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AK

We wish to prove the second part of the following Theorem:

Theorem 1 (Spronk – Turowska [3]) *A closed set $E \subseteq G$ is an S-set (i.e. satisfies spectral synthesis) if and only if the set*

$$E^* = \{(s,t) \in G \times G : st^{-1} \in E\}$$

is operator synthetic.

Preliminaries. Recall for $\tau \in A(G)^*$:

$$\operatorname{supp} \tau = \left(\bigcup \{ V \subseteq G \text{ open } : \tau|_V = 0 \} \right)^c$$

where $\tau|_V = 0$ means: $u \in A(G), \operatorname{supp} u \subseteq V \Rightarrow \tau(u) = 0$.

Also recall for $E \subseteq G$ closed:

$$\mathcal{J}(E) = \{u \in \mathcal{A} : \operatorname{supp} u \cap E = \emptyset\}$$

i.e. u vanishes near E.

Remark 2 $\mathcal{J}(E)^\perp = \{\tau \in A(G)^* : \operatorname{supp} \tau \subseteq E\}$.

Proof Take $\tau \in \mathcal{J}(E)^\perp$.

Let V be open, $V \cap E = \emptyset$. Then for all $u \in A(G)$ with $\operatorname{supp} u \subseteq V$ we have $u \in \mathcal{J}(E)$ so $\tau(u) = 0$. Thus $V \cap \operatorname{supp} \tau = \emptyset$.

We have shown that $\operatorname{supp} \tau \subseteq E$.

Conversely suppose that $\operatorname{supp} \tau \subseteq E$. Let $u \in A(G)$ be s.t. $\operatorname{supp} u \cap E = \emptyset$. Hence there is an open neighbourhood V of $\operatorname{supp} u$ with $V \cap E = \emptyset$. (compactness)

Thus τ vanishes on V and so $\tau(u) = 0$. We have shown that $\tau \in \mathcal{J}(E)^\perp$. □

Let

$$VN(G) = \{\lambda_s : s \in G\}'' \subseteq B(L^2(G))$$

Note that, since the set of generators $\{\lambda_s : s \in G\}$ is a semigroup, the algebra $VN(G)$ equals the WOT-closed linear span of the set of generators.

It is known that this von Neumann algebra is isometrically and w*-homeomorphically isomorphic to the Banach space dual of $(A(G), \|\cdot\|_A)$.

Sketch For $A = \lambda_s$ and $u(t) = \langle \lambda_t f, g \rangle \in A(G)$ define

$$\tau_A(u) = \langle Af, g \rangle .$$

Note that this is independent of f, g and depends only on u. Further $|\tau_A(u)| \leq \|A\| \, \|f\|_2 \, \|g\|_2$ for each such representation of u so $|\tau_A(u)| \leq \|A\| \, \|u\|_A$. Thus $\tau_A \in (A(G)^*$ and $\|\tau_A\|_{A^*} \leq \|A\|$. Since the map $A \to \tau_A$ is linear and WOT-w*-continuous it extends to a contraction $A \to \tau_A : VN(G) \to A(G)^*$.

Conversely given $\tau \in A(G)^*$ we consider the sesquilinear from $B : L^2(G) \times L^2(G) \to \mathbb{C}$ given by $B(f,g) = \tau(u)$ where $u(s) = \langle \lambda_s f, g\rangle$. Since $|B(f,g)| = |\tau(u)| \le \|\tau_A\|_{A^*} \|f\|_2 \|g\|_2$ there is a unique $A \in B(L^2(G)$ with $\|A\| \le \|\tau_A\|_{A^*}$ so that $B(f,g) = \langle Af, g\rangle$. One shows that A must commute with all right translations and hence must be in $VN(G)$.

Lemma 3 *If* $A \in VN(G)$*, then* $\operatorname{supp} A \subseteq ((\operatorname{supp}\tau_A)^{-1})^*$ *i.e. if* $(t,s) \in \operatorname{supp} A$ *then* $st^{-1} \in \operatorname{supp}\tau_A$.

Proof Let $(t,s) \in \operatorname{supp} A$. If $st^{-1} \notin \operatorname{supp}\tau_A$ there is an open set W with $st^{-1} \in W$ but $W \cap \operatorname{supp}\tau_A = \emptyset$. Find U, V open in G with $t \in U$, $s \in V$ and $\overline{VU^{-1}} \subseteq W$.

Since $(t,s) \in \operatorname{supp} A$, we have $P(V)AP(U) \neq 0$, so there are $f, g \in L^2(G)$ with $\operatorname{supp} f \subseteq U$, $\operatorname{supp} g \subseteq V$ and $\langle Af, g\rangle \neq 0$.

But if $u \in A(G)$ is given by $u(s) = \langle \lambda_s f, g\rangle$ then $u \in A(G)$ and $\langle Af, g\rangle = \tau_A(u)$.

If $u(s) \neq 0$ then $\int_G f(s^{-1}y)\bar{g}(y)dy \neq 0$ so there must exist $y \in V$ so that $s^{-1}y \in U$, i.e. $y^{-1}s \in U^{-1}$ i.e. $s \in yU^{-1}$ hence $s \in VU^{-1}$. It follows that $\operatorname{supp} u \subseteq \overline{VU^{-1}} \subseteq W$; but $W \cap \operatorname{supp}\tau_A = \emptyset$ and so $\tau_A(u) = 0$, a contradiction. □

Lemma 4 *If* $T \in B(H)$ *is supported in a closed set* $F \subseteq G \times G$*, then* $T \in \Psi(F)^\perp$*, i.e. if* $\omega \in T(G)$ *vanishes m.a.e.* in a neighbourhood of F*, then* $\omega(T) = 0$.

Proof Changing ω on a marginally null set, if necessary, (this doesn't affect $\omega(T)$) we may assume that ω vanishes everywhere in some neighbourhood V of F. We may cover the closed set V^c by a finite number of open rectangles $U_i \times V_i$ each disjoint from F (compactness of V^c). Thus

$$\begin{aligned}\omega(s,t) &= \left(\sum_i \chi_{U_i\times V_i}(s,t)\right)\omega(s,t) = \sum_i\sum_n \chi_{U_i}(s)\chi_{V_i}(t) f_n(s)\bar{g}_n(t) \\ &= \sum_i\sum_n (\chi_{U_i} f_n)(s)(\overline{\chi_{V_i} g_n})(t).\end{aligned}$$

Since $(U_i \times V_i) \cap F = 0$ we have $P(V_i)TP(U_i) = 0$ for all i and so

$$\begin{aligned}\omega(T) &= \sum_i\sum_n \langle T(\chi_{U_i} f_n), (\chi_{V_i} g_n)\rangle = \sum_i\sum_n \langle TP(U_i)f_n, P(V_i)g_n\rangle \\ &= \sum_i\sum_n \langle P(V_i)TP(U_i)f_n, g_n\rangle = 0\end{aligned}$$

as claimed. □

Conclusion of the proof of the Theorem Let $E \subseteq G$ be closed. Assume $E^* \subseteq G \times G$ is operator synthetic. We have to show that E is an S-set.

Fix $u \in \mathcal{K}(E)$ (i.e. $u \in A(G)$ vanishes on E) and show that $u \in \overline{\mathcal{J}(E)}$. By Remark 2, this is equivalent to showing that if $\tau \in A(G)^*$ has $\operatorname{supp}\tau \subseteq E$ then $\tau(u) = 0$. Now τ is of the form $\tau = \tau_T$ where $T \in VN(G)$ satisfies $\operatorname{supp} T \subseteq ((\operatorname{supp}\tau_T)^{-1})^* \subseteq (E^{-1})^*$ (Lemma 3). We have to prove that $\tau_T(u) = 0$.

Recall [1] that $A(G)$ can be embedded isometrically into $V(G)$ by $u \to Nu$ where $(Nu)(s,t) = u(st^{-1})$. If

$$Nu(s,t) = \sum_i \phi_i(t)\psi_i(s)$$

[1] The proof (which uses the Peter – Weyl theorem) is in [3, Theorem 2.2]

is any representation of Nu, consider the map (recall that u is fixed)

$$\Phi : B(H) \to B(H) : A \to \sum_i M_{\phi_i} A M_{\psi_i}.$$

Note that Φ depends only on Nu (hence on u) and not on the particular representation of Nu. One can show that the sum converges in the weak* topology, because the sums $\sum_i |\phi_i(t)|^2$ and $\sum_i |\psi_i(t)|^2$ converge uniformly in t, hence are uniformly bounded and converge in the L^2 norm. It follows also that the map Φ is w*-w*-continuous. (See for example [1, Section 3] or [2]).

Claim 5 *For all $A \in VN(G)$,*

$$\tau_{\Phi(A)}(v) = \tau_A(uv), \qquad (v \in A(G)).$$

Proof Assume first $A = \lambda_s$ ($s \in G$ fixed). Then for all $v \in A(G)$ of the form $v(t) = \langle \lambda_t f, g\rangle$ we have

$$\begin{aligned}
\tau_{\Phi(A)}(v) &:= \langle \Phi(A)f, g\rangle = \left\langle \sum_i M_{\phi_i} A M_{\psi_i} f, g \right\rangle \\
&= \left\langle \sum_i M_{\phi_i} A(\psi_i f), g \right\rangle = \int \sum_i \phi_i(t)(\lambda_s(\psi_i f))(t)\bar{g}(t)dt \\
&= \int \sum_i \phi_i(t)\psi_i(s^{-1}t) f(s^{-1}t)\bar{g}(t)dt \\
&= \int (Nu)(t, s^{-1}t) f(s^{-1}t)\bar{g}(t)dt = \int u(tt^{-1}s) f(s^{-1}t)\bar{g}(t)dt \\
&= \int u(s)(\lambda_s f)(t)\bar{g}(t)dt = \langle u(s)\lambda_s f, g\rangle = u(s)\langle \lambda_s f, g\rangle \\
&= u(s)v(s) = (uv)(s).
\end{aligned}$$

But by definition, when $w(s) = \langle \lambda_s \xi, \eta\rangle$ is in $A(G)$ then for $A = \lambda_s$ we have $\tau_A(w) = \langle A\xi, \eta\rangle = \langle \lambda_s \xi, \eta\rangle = w(s)$; thus $(uv)(s) = \tau_A(uv)$ and the Claim is proved for the generators $A = \lambda_s$ of $VN(G)$.

Thus, for each fixed $v \in A(G)$, the maps $A \to \tau_{\Phi(A)}(v)$ and $A \to \tau_A(uv)$ agree on the generators of $VN(G)$. Since both maps are linear and weak*-continuous and $VN(G)$ is the w*-closed linear span of the set $\{\lambda_s : s \in G\}$, these two maps must agree on the whole of $VN(G)$. This proves the Claim. □

Claim 6 $\Phi(T) = 0$.

Proof For $A \in B(H)$, let $\omega(A) = \langle Af, g\rangle$ where $f, g \in L^2(G)$ are arbitrary. Then

$$\begin{aligned}
\omega(\Phi(A)) &= \left\langle \sum_i M_{\phi_i} A M_{\psi_i} f, g \right\rangle = \sum_i \left\langle A M_{\psi_i} f, M_{\phi_i}^* g\right\rangle \\
&= \sum_i \left\langle A(\psi_i f), \overline{\phi_i} g\right\rangle = \omega_1(A)
\end{aligned}$$

so, when ω comes from the function $f(t)\bar{g}(s)$, the functional $\omega_1 := \omega \circ \Phi$ comes from the function

$$\sum_i \psi_i(t) f(t)\phi_i(s)\bar{g}(s) = \sum_i \psi_i(t)\phi_i(s) f(t)\bar{g}(s) = (Nu)(t,s) f(t)\bar{g}(s) = u(ts^{-1}) f(t)\bar{g}(s).$$

But u vanishes on E, so $u(ts^{-1}) = 0$ when $(s,t) \in (E^{-1})^*$, which is a set of operator synthesis, by assumption. Thus ω_1 can be approximated, in $\|\cdot\|_1$, by a sequence of functions $\omega_n \in T(G)$ vanishing in a neighbourhood of $(E^{-1})^*$. On the other hand $\operatorname{supp} T \subseteq (E^{-1})^*$ and therefore $\omega_n(T) = 0$ for all n by Lemma 4. Therefore

$$\omega_1(T) = 0.$$

But $\omega_1(T) = \omega(\Phi(T))$ and so $\langle \Phi(T)f, g\rangle = 0$. Since f, g are arbitrary, we have shown that $\Phi(T) = 0$. □

Now using Claim 5 we have

$$\tau_T(uv) = \tau_{\Phi(T)}(v) = 0 \qquad \text{for all} \quad v \in A(G)$$

and so in particular $\tau_T(u) = \tau_T(u\mathbf{1}) = 0$, ce qu'il fallait démontrer.

References

[1] Aristides Katavolos and Vern I. Paulsen. On the ranges of bimodule projections. *Canad. Math. Bull.*, 48(1):97–111, 2005.

[2] Jean Ludwig and Lyudmila Turowska. On the connection between sets of operator synthesis and sets of spectral synthesis for locally compact groups. *J. Funct. Anal.*, 233(1):206–227, 2006.

[3] N. Spronk and L. Turowska. Spectral synthesis and operator synthesis for compact groups. *Journal of the London Mathematical Society*, 66(2):361, 2002.