VON NEUMANN ALGEBRAS AND TOMITA-TAKESAKI THEORY

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Contents

1. The von Neumann algebra of a (locally compact) group

1.1. **General definition.** Let G be a locally compact group; thus G is equipped with a locally compact Hausdorff topology [1] for which $(s,t) \to st^{-1} : G \times G \to G$ is continuous. Then G has a (left) Haar measure: that is, a Borel regular measure m which is *left-invariant*, i.e. satisfies

$$\int_G f(st)dm(t) = \int_G f(t)dm(t) \quad \forall s \in G, f \in L^1(G).$$

Consider the Hilbert space $L^2(G,m) = L^2(G)$ with the norm

$$\|f\|_2^2 = \int |f(t)|^2 dm(t) = \int |f(t)|^2 dt.$$

If $f \in C_c(G)$ (i.e. f is continuous with compact support) then its *left translate* f_s ($s \in G$) where $f_s(t) = f(s^{-1}t)$ is in $C_c(G)$; but also

$$\int |f_s(t)|^2 dt = \int |f(t)|^2 dt$$

(by left invariance of m). Hence the map

$$\lambda_s : f \to f_s$$

is an L^2 isometry and maps $C_c(G)$ onto $C_c(G)$ because $\lambda_s \lambda_t = \lambda_{st}$ hence $\lambda_s \lambda_{s^{-1}} = I$. Thus λ_s extends to a unitary operator on $L^2(G)$ (denoted by the same symbol) and the map $s \to \lambda_s$ is a group homomorphism of G into the group of unitary operators on $L^2(G)$, that is, a unitary representation of G, called *the left regular representation.*

Definition 1.1. *The von Neumann algebra generated by the set of unitaries*

$$\{\lambda_t : t \in G\}$$

is called the von Neumann algebra *of the group and is denoted* vN(G) *or* $\mathcal{L}(G)$*.*

Note that the set $\{\lambda_t : t \in G\}$ is already a group of unitary operators, hence its linear span is a selfadjoint unital algebra. Thus by the bicommutant theorem we have

$$\mathrm{vN}(G) = \overline{\mathrm{span}\{\lambda_t : t \in G\}}^{WOT}$$

[1] i.e. distinct points have distinct open neighbourhoods (Hausdorff), and every open neighbourhood of a point x contains a compact neighbourhood of x

1.2. **The case of a discrete group.** When G has the discrete topology, *counting measure* is left-invariant and so

$$L^2(G) = \ell^2(G) = \{f : G \to \mathbb{C} : \sum_{t \in G} |f(t)|^2 < \infty\}.$$

Then $\ell^2(G)$ has an orthonomal basis $\{\delta_t : t \in G\}$ (where $\delta_t(s) = 1$ when $s = t$ and $\delta_t(s) = 0$ otherwise). Consider a linear combination of the generators of $\mathrm{vN}(G)$, i.e. a *finite sum*

$$A = \sum_{u \in G} f_A(u)\lambda_u \tag{1}$$

(where $f_A(u) \in \mathbb{C}$ and $f_A(u) = 0$ except for finitely many $u \in G$). Then its matrix has the form

$$a_{s,t} = \langle A\delta_t, \delta_s\rangle = \sum_{u \in G} f_A(u) \langle \delta_{ut}, \delta_s\rangle = f_A(st^{-1}).$$

Note that in the case $G = \mathbb{Z}$ this matrix is constant along diagonals.

It is not hard to show [2] that for any $A \in \mathrm{vN}(G)$ the matrix elements $a_{s,t}$ depend only on st^{-1}, hence can be written $a_{s,t} = f_A(st^{-1})$ for some function $f_A : G \to \mathbb{C}$; in fact now $f_A \in \ell^2(G)$ (because $A\delta_e = \sum_u f_A(u)\delta_u \in \ell^2(G)$).[3]

Exercise 1. *In the case $G = \mathbb{Z}$,*

(a) identify explicitly the set of functions $\{f_A : A \in \mathrm{vN}(G)\}$ and

(b) examine whether the formal series $A = \sum_{u \in G} f_A(u)\lambda_u$ converges or is summable in some sense.

1.3. **The commutant, the trace.** Let us remain in the situation when G *is a discrete group.*

The **commutant** $(\mathcal{L}(G))'$ of $\mathcal{L}(G)$, namely

$$(\mathcal{L}(G))' := \{T \in \mathcal{B}(L^2(G)) : TA = AT \ \forall A \in \mathcal{L}(G)\}$$

can be shown to equal $\mathcal{R}(G)$, the von Neumann algebra generated by all *right translations* ρ_t, $t \in G$ where $(\rho_t f)(s) = f(st)$, $f \in C_c(G)$. In the present case where G is discrete, the map ρ_s does extend to a bounded operator on $L^2(G)$.

Exercise 2. *What happens in the general (non-compact, non-abelian) case?*

[2] use the fact that A must commute with right translations (see 1.3)

[3] But, when G is infinite, not all ℓ^2 functions define elements of $\mathrm{vN}(G)$ - see Exercise 1.

Consider the linear functional τ defined on $\mathrm{vN}(G)$ by

$$\tau(A) = \langle A\delta_e, \delta_e\rangle \quad \text{for all } A \in \mathrm{vN}(G).$$

It is a WOT-continuous *state*,[4] it is *faithful* in the sense that $\tau(A^*A) = 0 \iff A = 0$ [5] and it is *tracial*, i.e.

$$\tau(AB) = \tau(BA) \qquad \text{for all } \; A, B \in \mathrm{vN}(G).$$

To prove this, note that (by linearity and WOT-continuity) it is enough to check it when $A = \lambda_s$, $B = \lambda_t$; and in this case, the result is obvious!

2. Example of a non-type I factor: $\mathrm{vN}(F_2)$

When G is a discrete group, the *centre* $\mathcal{L}(G) \cap (\mathcal{L}(G))'$ of $\mathcal{L}(G)$ consists of all $A \in \mathcal{L}(G)$ such that f_A is constant on all conjugacy classes $C_t = \{sts^{-1} : s \in G\}$. (This is an easy calculation.)

Example Let $G = F_2$, the free group in two generators. This consists of all (finite) *words* in the generators a and b and their inverses, together with the empty word (corresponding to the identity e) subject to no relations other than $aa^{-1} = a^{-1}a = e$ and similarly for b. Here all conjugacy classes C_t (for $t \neq e$) are infinite. It follows that if A is in the centre of $\mathcal{L}(G)$ then, since f_A is square-summable, $f_A(t)$ must vanish unless $t = e$, so $A = f_A(e)I$!

Conclusion: The centre of $\mathrm{vN}(F_2)$ is trivial, it consists only of multiples of the identity operator.

Definition 2.1. *A* factor *is a von Neumann algebra $\mathcal{M}$ whose centre $\mathcal{M} \cap \mathcal{M}'$ is trivial, i.e. equal to $\mathbb{C}I$.* [6]

Thus $\mathrm{vN}(F_2)$ is a factor, like $\mathcal{B}(\ell^2)$. But it has a *finite* faithful trace, unlike $\mathcal{B}(\ell^2)$. In some sense it seems have more in common with $\mathcal{B}(\mathbb{C}^n)$ (although of course it is infinite -dimensional).

For example, any isometry $u \in \mathrm{vN}(F_2)$ must be unitary.[7] Thus isometries like the unilateral shift $S : e_n \to e_{n+1}$ cannot belong to $\mathrm{vN}(F_2)$.

[4] that is, a positive linear functional of norm 1

[5] *Proof:* $\tau(A^*A) = 0 \iff A\delta_e = 0$; but the latter condition implies that $A\delta_s = 0$ for all $s \in G$ (because $\rho_s A = A\rho_s$) and so $A = 0$ (one says that δ_e *separates* $\mathrm{vN}(G)$)

[6] Equivalently, a factor is a von Neumann algebra that cannot be decomposed as a direct sum of (nozero) von Neumann subalgebras. Factors are 'building blocks' for general von Neumann algebras.

[7] *Proof:* If $u^*u = I$ then $\tau(uu^*) = \tau(u^*u) = 1$ so $\tau(I - uu^*) = 0$ hence $I - uu^* = 0$ because $I - uu^* \geq 0$ and τ is faithful.

3. The type classification

Murray and von Neumann classified factors into three 'types' according to the 'kinds' of projections that they contain.

Definition 3.1. *Let $\mathcal{P}(\mathcal{M})$ be the set of projections in a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space H.*

(i) *A projection $p \in \mathcal{P}(\mathcal{M})$ is said to be* minimal in $\mathcal{M}$ *if $p \neq 0$ and $\mathcal{M}$ contains no proper nonzero subprojections of p, i.e. if the only projections $q \in \mathcal{P}(\mathcal{M})$ such that $q \leq p$ are $q = 0$ and $q = p$.*

(ii) *Two projections $p, q \in \mathcal{P}(\mathcal{M})$ are said to be* (Murray - von Neumann) equivalent *(in $\mathcal{M}$) if there exists $u \in \mathcal{M}$ such that $u^*u = p$ and $uu^* = q$.*

(iii) *A projection $p \in \mathcal{P}(\mathcal{M})$ is said to be* finite *(in $\mathcal{M}$) if it is not equivalent to a proper subprojection.*

Remarks (i) In an abelian algebra, Murray - von Neumann equivalence is simply equality. In $\mathcal{B}(H)$, two projections are equivalent if and only if their ranges have the same dimension.
(ii) In any von Neumann algebra, a minimal projection is necessarily finite.

Definition 3.2. *A factor $\mathcal{M}$ (acting on a separable Hilbert space) is said to be*

- *of* type (I) *if $\mathcal{M}$ has a minimal projection*
- *of* type (II) *if $\mathcal{M}$ has no minimal projections but has a finite projection*
- *of* type (III) *if $\mathcal{M}$ has no finite projections.*

More precisely, a type (I) *factor is said to be of type* (I_n) *(for some $n \in \mathbb{N} \cup \{\infty\}$) if the identity $\mathbf{1}$ is the sum of n minimal projections (necessarily orthogonal).*

A type (II) *factor is said to be of type* (II_1) *if the identity $\mathbf{1}$ is finite, and of type* (II_∞) *otherwise.*

Murray and von Neumann proved that any factor $\mathcal{M}$ must be of one and only one of the above types. [8] However the existence of non-type I factors was not obvious.[9]

In order to give examples of factors of all three types, they introduced the so called *group – measure space construction.*

[8] Much later, Connes refined the classification by classifying type (III) factors into types (III_λ) for $\lambda \in [0, 1]$.

[9] In fact $\mathrm{vN}(F_2)$ is a finite factor, because $\mathbf{1}$ is a finite projection; and it cannot be type I because finite type I factors can be shown to be isomorphic to M_n for some $n \in \mathbb{N}$. *Conclusion:* $\mathrm{vN}(F_2)$ is a type (II_1) factor.

4. The group – measure space construction

In modern terminology, this construction is a special case of the crossed product of a von Neumann algebra by a group.

4.1. **The crossed product of L^∞ by a group.** Let $(X, \mathcal{S}, \mu)$ be a *countably separated* measure space. This means that there is a sequence $\{S_n\} \subseteq \mathcal{S}$ with $0 < \mu(S_n) < \infty$ for each n, such that if $x \neq y$ are in X there exists an S_n containing x and not y. [10]

Let G be a countable (discrete) group acting on X by measurable bijections $\{\phi_t : t \in G\}$ preserving μ-null sets.

This means that for all $t \in G$ the measures μ and $\mu_t := \mu \circ \phi_t^{-1}$ are *equivalent* (if $S \in \mathcal{S}$, then $\mu(S) = 0$ if and only if $\mu(\phi_t^{-1}(S)) = 0$). Therefore the Radon-Nikodym derivative $\dfrac{d\mu_t}{d\mu}$ is defined and is positive μ-a.e. Ones says that μ is *quasi-invariant* under the action of G.

For example G might preserve the measure μ. This happens for instance when $X = G$, μ is left Haar measure and G acts by left translations on itself.

If μ is quasi-invariant under G, then the weighted composition operator

$$U_t : f \to r_t(f \circ \phi_t^{-1}) \quad \text{where } \ r_t = \sqrt{\frac{d\mu_t}{d\mu}}$$

maps $L^2(X, \mu)$ isometrically onto itself.

We consider the Hilbert space $H = L^2(X \times G, \mu \times m)$ of functions of two variables. This space can be identified with the 'Hilbert space tensor product' $H = L^2(X, \mu) \otimes \ell^2(G)$ by identifying $f \otimes g$ with the function $h(x, t) = f(x)g(t)$.[11]

This allows us to represent both $L^\infty(X, \mu)$ and G on the same space H; for $\psi \in L^\infty(X, \mu)$ and $t \in G$ we define operators $\pi(\psi)$ and W_t on H as follows: if $h \in L^2(X \times G)$ then for all $(x, s) \in X \times G$ we set

$$(\pi(\psi)h)(x, s) = \psi(x)h(x, s) \quad \text{and} \quad (W_t h)(x, s) = r_t(x)h(\phi_t(x), t^{-1}s)\,.$$

Thus if $h = f \otimes g$ then $\pi(\psi)h = (M_\psi f) \otimes g$, i.e. $\pi(\psi)$ acts as a multiplication operator on f and as the identity on g; also $(W_t h)(x, s) = (U_t f)(x)(\lambda_t g)(s)$. Thus we write

$$\pi(\psi) = M_\psi \otimes I \quad \text{and} \quad W_t = U_t \otimes \lambda_t\,.$$

[10] This assumption can be weekened; it ensures that $L^2(X, \mu)$ is separable, and also simplifies the definition of a free action below.

[11] We use the tensor product only as a notational convenience.

Definition 4.1. [12] *The* crossed product $\mathcal{A} = L^\infty(\mu) \rtimes_\phi G$ *is the von Neumann algebra on H generated by the images of $L^\infty(X,\mu)$ and of G under these representations:*

$$L^\infty(X,\mu) \rtimes_\phi G := \{\pi(f), W_t : f \in L^\infty(X,\mu), t \in G\}''.$$

Remarks (i) When X consists of one point, and so $L^\infty(X,\mu) \simeq \mathbb{C}$ and ϕ_t is trivial, then the crossed product is just the von Neumann algebra $\mathrm{vN}(G)$ defined earlier.
(ii) In general the representations π and W are related by the following *covariance relation*

$$W_t\pi(f) = \pi(f \circ \phi_t^{-1})W_t, \quad f \in L^\infty(X,\mu), t \in G$$

(the proof is an easy calculation). Thus the representations π and W only commute when the action ϕ of G is trivial. The crossed product is commutative if and only if the action is trivial *and* G is abelian.
(iii) On the other hand the covariance relation makes it possible to express all products such as $W_t\pi(f)\ldots W_s\pi(g)$ in the form $\pi(h)W_r$ for suitable h and r. [13] Similarly the adjoint $(\pi(f)W_t)^*$ can also be witten in the form $\pi(h)W_r$ (with $h = \overline{h \circ \phi_t}$ and $r = t^{-1}$). This shows that the linear span of all 'monomials' $\pi(h)W_r$, i.e. the set

$$\mathcal{A}_0 = \{\sum_k \pi(f_k)W_{t_k} : f_k \in L^\infty(X,\mu), t_k \in G\}$$

is already a unital selfadjoint algebra. Thus, by the bicommutant theorem, it is WOT-dense in the crossed product:

$$L^\infty(X,\mu) \rtimes_\phi G = \overline{\{\sum_k \pi(f_k)W_{t_k} : f_k \in L^\infty(X,\mu), t_k \in G\}}^{WOT}.$$

(iv) Observe that a typical element of the WOT-dense subalgebra $\mathcal{A}_0$ is a 'linear combination' of the unitaries W_r, just like elements of the form (1) in the group von Neumann algebra, but this time with 'coefficients' from $L^\infty(X,\mu)$, not from $\mathbb{C}$. This shows again that the crossed product is a generalisation of the group von Neumann algebra; notice however that the multiplication (as well as the adjoint operation) is 'twisted' by the covariance relation.
(v) This construction can be generalized to the case of a locally compact (non-discrete) group, provided the action of G is continuous in a suitable sense. It can even be generalized to the crossed product

[12] This is not the usual definition; however it is unitarily equivalent to it in our case.

[13] for example $W_t\pi(f)W_s\pi(g) = \pi(f \circ \phi_t^{-1})W_tW_s\pi(g) = \pi(f \circ \phi_t^{-1})W_{ts}\pi(g) = \pi((f \circ \phi_t^{-1})(g \circ \phi_{ts}^{-1}))W_{ts}$.

$\mathcal{M} \rtimes_\alpha G$ where $\mathcal{M}$ is a (possibly non-abelian) von Neumann algebra on which G acts by automorphisms α_t in a suitably continuous fashion. This is crucial for the study of type III factors.

4.2. **Examples of factors.** We now use the crossed product construction to give examples of all types of factors. Notice, however, that $\mathcal{A} = L^\infty(X,\mu) \rtimes G$ is not always a factor. For instance if some non-constant $f \in L^\infty(X,\mu)$ is fixed by all ϕ_t, then it is easy to verify that $\pi(f) = \pi(f)W_e$ will be a non-scalar element of the centre of $\mathcal{A}$.

Thus some restriction on the action is needed.

Definition 4.2. (i) *The action of G on (X,μ) is called an (essentially)* free action *if for each $t \in G$, $t \neq e$, the fixed point set $F_t := \{x \in X : \phi_t(x) = x\}$ is negligible, i.e. $\mu(F_t) = 0$.*

(ii) *The action of G on (X,μ) is called* ergodic *if the only $L^\infty(X,\mu)$ functions f which are fixed by G in the sense that $f \circ \phi_t = f$ a.e. are the (a.e.-)constant functions.*

Note The action is ergodic if and only if the only (almost) invariant measurable sets are null or conull: the action is ergodic if and only if whenever $S \in \mathcal{S}$ satisfies $\mu(S\Delta\phi_t(S)) = 0$ for all $t \in G$, necessarily either $\mu(S) = 0$ or $\mu(S^c) = 0$.

Also note that when G is abelian, ergodicity implies freeness. [14]

Proposition 1. *If the action of G is free and ergodic, then $\mathcal{A} = L^\infty(\mu) \rtimes_\phi G$ is a factor (its centre is trivial).*
In this case, if there exists a σ-finite Borel measure ν which is G-invariant and equivalent to μ, then
(i) $\mathcal{A}$ is of type I if and only if the measure space $(X,\mathcal{S},\mu)$ has atoms.
(ii) $\mathcal{A}$ is of type II if and only if the measure space $(X,\mathcal{S},\mu)$ has no atoms; it is of type II_1 when ν can be chosen finite and of type II_∞ otherwise.
Finally, $\mathcal{A}$ is of type III if no such measure ν exists.

We can now exhibit examples of all types:

Type (I_∞) Let $X = \mathbb{Z}$ with counting measure, let $G = \mathbb{Z}$ and define the action of G on X by $\phi_n(k) = k + n$; then $\mathcal{A}$ is in fact isomorphic (as a von Neumann algebra) to $\mathcal{B}(\ell^2(\mathbb{Z}))$.

[14] Indeed, observe that each F_t is invariant under all $s \in G$; for if $x \in F_t$ then for all $s \in G$ we have $\phi_t(\phi_s(x)) = \phi_s(\phi_t(x)) = \phi_s(x)$ and so $\phi_s(x) \in F_t$. Thus by ergodicity either $\mu(F_t) = 0$ or else $\mu(F_t^c) = 0$. But if $\mu(F_t^c) = 0$ then almost all points of X are fixed under ϕ_t hence $t = e$; thus, if $t \neq e$ then $\mu(F_t) = 0$ and so the action is free.

Type (I_n) (Variation of previous example): Letting $X = G = \mathbb{Z}_n$ (finite cyclic group), we obtain a finite type (I) factor:

$$\mathcal{A} \simeq \mathcal{B}(\ell^2(\mathbb{Z}_n)) \simeq M_n.$$

Type (II_1) Let $(X, \mu) = (\mathbb{T}, m)$, (the unit circle with normalized Lebesgue measure) let $G = \mathbb{Z}$ and $\phi_n(z) = e^{2\pi i n\theta} z$ where $\theta \notin \mathbb{Q}$.

Note that G preserves m, a finite measure.

Using this, it can be shown that $\mathcal{A}$ has a normal faithful finite trace τ with $\tau(\mathcal{P}(\mathcal{A})) = [0, 1]$.

Type (II_∞) Let $(X, \mu) = (\mathbb{R}, m)$, let $G = \mathbb{Q}$ and $\phi_q(x) = x + q$.

Here G preserves m, an infinite but σ-finite non-atomic measure. [15]

It can be shown that $\mathcal{A}$ has a normal faithful semifinite[16] trace τ with $\tau(\mathcal{P}(\mathcal{A})) = [0, \infty]$.

Type (III) Let $(X, \mu) = (\mathbb{R}, m)$. Fix a number $a > 1$ and define $\phi_{n,q}(x) = a^n x + q$. Let $G = \{\phi_{n,q} : n \in \mathbb{Z}, q \in \mathbb{Q}\}$. (Note that this is a non-abelian group.) Now there is no σ-finite measure equivalent to m which is preserved by G. [17]

5. The standard form of a II_1 factor M

Suppose $\mathcal{M}$ is a von Neumann algebra equipped with a normal faithful tracial state τ. [18] Do a GNS on $(\mathcal{M}, \tau)$: that is, form the scalar product

$$\langle a, b\rangle = \tau(b^* a) \quad \text{and complete to get} \quad H = L^2(\mathcal{M}, \tau).$$

For each $a \in \mathcal{A}$, the map $\pi(a) : b \to ab$ extends to a bounded operator on H. Thus we have an action $\pi : \mathcal{A} \to \mathcal{B}(H)$. The vector $\xi = \mathbf{1}$ is *cyclic* (i.e. $\pi(\mathcal{M})\xi$ is dense in H) and also *separating* (because τ is faithful).

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[15] Lebesgue measure is (up to constant multiples) the only Borel regular measure preserved by all rational translations; hence no finite measure can be preserved by G.

[16] i.e. there exists a map $\tau : \mathcal{A}_+ \to [0, \infty]$ which is additive, homogeneous under positive scalars, unitarily invariant, vanishing only at 0 and such that the set $\{A \in \mathcal{A}_+ : \tau(A) < \infty\}$ is WOT-dense in $\mathcal{A}_+$

[17] Indeed if there were such a measure ν, then it would be preserved by the subgroup $G_0 := \{\phi_{0,q} : q \in \mathbb{Q}\}$ of rational translations; hence ν would be a multiple of Lebesgue measure, hence could not be preserved by dilations $x \to a^n x$, a contradiction.

[18] Then $\mathcal{M}$ is a finite von Neumann algebra, that is, $\mathbf{1}_{\mathcal{M}}$ is a finite projection in $\mathcal{M}$. If $\mathcal{M}$ is a II_1 factor, such a tracial state τ exists and is unique.

Let us identify $\mathcal{M}$ with its image $\pi(\mathcal{M})$ in the sequel. The densely defined map

$$S_0 : \mathcal{M}\xi \to \mathcal{M}\xi : a\xi \to a^*\xi$$

is antilinear and has the magical property:

$$\langle S_0(a\xi), S_0(b\xi)\rangle = \langle b\xi, a\xi\rangle$$

for all $a, b \in \mathcal{M}$. Indeed, since τ is a trace (!),

$$\langle S_0(a\xi), S_0(b\xi)\rangle = \langle a^*\xi, b^*\xi\rangle = \tau(ba^*) \stackrel{!}{=} \tau(a^*b) = \langle b\xi, a\xi\rangle \,.$$

Therefore S_0 is $\|\cdot\|_2$-isometric; also obviously $S_0^2 a\xi = a\xi$ for all $a \in \mathcal{M}$, so S_0 has dense range. Hence S_0 extends to an antilinear bijection S on H which satisfies

$$(*) \qquad \langle S\eta, S\zeta\rangle = \langle \zeta, \eta\rangle \quad \text{for all } \eta, \zeta \in H.$$

Theorem 2. *$S\mathcal{M}S = \mathcal{M}'$.*

Thus, *in this (faithful) representation* (called *the standard form for* $\mathcal{M}$) the algebra $\mathcal{M}$ is anti-isomorphic to its commutant; the map $a \to Sa^*S$ is a linear bijection between $\mathcal{M}$ and $\mathcal{M}'$ that reverses the products.

Example: If $\mathcal{M} = M_n$ then π acts (not on $\mathbb{C}^n$, but) on $H = M_n$ equipped with the Hilbert-Schmidt norm.

To prove the Theorem, we need two lemmas:

Lemma 3. *For all $x, a \in \mathcal{M}$,*

$$SxSa\xi = ax^*\xi.$$

Proof. $Sa\xi = a^*\xi$, so

$$Sx(Sa\xi) = Sxa^*\xi = S(xa^*\xi) = (xa^*)^*\xi = ax^*\xi. \qquad \square$$

Corollary 4. *$S\mathcal{M}S \subseteq \mathcal{M}'$.*

Proof. Given $x, b \in \mathcal{M}$, we need to prove that $(SxS)b = b(SxS)$. So let $a \in \mathcal{M}$ and calculate

$$(SxS)b(a\xi) = (SxS)(ba\xi) \stackrel{L3}{=} (ba)x^*\xi$$

$$\text{and} \quad b(SxS)(a\xi) \stackrel{L3}{=} (ba)x^*\xi.$$

Thus the *bounded* operators $(SxS)b$ and $b(SxS)$ agree on the dense subset $\{a\xi : a \in \mathcal{M}\}$ of H, hence they coincide. $\square$

Lemma 5. *If $x \in \mathcal{M}'$,*

$$Sx\xi = x^*\xi.$$

Proof. Let $a \in \mathcal{M}$. Using relation $(*)$ and the fact that $S^2 = I$, we have

$$\langle Sx\xi, a\xi\rangle = \langle Sa\xi, x\xi\rangle \stackrel{(a\in\mathcal{M})}{=} \langle a^*\xi, x\xi\rangle = \langle \xi, ax\xi\rangle = \langle \xi, xa\xi\rangle = \langle x^*\xi, a\xi\rangle$$

since $ax = xa$. This shows that $Sx\xi - x^*\xi$ is orthogonal to the dense set $\mathcal{M}\xi$, hence must vanish. □

It follows that the functional

$$\tau' : \mathcal{M}' \to \mathbb{C} : \tau'(x) = \langle x\xi, \xi\rangle$$

is a *trace* on $\mathcal{M}'$. Indeed, for all $x, y \in \mathcal{M}'$,

$$\langle xy\xi, \xi\rangle = \langle y\xi, x^*\xi\rangle \stackrel{L5}{=} \langle y\xi, Sx\xi\rangle \stackrel{(*)}{=} \langle x\xi, Sy\xi\rangle \stackrel{L5}{=} \langle x\xi, y^*\xi\rangle = \langle yx\xi, \xi\rangle\,.$$

But, if we define

$$F_0 : \mathcal{M}'\xi \to \mathcal{M}'\xi : x\xi \to x^*\xi$$

then as before this densely defined antilinear map is isometric, hence extends to an antiunitary operator F on H, and Lemma 3 as well as Corollary 4 are true for the pair $(\mathcal{M}', F)$ on H. Applying Corollary 4 we obtain

$$F\mathcal{M}'F \subseteq (\mathcal{M}')' = \mathcal{M}$$

by the bicommutant theorem. But Lemma 5 shows that in fact the bounded operator F coincides with S on the dense subspace $\mathcal{M}'\xi$, hence everywhere. Thus the previous inclusion becomes

$$S\mathcal{M}'S \subseteq \mathcal{M}$$

so, remembering that $S^2 = I$,

$$\mathcal{M}' \subseteq S\mathcal{M}S.$$

Combined with Corollary 4, this gives $\mathcal{M}' = S\mathcal{M}S$ and completes the proof of the Theorem.

6. Brief Description of Tomita-Takesaki theory

What to do when there is no trace? Assume (for simplicity) that there is a cyclic and separating vector ξ for $(\mathcal{M}, H)$ (equivalently: there is a faithful normal state ω on $\mathcal{M}$).[19] But now the (antilinear) densely

[19] A positive linear functional is called *normal* when $a_i \nearrow a$ in $\mathcal{M}_+$ implies $\omega(a_i) \to \omega(a)$. Such an ω always exists when $\mathcal{M}$ acts on a separable Hilbert space. If no faithful normal state exists, one uses a (normal faithful semifinite) 'weight'. This is a map $\varphi : \mathcal{M}_+ \to [0, \infty]$ which is additive, homogeneous under positive scalars, vanishing only at 0 and such that the set $\{a \in \mathcal{M}_+ : \varphi(a) < \infty\}$ is WOT-dense in $\mathcal{M}_+$. A weight can be thought of as a noncommutative generalisation of an infinite measure.

defined map
$$S_0 : \mathcal{M}\xi \to \mathcal{M}\xi : a\xi \to a^*\xi$$
need no longer be isometric, in fact not even bounded! However it can be verified that S_0 is *closable*, i.e. the closure, in $H \oplus H$, of its graph $\{(u, S_0 u) : u \in D(S_0)\}$ is again the graph of an operator. This implies that S_0 has a densely defined adjoint S^* (which in fact satisfies $S^*(b\xi) = b^*\xi$ on $\mathcal{M}'\xi$) and $S := (S^*)^*$ is the 'closure' of S_0 [20].

Define $\Delta = S^*S$; this is a positive selfadjoint (usually unbounded) operator, called *the modular operator*. Then one can form the *polar decomposition*
$$S = J\Delta^{1/2}$$
where J is an antilinear isometric bijection.[21]

Remark 6. *Notice that if the state $a \to \omega(a) = \langle a\xi, \xi\rangle$ is tracial, then (relation $(*)$ of the previous section holds, hence) S is already isometric and so $\Delta = I$; the converse is also true. Thus the non-triviality of Δ expresses the fact that ω is not tracial.* [22]

The unitary group $\{\Delta^{it} : t \in \mathbb{R}\}$. Since Δ is selfadjoint and positive, so that its spectrum is contained in $\mathbb{R}_+$, for all $t \in \mathbb{R}$ one may form $U_t = \Delta^{it}$ using the functional calculus. Specifically, consider the spectral resolution of Δ:
$$\Delta = \int_0^\infty \lambda dE_\lambda \quad \text{i.e.} \quad \langle \Delta\eta, \zeta\rangle = \int_0^\infty \lambda d\mu_{\eta,\zeta}(\lambda)$$
where for η and ζ in the domain of Δ, the measure $\mu_{\eta,\zeta}$ is given by $\mu_{\eta,\zeta}(\Omega) = \langle E(\Omega)\zeta, \eta\rangle$ for all Borel sets $\Omega \subseteq \mathbb{R}$. Then define
$$U_t := \Delta^{it} = \int_0^\infty \lambda^{it} dE_\lambda.$$

[20] that is, the graph of S is the closure of the graph of S_0

[21] Sketch: for all η in a suitable dense subspace of H, we have
$$\|S\eta\|^2 = \langle S^*S\eta, \eta\rangle = \left\langle \Delta^{1/2}\Delta^{1/2}\eta, \eta\right\rangle = \left\langle \Delta^{1/2}\eta, \Delta^{1/2}\eta\right\rangle = \left\|\Delta^{1/2}\eta\right\|^2$$
so the map $J_0 : S\eta \to \Delta^{1/2}\eta$ is antilinear, densely defined and isometric, hence has an isometric extension J to H satisfying $S = J\Delta^{1/2}$ (one must verify that both operators have the same domain) whose range is dense because it contains $\mathcal{M}\xi$: indeed for each $a \in \mathcal{M}$ $a^*\xi = Sa\xi = J\Delta^{1/2}a\xi$ is in the range of J.

[22] The same phenomenon occurs in a locally compact group G; the modular operator of the von Neumann algebra $\mathrm{vN}(G)$ turns out to be multiplication by the modular function δ which is defined by $\int_G f(s^{-1})ds = \int_G f(s)\delta(s^{-1})ds$ for $f \in C_c(G)$; it measures how much the map $G \to G : s \to s^{-1}$ fails to preserve Haar measure.

It is not hard to see that since $\lambda \to \lambda^{it}$ is bounded on $\mathbb{R}$, U_t is everywhere defined and bounded. The properties of the exponential function show that each U_t is a unitary operator and $t \to U_t$ is a one-parameter group which is SOT-continuous, meaning that the function $\mathbb{R} \to H : t \to U_t\eta$ is continuous for all $\eta \in H$.

We may now formulate the main theorem:

Theorem 7. *If J and Δ are as defined above,*

$$J\mathcal{M}J = \mathcal{M}' \qquad \text{and} \qquad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \quad \forall t \in \mathbb{R}.$$

Thus Theorem 2 appears as a special case, with $\Delta = I$.

Notice that for each $t \in \mathbb{R}$ the map

$$\sigma_t : \mathcal{M} \to \mathcal{M} : \sigma_t(x) := \Delta^{it}x\Delta^{-it}$$

is a *-automorphism of $\mathcal{M}$.

Thus to every cyclic and separating vector ξ there corresponds a one-parameter automorphism group $\{\sigma_t : t \in \mathbb{R}\}$ of $\mathcal{M}$, called *the modular automorphism group of* $\mathcal{M}$. [23] It can be shown that this group acts trivially on $\mathcal{M}$ if and only if $\Delta = I$, equivalently (see Remark 6) if the vector ξ is tracial.

In other words, whenever the map $S : a\xi \to a^*\xi$ is not isometric, there comes up a non-trivial dynamical system $(\mathcal{M}, \sigma)$. And in fact, the fixed-point set of the modular group,

$$\mathcal{M}^\sigma := \{a \in \mathcal{M} : \sigma_t(a) = a \ \forall t \in \mathbb{R}\}$$

(which is in fact a von Neumann subalgebra of $\mathcal{M}$) is precisely the set on which ω is tracial:

Proposition 8. *An element $a \in \mathcal{M}$ belongs to the fixed-point algebra $\mathcal{M}^\sigma$ of the modular group if and only if*

$$\omega(ab) = \omega(ba) \quad \text{for all} \quad b \in \mathcal{M}.$$

Notice also that each σ_t leaves the state ω invariant: $\omega \circ \sigma_t = \omega$ for all $t \in \mathbb{R}$; equivalently, $\Delta^{it}\xi = \xi$ for all $t \in \mathbb{R}$.

6.1. **The KMS condition.** How much does the state

$$\omega(a) = \langle a\xi, \xi \rangle , \quad a \in \mathcal{M}$$

differ from being a trace? For fixed $a, b \in \mathcal{M}$, instead of just comparing

$$\omega(ab) \quad \text{and} \quad \omega(ba)$$

[23] It can be shown that this group is pointwise-weak* continuous; equivalently, that for all $a \in \mathcal{M}$ and every *normal* state φ the map $\mathbb{R} \to \mathbb{C} : t \to \varphi(\sigma_t(a))$ is continuous.

compare *the functions defined for* $t \in \mathbb{R}$ by:

$$f_{a,b}(t) = \omega(a\sigma_t(b)) \quad \text{and} \quad g_{a,b}(t) = \omega(\sigma_t(b)a).$$

They can be analytically interpolated:

Proposition 9. *There exists a function* $F_{a,b}$*,* defined and continuous *on the infinite closed strip*

$$\Omega := \{t + is \in \mathbb{C} : t \in \mathbb{R}, 0 \leq s \leq 1\}$$

and analytic *in the interior*

$$\Omega^\circ = \{t + is \in \mathbb{C} : t \in \mathbb{R}, 0 < s < 1\}$$

such that

$$F_{a,b}(t+i) = \omega(a\sigma_t(b)) \quad \textit{and} \quad F_{a,b}(t) = \omega(\sigma_t(b)a) \quad \textit{for all } \ t \in \mathbb{R}$$

in particular

$$F_{a,b}(i) = \omega(ab) \quad \textit{and} \quad F_{a,b}(0) = \omega(ba).$$

Definition 6.1 (The KMS condition)**.** *A state* ω *of* $\mathcal{M}$ *is said to satisfy the KMS condition with respect to a one-parameter automorphism group* $\{\phi_t : t \in \mathbb{R}\}$ *if for every* $a, b \in \mathcal{M}$ *there exists an analytic function for the pair* (ω, ϕ) *as in the previous Proposition.*

It is remarkable that the KMS condition actually characterises the modular automorphism group:

Proposition 10 (Uniqueness)**.** *Let* ω *be a faithful normal state on a von Neumann algebra* $\mathcal{M}$ *with associated modular group* $\{\sigma_t\}$*.*

Then the only pointwise-weak continuous* [24] *one-parameter group of automorphisms of* $\mathcal{M}$ *that satisfies the KMS condition with respect to* ω *is* $\{\sigma_t\}$*.*

6.2. **Application to type III factors.** Let $\mathcal{M}$ be a type III factor. To any faithful normal semifinite weight φ on $\mathcal{M}_+$ there corresponds a modular operator Δ_φ whose spectrum $\mathrm{spec}(\Delta_\varphi)$ is a closed subset of $\mathbb{R}_+$. It turns out that the intersection

$$\mathcal{S}(\mathcal{M}) = \bigcap \{\mathrm{spec}(\Delta_\varphi) : \varphi \ \text{ faithful normal semifinite weight on } \ \mathcal{M}\}$$

is an isomorphism invariant of $\mathcal{M}$. One can show that $\mathcal{S}(\mathcal{M})$ is a closed multiplicative semigroup of $\mathbb{R}_+$, so there are only three posibilities:

- $\mathcal{S}(\mathcal{M}) = [0, +\infty)$
- $\mathcal{S}(\mathcal{M}) = \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$, for some $\lambda \in (0, 1)$

[24] see the previous footnote

- $\mathcal{S}(\mathcal{M}) = \{0, 1\}$.

In the first case, $\mathcal{M}$ is said to be of type III_1; in the second, it is called type III_λ; and in the third case $\mathcal{M}$ is called type III_0.

These arise as follows:

Let $\mathcal{N}$ be a factor von Neumann algebra equipped with a faithful normal infinite but semifinite trace τ (thus $\mathcal{N}$ is a type II_∞ factor). Suppose that there exists $\lambda \in (0, 1)$ and a *-automorphism $\theta : \mathcal{N} \to \mathcal{N}$ which 'scales the trace by λ', that is

$$\tau(\theta(a)) = \lambda\tau(a) \quad \text{for all } a \in \mathcal{N}_+ \text{ such that } \; \tau(a) < \infty.$$

Then the crossed product

$$\mathcal{M} := \mathcal{N} \rtimes_\theta \mathbb{Z}$$

of $\mathcal{N}$ by the automorphism group $\{\theta^n : n \in \mathbb{Z}\}$ is a type III_λ factor.

Conversely, for every $\lambda \in (0, 1)$, every type III_λ factor arises as a crossed product in this way.

Any type III_0 factor $\mathcal{M}$ can also be written as a crossed product $\mathcal{M} := \mathcal{N} \rtimes_\theta \mathbb{Z}$ where $\mathcal{N}$ is a von Neumann algebra (not necessarily a factor) equipped with a faithful normal semifinite trace τ and θ is a *-automorphism of $\mathcal{N}$ having the property that $\tau \circ \theta \leq \lambda\tau$ for some $\lambda \in (0, 1)$.

These results were proved by A. Connes in his thesis.

The type III_1 case is due to Takesaki:

Any factor $\mathcal{M}$ of type III_1 arises as a crossed product

$$\mathcal{M} := \mathcal{N} \rtimes_\theta \mathbb{R}$$

where now $(\mathcal{N}, \tau)$ is a type II_∞ factor as in the III_λ case and $\{\theta_t : t \in \mathbb{R}\}$ is a one parameter group of automorphisms of $\mathcal{N}$ that scale the trace as follows: $\tau \circ \theta_t = e^{-t}\tau$ for all $t \in \mathbb{R}$.

7. Comments on the bibliography

The origins The papers *On Rings of Operators* [11, 12, 21, 13], 'where it all began' are well worth reading.

General texts *Murphy* [10] is a popular introductory book in Operator Theory.

Dixmier's book [4, 5] is a classic; it is a very lucid presentation of von Neumann algebra theory up to the Tomita-Takesaki era.

Kadison and Ringrose [8] and [9] develop the basic theory of (selfadjoint) operator algebras from scratch and in great detail.

Takesaki's three volume treatice [17, 18, 19] is a complete and advanced presentation of von Neumann algebra theory.

The course notes by *Vaughan Jones* [7] (one of the 'masters' of the subject) are rough but very interesting and lively.

Blackadar's book [2] is encyclopaedic; it attempts to give a comprehensive discussion of the whole theory of C* algebras and von Neumann algebras.

On the other hand, *Fillmore's* much shorter book [6] is really a 'guide'; it stresses the main points and examples, often with a sketch of the proofs.

Of course, the book *Non-commutative Geometry* [3] (by *Alain Connes*, another one of the 'masters') is the definitive reference that puts the theory of C* and von Neumann algebras in its proper context within the development of Mathematics and Mathematical Physics.

Tomita - Takesaki theory etc. *Takesaki's* lecture notes [16] marked the first full presentation of Tomita's theory which made the theory accessible to international audiences.

Sunder [15] is centered on Tomita – Takesaki theory and its consequences for type III factors. It contains a proof of theorem 7 under the very strong and restrictive assumption that the operator S (hence also Δ) is bounded.

In [1, Chapter 20], the theory is presented for the case of the von Neumann algebra $M_n(\mathbb{C})$.

These two special cases are helpful in order to get an idea of the validity of theorem 7.

The article [14] of *Rieffel and van Daele* develops a fairly elementary and readable method that yields a full proof of the theorem using only bounded operators.

Finally, *van Daele's* lecture notes [20] contain a very readable and careful presentation of crossed products of general (possibly non abelian) von Neumann algebras and their use in the analysis of type III factors.

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