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We wish to prove the following Theorem:

Theorem 1 (Spronk – Turowska [5]) *A closed set $E \subseteq G$ is a n S-set (i.e. satisfies spectral synthesis) if and only if the set*

$$E^* = \{(s,t) \in G \times G : st^{-1} \in E\}$$

is operator synthetic.

(1) The extended Varopoulos algebra. The Varopoulos algebra $V(G) = C(G) \otimes^h C(G)$ was identified with the space of all continuous functions $u : G \times G \to \mathbb{C}$ which can be represented in the form

$$u(s,t) = \sum_k e_k(s) f_k(t) \quad \text{with } e_k, f_k \in C(G) \text{ s.t. } \sum_k |e_k|^2 \text{ and } \sum_k |f_k|^2 \text{ converge uniformly.}$$

$V(G)$ is equipped with the norm

$$\|u\|_h = \inf \left\{ \left\| \sum_k |e_k|^2 \right\|_\infty^{1/2} \left\| \sum_k |f_k|^2 \right\|_\infty^{1/2} : \text{ all such repr's } u = \sum_k e_k \otimes f_k \right\}.$$

We now consider the algebra [1] $V^\infty(G)$ consisting of all bounded Borel functions $u : G \times G \to \mathbb{C}$ which can be represented in the form

$$u(s,t) = \sum_k e_k(s) f_k(t) \quad \text{with } e_k, f_k \in \mathcal{L}^\infty(G) \text{ s.t. } \left\| \sum_k |e_k|^2 \right\|_\infty \cdot \left\| \sum_k |f_k|^2 \right\|_\infty < \infty.$$

We identify [2] *two such functions if they coincide marginally almost everywhere.*

Note that the condition $\|\sum_k |e_k|^2\|_\infty < \infty$ just means that there is a $C_e < \infty$ so that $\sum_{n=1}^N |e_n(t)|^2 \le C_e$ for all N and all $t \in G$. If $u \in V^\infty(G)$ then for any $S \in B(L^2(G))$ the series

$$\sum_{k=1}^\infty M_{e_k} S M_{f_k} := T_u(S)$$

converges weak* to an element $T_u(S) \in B(L^2(G))$. The map $T_u : B(L^2(G)) \to B(L^2(G))$ is in fact w*-w* continuous and it can be shown that $\|T_u\| = \|u\|_h$, where $\|u\|_h$ is given by

$$\|u\|_h = \inf \left\{ \left\| \sum_k |e_k|^2 \right\|_\infty^{1/2} \left\| \sum_k |f_k|^2 \right\|_\infty^{1/2} : \text{ all such repr's } u = \sum_k e_k \otimes f_k \right\}.$$

[1] In fact $V^\infty(G)$ coincides with the w*-Haagerup tensor product $L^\infty \otimes^{w^* h} L^\infty$; see [1].

[2] $\mathcal{L}^\infty(G)$ consists of bounded measurable *functions*, as opposed to equivalence classes

It can be shown that $\|T_u\| = \|u\|_h$ [4] and that $T_u = 0$ if and only if u vanishes marginally almost everywhere if and only if u vanishes almost everywhere (see for example [3, Theorem 7]). Thus $\|u\|_h$ is in fact a norm. This also shows that the Varopoulos algebra $V(G)$ embeds isometrically into $V^\infty(G)$.

Proposition 2 *Any $u \in V^\infty(G)$ defines a* multiplier m_u *on $T(G) = L^2(G)\hat{\otimes}L^2(G)$: if $(m_u(\omega))(s,t) = u(s,t)\omega(s,t)$, the map m_u is a bounded operator $m_u : T(G) \to T(G)$ such that $\|m_u\| \le \|u\|_h$.* [3]

Proof Choose a representation $u(s,t) = \sum_k e_k(s)f_k(t)$ with $\|u\|_h = \|\sum_k |e_k|^2\|_\infty^{1/2} \|\sum_k |f_k|^2\|_\infty^{1/2}$ (such a representation exists). Consider a rank one $\omega(s,t) = \phi(t)\psi(s)$ in $T(G)$. Then

$$(m_u(\omega))(s,t) = \sum_k e_k(s)f_k(t)\phi(t)\psi(s) = \sum_k (f_k\phi)(t)(e_k\psi)(s)$$

$$\text{so} \quad \|m_u(\omega)\|_1 = \left\|\sum_k (f_k\phi)\otimes(e_k\psi)\right\|_1 \le \sum_k \|f_k\phi\|_2 \|e_k\psi\|_2$$

$$\le \left(\sum_k \|f_k\phi\|_2^2 \sum_k \|e_k\psi\|_2^2\right)^{1/2}$$

But

$$\sum_k \|f_k\phi\|_2^2 = \sum_k \int |f_k(t)\phi(t)|^2 dt = \int \sum_k |f_k(t)\phi(t)|^2 dt$$

$$= \int \left(\sum_k |f_k(t)|^2\right) |\phi(t)|^2 dt \le \left\|\sum_k |f_k|^2\right\|_\infty \|\phi\|_2^2$$

and similarly

$$\sum_k \|e_k\psi\|_2^2 \le \left\|\sum_k |e_k|^2\right\|_\infty \|\psi\|_2^2$$

so that

$$\|m_u(\omega)\|_1 \le \left\|\sum_k |f_k|^2\right\|_\infty^{1/2} \left\|\sum_k |e_k|^2\right\|_\infty^{1/2} \|\phi\|_2 \|\psi\|_2 = \|u\|_h \|\omega\|_1 .$$

By linearity and continuity, the same inequality holds for any $\omega \in T(G)$. $\square$

Corollary 3 *The map $J : V^\infty(G) \to T(G) : u \to m_u(\mathbf{1})$ (i.e. $(Ju)(s,t) = u(s,t)$ considered as an element of $T(G)$) is contractive and injective.*

Remarks 4 *Here $\mathbf{1}(s,t) = 1$ for all $s,t \in G$. This is in $T(G)$ only because G is compact!*

Note also that by our conventions if $u = e\otimes f \in V^\infty(G)$, i.e. $u(s,t) = e(s)f(t)$, then $(Ju)(s,t) = e(s)f(t)$ should be written as $Ju = f \otimes e$ in $T(G)$!

Proof Obviously,

$$\|Ju\|_1 = \|m_u(\mathbf{1})\|_1 \le \|u\|_h \|\mathbf{1}\|_1 = \|u\|_h .$$

For the injectivity, if $Ju = 0$ then u is zero as an element of $T(G)$ so $u(s,t) = 0$ marginally a.e. and so $u = 0$ as an element of $V^\infty(G)$. $\square$

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[3] in fact equality holds - see [5]

(2) Isometric embedding of $A(G)$ into $T(G)$. If $u \in A(G)$, we define $\tilde{N}u : G \times G \to \mathbb{C}$ by

$$(\tilde{N}u)(s,t) = u(st^{-1}).$$

We will use the fact that the map $N : u \to Nu$ given by $(Nu)(s,t) = u(st^{-1})$ maps $A(G)$ contractively [4] into $V(G)$. Note that $\tilde{N}u = JNu$. Therefore $\tilde{N}$ maps $A(G)$ contractively into $T(G)$.

If $\omega \in T(G)$ we define $Q\omega : G \to \mathbb{C}$ by

$$(Q\omega)(s) = \int_G \omega(sr, r)dr$$

Proposition 5 *The map Q is a well-defined contraction $Q : T(G) \to A(G)$ and*

$$(Q \circ \tilde{N})u = u \qquad \text{for all } u \in A(G).$$

Thus $\tilde{N}$ is in fact an isometric embedding *of $A(G)$ into $T(G)$.*

Proof First consider $\omega \in L^2(G) \otimes L^2(G)$ of the form $\omega(s,t) = \sum_{k=1}^{n} f_k(t)g_k(s)$. Then

$$\begin{aligned}(Q\omega)(s) &= \int_G \sum_{k=1}^{n} f_k(r)g_k(sr)dr = \sum_{k=1}^{n} \int_G f_k(s^{-1}t)g_k(t)dt = \sum_{k=1}^{n} \int_G (\lambda_s f_k)(t)g_k(t)dt \\ &= \sum_{k=1}^{n} \langle (\lambda_s f_k), \bar{g}_k \rangle = \sum_{k=1}^{n} u_k(s).\end{aligned}$$

By the definition of $A(G)$, each u_k is in $A(G)$, hence so is $Q\omega$. Furthermore

$$\|Q\omega\|_A = \left\| \sum_{k=1}^{n} u_k \right\|_A \le \sum_k \|u_k\|_A \le \sum_k \|f_k\|_2 \|\bar{g}_k\|_2$$

and this holds for every representation $\omega(s,t) = \sum_{k=1}^{n} f_k(t)g_k(s)$. Thus

$$\|Q\omega\|_A \le \inf \left\{ \sum_k \|f_k\|_2 \|g_k\|_2 : \omega = \sum_{k=1}^{n} f_k \otimes g_k \right\} = \|\omega\|_1$$

Hence Q maps the algebraic tensor product $L^2(G) \otimes L^2(G)$ contractively into $A(G)$, and so the first claim follows by continuity.

The second claim is easy: If $u \in A(G)$ then

$$(Q(\tilde{N}u))(s) = \int (\tilde{N}u)(sr, r)dr = \int u(srr^{-1})dr = u(s)$$

because the Haar measure of G is 1 (compactness!). $\square$

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[4] Proof next time!

(3) The right action of G and of $L^1(G)$ on $T(G)$. For $r \in G$ we define a map

$$\omega \to r \bullet \omega : T(G) \to T(G)$$
$$\text{by} \quad (r \bullet \omega)(s,t) = \omega(sr, tr).$$

Using the fact that $\int |f(tr)|^2 dt = \int |f(t)|^2 dt$ (*right*-invariance of Haar measure on a compact group) for all $f \in L^2(G)$, it is readily verified that $\|r \bullet \omega\|_1 = \|\omega\|_1$ and so this map is an isometric action of G on $T(G)$. Moreover, it is strongly continuous. Indeed, if $\omega = \sum_{k=1}^n f_k \otimes g_k$ then, denoting provisionally by f^r the right translate of f, we see that

$$r \bullet \omega - \omega = \sum_k (f_k^r \otimes g_k^r - f_k \otimes g_k) = \sum_k [(f_k^r - f_k) \otimes g_k^r + f_k \otimes (g_k^r - g_k)]$$
$$\text{so} \quad \|r \bullet \omega - \omega\|_1 \le \sum_k (\|f_k^r - f_k\|_2 \|g_k^r\|_2 + \|f_k\|_2 \|g_k^r - g_k\|_2)$$

and therefore $\|r \bullet \omega - \omega\|_1 \to 0$ as $r \to e$.

In fact the action $\omega \to r \bullet \omega$ extends to a contractive action of (the convolution algebra) $L^1(G)$, defined as follows: [5]

$$h \bullet \omega = \int_G h(r)(r \bullet \omega) dr \quad \text{i.e.} \quad (h \bullet \omega)(s,t) = \int_G h(r)(r \bullet \omega)(s,t) dr = \int_G h(r) \omega(sr, tr) dr, \quad h \in L^1(G).$$

Thus

$$\|h \bullet \omega\|_1 = \left\| \int_G h(r)(r \bullet \omega) dr \right\|_1 \le \int_G |h(r)| \, \|r \bullet \omega\|_1 \, dr = \int_G |h(r)| dr \, \|\omega\|_1 = \|h\|_{L^1} \|\omega\|_1 .$$

(4) The image of $A(G)$ in $T(G)$. If

$$P\omega = \int_G (r \bullet \omega) dr \quad \text{i.e.} \quad (P\omega)(s,t) = \int_G \omega(sr, tr) dr$$

then $P\omega \in T(G)$ and in fact clearly $P\omega$ is *invariant*, i.e. $r \bullet (P\omega) = P\omega$ for all $r \in G$.

Note also that,

$$\|P\omega\|_1 \le \int_G \|r \bullet \omega\|_1 \, dr = \int_G \|\omega\|_1 \, dr = \|\omega\|_1 .$$

We put

$$T_{inv}(G) = \{\omega \in T(G) : \omega(sr, tr) = \omega(s,t) \text{ for all } s, t, r \in G\}$$

and it is clear that P is a contractive projection onto $T_{inv}(G)$.

Proposition 6 *The range of $\tilde{N}$ consists of all* invariant *elements of $T(G)$:*

$$\tilde{N}(A(G)) = T_{inv}(G).$$

Proof It is clear that $\tilde{N}u$ is invariant; indeed for all $u \in A(G)$ we have

$$(\tilde{N}u)(sr, tr) = u(sr(tr)^{-1}) = u(st^{-1}) = (\tilde{N}u)(s,t)$$

[5] The integral converges in the norm of $T(G)$ for every $h \in L^1(G)$.

for all $s,t \in G$. For the converse, let $\omega \in T_{inv}(G)$ and $\epsilon > 0$ be given. First choose $\omega_1(s,t) = \sum_{k=1}^{n} f_k(t)g_k(s)$ such that $\|\omega - \omega_1\|_1 < \epsilon$. Then approximate each f_k, g_k by continuous functions: choose $e_k, h_k \in C(G)$ such that

$$\|e_k - g_k\|_2 < \frac{\epsilon}{2n\,\|f_k\|_2} \quad \text{and} \quad \|h_k - f_k\|_2 < \frac{\epsilon}{2n\,\|e_k\|_2}, \quad k = 1, \dots n.$$

Then setting $u(s,t) = \sum_{k=1}^{n} e_k(s)h_k(t)$ so that $u \in C(G) \otimes C(G) \subseteq V(G)$ we have

$$Ju - \omega_1 = \sum_k (h_k \otimes e_k - f_k \otimes g_k) = \sum_k [(h_k - f_k) \otimes e_k + f_k \otimes (e_k - g_k)]$$

$$\text{so} \quad \|Ju - \omega_1\|_1 \le \sum_k (\|h_k - f_k\|_2 \|e_k\|_2 + \|f_k\|_2 \|e_k - g_k\|_2) < \epsilon$$

and so $\|Ju - \omega\|_1 < 2\epsilon$. It follows that $\|PJu - P\omega\|_1 < 2\epsilon$.

Now put

$$v(s) = \int_G \sum_{k=1}^{n} e_k(sr)h_k(r)dr = \sum_{k=1}^{n} \int_G e_k(x)h_k(s^{-1}x)dx = \sum_{k=1}^{n} \langle \lambda_s h_k, \bar{e}_k \rangle$$

hence $v \in A(G)$ and

$$(\tilde{N}v)(s,t) = v(st^{-1}) = \int_G \sum_{k=1}^{n} e_k(st^{-1}x)h_k(x)dx$$
$$= \int_G \sum_{k=1}^{n} e_k(sr)h_k(tr)dr = (PJu)(s,t)$$

hence $PJu = \tilde{N}v$ and so $\left\|\tilde{N}v - \omega\right\|_1 = \|PJu - P\omega\|_1 < 2\epsilon$ since $P\omega = \omega$ because ω is invariant. Thus $\omega \in \overline{\tilde{N}(A(G))} = \tilde{N}(A(G))$ since $A(G)$ is complete and $\tilde{N}$ is isometric. $\square$

(5) Proof of the Theorem: First part.

Proposition 7 *Let $E \subset G$ be a closed set. If E is a set of synthesis, then $E^* = \{(s,t) : st^{-1} \in E\}$ is a set of operator synthesis.*

Proof Let $\omega \in \Phi(E^*)$, i.e. $\omega \in T(G)$ vanishes marginally almost everywhere in E^*. It is to be shown that $\omega \in \Phi_0(E^*)$, i.e. that ω can be approximated in $\|\cdot\|_1$ by elements of $T(G)$ that vanish m.a.e. *in a neighbourhood* of E^*.

Step 1 Assume additionally that $\omega \in T_{inv}(G)$. Then by Proposition 6 there exists $u \in A(G)$ such that $\tilde{N}u = \omega$. Thus ω is continuous and hence vanishes everywhere on E^*. It follows that u vanishes in E (i.e. $u \in \mathcal{K}(E)$). Indeed, if $x \in E$ then $(x,e) \in E^*$ and so $u(x) = (\tilde{N}u)(x,e) = 0$.

Since E is an S-set, u can be approximated in $\|\cdot\|_A$ by a sequence (u_n) vanishing *near* E. If $\omega_n = \tilde{N}u_n$ then $\text{supp}(\omega_n) \subseteq (\text{supp}\, u_n)^*$, because if $\omega_n(s,t) \neq 0$ then $u_n(st^{-1}) \neq 0$. The complement U_n of $(\text{supp}\, u_n)^*$ is an open neighbourhood of E^* and ω_n vanishes in U_n. In the notation of the previous talk, $\omega_n \in \Psi(E^*)$. But

$$\|\omega_n - \omega\|_1 = \|\tilde{u}_n - \tilde{u}\|_1 \le \|u_n - u\|_A \to 0$$

showing that $\omega \in \Phi_0(E^*)$ as required.

Step 2 Now let $\omega \in \Phi(E^*)$ be arbitrary.

For each irreducible representation (π, H_π) of G let $\{e_i^\pi : i = 1, \ldots, d_\pi\}$ be an orthonormal basis of H_π (it is known that $\dim H_\pi = d_\pi$ is always finite when G is compact) and consider the coefficients of the matrix $\pi(s) \in B(H_\pi)$ given by

$$u_{ij}^\pi(s) = \langle \pi(s) e_j, e_i \rangle_{H_\pi}, \quad i, j = 1, \ldots, d_\pi,\ s \in G.$$

These functions of course depend only on the unitary equivalence class $[\pi]$ of π.
[*If G were abelian as well as compact then $d_\pi = 1$ for all π and u^π would be the character corresponding to π. By Plancherel, these characters would form an orthonormal basis of $L^2(G)$.*]

If $\widehat{G}$ denotes the set of unitary equivalence classes of irreducible representations of G, the set

$$\mathcal{S} = \{\sqrt{d_\pi} u_{ij}^\pi : i, j = 1, \ldots, d_\pi,\ [\pi] \in \widehat{G}\}$$

forms an orthonormal basis of $L^2(G)$. This is *the Peter-Weyl Theorem* [2, Theorem 27.40].

For each $[\pi] \in \widehat{G}$ we define

$$\omega^\pi(s,t) = \int_G \omega(sr, tr)\pi(r) dr$$
$$\text{and} \qquad \tilde{\omega}^\pi(s,t) = \pi(s)\omega^\pi(s,t), \qquad (s,t) \in G \times G.$$

Since each $\pi(r)$ is a unitary operator on H_π, these are elements of $B(H_\pi)$, i.e. each ω^π is a $d_\pi \times d_\pi$-matrix-valued function on $G \times G$. Since $\omega \in \Phi(E^*)$, it follows that the matrix $\omega^\pi(s,t)$ vanishes for marginally almost all $(s,t) \notin E^*$ and hence so does $\tilde{\omega}^\pi(s,t)$ (multiplying by $\pi(s)$ cannot increase the support).

Note that $\tilde{\omega}^\pi$ is invariant:

$$\begin{aligned}
\tilde{\omega}^\pi(sx, tx) &= \pi(sx)\omega^\pi(sx, tx) = \pi(sx) \int_G \omega(sxr, txr)\pi(r) dr \\
&= \pi(s) \int_G \omega(sxr, txr)\pi(x)\pi(r) dr = \pi(s) \int_G \omega(sy, ty)\pi(y) dy \\
&= \pi(s)\omega^\pi(s,t) = \tilde{\omega}^\pi(s,t)
\end{aligned}$$

where we have used the fact that π is a group morphism and that $\pi(r)\omega(s,t) = \omega(s,t)\pi(r)$ since $\omega(s,t) \in \mathbb{C}$. It follows that the matrix coefficients

$$\tilde{\omega}_{ij}^\pi(s,t) = \langle \tilde{\omega}^\pi(s,t) e_j, e_i \rangle$$

are also invariant: $\tilde{\omega}_{ij}^\pi \in T_{inv}(G)$; since they also vanish for marginally almost all $(s,t) \notin E^*$, by the first part $\tilde{\omega}_{ij}^\pi \in \Phi_0(E^*)$.

However since $\tilde{\omega}^\pi(s,t) = \pi(s)\omega^\pi(s,t)$, we have $\omega^\pi(s,t) = \pi(s^{-1})\tilde{\omega}^\pi(s,t)$ and therefore the matrix coefficients satisfy

$$\begin{aligned}
\omega_{ij}^\pi(s,t) := \langle \omega^\pi(s,t) e_j, e_i \rangle &= \sum_{k=1}^{d_\pi} \left\langle \pi(s^{-1}) e_k, e_i \right\rangle \langle \tilde{\omega}^\pi(s,t) e_j, e_k \rangle \\
&= \sum_k \check{u}_{i,k}^\pi(s) \tilde{\omega}_{kj}^\pi(s,t)
\end{aligned}$$

where $\check{u}(s) = u(s^{-1})$. If we denote by $u \otimes \mathbf{1}$ the function $(u \otimes \mathbf{1})(s,t) = u(s)\mathbf{1}(t)$, the last formula may be written

$$\omega_{ij}^\pi = \sum_k (\check{u}_{i,k}^\pi \otimes \mathbf{1}) \tilde{\omega}_{kj}^\pi$$

(pointwise multiplication). Since $\tilde{\omega}^{\pi}_{ij} \in \Phi_0(E^*)$, it follows from this that $\omega^{\pi}_{ij} \in \Phi_0(E^*)$.

Thus for each π and i, j the function

$$(s,t) \to \int_G u^{\pi}_{ij}(r)\omega(sr,tr)dr = \int_G \omega(sr,tr) \langle \pi(r)e_j, e_i \rangle \, dr = \left\langle \left(\int_G \omega(sr,tr)\pi(r)dr \right) e_j, e_i \right\rangle = \omega^{\pi}_{ij}(s,t)$$

which we denote by $u^{\pi}_{ij} \bullet \omega$, belongs to $\Phi_0(E^*)$. Therefore if u is a linear combination of the functions u^{π}_{ij}, i.e. if u belongs to $[\mathcal{S}]$, then $u \bullet \omega \in \Phi_0(E^*)$. Since $\Phi_0(E^*)$ is closed, to prove that $\omega \in \Phi_0(E^*)$ it therefore remains to prove the

Claim *Given $\epsilon > 0$ there exists $u = u_\epsilon \in [\mathcal{S}]$ such that $\|u \bullet \omega - \omega\|_1 < 2\epsilon$.*

Indeed, since $r \to r \bullet \omega$ is continuous, there is a neighbourhood U of $e \in G$ such that $\|r \bullet \omega - \omega\|_1 < \epsilon$ for all $r \in U$. Then letting $\chi := \dfrac{\chi_U}{m(U)}$ (here $m(U)$ is the Haar measure of U) we have, since $\int_G \chi(r)dr = 1$,

$$\|\chi \bullet \omega - \omega\|_1 = \left\| \int_G \chi(r)(r \bullet \omega - \omega)dr \right\|_1 \leq \int_G \chi(r) \|r \bullet \omega - \omega\|_1 \, dr \leq \epsilon$$

But observe that since $\mathcal{S}$ is an orthonormal basis of $L^2(G)$, the linear span $[\mathcal{S}]$ is dense in $L^2(G)$, hence also in $L^1(G)$ (recall that $\|\cdot\|_1 \leq \|\cdot\|_2$ since Haar measure of a compact group is finite). Thus there is $u \in [\mathcal{S}]$ such that $\|\chi - u\|_{L^1} < \dfrac{\epsilon}{\|\omega\|_1}$ and therefore

$$\|\chi \bullet \omega - u \bullet \omega\|_1 \leq \|\chi - u\|_{L^1} \|\omega\|_1 < \epsilon$$

and the claim follows.

This completes the proof of the first part of Theorem 1.

Recall for $\tau \in A(G)^*$:

$$\operatorname{supp}\tau = \Big(\bigcup\{V \subseteq G \text{ open } : \tau|_V = 0\}\Big)^c$$

where $\tau|_V = 0$ means: $u \in A(G), \operatorname{supp} u \subseteq V \Rightarrow \tau(u) = 0$.

Also recall for $E \subseteq G$ closed:

$$\mathcal{J}(E) = \{u \in \mathcal{A} : \operatorname{supp} u \cap E = \emptyset\}$$

i.e. u vanishes near E.

Remark 8 $\mathcal{J}(E)^\perp = \{\tau \in A(G)^* : \operatorname{supp}\tau \subseteq E\}$.

Proof Take $\tau \in \mathcal{J}(E)^\perp$.

Let V be open, $V \cap E = \emptyset$. Then for all $u \in A(G)$ with $\operatorname{supp} u \subseteq V$ have $u \in \mathcal{J}(E)$ so $\tau(u) = 0$. Thus $V \cap \operatorname{supp}\tau = \emptyset$.

We have shown that $\operatorname{supp}\tau \subseteq E$.

Cvsly spose $\operatorname{supp}\tau \subseteq E$. Let $u \in A(G)$ be s.t. $\operatorname{supp} u \cap E = \emptyset$. Hence there is V open nhd of $\operatorname{supp} u$ with $V \cap E = \emptyset$. (compactness)

Thus τ vanishes on V and so $\tau(u) = 0$. We have shown that $\tau \in \mathcal{J}(E)^\perp$. □

Let

$$VN(G) = \{\lambda_s : s \in G\}'' \subseteq B(L^2(G))$$

For $u(s) = \langle \lambda_s f, g\rangle \in A(G)$ cal $S = \lambda_s$ and define $\tau_S(u) = \langle Sf, g\rangle$ Note that this is independent of f, g and depends only on u. Further $|\tau_S(u)| \leq \|S\| \, \|f\|_2 \, \|g\|_2$ for each such repr. of u so $|\tau_S(u)| \leq \|S\| \, \|u\|_A$. Thus $\tau_S \in (A(g)^*$ and $\|\tau_S\|_{A^*} \leq \|S\|$. Since the map $S \to \tau_S$ is WOT-w*-continuous it extends to a contraction $S \to \tau_S : VN(G) \to A(G)^*$. In fact this is an onto isometry and a w*-homeo (?).

Lemma 9 *If* $S \in VN(G)$, *then* $\operatorname{supp} S \subseteq ((\operatorname{supp}\tau_S)^{-1})^*$ *i.e. if* $(t,s) \in \operatorname{supp} S$ *then* $st^{-1} \in \operatorname{supp}\tau_S$.

Proof Let $(t,s) \in \operatorname{supp} S$. If $st^{-1} \notin \operatorname{supp}\tau_S$ there is W open with $st^{-1} \in W$ but $W \cap \operatorname{supp}\tau_S = \emptyset$. Find U, V open in G with $t \in U$, $s \in V$ and $\overline{VU^{-1}} \subseteq W$.

Since $(t,s) \in \operatorname{supp} S$, have $P(V)SP(U) \neq 0$, so there are $f, g \in L^2(G)$ with $\operatorname{supp} f \subseteq U$, $\operatorname{supp} g \subseteq V$ and $\langle Sf, g\rangle \neq 0$.

But if $u(s) = \langle \lambda_s f, g\rangle$ then $u \in A(G)$ and $\langle Sf, g\rangle = \tau_S(u)$.

If $u(s) \neq 0$ then $\int_G f(s^{-1}y)\bar{g}(y)dy \neq 0$ so there must exist $y \in V$ so that $s^{-1}y \in U$, i.e. $y^{-1}s \in U^{-1}$ i.e. $s \in yU^{-1}$ or $s \in VU^{-1}$. It follows that $\operatorname{supp} u \subseteq \overline{VU^{-1}} \subseteq W$; but $W \cap \operatorname{supp}\tau_S = \emptyset$ and so $\tau_S(u) = 0$, a contradiction. □

Proof of Theorem Let $E \subseteq G$ be closed. Assume $E^* \subseteq G \times G$ is operator synthetic. To show that E is an S-set, fix $u \in \mathcal{K}(E)$ and show that $u \in \overline{\mathcal{J}(E)}$. By Remark 8, this is equivalent to showing that if $\tau \in A(G)^*$ has $\operatorname{supp}\tau \subseteq E$ then $\tau(u) = 0$. Now $\tau = \tau_S$ where $S \in VN(G)$ satisfies $\operatorname{supp} S \subseteq ((\operatorname{supp}\tau_S)^{-1})^* \subseteq (E^{-1})^*$ (Lemma 9).

Recall that we have embedded $A(G)$ isometrically into $V(G)$ by $u \to Nu$ where $(Nu)(s,t) = u(st^{-1})$ (proof missing!). If

$$Nu(s,t) = \sum_i \phi_i(t)\psi_i(s)$$

is a representation of Nu, then we have defined a map

$$T_{Nu} : B(H) \to B(H) : A \to \sum_i M_{\phi_i} A M_{\psi_i}$$

where the sum converges an the weak* topology (and boundedly? mallon). We shorten $T_{Nu}(A)$ to $\Phi(A)$ (recall u is fixed).

Claim For all $A \in VN(G)$,

$$\tau_{\Phi(A)}(v) = \tau_A(uv), \qquad (v \in A(G)).$$

Proof Assume first $A = \lambda_s$ ($s \in G$ fixed). Then for all $v \in A(G)$ of the form $v(s) = \langle \lambda_s f, g\rangle$ we have

$$\begin{aligned}
\tau_{\Phi(A)}(v) &:= \langle \Phi(\lambda_s) f, g\rangle = \left\langle \sum_i M_{\phi_i} \lambda_s M_{\psi_i} f, g \right\rangle \\
&= \left\langle \sum_i M_{\phi_i} \lambda_s (\psi_i f), g \right\rangle = \int \sum_i \phi_i(t)(\lambda_s(\psi f))(t)\bar{g}(t)dt \\
&= \int \sum_i \phi_i(t)\psi(s^{-1}t)f(s^{-1}t)\bar{g}(t)dt \\
&= \int (Nu)(t, s^{-1}t) f(s^{-1}t)\bar{g}(t)dt = \int u(tt^{-1}s)f(s^{-1}t)\bar{g}(t)dt \\
&= \int u(s)(\lambda_s f(t)\bar{g}(t)dt = \langle u(s)\lambda_s f, g\rangle = u(s)\langle \lambda_s f, g\rangle \\
&= u(s)v(s) = (uv)(s).
\end{aligned}$$

But by definition, when $w(s) = \langle \lambda_s \xi, \eta\rangle$ is in $A(G)$ then for $A = \lambda_s$ we have $\tau_A(w) = \langle A\xi, \eta\rangle = \langle \lambda_s \xi, \eta\rangle = w(s)$; thus $(uv)(s) = \tau_A(uv)$ and the Claim is proved for the generators $A = \lambda_s$ of $VN(G)$.

Thus, for each fixed $v \in A(G)$, the maps $A \to \tau_{\Phi(A)}(v)$ and $A \to \tau_A(uv)$ agree on the generators. Since they are both weak*-continuous (or WOT continuous on the unit ball) and $VN(G)$ is the w*-closed linear span of the set $\{\lambda_s : s \in G\}$ (it is already a semigroup), these two maps must agree on the whole of $VN(G)$.

Now let $\omega(A) = tr(\Theta_{fg^*}A)$ where Θ_{fg^*} is the rank one operator $\Theta_{fg^*}\xi = \langle \xi, g\rangle f$. Then

$$\begin{aligned}
\omega(\Phi(A)) &= tr(\Theta_{fg^*} \sum_i M_{\phi_i} A M_{\psi_i}) = tr(\sum_i M_{\psi_i} \Theta_{fg^*} M_{\phi_i} A M_{\psi_i}) \\
&= tr(\Psi(\Theta_{fg^*})A) = \omega_1(A)
\end{aligned}$$

where

$$\Psi(B) = \sum_i M_{\psi_i} B M_{\phi_i}$$

so, when ω comes from the function $f(t)\bar{g}(s)$, ω_1 comes from the function

$$\sum_i \psi_i(t)f(t)\bar{g}(s)\phi_i(s) = \sum_i \psi_i(t)f(t)\bar{g}(s)\phi_i(s)f(t)\bar{g}(s) = (Nu)(t,s)f(t)\bar{g}(s)$$

i.e. $\omega_1 = m_{N\check{u}}\omega$ (recall $\check{u}(s) = u(s^{-1})$). But u vanishes on E, so $\check{u}$ vanishes on E^{-1} and hence $N\check{u}$ vanishes on $(E^{-1})^*$, which is a set of operator synthesis, by assumption. On the other hand $\operatorname{supp} S \subseteq (E^{-1})^*$ and therefore

$$\omega_1(S) = 0$$

(you need a Lemma like Lemma 9 for that). By the previous calculation it follows that $\omega(\Phi(S)) = 0$ or $\langle \Phi(S)f, g\rangle = 0$. Since f, g are arbitrary, we have shown that $\Phi(S) = 0$.

Now form the claim we have

$$\tau_S(uv) = \tau_{\Phi(S)}(v) = 0 \qquad \text{for all} \;\; v \in A(G)$$

and so in particular $\tau_S(u) = \tau_S(u\mathbf{1}) = 0$, ce qu'il fallait démontrer.

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