \pi(a_i^* a_j)\xi_j, \xi_i \rangle = \sum_{i,j=1}^{n} \langle \pi(a_j)\xi_j, \pi(a_i)\xi_i \rangle = \left\| \sum_i \pi(a_i)\xi_i \right\|^2 \geq 0.$$

(ii) Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, let $\mathcal{K}$ be another Hilbert space and $V : \mathcal{H} \to \mathcal{K}$ be a bounded linear map. Define $\phi(a) = V^* a V$. Then ϕ is CP:

$$\sum_{i,j=1}^{n} \langle \phi(a_i^* a_j)\xi_j, \xi_i \rangle = \sum_{i,j=1}^{n} \langle V^* a_i^* a_j V\xi_j, \xi_i \rangle = \sum_{i,j=1}^{n} \langle a_j V\xi_j, a_i V\xi_i \rangle = \left\| \sum_i a_i V\xi_i \right\|^2 \geq 0.$$

The suprising fact is that *every* completely positive map on a unital Banach *-algebra is a combination of these two types:

Theorem 2.2 *Let $\mathcal{A}$ be a unital Banach *-algebra with $\|\mathbf{1}\| = 1$. If $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely positive map, then there exists a triple $(V, \pi, \mathcal{K})$ where $\mathcal{K}$ is a Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ a *-representation and $V : \mathcal{H} \to \mathcal{K}$ a bounded linear map satisfying*

$$\phi(a) = V^* \pi(a) V, \quad a \in \mathcal{A}.$$

In fact ϕ is automatically continuous with $\|\phi\| = \|V\|^2 = \|\phi(\mathbf{1})\|$.

This theorem was proved (for the case of a C*-algebra $\mathcal{A}$) by Stinespring. For the proof see Theorem 3.2 below.

[3] We will see later (see Theorem 3.5) that for any unital Banach *-algebra $\mathcal{A}$ there exists an 'enveloping C*-algebra' $C^*(\mathcal{A})$ and a contractive *-homomorphism $\mathcal{A} \to C^*(\mathcal{A})$ with dense range.

Minimality A 'Stinespring triple' $(V,\pi,\mathcal{K})$ as above is said to be **minimal** if

$$\overline{[\pi(a)V\xi : a\in\mathcal{A},\xi\in\mathcal{H}]}=\mathcal{K}.$$

Remarks 2.3 *(i) Given any Stinespring triple $(V,\pi,\mathcal{K})$ one may find a minimal one by restricting each $\pi(a)$ to the (invariant) subspace*

$$\mathcal{K}_1=\overline{[\pi(a)V\xi : a\in\mathcal{A},\xi\in\mathcal{H}]}.$$

(ii) Moreover if $(V,\pi,\mathcal{K})$ is minimal and $\mathcal{A}$ has a unit $\mathbf{1}$ then necessarily $\pi(\mathbf{1})=I_{\mathcal{K}}$.

Proposition 2.4 *If $(V_i,\pi_i,\mathcal{K}_i)$ $(i=1,2)$ are two minimal pairs for ϕ, then they are equivalent in the following sense: there exists a unitary $W:\mathcal{K}_1\to\mathcal{K}_2$ such that $WV_1=V_2$ and $W\pi_1(a)=\pi_2(a)W$ for all $a\in\mathcal{A}$.*

Proof. Let $a,b\in\mathcal{A}$ and $\xi,\eta\in\mathcal{H}$. Then

$$\begin{aligned}\langle\pi_2(a)V_2\xi,\pi_2(b)V_2\eta\rangle &= \langle V_2^*\pi_2(b^*a)V_2\xi,\eta\rangle=\langle\phi(b^*a)\xi,\eta\rangle\\ &=\langle V_1^*\pi_1(b^*a)V_1\xi,\eta\rangle=\langle\pi_1(a)V_1\xi,\pi_1(b)V_1\eta\rangle\,.\end{aligned}$$

This shows that the map

$$W_0:\pi_1(a)V_1\xi\to\pi_2(a)V_2\xi$$

extends by linearity and continuity to an isometry W between the closed linear span of $\{\pi_1(a)V_1\xi : a\in\mathcal{A},\xi\in\mathcal{H}\}$, which is $\mathcal{K}_1$ and the closed linear span of $\{\pi_2(a)V_2\xi : a\in\mathcal{A},\xi\in\mathcal{H}\}$ which is $\mathcal{K}_2$ (by minimality). So this extension is onto, i.e. a unitary, and it is easy to verify that it has the stated properties. □

Remark 2.5 Suppose additionally that $\phi(\mathbf{1})=I_{\mathcal{H}}$. If $(V,\pi,\mathcal{K})$ is minimal for ϕ, then

$$V^*V=V^*\pi(\mathbf{1})V=\phi(\mathbf{1})=I_{\mathcal{H}}$$

showing that V is in this case an isometry. Therefore we may identify $\mathcal{H}$ with its image, a closed subspace of $\mathcal{K}$. Then V becomes the identity mapping of H into K and so $V^*:\mathcal{K}\to\mathcal{H}$ is simply the projection onto $\mathcal{H}$ and the formula $\phi(a)=V^*\pi(a)V$ becomes

$$\phi(a)=P_{\mathcal{H}}\pi(a)|_{\mathcal{H}},\quad a\in\mathcal{A}.$$

In other words, each $\phi(a)$ is the *compression* of $\pi(a)$ to the subspace $\mathcal{H}$; equivalently, $\pi(a)$ is the (simultaneous) *dilation* of $\phi(a)$ (for all $a\in\mathcal{A}$) to the larger space $\mathcal{K}$.

Unitary (power) dilation of a contraction We wish to prove the following result of B. Sz.-Nagy:

Theorem 2.6 *If $T\in\mathcal{B}(\mathcal{H})$ is a contraction, there is a Hilbert space $\mathcal{K}\supseteq\mathcal{H}$ and a unitary $U\in\mathcal{B}(\mathcal{K})$ such that*

$$T^n=P_{\mathcal{H}}U^n|_{\mathcal{H}},\quad n\in\mathbb{Z}_+.$$

The idea of the proof is the following: If such a unitary exists, then, for $n\geq 1$,

$$P_{\mathcal{H}}U^{-n}|_{\mathcal{H}}=P_{\mathcal{H}}U^{*n}|_{\mathcal{H}}=T^{*n}.$$

Thus if we define

$$T(n)=\begin{cases}T^n, & n\geq 0\\ (T^*)^{|n|}, & n<0\end{cases}\qquad(n\in\mathbb{Z})$$

then for every 'trigonometric polynomial' $a = \sum\limits_{n=-N}^{N} a(n)\delta_n$ we would have

$$P_{\mathcal{H}}\left(\sum_{n=-N}^{N} a(n)U^n\right)\Big|_{\mathcal{H}} = \sum_{n=-N}^{N} a(n)T(n).$$

Now, since U is unitary, the map

$$\sum_{n=-N}^{N} a(n)\delta_n \to \sum_{n=-N}^{N} a(n)U^n$$

extends to a *-representation of $\ell^1(\mathbb{Z})$ on $\mathcal{K}$. Therefore, to prove the existence of U, it suffices to *dilate* the linear map [4]

$$\begin{aligned} \phi : \ell^1(\mathbb{Z}) &\to B(H) \\ a \to \phi(a) &= \sum_{n=-\infty}^{\infty} a(n)T(n) \end{aligned}$$

to a *-representation π of $\ell^1(\mathbb{Z})$ on some larger Hilbert space. The required unitary U will then be given by $\pi(\delta_1)$.

To apply Stinespring's theorem, we need to prove that ϕ is *completely positive*. However,

Theorem 2.7 *A positive linear map defined on an* abelian *Banach *-algebra with an identity of norm one is* automatically completely positive.

(For the proof, see Theorem 3.10 below).

Thus it remains to prove that ϕ is positive.

Proof of positivity of ϕ We have to prove that for every $a \in \ell^1$ and $\xi \in H$,

$$\langle \phi(a^* a)\xi, \xi\rangle \geq 0.$$

By continuity of ϕ (and the operations on $\ell^1(\mathbb{Z})$), it suffices to prove this when $a \in c_{00}(\mathbb{Z})$, i.e. when there exists $N \in \mathbb{Z}_+$ s.t.

$$a = \sum_{n=-N}^{N} a(n)\delta_n.$$

But notice that then

$$(a * \delta_N)^* * (a * \delta_N) = a^* * \delta_{-N} * a * \delta_N = a^* * a$$

by commutativity, i.e. $a^* * a = b^* * b$, where

$$b = a * \delta_N = \sum_{n=-N}^{N} a(n)\delta_{n+N} = \sum_{k=0}^{2N} a(k-N)\delta_k = \sum_{k=0}^{M} b(k)\delta_k$$

is an 'analytic polynomial' (we have set $M = 2N$ and $b(k) = a(k-N)$).

Now $b^* * b = \sum\limits_m \overline{b(m)}\delta_{-m} \sum\limits_n b(n)\delta_n = \sum\limits_{n,m} \overline{b(m)} b(n)\delta_{n-m}$ and so

[4]the series below converges absolutely since $\|T(n)\| \leq 1$ for all $n \in \mathbb{Z}$

$$\langle \phi(a^*a)\xi, \xi\rangle = \langle \phi(b^*b)\xi, \xi\rangle = \sum_{n,m=0}^{M} \overline{b(m)}b(n) \langle T(n-m)\xi, \xi\rangle$$
$$= \sum_{n,m=0}^{M} \left\langle T(n-m)\overline{b(m)}\xi, \overline{b(n)}\xi \right\rangle .$$

Now consider the Hilbert space $K = H^{M+1}$ of 'columns' $[\xi_0, \ldots, \xi_M]^{\mathtt{T}}$ with scalar product

$$\left\langle [\xi_1, \ldots, \xi_M]^{\mathtt{T}}, [\eta_1, \ldots, \eta_M]^{\mathtt{T}} \right\rangle = \sum_{k=0}^{M} \langle \xi_k, \eta_k \rangle_H .$$

The matrix $[T(n-m)]$ defines an operator A on K satisfying

$$\left\langle A \begin{bmatrix} \xi_0 \\ \vdots \\ \xi_M \end{bmatrix}, \begin{bmatrix} \eta_0 \\ \vdots \\ \eta_M \end{bmatrix} \right\rangle = \sum_{m,n} \langle T(n-m)\xi_m, \eta_n \rangle$$

and therefore

$$\langle \phi(a^*a)\xi, \xi\rangle = \left\langle A \begin{bmatrix} \overline{b(0)}\xi \\ \vdots \\ \overline{b(M)}\xi \end{bmatrix}, \begin{bmatrix} \overline{b(0)}\xi \\ \vdots \\ \overline{b(M)}\xi \end{bmatrix} \right\rangle .$$

Hence it is enough to show that the matrix A defines a positive operator on K. This matrix is

$$A = \begin{bmatrix} I & T^* & T^{*2} & \ldots & T^{*(M-1)} & T^{*M} \\ T & I & T^* & \ldots & T^{*(M-2)} & T^{*(M-1)} \\ T^2 & T & I & \ldots & \ldots & \\ \vdots & & & & \ldots & \\ \vdots & & & & \ldots & \\ T^M & T^{M-1} & T^{M-2} & \ldots & T & I \end{bmatrix}$$

If we let R be the operator matrix

$$R = \begin{bmatrix} 0 & & \ldots & & 0 \\ T & \ddots & & & \\ & T & & & \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & & T & 0 \end{bmatrix}$$

then A is given by

$$A = I + R + \cdots + R^M + R^* + \cdots + R^{*M}.$$

But $R^{M+1} = 0$ and so

$$(I + R + \cdots + R^M)(I - R) = (I - R)(I + R + \cdots + R^M) = I - R^{M+1} = I$$

hence $I - R$ is invertible, $(I-R)^{-1} = I + R + \cdots + R^M$ and thus

$$A = (I-R)^{-1} + (I-R^*)^{-1} - I.$$

Therefore, for all $x \in K$, if we set $y = (I-R)^{-1}x$ we have

$$\begin{aligned} \langle Ax, x\rangle &= \left\langle (I-R)^{-1}x, x \right\rangle + \left\langle (I-R^*)^{-1}x, x \right\rangle - \langle x, x\rangle \\ &= \langle y, (I-R)y\rangle + \langle (I-R)y, y\rangle - \langle (I-R)y, (I-R)y\rangle \\ &= \|y\|^2 - \|Ry\|^2 \geq 0 \end{aligned}$$

because $\|R\| = \|T\| \leq 1$.

This shows that the operator matrix A is positive and therefore

$$\langle \phi(a^*a)\xi, \xi\rangle \geq 0.$$

for all $a \in c_{00}(\mathbb{Z})$ and all $\xi \in H$. $\square$

Aside: Connection with the Poisson kernel. If $a \in \ell^1(\mathbb{Z})$ the formula

$$f_a(z) = \sum_n a(n) z^n, \qquad z = e^{it} \in \mathbb{T}$$

defines a continuous function f_a on the circle, the (inverse) 'Fourier transform' of a. The Poisson extension of this to the disc $\mathbb{D}$ is given by

$$\tilde{f}_a(w) = \sum_n a(n) r^{|n|} e^{int}, \quad w = re^{it},\ 0 \leq r < 1$$

which is the 'convolution' of f_a with the 'Poisson kernel' $\sum_n r^{|n|} e^{int}$. The map $f \to \tilde{f}$ is positive: if $f(z) \geq 0$ everywhere on the circle, then $\tilde{f}(w) \geq 0$ everywhere on the disc. This is because the Poisson kernel is nonnegative: indeed, if $w = re^{it}$ with $0 < r < 1$, then

$$\sum_n r^{|n|} e^{int} = \sum_{n \geq 0} w^n + \sum_{m \geq 1} \bar{w}^m = (1-w)^{-1} + (1-\bar{w})^{-1} - 1$$

and the latter quantity is always nonnegative: it equals $\dfrac{1-r^2}{|1-w|^2}$.

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Dilation of operator-valued measures. [5]

A *positive operator - valued measure* is a map $E : \mathcal{S} \to \mathcal{B}(H)_+$ defined on a σ-algebra $\mathcal{S}$ of subsets of some set X and taking values in the set of positive operators on a Hilbert space H, which is *weakly countably additive* in the sense that for each $\xi \in H$ the map

$$\mu_{\xi,\xi}(S) := \langle E(S)\xi, \xi\rangle, \quad S \in \mathcal{S}$$

is a positive (countably additive) measure. We will assume that $E(X) = I$; it then follows that all $E(S)$ are positive contractions. [6] Note that the operators $E(S)$ need not commute.

$E(\cdot)$ is said to be a *spectral or projection-valued measure* or a *resolution of the identity* in case it additionally satisfies $E(S_1 \cap S_2) = E(S_1)E(S_2)$. Then each $E(S)$ is a positive idempotent, hence an orthogonal projection.

When $\mathcal{S}$ is the Borel σ-algebra of a compact Hausdorff space X, we say $E(\cdot)$ is *regular* when for each $\xi \in H$ the scalar measure $\mu_{\xi,\xi}$ is regular.

One version of the *spectral theorem* states that, if X is a compact Hausdorff space, every unital *-representation $\pi : C(X) \to \mathcal{B}(H)$ gives rise to a unique regular Borel spectral measure $F(\cdot)$ such that

$$\pi(f) = \int_X f dF, \qquad f \in C(X)$$

[5] Neumark, M. A. On a representation of additive operator set functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, (1943). 359–361.

[6] Indeed, additivity implies that $0 \leq \langle E(S)\xi, \xi\rangle \leq \mu_{\xi,\xi}(S) \leq \mu_{\xi,\xi}(X) = \langle I\xi, \xi\rangle$ for all $\xi \in H$, hence $0 \leq E(S) \leq I$.

where the integral may be defined in the 'weak' sense:

$$\langle \pi(f)\xi, \eta\rangle_K = \int_X f d\nu_{\xi,\eta}$$

for all $\xi, \eta \in H$, where $\nu_{\xi,\eta}(S) = \langle F(S)\xi, \eta\rangle$.

Theorem 2.8 (Naimark) *Let $E : \mathcal{S} \to \mathcal{B}(H)$ be a positive regular operator – valued measure defined on the Borel σ-algebra $\mathcal{S}$ of a compact Hausdorff space X.*

Then there is a Hilbert space $K \supseteq H$ and a regular spectral measure $F : \mathcal{S} \to \mathcal{B}(K)$ which dilates *$E(\cdot)$ in the sense that*

$$E(S) = P_H F(S)|_H \quad \textit{for all } \ S \in \mathcal{S}.$$

Proof. For each $\xi, \eta \in H$ the map

$$\mathcal{S} \to \mathbb{C} : \ S \to \mu_{\xi,\eta}(S) = \langle E(S)\xi, \eta\rangle_H \, ,$$

is a complex regular Borel measure on X. Actually, it is a linear combination of four positive regular Borel measures: $\mu_{\xi,\eta} = \frac{1}{4}\sum_{n=1}^{4} i^n \mu_{\xi_n,\xi_n}$ where $\xi_n = \xi + i^n\eta$; so if $f \in C(X)$, the integral $\int_X f d\mu_{\xi,\eta}$ may be defined as a linear combination of four integrals over positive measures. Now for $\xi = \eta$ we have

$$\left|\int_X f d\mu_{\xi,\xi}\right| \le \|f\|_\infty \mu_{\xi,\xi}(X) = \|f\|_\infty \langle E(X)\xi, \xi\rangle = \|f\|_\infty \|\xi\|^2 .$$

Therefore, if $\|\xi\| \le 1$ and $\|\eta\| \le 1$

$$\begin{aligned}\left|\int_X f d\mu_{\xi,\eta}\right| &= \frac{1}{4}\left|\sum_{n=1}^{4} i^n \int_X f d\mu_{\xi_n,\xi_n}\right| \le \frac{1}{4}\sum_{n=1}^{4}\int_X |f| d\mu_{\xi_n,\xi_n} \\ &\le \frac{1}{4}\sum_{n=1}^{4} \|f\|_\infty \|\xi + i^n\eta\|^2 = \|f\|_\infty (\|\xi\|^2 + \|\eta\|^2) \le 2\|f\|_\infty\end{aligned}$$

(we have used the parallelogram law). Consider the map

$$H \times H \to \mathbb{C} : (\xi, \eta) \to \int_X f d\mu_{\xi,\eta}.$$

This is sesquilinear, since the map $(\xi, \eta) \to \mu_{\xi,\eta}(S)$ is sesquilinear for each S (*Proof: Exercise!*). Moreover, we have just shown that it is bounded. Therefore there exists a unique $\phi_E(f) \in \mathcal{B}(H)$ such that

$$\langle \phi_E(f)\xi, \eta\rangle_H = \int_X f d\mu_{\xi,\eta} \quad \text{for all } \ \xi, \eta \in H.$$

The map $\phi_E : C(X) \to \mathcal{B}(H)$ is linear and unital because

$$\langle \phi_E(\mathbf{1})\xi, \eta\rangle_H = \int_X \mathbf{1} d\mu_{\xi,\eta} = \langle E(X)\xi, \eta\rangle_H = \langle \xi, \eta\rangle_H$$

since $E(X) = I$. Moreover, it is positive because the measures $\mu_{\xi,\xi}$ are all positive. Since the domain of ϕ_E is abelian, [7] ϕ_E is completely positive (Theorem 3.10). Therefore, by Stinespring's theorem there is a *-representation π of $C(X)$ such that

$$\phi_E(f) = P_H \pi(f)|_H \quad \text{for all } \ f \in \mathcal{C}(X).$$

[7] but the range $\phi_E(C(X))$ is neither an algebra nor commutative in general

But now, as noted before the theorem, π defines a unique regular (projection-valued) spectral measure $F : \mathcal{S} \to \mathcal{B}(K)$ such that, if $\nu_{x,y}(S) = \langle F(S)x, y\rangle_K$,

$$\langle \pi(f)x, y\rangle_K = \int_X f d\nu_{x,y} \quad \text{for all} \ \ f \in \mathcal{C}(X).$$

We have to show that, for all $S \in \mathcal{S}$,

$$E(S) = P_H F(S)|_H,$$

equivalently that, for all $\xi, \eta \in H \subseteq K$,

$$\langle E(S)\xi, \eta\rangle = \langle P_H F(S)\xi, \eta\rangle = \langle F(S)\xi, \eta\rangle \,.$$

By polarisation, it is enough to prove the equality $\langle E(S)\xi, \xi\rangle = \langle F(S)\xi, \xi\rangle$. In other words we need to show that for all $\xi \in H \subseteq K$ the positive measures $\mu_{\xi,\xi}$ and $\nu_{\xi,\xi}$ are the same. Indeed, for all $f \in C(X)$,

$$\int_X f d\nu_{\xi,\xi} = \langle \pi(f)\xi, \xi\rangle = \langle \pi(f)P_H\xi, P_H\xi\rangle = \langle P_H\pi(f)P_H\xi, \xi\rangle = \langle \phi_E(f)\xi, \xi\rangle = \int_X f d\mu_{\xi,\xi}$$

so that

$$\nu_{\xi,\xi}(S) = \mu_{\xi,\xi}(S) \quad \text{far all} \ \ S \in \mathcal{S}$$

because both $\mu_{\xi,\xi}$ and $\nu_{\xi,\xi}$ are regular measures. $\square$

3 Proofs of the main theorems

3.1 Stinespring's Dilation Theorem

We will need the following result, which is well known for C* algebras.

Proposition 3.1 *Any positive linear form ρ defined on a unital Banach *-algebra with $\|\mathbf{1}\| = 1$ is bounded with $\|\rho\| = \rho(\mathbf{1})$.*

Proof. The inequality $\rho(a^*a) \geq 0$ for all $a \in \mathcal{A}$ implies (in the standard way) the Cauchy - Schwarz inequality

$$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b) \qquad \text{and so} \qquad |\rho(a)|^2 \leq \rho(a^*a)\rho(1).$$

It is therefore enough to prove the inequality

$$\rho(a^*a) \leq \rho(1)\,\|a\|^2 \,.$$

Take $a \in \mathcal{A}$ with $\|a\| \leq 1$; we have to prove that $\rho(a^*a) \leq \rho(1)$.

By Taylor's theorem, for all $z \in \mathbb{C}$ with $|z| \leq 1$ we have

$$(1-z)^{1/2} = 1 - \sum_{n=1}^{\infty} \lambda_n z^n$$

where one can verify that all $\lambda_n > 0$ and moreover that [8] $\sum_{n=1}^{\infty} \lambda_n < \infty$, i.e. $(\lambda_n) \in \ell^1$. It follows that if $b \in \mathcal{A}$ has $\|b\| \leq 1$ then the series $\sum_n \lambda_n b^n$ converges absolutely to some $c \in \mathcal{A}$ such that

$$(\mathbf{1} - c)^2 = \left(\mathbf{1} - \sum_n \lambda_n b^n\right)^2 = \mathbf{1} - b.$$

[8] indeed, for all $N \in \mathbb{N}$ and all $t \in (0,1)$ we have $0 < \sum\limits_{n=1}^{N} \lambda_n t^n \leq \sum\limits_{n=1}^{\infty} \lambda_n t^n = 1 - (1-t)^{1/2} < 1$ and hence $0 < \sum\limits_{n=1}^{N} \lambda_n = \sup\limits_t (\sum\limits_{n=1}^{N} \lambda_n t^n) \leq 1$.

Apply this to $b = a^*a$: note that $b^* = b$ and so $c^* = c$ (the coefficients of the series are real) and

$$\begin{aligned} \mathbf{1} - a^*a &= (\mathbf{1}-c)^2 = (\mathbf{1}-c)^*(\mathbf{1}-c) \\ \text{hence} \quad \rho(\mathbf{1} - a^*a) &= \rho((\mathbf{1}-c)^*(\mathbf{1}-c)) \geq 0 \end{aligned}$$

which shows that $\rho(a^*a) \leq \rho(\mathbf{1})$ as required. □

Theorem 3.2 *Let $\mathcal{A}$ be a unital Banach *-algebra with $\|\mathbf{1}\| = 1$. If $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely positive map, then there exists a triple $(V, \pi, \mathcal{K})$ where $\mathcal{K}$ is a Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ a *-representation and $V : \mathcal{H} \to \mathcal{K}$ a bounded linear map satisfying*

$$\phi(a) = V^*\pi(a)V, \qquad a \in \mathcal{A}.$$

In fact ϕ is automatically continuous with $\|\phi\| = \|V\|^2 = \|\phi(\mathbf{1})\|$.

Proof. Define a sesquilinear form on the algebraic tensor product [9] $\mathcal{A} \otimes H$ by the formula

$$\left\langle \sum_i a_i \otimes \xi_i, \sum_j b_j \otimes \eta_j \right\rangle_o = \sum_{i,j} \left\langle \phi(b_j^* a_i)\xi_i, \eta_j \right\rangle_H \quad (a_i, b_i \in \mathcal{A},\ \xi_i, \eta_i \in H).$$

Since the map $(a, \xi) \to a \otimes \xi$ is bilinear, this is clearly a sesquilinear form. The complete positivity of ϕ is exactly what is needed to ensure that $\langle \cdot, \cdot \rangle_o$ is positive semidefinite:

$$\left\langle \sum_i a_i \otimes \xi_i, \sum_j a_j \otimes \xi_j \right\rangle_o = \sum_{i,j} \left\langle \phi(a_j^* a_i)\xi_i, \xi_j \right\rangle \geq 0.$$

Therefore $\langle \cdot, \cdot \rangle_o$ satisfies the Cauchy-Schwarz inequality, and so

$$\left\| \sum_i a_i \otimes \xi_i \right\|_o := \left(\left\langle \sum_i a_i \otimes \xi_i, \sum_j a_j \otimes \xi_j \right\rangle_o \right)^{1/2}$$

is a seminorm on $\mathcal{A} \otimes H$.

For all $a \in \mathcal{A}$ define a map

$$\begin{aligned} \pi_o(a) : \mathcal{A} \otimes H &\to \mathcal{A} \otimes H : \\ \sum_j b_j \otimes \xi_j &\to \sum_j (ab_j) \otimes \xi_j. \end{aligned}$$

Clearly each $\pi_o(a)$ is a linear map. Moreover, it is immediate that

$$\pi_o(a+b) = \pi_o(a) + \pi_o(b), \;\; \pi_o(ab) = \pi_o(a)\pi_o(b), \;\; \pi_o(\mathbf{1}) = I. \tag{1}$$

Claim 1 For all $u, v \in \mathcal{A} \otimes H$ and all $a \in \mathcal{A}$ we have

$$\langle \pi_o(a)u, v \rangle_o = \langle u, \pi_o(a^*)v \rangle_o .$$

[9] This is isomorphic to the linear space of all continuous finite rank *antilinear* maps $f : H \to \mathcal{A}$: if the range $f(H)$ is spanned by $\{a_1, \ldots, a_n\}$, then there are linearly independent $\xi_1, \ldots, \xi_n \in H$ such that $f(\xi) = \sum_{i=1}^n a_i \langle \xi_i, \xi \rangle$; we write $f = \sum_i a_i \otimes \xi_i$.

Proof If $u = \sum_i b_i \otimes \xi_i$ and $v = \sum_j c_j \otimes \eta_j$ then

$$\begin{aligned}
\langle \pi_o(a)u, v\rangle_o &= \left\langle \pi_o(a)\sum_i b_i \otimes \xi_i, \sum_j c_j \otimes \eta_j \right\rangle_o = \left\langle \sum_i (ab_i) \otimes \xi_i, \sum_j c_j \otimes \eta_j \right\rangle_o \\
&= \sum_{i,j} \langle \phi(c_j^* a b_i)\xi_i, \eta_j\rangle_H = \sum_{i,j} \langle \phi((a^*c_j)^* b_i)\xi_i, \eta_j\rangle_H = \sum_{i,j} \langle b_i \otimes \xi_i, (a^*c_j) \otimes \eta_j\rangle_o \\
&= \left\langle \sum_i b_i \otimes \xi_i, \sum_j (a^*c_j) \otimes \eta_j \right\rangle_o = \langle u, \pi_o(a^*)v\rangle_o . \qquad \square
\end{aligned}$$

It follows from the claim that for each $u \in \mathcal{A} \otimes H$ and $a \in \mathcal{A}$,

$$\langle \pi_o(a^*a)u, u\rangle_o = \langle \pi_o(a^*)\pi_o(a)u, u\rangle_o = \langle \pi_o(a)u, \pi_o(a)u\rangle_o \geq 0$$

which means that the functional $\rho_u : \mathcal{A} \to \mathbb{C}$ given by

$$\rho_u(b) = \langle \pi_o(b)u, u\rangle_o , \quad b \in \mathcal{A}$$

is a positive linear form. By Proposition 3.1, it is bounded with $\|\rho_u\| = \rho_u(\mathbf{1}) = \langle u, u\rangle_o$. Thus

$$\rho_u(a^*a) \leq \|\rho_u\| \, \|a^*a\| \leq \|\rho_u\| \, \|a^*\| \, \|a\| = \|\rho_u\| \, \|a\|^2$$

and therefore

$$\|\pi_o(a)u\|_o^2 = \langle \pi_o(a)u, \pi_o(a)u\rangle_o = \rho_u(a^*a) \leq \|\rho_u\| \, \|a\|^2 = \|u\|_o^2 \, \|a\|^2 . \tag{2}$$

It follows from this that if

$$\mathcal{N} := \{u \in \mathcal{A} \otimes H : \|u\|_o = 0\}$$

then $\mathcal{N}$ is invariant under $\pi_o(a)$ (if $\|u\|_o = 0$ then $\|\pi_o(a)u\|_o = 0$), and so $\pi_o(a)$ factors to a linear map $\pi(a)$ from the quotient space $K_o := (\mathcal{A} \otimes H)/\mathcal{N}$ to itself given by

$$\pi(a)(u + \mathcal{N}) = (\pi_o(a)u) + \mathcal{N}.$$

Also, if $\|u + \mathcal{N}\| := \|u\|_o$ is the quotient norm, then

$$\|\pi(a)(u + \mathcal{N})\| = \|\pi_o(a)u\|_o \leq \|u\|_o \, \|a\| = \|a\| \, \|u + \mathcal{N}\|$$

by (2). Therefore $\pi(a)$ is bounded on K_o by $\|a\|$, hence extends to a bounded operator (also denoted by $\pi(a)$) on the Hilbert space completion K of K_o satisfying $\|\pi(a)\| \leq \|a\|$. It is immediate that the map

$$\pi : \mathcal{A} \to \mathcal{B}(K) : \; a \to \pi(a)$$

is a unital algebra morphism; it follows from Claim 1 that it is also *-preserving: for all $u, v \in \mathcal{A} \otimes H$,

$$\begin{aligned}
\langle (u + \mathcal{N}), \pi(a)^*(v + \mathcal{N})\rangle &= \langle \pi(a)(u + \mathcal{N}), (v + \mathcal{N})\rangle = \langle \pi_o(a)u, v\rangle_o = \langle u, \pi_o(a^*)v\rangle_o \\
&= \langle (u + \mathcal{N}), \pi(a^*)(v + \mathcal{N})\rangle .
\end{aligned}$$

Thus the bounded operators $\pi(a)^*$ and $\pi(a^*)$ coincide on the dense subspace K_o of K, hence they are equal.

We have defined the Hilbert space K and the *-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$. Now we define

$$V : H \to K : \ \xi \to (\mathbf{1} \otimes \xi) + \mathcal{N}.$$

Note that the (obviously linear) map V is bounded; indeed,

$$\|V\xi\|^2 = \langle (\mathbf{1} \otimes \xi) + \mathcal{N}, (\mathbf{1} \otimes \xi) + \mathcal{N} \rangle = \langle \mathbf{1} \otimes \xi, \mathbf{1} \otimes \xi \rangle_o = \langle \phi(\mathbf{1})\xi, \xi \rangle_o \leq \|\phi(\mathbf{1})\| \, \|\xi\|_H^2$$

so that

$$\|V\|^2 \leq \|\phi(\mathbf{1})\| . \tag{3}$$

Now for all $a \in \mathcal{A}$ and $\xi, \eta \in H$ we have

$$\begin{aligned} \pi(a)V\xi &= \pi(a)(\mathbf{1} \otimes \xi + \mathcal{N}) = a \otimes \xi + \mathcal{N} \\ \text{hence} \quad \langle V^*\pi(a)V\xi, \eta \rangle_H &= \langle \pi(a)V\xi, V\eta \rangle_K = \langle (a \otimes \xi) + \mathcal{N}, (\mathbf{1} \otimes \eta) + \mathcal{N} \rangle_K \\ &= \langle a \otimes \xi, \mathbf{1} \otimes \eta \rangle_o = \langle \phi(\mathbf{1}^* a)\xi, \eta \rangle_H = \langle \phi(a)\xi, \eta \rangle_H \\ \text{so that} \quad V^*\pi(a)V &= \phi(a) \end{aligned}$$

as required.

Using this equality we have

$$\|\phi(a)\| \leq \|V^*\| \, \|\pi(a)\| \, \|V\| \leq \|V\|^2 \, \|a\| .$$

This shows that the map $\phi : \mathcal{A} \to \mathcal{B}(H)$ is automatically bounded with $\|\phi\| \leq \|V\|^2$; but since $\|V\|^2 \leq \|\phi(\mathbf{1})\| \leq \|\phi\|$ by (3), we have $\|\phi\| = \|V\|^2 = \|\phi(\mathbf{1})\|$. $\square$

Remarks 3.3 (i) *The Stinespring triple (V, π, K) that we have constructed is* minimal *in the sense that*

$$K = \overline{[\pi(a)V\xi : a \in \mathcal{A}, \xi \in H]}$$

(indeed we saw that $[\pi(a)V\xi : a \in \mathcal{A}, \xi \in H\} = [a \otimes \xi + \mathcal{N} : a \in \mathcal{A}, \xi \in H] = K_o$). Recall that all minimal Stinespring triples for ϕ are unitarily equivalent (Proposition 2.4).

(ii) *Note that $V^*V = V^*\pi(\mathbf{1})V = \phi(\mathbf{1})$. Thus, if $\phi(\mathbf{1}) = I_H$, then $V^*V = I_H$ and so $V : H \to K$ is an isometry.*

The special case $\dim H = 1$ yields the celebrated *Gelfand - Naimark - Segal representation*:

Theorem 3.4 (GNS) *Let $\mathcal{A}$ be a unital Banach *-algebra with $\|\mathbf{1}\| = 1$. If $\rho : \mathcal{A} \to \mathbb{C}$ is a* positive *linear form, there exists a Hilbert space H_ρ, a vector $\xi_\rho \in H_\rho$ and a *-representation $\pi_\rho : \mathcal{A} \to \mathcal{B}(H_\rho)$ such that*

$$\rho(a) = \langle \pi_\rho(a)\xi_\rho, \xi_\rho \rangle , \qquad a \in \mathcal{A}.$$

Moreover π_ρ is a cyclic representation with cyclic vector ξ_ρ, that is, $\{\pi_\rho(a)\xi_\rho : a \in \mathcal{A}\}$ is dense in H_ρ.

Proof. We apply Theorem 3.2: The Hilbert space H is $\mathbb{C}$ and hence $\mathcal{B}(H) = \mathbb{C}$. Since ρ takes values in $\mathbb{C}$, positivity implies complete positivity (Theorem 3.10). Thus Stinespring's theorem applies: There is a Hilbert space K, a bounded operator $V : \mathbb{C} \to K$ and a *-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ such that

$$\rho(a) = V^*\pi(a)V, \qquad a \in \mathcal{A}.$$

Now setting $\xi = V1$ we have $V^*\eta = \langle \eta, \xi \rangle, \eta \in K$. Indeed,

$$\langle \eta, \xi \rangle_K = \langle \eta, V1 \rangle_K = \langle V^*\eta, 1 \rangle_{\mathbb{C}} = V^*\eta.$$

Therefore,

$$\rho(a) = \rho(a)1 = V^*\pi(a)V1 = V^*(\pi(a)\xi) = \langle \pi(a)\xi, \xi \rangle$$

as required. We now set $\xi_\rho = \xi$, $H_\rho = \overline{\{\pi(a)\xi : a \in \mathcal{A}\}}$ and $\pi_\rho(a) = \pi(a)|_{H_\rho}$ to get the required GNS triple $(\pi_\rho, H_\rho, \xi_\rho)$. $\square$

3.2 The enveloping C* algebra

Theorem 3.5 *Let $\mathcal{A}$ be a Banach *-algebra with identity of norm 1. Then there exists a C*-algebra $C^*(\mathcal{A})$ and a contractive *-homomorphism*

$$\iota : \mathcal{A} \to C^*(\mathcal{A})$$

with dense range, having the following universal property:

For every *-representation (π, H) of $\mathcal{A}$ there is a unique *-representation $(\tilde{\pi}, H)$ of $C^*(\mathcal{A})$ such that

$$\tilde{\pi} \circ \iota = \pi.$$

*In particular, $C^*(\mathcal{A})$ is unique up to *-isomorphism.*

Proof. Let

$$\mathcal{S} = \{\rho : \mathcal{A} \to \mathbb{C} \quad \text{positive linear form s.t.} \quad \|\rho\| \le 1\}.$$

Recall that each $\rho \in \mathcal{S}$ defines a cyclic *-representation $(\pi_\rho, H_\rho, \xi_\rho)$ of $\mathcal{A}$ such that

$$\rho(a) = \langle \pi_\rho(a)\xi_\rho, \xi_\rho \rangle \quad (a \in \mathcal{A})$$

hence in particular $\|\xi_\rho\|^2 = \rho(\mathbf{1}) \le 1$. Conversely every cyclic *-representation (π, H, ξ) defines a $\rho \in \mathcal{S}$ by the formula $\rho(a) = \langle \pi(a)\xi, \xi \rangle$, and π is unitarily equivalent to π_ρ.

For $a \in \mathcal{A}$, define

$$\|a\|_o^2 = \sup\{\rho(a^*a) : \rho \in \mathcal{S}\}.$$

Since $\rho(a^*a) \le \|\rho\| \|a^*a\| \le \|a^*a\| \le \|a^*\| \|a\| = \|a\|^2$, it is clear that

$$\|a\|_o \le \|a\|.$$

Claim 1 $\qquad \|a\|_o = \sup\{\|\pi_\rho(a)\| : \rho \in \mathcal{S}\}.$

Proof Write b for the right hand side of the claimed equality. For every $\rho \in \mathcal{S}$ we have

$$\rho(a^*a) = \langle \pi_\rho(a^*a)\xi_\rho, \xi_\rho \rangle = \|\pi_\rho(a)\xi_\rho\|^2 \le \|\pi_\rho(a)\|^2 \le b^2$$

(since $\|\xi_\rho\| \le 1$) and so, taking sup over all $\rho \in \mathcal{S}$,

$$\|a\|_o^2 \le b^2.$$

For the reverse inequality, take any $\rho \in \mathcal{S}$ and calculate

$$\begin{aligned} \|\pi_\rho(a)\|^2 &= \sup\{\|\pi_\rho(a)\eta\|^2 : \eta \in H, \|\eta\| \le 1\} \\ &= \sup\{\langle \pi_\rho(a^*a)\eta, \eta \rangle : \eta \in H, \|\eta\| \le 1\}. \end{aligned}$$

But the forms ρ_η given on $\mathcal{A}$ by $\rho_\eta(a) = \langle \pi_\rho(a)\eta, \eta\rangle$ belong to $\mathcal{S}$ (because $\|\eta\| \le 1$), and so the last supremum is no larger than $\|a\|_o^2$. Therefore

$$\|\pi_\rho(a)\|^2 \le \|a\|_o^2$$

for each ρ, and so $b^2 \le \|a\|_o^2$. $\square$

The claim shows that $\|a\|_o$ is indeed an algebra seminorm on $\mathcal{A}$, being the supremum of algebra seminorms. It also satisfies the C*-identity, because each $\pi_\rho(a)$ is a Hilbert space operator, hence $\|\pi_\rho(a)\|^2 = \|\pi_\rho(a)^*\pi_\rho(a)\| = \|\pi_\rho(a^*a)\|$.

Thus if we define $\mathcal{N}$ by

$$\mathcal{N} = \{a \in \mathcal{A} : \|a\|_o = 0\}$$

then $\mathcal{N}$ is a two-sided selfadjoint ideal, hence the quotient space $\mathcal{A}/\mathcal{N}$ is a unital *-algebra; moreover $\|\cdot\|_o$ induces an algebra norm $\|\cdot\|_*$ on $\mathcal{A}/\mathcal{N}$ which satisfies the C*-identity.

The quotient map $\iota : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ is a *-epimorphism and is contractive because $\|\iota(a)\|_* = \|a\|_o \le \|a\|_{\mathcal{A}}$. Thus if we define $C^*(\mathcal{A})$ to be the completion of $(\mathcal{A}/\mathcal{N}, \|\cdot\|_*)$, we obtain a C*-algebra and a contractive *-homomorphism $\iota : \mathcal{A} \to C^*(\mathcal{A})$ whose range $\mathcal{A}/\mathcal{N}$ is dense in $C^*(\mathcal{A})$.

It remains to prove the universal property. We show that it suffices to consider cyclic representations:

Claim Any *-representation π of $\mathcal{A}$ is the direct sum of cyclic representations.

Proof Call two unit vectors $\xi, \eta \in H$ *very orthogonal* if $\langle \pi(a)\xi, \pi(b)\eta\rangle = 0$ for all $a, b \in \mathcal{A}$ [10]. This is equivalent to requiring that the cyclic subspaces $H_\xi = \overline{\{\pi(a)\xi : a \in \mathcal{A}\}}$ and H_η be orthogonal. Of course H_ξ is $\pi(\mathcal{A})$-invariant (hence reducing, because $\pi(\mathcal{A})$ is selfadjoint) and the map $\pi_\xi : a \to \pi(a)|_{H_\xi}$ is a cyclic *-representation of $\mathcal{A}$ (with cyclic vector ξ).

Let $\{\xi_i : i \in I\}$ be a maximal family of very orthogonal unit vectors in H (Zornication), let H_i be the cyclic subspace corresponding to ξ_i, and let π_i be the corresponding subrepresentation of π. We have to show that the (pairwise orthogonal) invariant subspaces H_i satisfy $\oplus_i H_i = H$.

Indeed, if there existed a unit vector $\xi \in H$ orthogonal to $\oplus_i H_i$, then for all $i \in I$ and all $a, b \in \mathcal{A}$ we would have

$$\langle \pi(a)\xi, \pi(b)\xi_i\rangle = \langle \xi, \pi(a^*b)\xi_i\rangle = 0$$

because $\pi(a^*b)\xi_i$ is in H_i while ξ is orthogonal to H_i. This shows that H_ξ is orthogonal to H_i, for all $i \in I$, contradicting the maximality of $\{\xi_i : i \in I\}$. $\square$

Therefore, to show that any *-representation of $\mathcal{A}$ induces a *-representation of $C^*(\mathcal{A})$ as claimed, it is sufficient to consider cyclic representations. Indeed, if $\pi = \oplus_i \pi_i$, and each π induces a *-representation $\tilde{\pi}_i$ of $C^*(\mathcal{A})$ such that $\tilde{\pi}_i \circ \iota = \pi_i$, then $\tilde{\pi} := \oplus_i \tilde{\pi}_i$ satisfies $\tilde{\pi} \circ \iota = \pi$.

So let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a *-representation of $\mathcal{A}$ with unit cyclic vector ξ .

If $a \in \mathcal{A}$ then, for all $b \in \mathcal{A}$ such that $\|\pi(b)\xi\| \le 1$,

$$\|\pi(a)\pi(b)\xi\|^2 = \langle \pi(ab)\xi, \pi(ab)\xi\rangle = \langle \pi(b^*a^*ab)\xi, \xi\rangle = \rho_b(a^*a)$$

where $\rho_b(x) = \langle \pi(b^*xb)\xi, \xi\rangle$. Clearly $\rho_b \in \mathcal{S}$, so $\rho_b(a^*a) \le \|a\|_o^2$. Thus $\|\pi_\rho(a)\pi_\rho(b)\xi\| \le \|a\|_o$ for all $b \in \mathcal{A}$ of norm at most 1. Since $\{\pi_\rho(b)\xi : b \in \mathcal{A}, \|\pi(b)\xi\| \le 1\}$ is dense in the unit ball of H, this shows that $\|\pi_\rho(a)\| \le \|a\|_o$. In particular if $a \in \mathcal{N}$ then $\pi_\rho(a) = 0$. Thus π_ρ factors through $\mathcal{N}$ to a map $\tilde{\pi}_\rho : \mathcal{A}/\mathcal{N} \to \mathcal{B}(H_\rho)$ given by $\tilde{\pi}_\rho(\iota(a)) = \pi_\rho(a)$ for $a \in \mathcal{A}$, and we have $\|\tilde{\pi}_\rho(\iota(a))\| = \|\pi_\rho(a)\| \le \|a\|_o = \|\iota(a)\|_*$.

[10] actually, it is enough to take $b = \mathbf{1}$

Thus $\tilde{\pi}_\rho$ induces a *-representation of $C^*(\mathcal{A})$ (which is contractive).

Taking direct sums, it follows that there exists a *-representation $\tilde{\pi}$ of $C^*(\mathcal{A})$ satisfying $\tilde{\pi}(\iota(a)) = \pi(a)$ for all $a \in \mathcal{A}$.

Finally, the relation $\tilde{\pi} \circ \iota = \pi$ uniquenely determines $\tilde{\pi}$ since $\iota(\mathcal{A})$ is dense in $C^*(\mathcal{A})$.

In particular, suppose $(\mathcal{C}, \kappa)$ is a C*-algebra and $\kappa : \mathcal{A} \to \mathcal{C}$ is a unital *-morphism with dense range, satisfying the analogous universal property. Then the *-representation κ induces a *-representation $\tilde{\kappa} : C^*(\mathcal{A}) \to \mathcal{C}$ satisfying $\tilde{\kappa} \circ \iota = \kappa$; but the *-representation ι also induces a *-representation $\tilde{\iota} : \mathcal{C} \to C^*(\mathcal{A})$ satisfying $\tilde{\iota} \circ \kappa = \iota$. Now we have

$$\tilde{\kappa} \circ \iota = \kappa \quad \Rightarrow \quad \tilde{\kappa} \circ (\tilde{\iota} \circ \kappa) = \kappa$$

so $(\tilde{\kappa} \circ \tilde{\iota}) \circ \kappa = \kappa$ and thus $\tilde{\kappa} \circ \tilde{\iota}$ is the identity on $\kappa(\mathcal{A})$ and hence on its closure $\mathcal{C}$. By the same argument, $\tilde{\iota} \circ \tilde{\kappa}$ is the identity on $C^*(\mathcal{A})$. Thus $\tilde{\iota} : \mathcal{C} \to C^*(\mathcal{A})$ is a *-isomorphism such that $\tilde{\iota} \circ \kappa = \iota$: this proves the uniqueness of $C^*(\mathcal{A})$.

Proposition 3.6 *Every positive linear map $\rho : \mathcal{A} \to \mathcal{B}(H)$ defined on an Banach *-algebra with unit of norm $\|\mathbf{1}\| = 1$ induces a unique positive linear map $\tilde{\rho} : C^*(\mathcal{A}) \to \mathcal{B}(H)$ such that*

$$\tilde{\rho} \circ \iota = \rho.$$

Moreover ρ is completely positive if and only if $\tilde{\rho}$ is.

Proof. We use the notation of the previous proof. It suffices to assume that $\|\rho\| \leq 1$, so that $\rho \in \mathcal{S}$. Now, for all $a \in \mathcal{A}$,

$$|\rho(a)|^2 \leq \rho(a^*a)\rho(\mathbf{1}) \leq \|a\|_o^2$$

by the definition of $\|a\|_o$. Thus ρ leaves $\mathcal{N}$ invariant, hence induces a linear functional $\tilde{\rho} : \mathcal{A}/\mathcal{N} \to \mathbb{C}$ by $\tilde{\rho}(a + \mathcal{N}) = \rho(a)$, i.e. $\tilde{\rho}(\iota(a)) = \rho(a)$ for all $a \in \mathcal{A}$. Thus

$$|\tilde{\rho}(\iota(a))| = |\rho(a)| \leq \|a\|_o = \|\iota(a)\|_* .$$

Thus $\tilde{\rho}$ is bounded, hence extends (uniquely, since $\iota(\mathcal{A})$ is dense in $C^*(\mathcal{A})$) to a linear form on $C^*(\mathcal{A})$.

We show $\tilde{\rho}$ is positive: if $b \in C^*(\mathcal{A})$, there is a sequence (a_n) in $\mathcal{A}$ such that $\iota(a_n) \to b$; then$\iota(a_n^* a_n) \to b^*b$. It follows that

$$\tilde{\rho}(b^*b) = \lim \tilde{\rho}(\iota(a_n^* a_n)) = \lim \rho(a_n^* a_n) \geq 0.$$

An obvious modification of this argument, considering an m-tuple $b_1, \ldots, b_m$ in $C^*(\mathcal{A})$, shows that if ρ is completely positive, then so is $\tilde{\rho}$. The converse is obvious.

Finally, since $\iota(\mathbf{1}) = \mathbf{1}$,

$$\|\tilde{\rho}\| = \tilde{\rho}(\mathbf{1}) = \tilde{\rho}(\iota(\mathbf{1})) = \rho(\mathbf{1}) = \|\rho\| .$$

3.3 Positive maps on abelian algebras

Let us repeat the definition:

Definition 2 *Let $\mathcal{A}$ be a Banach *-algebra with identity of norm 1, and $\mathcal{B} = \mathcal{B}(H)$, the bounded operators on a Hilbert space H. A map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be* positive *if for all $a \in \mathcal{A}$,*

$$\phi(a^*a) \geq 0 \quad \textit{i.e.} \quad \langle \phi(a^*a)\xi, \xi \rangle \geq 0 \quad \textit{for all } \xi \in H.$$

It is said to be completely positive *if for all n and all $a_1, a_2, \ldots, a_n \in \mathcal{A}$, the operator-valued matrix*

$$\phi_n([a_i^* a_j]) := [\phi(a_i^* a_j)]$$

is positive as an operator on H^n, that is if, for all $\xi_1 \ldots, \xi_n \in H$,

$$\left\langle \phi_n([a_i^* a_j]) \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle = \sum_{m,n} \langle \phi(a_n^* a_m)\xi_m, \xi_n \rangle \geq 0\,.$$

Example 3.7 *If $\mathcal{A} = \mathcal{B} = M_2(\mathbb{C})$, the transpose map $\phi : A \to A^{\mathrm{T}}$ is positive but not completely positive.*

The transpose map on M_n is positive: if $A = [a_{ij}] \in M_n$ is a positive matrix then for each $\xi = [\lambda_1, \ldots, \lambda_n]^{\mathrm{T}}$ we must have

$$\langle A\xi, \xi \rangle = \sum_{i,j} a_{ij} \lambda_j \bar{\lambda}_i \geq 0.$$

But then, setting $\eta = [\mu_1, \ldots, \mu_n]$ where $\mu_k = \bar{\lambda}_k$, we see that

$$\left\langle A^{\mathrm{T}}\xi, \xi \right\rangle = \sum_{i,j} a_{ji} \lambda_j \bar{\lambda}_i = \sum_{i,j} a_{ji} \bar{\mu}_j \mu_i = \langle A\eta, \eta \rangle \geq 0.$$

However the transpose map is not even 2-positive on M_2: if $E = [a_i^* a_j] \in M_2(\mathcal{A})$ where $a_1 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $a_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then

$$E = \begin{bmatrix} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \phi_2(E) = \begin{bmatrix} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so E is positive (it is a multiple of a projection) while $\phi_2(E)$ is not positive. [11]

Remark 3.8 *If $\phi : \mathcal{A} \to \mathbb{C}$ is positive, then it is completely positive.*

Proof. Indeed, if $a_1 \ldots, a_n \in \mathcal{A}$ and $\lambda_1 \ldots, \lambda_n \in \mathbb{C}$,

$$\begin{aligned} \sum_{i,j} \langle \phi(a_i^* a_j)\lambda_j, \lambda_i \rangle_{\mathbb{C}} &= \sum_{i,j} \phi(a_i^* a_j)\lambda_j \bar{\lambda}_i = \sum_{i,j} \phi((\lambda_i a_i)^* (\lambda_j a_j)) \\ &= \phi\left(\left(\sum_i \lambda_i a_i \right)^* \left(\sum_j \lambda_j a_j \right) \right) \geq 0\,. \end{aligned}$$

Proposition 3.9 *Every positive linear map defined on an abelian (unital) C^*-algebra $\mathcal{A}$ is automatically completely positive.*

Proof. By Gelfand theory, we may assume that $\mathcal{A} = C(X)$ where X is a a compact Hausdroff space. Let

$$\phi : C(X) \to B(H)$$

be a positive linear map, where H is a Hilbert space. If $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in C(X)$, write F for the matrix $F = [f_i^* f_j]$. We need to prove that

$$\phi_n(F) \geq 0$$

[11] For example, $\langle \phi_2(E)(e_2 - e_3), (e_2 - e_3) \rangle < 0$.

as an operator on H^n. Now the map

$$X \to M_n(\mathbb{C}) : t \to F(t) = [\overline{f_i(t)} f_j(t)]$$

is continuous. Therefore, given $\varepsilon > 0$ there exists a finite open cover $U_1, \ldots, U_m$ of X and points $t_k \in U_k$ so that

$$t \in U_k \;\Rightarrow\; \|F(t) - F(t_k)\|_{M_n} < \varepsilon \qquad (k = 1, \ldots, m)$$

(here $\|\cdot\|_{M_n}$ is the operator norm on $M_n(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^n)$). Write $F(t_k) = T_k$ for short; so $T_k \in M_n(\mathbb{C})$.

Let $\{u_1, \cdots, u_m\} \subseteq C(X)$ be a partition of unity [12] subordinate to the cover $U_1, \ldots, U_m$: this means that $0 \le u_k(t) \le 1$ for all k and t, $\sum_k u_k(t) = 1$ for all t and $\operatorname{supp} u_k \subseteq U_k$ for all k.

For all $y \in X$ we have

$$\begin{aligned}
\left\| F(y) - \sum_{l=1}^m u_l(y) T_l \right\|_{M_n} &= \left\| \left(\sum_{l=1}^m u_l(y) \right) F(y) - \sum_{l=1}^m u_l(y) T_l \right\|_{M_n} \\
&\le \sum_{l=1}^m u_l(y) \left\| F(y) - T_l \right\|_{M_n}.
\end{aligned}$$

Now each term $u_l(y) \|F(y) - T_l\|_{M_n}$ is less than $u_l(y)\varepsilon$; for either $y \in U_l$ in which case $\|F(y) - T_l\|_{M_n} < \varepsilon$ or $y \notin U_l$ in which case $u_l(y) = 0$. Therefore

$$\left\| F(y) - \sum_{l=1}^m u_l(y) T_l \right\|_{M_n} < \sum_{l=1}^m u_l(y) \varepsilon = \varepsilon\,.$$

Taking sup over $y \in X$,

$$\left\| F - \sum_{l=1}^m u_l T_l \right\|_{M_n(\mathcal{A})} \le \varepsilon.$$

Since ϕ_n is continuous,[13] it follows that

$$\left| \phi_n(F) - \sum_{l=1}^m \phi_n(u_l T_l) \right| \le \|\phi_n\| \, \varepsilon.$$

Therefore, since $\varepsilon > 0$ is arbitrary, in order to prove that $\phi_n(F) \ge 0$ it suffices to prove that each term $\phi_n(u_l T_l)$ is a positive operator. Droping the index l for clarity, we have a continuous function $u = u_l : X \to \mathbb{R}_+$ and a matrix $T = [\overline{f_i(t)} f_j(t)] \in M_n(\mathbb{C})$. Now

$$\phi_n(uT) = \phi_n([u\overline{f_i(t)} f_j(t)]) = [\phi(u\overline{f_i(t)} f_j(t))] = [\phi(u)\overline{f_i(t)} f_j(t)]$$

since each $f_j(t)$ is a scalar. Thus for each $\xi = [\xi_1, \ldots, \xi_n]^{\mathrm{T}} \in H^n$ we have

$$\begin{aligned}
\langle \phi_n(uT)\xi, \xi \rangle_{H^n} &= \left\langle [\phi(u)\overline{f_i(t)} f_j(t)]\xi, \xi \right\rangle_{H^n} \\
&= \sum_{i,j} \left\langle \phi(u)\overline{f_i(t)} f_j(t)\xi_j, \xi_i \right\rangle_H = \sum_{i,j} \langle \phi(u) f_j(t)\xi_j, f_i(t)\xi_i \rangle_H \\
&= \left\langle \phi(u) \left(\sum_j f_j(t)\xi_j \right), \left(\sum_i f_i(t)\xi_i \right) \right\rangle_H
\end{aligned}$$

[12] see for example Rudin, Real and Complex Analysis, Theorem 2.13

[13] in fact $\|\phi_n\| \le n \|\phi\|$

which is nonnegative, because $\phi(u) \in \mathcal{B}(H)$ is positive since u is positive in $C(X)$ and ϕ is a positive linear form.

Alternatively, the matrix $\phi_n(uT) \in M_n(\mathcal{B}(H))$ may be factorised as follows

$$\phi_n(uT) = \begin{bmatrix} \overline{f_1(t)} & \dots & 0 \\ \vdots & & \vdots \\ \overline{f_n(t)} & \dots & 0 \end{bmatrix} \begin{bmatrix} \phi(u) & & 0 \\ & \ddots & \\ 0 & & \phi(u) \end{bmatrix} \begin{bmatrix} f_1(t) & \dots & f_n(t) \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} = A^*BA$$

where $B \in M_n(\mathcal{B}(H)$ is positive and $A \in M_n(\mathbb{C})$.

Theorem 3.10 *Every positive linear map $\phi : \mathcal{A} \to \mathcal{B}(H)$ defined on an abelian Banach *-algebra with unit of norm $\|\mathbf{1}\| = 1$ is automatically completely positive.*

Proof. The enveloping C*-algebra $\mathcal{C} = C^*(\mathcal{A})$ is abelian and unital. Therefore every positive linear map defined on $\mathcal{C}$ is completely positive. Thus the theorem follows from Proposition 3.6.

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