Seminar 28 Feb 2011

AK

(1) Reminder: Synthesis in abelian Banach algebras. Let[1] $(\mathcal{A}, \|\cdot\|_A)$ be a unital regular Banach algebra with spectrum (=set of nonzero characters) K. Thus $\mathcal{A}$ can (and will) be identified with a subalgebra of $C(K)$ and $\|f\|_\infty \le \|f\|_A$ for all $f \in \mathcal{A}$. Regularity means that if $F \subseteq K$ is closed and $x \notin F$ there is $f \in \mathcal{A}$ with $f(x) = 1$ and $f|_K = 0$. Thus for instance the disk algebra is not regular (if f vanishes on a set of positive measure in the circle it must vanish identically).

If $\mathcal{J} \subseteq \mathcal{A}$ is a closed ideal, set

$$Z(\mathcal{J}) = \{t \in K : f \in \mathcal{J} \Rightarrow f(t) = 0\}$$

a closed subset of K. If $E \subseteq K$ is closed, set

$$\mathcal{K}(E) = \{f \in \mathcal{A} : t \in E \Rightarrow f(t) = 0\}.$$

a closed ideal of $\mathcal{A}$. Note that

$$Z(\mathcal{K}(E)) = E.$$

Indeed, obviously if $t \in E$ then every $f \in \mathcal{K}(E)$ vanishes at t, so $E \subseteq Z(\mathcal{K}(E))$; and if $t \notin E$ regularity gives $f \in \mathcal{A}$ s.t. $f(t) \neq 0$ but $f|_E = 0$, i.e. $f \in \mathcal{K}(E)$; hence $t \notin Z(\mathcal{K}(E))$ and equality holds.

Definition 1 *A closed set $E \subseteq K$ is called* **synthetic** *if for any closed ideal $\mathcal{J} \subseteq \mathcal{A}$,*

$$Z(\mathcal{J}) = E \quad \Rightarrow \quad \mathcal{J} = \mathcal{K}(E).$$

Reformulation: Let

$$\mathcal{J}(E) = \{f \in \mathcal{A} : \operatorname{supp} f \cap E = \emptyset\}$$

be the (possibly non-closed) ideal of all functions that vanish *near* E (f vanishes on the open neighbourhood $(\operatorname{supp} f)^c$ of E). Then

$$E \text{ is synthetic} \quad \Longleftrightarrow \quad \overline{\mathcal{J}(E)}^{\|\cdot\|_A} = \mathcal{K}(E).$$

2nd Reformulation: Say a continuous linear functional $\phi \in \mathcal{A}^*$ **vanishes on** an open set $U \subseteq K$ if $f \in \mathcal{A}$ and $\operatorname{supp} f \subseteq U$ implies $\phi(f) = 0$. Then **the support** of ϕ is the complement of the largest open set on which it vanishes. Equivalently, say ϕ **is supported in** a closed set $E \subseteq K$ if $f \in \mathcal{A}$ and $\operatorname{supp} f \cap E = \emptyset$ implies $\phi(f) = 0$. Note that a Dirac functional δ_t is supported in E iff $t \in E$. In this terminology,

$$E \text{ is synthetic} \quad \Longleftrightarrow \quad (\operatorname{supp}\phi \subseteq E \Rightarrow \phi \in \overline{\{\delta_t : t \in E\}}^{w*}).$$

[1]sprtur, March 4, 2011

(2) Reminder: The support of an operator. Let $H = \ell^2(I)$, $H' = \ell^2(J)$ where I, J are index sets. Every $T \in \mathcal{B}(H, H')$ gives a matrix $[a_{ji}] \in \ell^\infty(J \times I)$ so that $(Tx)_j = \sum_i a_{ji} x_i$, or $\langle Te_i, e_j \rangle = a_{ji}$. The *support* of the operator T is the set $\{(i,j) \in I \times J : a_{ji} \neq 0\}$. (note the flip $(j,i) \rightsquigarrow (i,j)$)

Now let $H = L^2(X, m)$, $H' = L^2(Y, n)$ where (for now) X, Y are compact metric spaces and m, n regular Borel measures.

Definition 2 *Say an operator $T \in \mathcal{B}(H, H')$* **vanishes on** *an open rectangle $U \times V \subseteq X \times Y$ if $P(V)TP(U) = 0$, where $P(U) = M_{\chi_U}$. Equivalently, T vanishes on $U \times V$ if $\langle Tf, g \rangle = 0$ for all $f \in L^2(X, m)$ s.t. $f(s) = 0$ a.e on U^c and all $g \in L^2(Y, n)$ s.t. $g(t) = 0$ a.e on V^c.*

Say T **is supported in** *a closed set $K \subseteq X \times Y$ if T vanishes on any open rectangle $U \times V$ disjoint from K:*

$$\operatorname{supp} T \subseteq K \qquad \iff \qquad [(U \times V) \cap K = \emptyset \Rightarrow P(V)TP(U) = 0].$$

For example, if T is an integral operator with kernel $h \in L^2(Y \times X, n \otimes m)$,

$$\langle Tf, g \rangle = \int_Y \left(\int_X h(y,x) f(x) dm(x) \right) \overline{g(y)} dn(y) \quad f \in L^2(X, m),\ g \in L^2(Y, n)$$

then $P(V)TP(U) = 0$ iff $\chi_V(y) h(y,x) \chi_U(x) = 0$ a.e., that is, iff $h(y,x) = 0$ for a.a. $(x,y) \in U \times V$ (note the flip $(y,x) \rightsquigarrow (x,y)$). [2]

More generally, if μ is a (regular, Borel, complex) measure on $Y \times X$ such that the operator T_μ defined by the sesquilinear form

$$\langle T_\mu f, g \rangle = \int_{Y \times X} f(x) \overline{g(y)} d\mu(y,x) \quad f \in L^2(X, m),\ g \in L^2(Y, n)$$

is bounded [3], then $P(V) T_\mu P(U) = 0$ iff $|\mu|(V \times U) = 0$.

(3) Reminder: The support of a convolution operator. Now specialise to the case where $X = Y = G$ is a compact (metrisable) abelian group (for example, $G = \mathbb{T}$) and $m = n$ is Haar measure.

If $\phi \in L^1(G)$ the convolution operator is given by

$$\langle C_\phi f, g \rangle = \int_G \left(\int_G \phi(y - x) f(x) dx \right) \overline{g(y)} dy = \int_G \left(\int_G f(y - x) \phi(x) dx \right) \overline{g(y)} dy \quad f, g \in L^2(G).$$

Suppose that ϕ is continuous and let $E = \operatorname{supp} \phi = \overline{\{t \in G : \phi(t) \neq 0\}}$; so E is the complement of the largest *open* set on which ϕ vanishes. Let $U \times V \subseteq X \times Y$ be an open rectangle. If $P(V) C_\phi P(U) = 0$ then $\chi_V(y) \phi(y - x) \chi_U(x) = 0$, so if $E^* = \{(x,y) : y - x \in E\}$ then $(U \times V) \cap E^* = \emptyset$: for if $(x,y) \in U \times V$ then $\phi(y - x) = 0$, so $y - x \notin E$, i.e. $(x,y) \notin E^*$. And conversely, if $U \times V$ is an open rectangle disjoint from E^* then $\chi_V(y) \phi(y - x) \chi_U(x) = 0$ for all (x,y), so $P(V) C_\phi P(U) = 0$.

Conclusion: If $E \subseteq G$ is the support of ϕ, then the support of C_ϕ is

$$E^* = \{(x,y) : y - x \in E\}.$$

More generally if $\mu \in M(G)$ (i.e. μ is a (complex) regular Borel measure on G) define

$$\langle C_\mu f, g \rangle = \int_G \left(\int_G f(y - x) d\mu(x) \right) \overline{g(y)} dy \quad f, g \in L^2(G).$$

If $E = \operatorname{supp} \mu$, then the support of C_μ is E^*.

[2]This is really a matter of convention: for example Davidson in 'Nest Algebras' [3], p. 343 uses $P(U)TP(V) = 0$ in his definition of support.

[3]the condition used by Arveson [1] is: there exists $C < \infty$ s.t. for all Borel $E \subseteq X$ and $F \subseteq Y$ we have $|\mu|(Y \times E) \leq Cm(E)$ and $|\mu|(F \times X) \leq Cn(F)$

(4) The case of $A(\mathbb{T})$. Specialise further to the case $G = \mathbb{T}$ and recall the definition

$$A(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\}, \qquad \|f\|_A := \left\|\hat{f}\right\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|.$$

So $(A(\mathbb{T}), \cdot)$ is isometrically isomorphic, as a Banach algebra, with $(\ell^1(\mathbb{Z}), *)$.

Now observe that any $a \in \ell^1$ can be factorised as the pointwise product of two ℓ^2 sequences. So if $f \in A(\mathbb{T})$, writing $\hat{f} = b \cdot c$ where $b, c \in \ell^2(\mathbb{Z})$, we have $f = g * h$ where $g = \mathcal{F}^{-1}b$, $h = \mathcal{F}^{-1}c$ are in $L^2(\mathbb{T})$ (recall that the Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is an onto isometry).

Explicitly,

$$\begin{aligned} f(s) &= \int_G g(s-t)h(t)dt = \int g^\flat(t-s)h(t)dt \qquad (g^\flat(x) = g(-x)) \\ &= \int (\lambda_s g^\flat)(t)h(t)dt \qquad (\lambda_s\phi(x) = \phi(x-s)) \\ &= \left\langle \lambda_s g^\flat, \bar{h}\right\rangle_{L^2(\mathbb{T})}. \end{aligned}$$

Note here that the maps $g \to g^\flat$ and λ_s are unitary.

Conclusion: $f \in A(\mathbb{T})$ if and only if there are $\phi_1, \phi_2 \in L^2(\mathbb{T})$ s.t. $f(s) = \langle \lambda_s\phi_1, \phi_2\rangle_{L^2(\mathbb{T})}$ for all $s \in \mathbb{T}$.

But this latter property does not use the fact that $\mathbb{T}$ is abelian; so we may define:

Definition 3 (Eymard) *If G is any locally compact group,*

$$A(G) = \{f : G \to \mathbb{C} : \text{ there are } \phi_1, \phi_2 \in L^2(G) \text{ s.t. } f(s) = \langle \lambda_s\phi_1, \phi_2\rangle_{L^2(G)} \text{ for all } s \in G\}.$$

Remark 1 *If $f \in A(\mathbb{T})$ then $\|f\|_A = \inf\{\|\phi\|_2 \|\psi\|_2 : \phi_1, \phi_2 \in L^2(\mathbb{T}), f(s) = \langle \lambda_s\phi_1, \phi_2\rangle\}$.*

Proof By the calculation above, $f(s) = \langle \lambda_s\phi_1, \phi_2\rangle$ for all s iff $f = \phi * \psi$ where $\phi_1 = \phi^\flat$ and $\phi_2 = \bar{\psi}$.

Now if $f = \phi * \psi$ then

$$\begin{aligned} \|f\|_A &= \left\|\hat{f}\right\|_1 = \left\|\widehat{\phi * \psi}\right\|_1 = \left\|\hat{\phi}\hat{\psi}\right\|_1 \\ &\le \left\|\hat{\phi}\right\|_{\ell^2} \left\|\hat{\psi}\right\|_{\ell^2} = \|\phi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

For the reverse inequality, write $\hat{f} = u|\hat{f}|$ where $|u| = 1$ and put $b(n) = u(n)|\hat{f}(n)|^{1/2}$ and $c(n) = |\hat{f}(n)|^{1/2}$. Then $b, c \in \ell^2(\mathbb{Z})$ so $g = \mathcal{F}^{-1}b$ and $h = \mathcal{F}^{-1}c$ are in $L^2(\mathbb{T})$. Moreover $\|g\|_{L^2}^2 = \|b\|_{\ell^2}^2 = \sum |\hat{f}(n)| = \left\|\hat{f}\right\|_1$ and $\|h\|_{L^2}^2 = \left\|\hat{f}\right\|_1$ tambien. It follows that

$$\|f\|_A = \left\|\hat{f}\right\|_1 = \|g\|_{L^2} \|h\|_{L^2}. \qquad \square$$

(5) The paper of Spronk-Turowska [6]. Let G be a compact metrisable group (possibly non-abelian). Define $A(G)$ as in Definition 3. Also define *the Varopoulos algebra*:

$$V(G) = C(G) \otimes^h C(G)$$

where $C(G) = \{f : G \to \mathbb{C} : f \text{ continuous}\}$. This is the (complete) projective tensor product, but equipped with the Haagerup norm. More precisely: $V(G)$ is defined to be the completion of the algebraic tensor product $C(G)\otimes C(G)$ with respect to the norm

$$\|u\|_h = \inf\left\{ \left\|\sum_k e_k e_k^*\right\|^{1/2} \left\|\sum_k f_k^* f_k\right\|^{1/2} \;:\; \text{all repr's } \; u=\sum_k e_k\otimes f_k \right\}$$
$$= \inf\left\{ \|(e_1,e_n,\dots,e_n)\| \left\| \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \right\| \;:\; \text{all repr's } \; u=\sum_{k=1}^n e_k\otimes f_k \right\}.$$

Since $\|e\otimes f\|_h \le \|ee^*\|^{1/2}\|f^*f\|^{1/2} = \|e\|\,\|f\|$, it is easy to see that $\|u\|_h \le \sum_k \|e_k\|\,\|f_k\|$ for every representation $u=\sum_{k=1}^n e_k\otimes f_k$, and so

$$\|u\|_h \le \|u\|_\gamma := \inf\left\{ \sum_k \|e_k\|\,\|f_k\| : u=\sum_{k=1}^n e_k\otimes f_k \right\}.$$

Grothendieck's inequality states that there is a universal constant K_G so that $\|u\|_\gamma \le K_G\|u\|_h$, and so the two norms are in fact equivalent, hence they give the same completion.

It is a fact (see Blecher - Le Merdy [2, Proposition 1.5.6]) that each $u\in V(G)$ can be written as a $\|\cdot\|_h$-convergent series

$$u=\sum_{k=1}^\infty e_k\otimes f_k \quad\text{where } \sum_k |e_k|^2 \text{ and } \sum_k |f_k|^2 \text{ converge[4] uniformly.}$$

Let $u=\sum_{k=1}^\infty e_k\otimes f_k \in V(G)$. Given any $S\in B(L^2(G))$ the sum

$$\sum_{k=1}^\infty M_{e_k} S M_{f_k}$$

converges in the norm of $B(L^2(G))$ to an element $T_u(S)\in B(L^2(G))$. The map $T_u : B(L^2(G)) \to B(L^2(G))$ is in fact w*-w* continuous and it can be shown that $\|T_u\|=\|u\|_h$. In particular, $u\to T_u$ is injective.

On the other hand, the uniform convergence of $\sum_k |e_k|^2$ and $\sum_k |f_k|^2$ shows, by Cauchy-Schwarz, that the series $\sum_k e_k(s)f_k(t)$ also converges uniformly in $(s,t)\in G\times G$ and thus defines a function $\tilde u(s,t)=\sum_k e_k(s)f_k(t)$ which is continuous on $G\times G$.

I claim that the (clearly linear) map $u\to\tilde u$ is also injective.

Indeed, if S is the rank one operator $S\xi=\phi\,\langle\xi,\psi\rangle$, then for all $\eta\in L^2(G)$,

$$T_u(S)\xi = \sum_{k=1}^\infty M_{e_k}\phi\,\langle M_{f_k}\xi,\psi\rangle = \sum_{k=1}^\infty (e_k\phi)\,\langle (f_k\xi),\psi\rangle$$
$$\text{so}\quad \langle T_u(S)\xi,\eta\rangle = \sum_{k=1}^\infty \langle (e_k\phi)\,\langle (f_k\xi),\psi\rangle\,,\eta\rangle = \sum_{k=1}^\infty \langle (f_k\xi),\psi\rangle\,\langle (e_k\phi),\eta\rangle$$
$$= \sum_{k=1}^\infty \int f_k(t)\xi(t)\bar\psi(t)dt \int e_k(s)\phi(s)\bar\eta(s)ds$$
$$= \iint \left(\sum_{k=1}^\infty e_k(s)f_k(t)\right)\xi(t)\bar\psi(t)\phi(s)\bar\eta(s)dtds\,.$$

It follows that if $\tilde u(s,t)=0$ then $T_u=0$ and so $u=0$, which proves that $u\to\tilde u$ is injective.

Thus we may delete the tilde and identify $V(G)$ with its image in $C(G\times G)$.

[4] it is not claimed that $\sum_k e_k^2$ or $\sum_k f_k^2$ converge

Thus we identify $u = \sum_{k=1}^{\infty} e_k \otimes f_k$ *with* $u(s,t) = \sum_k e_k(s) f_k(t)$.

Note that $V(G)$ is a subalgebra of $C(G \times G)$; this follows from the fact that $T_{uv} = T_u T_v$ (since $u \to T_u$ is injective): indeed if $u(s,t) = e(s)f(t)$ and $v = g(s)h(t)$ then $uv(s,t) = (eg)(s)(fh)(t)$ so for all $S \in B(L^2(G))$,

$$T_u T_v(S) = T_u(M_g S M_h) = M_e M_g S M_h M_f = M_{eg} S M_{hf} = T_{uv}(S)$$

since $C(G)$ is abelian; the same argument works for general $u, v \in V(G)$.

The *invariant part* of $V(G)$ is defined to be

$$V_{inv}(G) = \{u \in V(G) : u(sr, tr) = u(s,t) \text{ for all } \ s,t,r \in G\}.$$

This is a closed subalgebra of $V(G)$.

Theorem 2 *The map*

$$N : A(G) \to V_{inv}(G)$$

given by

$$(Nu)(s,t) = u(st^{-1}) \qquad u \in A(G)$$

is an isometric isomorphism.

(6) Spectral synthesis and Operator Synthesis. Let $T(G) = L^2(G) \hat{\otimes} L^2(G)$. Thus $T(G)$ consists of all

$$u = \sum_i f_i \otimes g_i, \quad f_i, g_i \in L^2(G) \quad \text{such that} \qquad \sum_i \|f_i\|_2 \, \|g_i\|_2 < \infty$$

and the norm $\|u\|_\gamma = \|u\|_1$ is the infimum of these sums over all such representations of u.

Each $u \in T(G)$ gives rise to a w*-continuous linear functional ω_u on $B(L^2(G))$ defined by

$$\omega_u(T) = \sum_i \langle T f_i, \bar{g}_i \rangle \,, \qquad T \in B(L^2(G))$$

and in fact $\|\omega_u\| = \|\omega\|_1$. Each $\phi \in (L^2(G) \hat{\otimes} L^2(G))^*$ defines an operator $T_\phi \in B(L^2(G))$ through $\phi(f \otimes g) = \langle Tf, \bar{g} \rangle$, and the map $\phi \to T_\phi$ is an isometric isomorphism of the space $(L^2(G) \hat{\otimes} L^2(G))^*$ onto $B(L^2(G))$.

The element $u \in T(G)$ also gives rise to a function

$$\tilde{u}(s,t) = \sum_i f_i(s) g_i(t), \qquad (s,t) \in G \times G$$

which is well defined *marginally almost everywhere (m.a.e.)*; this means that $\tilde{u}$ is defined up to a *marginally null set*, i.e. a set of the form $(N \times G) \cup (G \times N)$ where $N \subseteq G$ is a null set. We will again use the same symbol for u and $\tilde{u}$.

For $u \in T(G)$ define

$$\operatorname{supp} u = \{(s,t) \in G \times G : u(s,t) \neq 0\}$$

(nb. no closure); this is defined up to a marginally null set.

Now given a closed set $F \subseteq G \times G$ we put

$$\begin{aligned}\Phi(F) &= \{u \in T(G) : \operatorname{supp} u \cap F = \emptyset\} \\ &= \{u \in T(G) : u\chi_F = 0 \text{ m.a.e.}\} \\ \Psi(F) &= \{u \in T(G) : \overline{\operatorname{supp} u} \cap F = \emptyset\} \\ \Phi_0(F) &= \overline{\Psi(F)}^{\|\cdot\|_1}.\end{aligned}$$

The subspace $\Phi(F)$ is $\|\cdot\|_1$-closed and contains the subspace $\Psi(F)$, which consists of all u which vanish m.a.e. *near* F (in the open neighbourhood $(\overline{\operatorname{supp} u})^c$ of F.)

Definition 4 (Shulman – Turowska [5]) *The set F is said to be* **operator synthetic** *if*

$$\Phi_0(F) = \Phi(F).$$

Theorem 3 (Spronk – Turowska) *A closed set $E \subseteq G$ is a n S-set (i.e. satisfies spectral synthesis) if and only if the set*

$$E^* = \{(s,t) \in G \times G : st^{-1} \in E\}$$

is operator synthetic.

This is due to Froelich [4] in the abelian case.

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