A note

Remark 1 *Let X* [1] *be a compact Hausdorff space and μ a regular Borel measure on X (i.e. the measure $\nu := |\mu|$ is regular). Then for every open set $U \subseteq X$,*

$$|\mu|(U) = \sup \left\{ \left| \int f d\mu \right| : f \in C(X),\ \mathrm{supp}\, f \subseteq U \,\text{compact},\ \|f\|_\infty \leq 1 \right\}.$$

Proof We write $\nu := |\mu|$. It is clear that if $f \in C(X)$ with $\|f\|_\infty \leq 1$ is supported in U then

$$\left| \int f d\mu \right| \leq \int |f| d\nu \leq \|f\|_\infty |\mu|(U) \leq |\mu|(U).$$

We prove the reverse inequality.
By the Radon-Nikodym theorem (see Rudin, Real and Complex Analysis, Theorem 6.12)

$$|\mu|(U) = \int \chi_U d\nu = \int \chi_U h d\mu$$

where $\bar{h} = \frac{d\mu}{d\nu}$ is in $L^1(X, \nu)$ and $|h| = 1$ a.e. wrt ν.
The space $C_c(U)$ of continuous functions $f : U \to \mathbb{C}$ with *compact* support (contained in U) is dense in $L^1(U, \nu)$ and $g := h|_U \in L^1(U, \nu)$; hence there exists $f_n \in C_c(U)$ with $\displaystyle\int_U |f_n - g| d\nu \to 0$. Extending f_n to $g_n : X \to \mathbb{C}$ by setting $g_n = 0$ on U^c, we obtain $g_n \in C_c(X)$ with compact support contained in U such that

$$\left| \int_X (g_n - h\chi_U) d\mu \right| \leq \int_X |g_n - h\chi_U| d|\mu| = \int_X |(g_n - h)\chi_U| d|\mu| = \int_U |f_n - g| d\nu \to 0$$

and so $\displaystyle\int g_n d\mu \to \int h\chi_U d\mu = \nu(U)$.

Remark 2 *Now take $X = \mathbb{T}$ (or any locally compact abelian group, for that matter). We claim that*

$$|\mu|(U) = \sup \left\{ \left| \int f d\mu \right| : f \in A(\mathbb{T}),\ \mathrm{supp}\, f \subseteq U \,\text{compact},\ \|f\|_\infty \leq 1 \right\}.$$

We will need the following lovely

[1] measur, Feb 18 2011

Lemma *Given a compact K and open U s.t. $K \subseteq U \subseteq \mathbb{T}$, there exists $\phi \in A(\mathbb{T})$ with $\chi_K \leq \phi \leq \chi_U$.* [2]

It follows that

$$A_c(U) := \{f \in A(\mathbb{T}) : \operatorname{supp} f \text{ is compact and contained in } U\}$$

has the properties:

(a) It does not vanish on U: indeed given $\lambda \in U$, choose an open V with $\overline{V}$ compact s.t. $\lambda \in V \subseteq \overline{V} \subseteq U$; since $\{\lambda\}$ is compact there is $\phi \in A(\mathbb{T})$ with $\phi(\lambda) = 1$ and $\phi|_{V^c} = 0$, so ϕ has compact support contained in U.

(b) It separates points of U: indeed given $\lambda, \lambda' \in U$ with $\lambda \neq \lambda'$, letting W be an open neighbourhood of λ' not containing λ and letting $U' = W \cup U$, choose an open V with $\overline{V}$ compact s.t. $\lambda \in V \subseteq \overline{V} \subseteq U'$; there is $\phi \in A(\mathbb{T})$ with $\phi(\lambda) = 1$ and $\phi|_{V^c} = 0$, so $\phi \in A_c(U)$, and $\phi(\lambda) = 1 \neq 0 = \phi(\lambda')$.

(c) It is a *self adjoint* subalgebra of $C_c(U) \subseteq C_0(U)$.

By the Stone-Weierstrass Theorem, $A_c(U)$ is $\|\cdot\|_\infty$-dense in $C_0(U)$.

Therefore: Given $\epsilon > 0$, by Remark 1 there is a compact $K \subseteq U$ and a continuous $f : \mathbb{T} \to \mathbb{C}$ with $\operatorname{supp} f \subseteq K$ such that $\left| |\mu|(U) - \int f d\mu \right| < \epsilon$. Now choosing $\phi \in A_c(U)$ so that $\|f - \phi\|_\infty < \epsilon$ we get

$$\left| |\mu|(U) - \int \phi d\mu \right| \leq \left| |\mu|(U) - \int f d\mu \right| + \int |\phi - f| d|\mu| < \epsilon + \epsilon|\mu|(U)$$

which proves Remark 2.

Proof of the Lemma Choose ϵ so that $K_{2\epsilon} := \{z \in \mathbb{T} : \operatorname{dist}(z, K) < 2\epsilon\}$ is contained in U. Let χ_ϵ be the characteristic function of $(-\epsilon, \epsilon)$ and define

$$f(s) = \frac{1}{2\epsilon}(\chi_\epsilon * \chi_{K_\epsilon})(s) = \frac{1}{2\epsilon} \int \chi_\epsilon(t) \chi_{K_\epsilon}(s - t) dt = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \chi_{K_\epsilon}(s - t) dt.$$

Clearly, $0 \leq f \leq 1$. If $s \in K$ then for all t with $|t| < \epsilon$ we have $s \in K_\epsilon + t$ and so $\chi_{K_\epsilon}(s - t) = 1$ hence $f(s) = 1$. If $f(s) > 0$ there must exist t with $|t| < \epsilon$ so that $s \in K_\epsilon + t \subseteq K_{2\epsilon}$, hence $s \in U$. Thus f vanishes in U^c.

Finally, since χ_ϵ and χ_{K_ϵ} are in $L^2(\mathbb{T})$, their Fourier transforms $\hat{\chi}_\epsilon$ and $\hat{\chi}_{K_\epsilon}$ are in $\ell^2(\mathbb{Z})$, and so the pointwise product $\hat{\chi}_\epsilon \hat{\chi}_{K_\epsilon}$ is in $\ell^1(\mathbb{Z})$. Hence

$$\hat{f} = \frac{1}{2\epsilon}(\widehat{\chi_\epsilon * \chi_{K_\epsilon}}) = \hat{\chi}_\epsilon \hat{\chi}_{K_\epsilon} \in \ell^1(\mathbb{Z})$$

which means that $f \in A(\mathbb{T})$.

Note *Essentially the same proof works in any locally compact abelian group, replacing the open set $(-\epsilon, \epsilon)$ with an open neighbourhood W of 0 of finite measure s.t. $W = -W$ and replacing $K_{2\epsilon}$ with $K + W + W$.*

[2] This means that $0 \leq f \leq 1$ everywhere, $f|_K = 1$ and $f|_{U^c} = 0$.