Fourier algebras on homogeneous spaces

K. Parthasarathy *, N. Shravan Kumar

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, India

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Abstract

Spectral synthesis and operator synthesis on a homogeneous space $G/K$, where $K$ is a compact subgroup of a locally compact group $G$, are studied. Injection theorem for sets of spectral synthesis for $A(G/K)$ is proved, extending the classical result of Reiter and more recent results of Kaniuth–Lau, Parthasarathy–Prakash and others. A simple direct image theorem for spectral synthesis is proved and an extension of the subgroup theorem and an alternate proof of the injection theorem are obtained as consequences. The relation between synthesis in the Fourier algebra $A(G/K)$ and an appropriate Varopoulos algebra is obtained, subsuming earlier results of Varopoulos, Spronk–Turowska and Parthasarathy–Prakash. Study of relations between spectral synthesis and operator synthesis pioneered by Arveson and carried forward recently by Shulman–Turowska, Parthasarathy–Prakash and Ludwig–Turowska is undertaken on homogeneous spaces. Operator space methods are needed for this study, and more specifically, a characterisation of completely bounded multipliers on $A(G/K)$ as the invariant part of a suitable weak* Haagerup tensor product (or the space of Schur multipliers) is given and is used for this study.

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1. Introduction

The Fourier algebra $A(G)$ of a (nonabelian) locally compact group has been extensively studied since the appearance, in 1964, of the pioneering work of Eymard [6]. In contrast, the Fourier algebra $A(G/K)$ on a homogeneous space $G/K$, where $K$ is a compact (nonnormal) subgroup of $G$, was introduced by Forrest [7] only about a decade ago, and does not seem to have re-
ceived so much attention since then. This paper studies spectral synthesis and operator synthesis for $A(G/K)$, and, as completely bounded multipliers of this algebra are needed in the study of operator synthesis, these are also discussed. Operator space techniques have come to play a central role in modern abstract harmonic analysis on nonabelian groups and we also make essential use of the operator space structure on the Fourier algebra, especially in our study of operator synthesis.

If $H$ is a closed subgroup of $G$ and if $E$ is a closed subset of $H$, then $E$ is a set of synthesis for $A(H)$ if and only if it is a set of synthesis for $A(G)$. This theorem is known as the injection theorem for sets of synthesis. In the abelian case this is a classical result of Reiter (see [21]). The nonabelian version is due to Kaniuth and Lau [12]. For more general versions, see Parthasarathy and Prakash [15,17]. Theorem 3.2 of this paper is the injection theorem for sets of (weak) spectral synthesis in the context of homogeneous spaces. The image of a set of synthesis under the canonical map $G \to G/K$ is again a set of synthesis (Proposition 3.6). As consequences, we get an extension of the subgroup theorem for synthesis (Corollary 3.7), as well as an alternative proof of our injection theorem.

Varopoulos [28] pioneered tensor product methods in harmonic analysis and, in particular, gave a tensor product proof of Malliavin’s famous theorem on the failure of spectral synthesis on nondiscrete abelian groups. For compact abelian groups, his proof is based on a relation that he obtained between spectral synthesis in $A(G)$ and spectral synthesis in the projective tensor product $V(G) = C(G) \otimes^\gamma C(G)$; this algebra $V(G)$ is now called the Varopoulos algebra. Recently, his results have been extended to nonabelian groups by Spronk and Turowska [26] and Parthasarathy and Prakash [15]. In Section 3 of this paper, we develop these in the context of the Fourier algebra $A(G/K)$ by introducing an appropriate Varopoulos algebra.

Arveson [1] introduced the concept of operator synthesis and obtained its relation to spectral synthesis on compact abelian groups. This was followed up by Froelich [9]. More recent work on operator synthesis and its relation to spectral synthesis on nonabelian groups can be found in the papers of Turowska co-authored with Ludwig, Shulman and Spronk [14,23,24,26] and the papers of Parthasarathy and Prakash [16,17]. In this context, noncompact groups were first considered by Ludwig and Turowska [14]. In Section 5, we present the relation between sets of spectral synthesis for $A(G/K)$ (for noncompact $G$) and sets of operator synthesis with respect to $m_{K \backslash G} \times m_G$, extending the results of earlier authors.

The inverse projection theorem for sets of synthesis on abelian groups is a well-known result due to Reiter [21]. In the nonabelian case, only partial results are known, both for sets of synthesis (Forrest [7]) and for sets of local synthesis (Lohoué [13], Derighetti [4]). As a consequence of our results on the connection between spectral synthesis and operator synthesis in Section 5, we obtain the complete inverse projection theorem for sets of local synthesis.

For our results on operator synthesis in Section 5, we need the characterisation of completely bounded multipliers on $A(G/K)$. We obtain this in Section 4, by considering the weak* Haagerup tensor product of suitable $L^\infty$ spaces and adapting Spronk’s arguments in [25] to our setting.

We begin with some of the required preliminaries in the next section.

2. Preliminaries

Let $G$ be a locally compact group and fix a left Haar measure $m_G$ on $G$. Let $\lambda$ be the left regular representation of $G$ acting on the Hilbert space $L^2(G)$ by left translations: $\lambda(s)f(t) = f(st^{-1})$ for $s, t \in G$ and $f \in L^2(G)$. The group von Neumann algebra $VN(G)$
is the smallest self-adjoint subalgebra in the operator algebra $\mathcal{B}(L^2(G))$ that contains all the $\lambda(t)$ and is closed in the weak operator topology. The Fourier algebra $A(G)$ is the predual of $VN(G)$. Each function $u \in A(G)$ has a representation $u(.) = \langle \lambda(.) f, g \rangle$, with $f, g \in L^2(G)$ and $\|u\|_{A(G)} = \|f\|_2 \|g\|_2$. The pairing between $A(G)$ and $VN(G)$ is then given by $\langle T, u \rangle = (Tf, g)$ where we have the inner product in $L^2(G)$ on the right side. The basic reference for Fourier algebras is the fundamental paper [6] of Eymard.

Let $K$ be a compact subgroup of $G$ and let $G/K$ be the homogeneous space of left cosets of $K$. We take the normalised Haar measure on $K$. Then there is a unique Radon measure $m_{G/K}$ on $G/K$ that is invariant for the left action of $G$ and which, using obvious notation, satisfies Weil's formula $\int_G f(t) dt = \int_{G/K} \int_K f(kt) dk dt$ for $f \in C_c(G)$. We also need the space $K \backslash G$ of right cosets and the right $G$-invariant Radon measure $m_{K \backslash G}$ on it. Let $q = q_K : G \rightarrow G/K$ and $p : G \rightarrow K \backslash G$ be the canonical quotient maps. These are both open maps and they are also closed maps by the compactness of $K$. For $t \in G$, we write $\bar{t}$ for the left coset $tK = q(t)$ and $\bar{t}$ for the right coset $K\bar{t} = p(t)$.

$A(G : K)$ is the closed subalgebra of $A(G)$ consisting of those functions in $A(G)$ that are constant on the left cosets of $K$ and is the range of the linear map $P_K$ defined on $A(G)$ by $P_K u(t) = \int_K u(kt) dk$, for $t \in G$. For a function $u \in A(G : K)$, $u(t)$ depends only on the left coset $t = tK$ and so gives a continuous function $\tilde{u}$ on $G/K$ given by $\tilde{u}(\bar{t}) = u(t)$. We write $A(G : K) := \{\tilde{u} : u \in A(G)\}$. It was introduced by Forrest [7] and is a regular, commutative, semisimple Banach algebra with Gelfand structure space $G/K$. We call it the Fourier algebra of $G/K$ and shall often identify it with $A(G : K)$. Any function in $A(G : K)$ has a representation of the form $u(.) = \langle \lambda(.) \tilde{f}, g \rangle$ with $\tilde{f} \in L^2(K \backslash G)$, $g \in L^2(G)$. The dual of $A(G/K)$ is $VN(G/K)$, the weak* closure of $\{\langle \lambda(.) \tilde{f} \rangle : \tilde{f} \in L^1(G/K)\}$ in $VN(G)$. Observe that, for $x \in G$ and $k \in K$ the restriction of $\lambda(\bar{x}k)$ to $L^2(K \backslash G)$ is independent of $k$. We write $\lambda(\bar{x}) \in \mathcal{B}(L^2(K \backslash G), L^2(G))$ for this restriction. Thus $VN(G/K)$ can be considered as the smallest subspace of $\mathcal{B}(L^2(K \backslash G), L^2(G))$ containing all the $\lambda(\bar{x})$ that is closed in the weak operator topology. As a linear functional on $A(G/K)$, $\lambda(\bar{x})$ acts as point evaluation.

Whenever meaningful, we shall use similar notation and convention for other function spaces also; thus, for example, we shall consider $L^1(G : K)$ and $L^1(G/K)$. We shall also adopt a similar convention for function spaces on the right coset space $K \backslash G$. The corresponding projection map is given by $\int_K u(kt) dk$.

When $G$ is compact and abelian, Varopoulos, in [28], considered the projective tensor product $V(G)$ of the Banach algebra $C(G)$ with itself. For nonabelian compact groups, Spronk and Turowska [26] renormed the algebra by considering the Haagerup tensor product of $C(G)$ with itself and called it the Varopoulos algebra on $G$. We consider $V(G, K \backslash G)$, the Haagerup tensor product of the $C^*$-algebras of continuous functions $C(G)$ and $C(K \backslash G)$: $V(G, K \backslash G) = C(G) \otimes^h C(K \backslash G)$ and call it the Varopoulos algebra of $G$ and $K \backslash G$. For our purposes, it is the algebra of continuous functions on $G \times K \backslash G$ which have representations of the form $w = \sum \varphi_i \otimes \tilde{\psi}_i$, $\varphi_i \in C(G)$, $\tilde{\psi}_i \in C(K \backslash G)$, with $\sum |\varphi_i|^2$ and $\sum |\tilde{\psi}_i|^2$ uniformly convergent. The Haagerup norm is given by

$$\|w\|_V = \inf \left\{ \left\| \sum |\varphi_i|^2 \right\|_\infty^{\frac{1}{2}} \left\| \sum |\tilde{\psi}_i|^2 \right\|_\infty^{\frac{1}{2}} : w = \sum \varphi_i \otimes \tilde{\psi}_i \right\}.$$  

$V(G, K \backslash G)$ is a commutative, semisimple, regular Banach algebra with Gelfand maximal ideal space $G \times K \backslash G$. It can be considered as a subalgebra of the Varopoulos algebra $V(G)$ consisting of those functions $w \in V(G)$ that satisfy the condition $w(s, t) = w(s, kt)$ for all $k \in K$, i.e. we
The algebras \( V(\mathbb{K},\mathbb{G}) \) of the form \( \tilde{B}(A(\mathbb{G})) \) are operator spaces. Therefore, both are operator spaces. Further, closed subspaces, duals and preduals of operator spaces are also operator spaces. Therefore, both \( A(\mathbb{G}) \) and \( A(\mathbb{G}/\mathbb{K}) \) are operator spaces. For our more specific needs we refer to Blecher and Smith [2], Smith [22] and Spronk [25]. The books of Effros and Ruan [5] and Pisier [20] are basic references on this topic. For our more specific needs we refer to Blecher and Smith [2], Smith [22] and Spronk [25].

Let \( I \) and \( J \) be any two index sets. We shall consider the space of bounded linear operators \( B(\ell^2(J), \ell^2(I)) \) and the space \( M_{I,J} \) of matrices representing these operators (with respect to the canonical orthonormal bases). A function \( u : I \times J \rightarrow \mathbb{C} \) is called a Schur multiplier if \([u(i, j)a_{ij}] \in M_{I,J} \) for every \([a_{ij}] \in M_{I,J} \). We shall denote by \( V^\infty(I, J) \), the space of all Schur multipliers. The following result, due to Pisier [19, Proposition 1.1], is basic.

**Theorem.** The space \( V^\infty(I, J) \) of Schur multipliers can be identified isometrically with each of the following spaces:

i) the weak*-Haagerup tensor product \( \ell^\infty(I) \otimes^{wh} \ell^\infty(J) \);

ii) the space of functions \( u = u_{\xi, \eta} : I \times J \rightarrow \mathbb{C} \) of the form \( u(i, j) = \langle \xi(i), \eta(j) \rangle_h \) for some Hilbert space valued, bounded functions \( \xi : I \rightarrow \mathbb{H}, \eta : J \rightarrow \mathbb{H} \) with \( \|u\| = \inf\{\|\xi\|_\infty \|\eta\|_\infty : u = u_{\xi, \eta}\} \).

For measure spaces \((X, \mu), (Y, \nu)\) satisfying the Radon–Nikodym theorem, Spronk [25] takes as the analogue of ii) above as the definition of \( V^\infty(X, Y) \). It is the space of all functions \( u = u_{\xi, \eta} \) as above, where, now, \( \xi \in L^\infty(X, \mu), \eta \in L^\infty(Y, \nu) \). This space is called the space of measurable Schur multipliers. Again this space can be identified with the weak*-Haagerup tensor product \( L^\infty(X, \mu) \otimes^{wh} L^\infty(Y, \nu) \).

The spaces of Schur multipliers that we shall need are \( V^\infty(\mathbb{G}) \) and \( V^\infty(\mathbb{K}, \mathbb{G}) \). For our purposes, elements of \( V^\infty(\mathbb{K}, \mathbb{G}) \) are functions (up to marginally null sets) on \( \mathbb{K} \times \mathbb{G} \) of the form \( \tilde{w} = \sum_{n=1}^{\infty} \tilde{\varphi}_n \otimes \psi_n \) where \( \tilde{\varphi}_n \) and \( \psi_n \) are in \( L^\infty(\mathbb{K}, \mathbb{G}) \) and \( L^\infty(\mathbb{G}) \), respectively, and the series is weak-* convergent. (For us, a **marginally null set** is a subset of a set of the form \( E \times \mathbb{G} \) where, \( E \) is locally null in \( \mathbb{G} \).) Moreover,

\[
\|w\|_{V^\infty} = \inf \left\{ \left\| \sum \tilde{\varphi}_n \otimes \psi_n \right\|_{1/2} \left\| \sum |\psi_n|^2 \right\|_\infty : w = \sum \tilde{\varphi}_n \otimes \psi_n \right\}
\]

with the series \( \sum |\tilde{\varphi}_n|^2 \) and \( \sum |\psi_n|^2 \) converging in the weak-* topology. The invariant parts, \( V^\infty_{inv}(\mathbb{G}) \) and \( V^\infty_{inv}(\mathbb{K}, \mathbb{G}) \), are defined in the obvious way, analogous to the case of Varopoulos algebras.

An element \( \omega = \sum_{n=1}^{\infty} \tilde{f}_n \otimes g_n \) of \( T(\mathbb{K}, \mathbb{G}) \), the projective tensor product \( L^2(\mathbb{K}) \otimes^\gamma L^2(\mathbb{G}) \), is considered as the function given by \( \omega(\tilde{y}, x) = \sum \tilde{f}_n(\tilde{y}) g_n(x) \) for marginally almost all \((\tilde{y}, x) \in \mathbb{K} \times \mathbb{G} \). For such an \( \omega \), \( supp \omega = \{ (\tilde{y}, x) \in \mathbb{K} \times \mathbb{G} : \omega(\tilde{y}, x) \neq 0 \} \) is defined up to marginally null sets. The dual of \( T(\mathbb{K}, \mathbb{G}) \) is identified with \( B(L^2(\mathbb{K}), L^2(\mathbb{G})) \) via the pairing given, for \( \tilde{f} \in L^2(\mathbb{K}), g \in L^2(\mathbb{G}) \) and \( S \in B(L^2(\mathbb{K}), L^2(\mathbb{G})) \), by \( \langle S, \tilde{f} \otimes g \rangle = \sum \tilde{f}_n(\tilde{y}) g_n(x) \).
\( \langle S \tilde{f}, \tilde{g} \rangle \), where on the right we have the \( L^2(G) \)-inner product. Moreover, \( V^\infty(K \backslash G, G) \) is the algebra of multipliers of \( T(K \backslash G, G) \):

\[
V^\infty(K \backslash G, G) = \left\{ \tilde{w} : \tilde{w} \text{ is a complex function on } K \backslash G \times G \text{ and } m_{\tilde{w}} : \omega \mapsto \tilde{w} \cdot \omega \right\}
\]

with \( \| \tilde{w} \|_{V^\infty(K \backslash G, G)} = \| m_{\tilde{w}} \| \). Here, \( m_{\tilde{w}} \) is given by \( m_{\tilde{w}}(\tilde{f} \otimes g)(i, s) = \tilde{w}(r, s) \tilde{f}(i) g(s) \) on elementary tensors. Two functions \( \tilde{w} \) and \( \tilde{w}' \) in \( V^\infty(K \backslash G, G) \) are identified if they differ on a marginally null set.

For a closed set \( F \subseteq K \backslash G \times G \), define

\[
\Phi(F) = \{ \omega \in T(K \backslash G, G) : \text{supp } \omega \cap F = \emptyset \},
\]

\[
\psi(F) = \{ \omega \in T(K \backslash G, G) : \text{supp } \omega \cap U = \emptyset \text{ for some open } U \supseteq F \},
\]

\[
\Psi(F) = \overline{\psi(F)}.
\]

If \( \Phi(F) = \Psi(F) \), then \( F \) is called a set of operator synthesis with respect to \( m_{K \backslash G} \times m_G \) (or is said to be \( m_{K \backslash G} \times m_G \)-synthetic).

A multiplier of \( A(G/K) \) is a continuous function \( \tilde{w} : G/K \rightarrow \mathbb{C} \) such that \( \tilde{w} \cdot \tilde{u} \in A(G/K) \) for each \( \tilde{u} \in A(G/K) \). The map \( m_{\tilde{w}} : A(G/K) \rightarrow A(G/K) \) given by \( m_{\tilde{w}}(\tilde{u}) = \tilde{w} \cdot \tilde{u} \) is a bounded linear map. Let \( M(A(G/K)) \) denote the space of all multipliers of \( A(G/K) \).

As \( A(G/K) \) is an operator space, it makes sense to speak about completely bounded maps on \( A(G/K) \). We denote by \( M_{cb}(A(G/K)) \) the set of all multipliers \( \tilde{w} \) of \( A(G/K) \) such that the corresponding map \( m_{\tilde{w}} \) is completely bounded. (This means that, for each \( n \), the map induced by \( m_{\tilde{w}} \) on the matrix space \( M_n(A(G/K)) \) is bounded.) The set of all completely bounded operators on \( A(G/K) \) is denoted by \( CB(A(G/K)) \). Thus

\[
M_{cb}(A(G/K)) = \left\{ \tilde{w} \in M(A(G/K)) : m_{\tilde{w}} \in CB(A(G/K)) \right\}.
\]

The operator space structure on \( M_{cb}(A(G/K)) \) can be described as follows: for \( [\tilde{w}_{ij}] \in M_n(M_{cb}(A(G/K))) \) let

\[
\left\| [\tilde{w}_{ij}] \right\|_{cb} = \left\| m_{[\tilde{w}_{ij}]} \right\|_{cb}
\]

where \( m_{[\tilde{w}_{ij}]} \in CB(A(G/K), M_n(A(G/K))) \), is defined as

\[
m_{[\tilde{w}_{ij}]}(\tilde{u}) = \left[ m_{\tilde{w}_{ij}}(\tilde{u}) \right] = [\tilde{w}_{ij} \cdot \tilde{u}]
\]

for \( \tilde{u} \in A(G/K) \). Since \( A(G/K) \) is a completely contractive Banach algebra, it embeds completely contractively into \( M_{cb}(A(G/K)) \).

3. Spectral synthesis

Malliavin’s theorem is a landmark in the study of the classical problem of spectral synthesis. Varopoulos [28] based his proof of this theorem on a relation that he obtained between spectral synthesis in \( A(G) \) and in \( V(G) \) for compact abelian \( G \). This was later extended to nonabelian groups by Spronk and Turowska [26] and Parthasarathy and Prakash [15]. In the second part of this section we consider the problem for \( A(G/K) \) and \( V(G, K \backslash G) \). More specifically, we consider the map \( N_K : A(G/K) \rightarrow V_{inv}(G, K \backslash G) \) analogous to the map \( N : A(G) \rightarrow V_{inv}(G) \) introduced and studied by Varopoulos [28] and Spronk and Turowska [26] and use it to obtain the relation between spectral synthesis in \( A(G/K) \) and \( V(G, K \backslash G) \). This gives back the earlier result when \( K \) reduces to the one element subgroup.
In the first part of the section, we present two approaches to an injection theorem for (weak) synthesis in the context of homogeneous spaces. The first proof follows arguments used by Parthasarathy and Prakash in [15]. We obtain a simple result on the image of a set of synthesis under the quotient map and our second approach to the injection theorem is based on this.

But first we set down some notation and recall some definitions.

Let $\mathcal{A}$ be a commutative, semisimple, regular Banach algebra with Gelfand space $\Delta(\mathcal{A})$. For a closed set $E$ in $\Delta(\mathcal{A})$, let

$$j_{\mathcal{A}}(E) = \{a \in \mathcal{A} : \widehat{a} \text{ has compact support disjoint from } E\},$$

$$J_{\mathcal{A}}(E) = j_{\mathcal{A}}(E),$$

$$I_{\mathcal{A}}(E) = \{a \in \mathcal{A} : \widehat{a} = 0 \text{ on } E\}.$$

These sets are all ideals in $\mathcal{A}$ with zero set $E$ and $j_{\mathcal{A}}(E) \subseteq I \subseteq I_{\mathcal{A}}(E)$ for any ideal $I$ with zero set $E$. $E$ is said to be a set of spectral synthesis (or a spectral set) for $\mathcal{A}$ if $I_{\mathcal{A}}(E) = J_{\mathcal{A}}(E)$. More generally, $E$ is said to be a set of weak spectral synthesis if there is a positive integer $n$ such that $u^n \in J_{\mathcal{A}}(E)$ for all $u \in I_{\mathcal{A}}(E)$. If this holds, the least such positive integer is denoted by $\xi_E(\mathcal{A})$. This weaker form of spectral synthesis was introduced by Warner [29] and has attracted much attention, see, for example, [18,11,17]. For instance, Malliavin’s theorem was extended to weak synthesis by Parthasarathy and Varma [18]. (For the analogous result on nonabelian groups, see Kaniuth [11] and Parthasarathy and Prakash [17].)

Let $I^*_A(E)$ denote the set of elements in $I_{\mathcal{A}}(E)$ with compactly supported Gelfand transforms. We say that $E$ is a set of local spectral synthesis if $I^*_A(E) \subseteq J_{\mathcal{A}}(E)$.

The Banach space dual $\mathcal{A}^*$ has a natural $\mathcal{A}$-module structure: $\langle u \cdot T, v \rangle = \langle T, uv \rangle$, for $u, v \in \mathcal{A}, T \in \mathcal{A}^*$. The support of $T \in \mathcal{A}^*$ is defined by $\text{supp } T = \{\chi \in \Delta(\mathcal{A}) : u \in \mathcal{A}, \widehat{u}(\chi) \neq 0 \Rightarrow u \cdot T \neq 0\}$.

**Lemma 3.1.** Let $K$ be a compact subgroup of a locally compact group $G$ and suppose $H$ is a closed subgroup of $G$ containing $K$. Let $E \subseteq H/K$ be closed and let $E = q^{-1}(E) \subseteq H$. For a function $\tilde{u}$ on $G/K$ (or $H/K$) let $u$ be the corresponding function on $G$ (or $H$) that is constant on left cosets of $K$: $u(t) = \tilde{u}(q(t))$. Then

i) the restriction map $r^*_K : \tilde{u} \mapsto \tilde{u}|_{H/K}$ is a linear continuous surjection of $A(G/K)$ onto $A(H/K)$;
ii) the adjoint $r^*_K : VN(H/K) \rightarrow VN(G/K)$ is an injection;
iii) $r^{-1}_K(I_{A(H/K)}(E)) = I_{A(G/K)}(\tilde{E})$;
iv) $r^{-1}_K(J_{A(H/K)}(E)) \supseteq J_{A(G/K)}(\tilde{E})$;
v) $u \in j_{A(G)}(E)$ if and only if $\tilde{u} \in j_{A(H/K)}(\tilde{E})$;
vi) $u \in I_{A(G)}(E)$ if and only if $\tilde{u} \in I_{A(H/K)}(\tilde{E})$;
vii) $u \in I_{A(G)}(E)$ if and only if $\tilde{u} \in I_{A(H/K)}(\tilde{E})$.

**Proof.** i) Considering $\tilde{u} \in A(G/K)$ as a function in $A(G)$ that is constant on left cosets of $K$, the restriction belongs to $A(H/K)$ because it is a function in $A(H)$ (by Herz [10]) that is constant on left cosets. Conversely, again by Herz [10], any function in $A(H)$ that is constant on left cosets extends to a function in $A(G)$ and then it can made left $K$-invariant by applying the map $P_K$. In other words, the restriction map is a (continuous) linear surjection of $A(G/K)$ onto $A(H/K)$.

ii) is a consequence of i) and iii) is clear.
iv) If \( \widetilde{V} \) is open in \( G/K \) containing \( \widetilde{E} \) and if \( \tilde{u} = 0 \) on \( \widetilde{V} \), then \( r_K \tilde{u} = 0 \) on \( \widetilde{V} \cap H/K \supseteq \widetilde{E} \). In other words, \( r_K (j_{A(G/K)}(\widetilde{E})) \subseteq j_{A(H/K)}(\widetilde{E}) \). By continuity of \( r_K \), iv) now follows.

v) Observe that if \( u(t) = 0 \) for all \( t \) in an open set \( V \supseteq \widetilde{E} \) in \( G \), then \( \tilde{u}(i) = 0 \) for all \( i \) in the open set \( g(V) \supseteq \widetilde{E} \) and if \( \tilde{u}(i) = 0 \) for all \( i \) in an open set \( \widetilde{V} \supseteq \widetilde{E} \) in \( G/K \), then \( u(t) = 0 \) for all \( t \in q^{-1}(V) \supseteq \widetilde{E} \).

vi) is a consequence of v) and the fact that \( u \leftrightarrow \tilde{u} \) is bicontinuous and vii) is clear. \( \square \)

We are now ready to prove the injection theorem for sets of (weak) spectral synthesis for Fourier algebras on homogeneous spaces. For \( A(G) \) and \( A(H) \), this result for synthesis can be found in Kaniuth and Lau [12]; more general versions for synthesis and weak synthesis are in Parthasarathy and Prakash [15,17]. For the injection theorem for sets of local synthesis, see Derighetti [4].

**Theorem 3.2 (Injection theorem for sets of (weak) synthesis).** Let \( K \) be a compact subgroup of a locally compact group \( G \) and suppose \( H \) is a closed subgroup containing \( K \). Let \( \widetilde{E} \subseteq H/K \) be closed. Then \( \widetilde{E} \) is of (weak) synthesis for \( A(G/K) \) if and only if it is of (weak) synthesis for \( A(H/K) \) with \( \xi_{A(G/K)}(\widetilde{E}) = \xi_{A(H/K)}(\widetilde{E}) \).

**Proof.** Suppose \( E \) is a set of weak synthesis for \( A(G/K) \) with \( \xi_{A(G/K)}(\widetilde{E}) = n \) and let \( \tilde{u} \in I_{A(G/K)}(\widetilde{E}) \). By Lemma 3.1 ii) there is a \( \tilde{u} \in I_{A(G/K)}(\widetilde{E}) \) such that \( r_K \tilde{u} = \tilde{u} \). Since by assumption \( \tilde{u}^n \in J_{A(G/K)}(\widetilde{E}) \), Lemma 3.1 iv) now implies that \( \tilde{u}^n = r_K \tilde{u}^n \in J_{A(H/K)}(\widetilde{E}) \) proving thereby that \( \widetilde{E} \) is of weak synthesis for \( A(H/K) \) with \( \xi_{A(H/K)}(\widetilde{E}) \leq n \).

Conversely, assume that \( \widetilde{E} \) is of weak synthesis for \( A(H/K) \) and suppose \( \xi_{A(H/K)}(\widetilde{E}) = n \). Let \( \tilde{u} \in I_{A(G/K)}(\widetilde{E}) \). Then \( r_K \tilde{u} \in I_{A(H/K)}(\widetilde{E}) \) by Lemma 3.1 i) and so, by the assumption on \( \widetilde{E} \), \( r_K \tilde{u}^n = (r_K \tilde{u})^n \in J_{A(H/K)}(\widetilde{E}) \). It follows that \( ru^n \in J_{A(H)}(E) \) by Lemma 3.1 vi) above (applied with \( H \) in place of \( G \) and hence \( u^n \in J_{A(G)}(E) \), since \( r^{-1}(J_{A(H)}(E)) = J_{A(G)}(E) \) by Theorem 3.1 of Parthasarathy and Prakash [15]. This, in turn, implies that \( \tilde{u} \in J_{A(G/K)}(\widetilde{E}) \) by Lemma 3.1 vi) again. We have proved that \( \widetilde{E} \) is of weak synthesis for \( A(G/K) \) with \( \xi_{A(G/K)}(\widetilde{E}) \leq n \) and the proof is complete. \( \square \)

**Remark 3.3.** The proof appears deceptively elementary, but we have used Theorem 3.1 of [15] whose proof uses the rather deep fact that a closed subgroup \( H \) of \( G \) is always of synthesis [27].

**Corollary 3.4.** With notation as in Theorem 3.2, \( H/K \) is a set of synthesis for \( A(G/K) \).

**Proof.** Apply (one part) of Theorem 3.2 with \( \widetilde{E} = H/K \). \( \square \)

**Remark 3.5.** This, of course, extends the subgroup theorem of Takesaki and Tatsuuma [27] mentioned above. Observe, however, that our proof of Corollary 3.4 uses the subgroup theorem.

It is hard to believe that the following rather simple, but quite nice, result has not been observed earlier. But we have been unable to find any reference. In conjunction with the subgroup theorem, it leads to another proof of the injection theorem for sets of synthesis on homogeneous spaces.

**Proposition 3.6 (Direct image theorem for spectral synthesis).** Let \( K \) be a compact subgroup of a locally compact group \( G \) and let \( E \subseteq G \) be closed. If \( E \) is a set of weak synthesis for \( A(G) \),
then the closed set \( q(E) \) is a set of weak synthesis for \( A(G/K) \) with \( \xi_{A(G/K)}(q(E)) \leq \xi_{A(G)}(E) \). In particular, if \( E \) is of synthesis for \( A(G) \), then so is \( q(E) \) for \( A(G/K) \).

**Proof.** Suppose \( E \) is a set of weak synthesis for \( A(G) \) with \( \xi_{A(G)}(E) = n \). Let \( \tilde{u} \in I_{A(G/K)}(q(E)) \). Then \( u \in I_{A(G)}(E) \) by Lemma 3.1 and, by the assumption on \( E \), \( u^n \in J_{A(G)}(E) \). This, in turn, gives that \( \tilde{u}^n \in J_{A(G/K)}(q(E)) \), again by Lemma 3.1.

The next result is Corollary 3.4 again, but with a different proof.

**Corollary 3.7.** Let \( K \subset H \subset G \) be locally compact groups with \( K \) compact. Then \( H/K \) is a set of synthesis for \( A(G/K) \).

**Proof.** Observe that \( H \) is a set of synthesis for \( A(G) \) by the result of Takesaki and Tatsuuma [27] and \( q(H) = H/K \). Now apply Proposition 3.6.

In view of Corollary 3.7, we can now once more prove the injection theorem just as in the group case (see Kaniuth and Lau [12]).

**Corollary 3.8 (Injection theorem for sets of synthesis).** With notation as above, let \( \tilde{E} \) be a closed subset of \( H/K \). Then \( \tilde{E} \) is a set of (weak) synthesis for \( A(G/K) \) if and only if it is a set of (weak) synthesis for \( A(H/K) \).

**Proof.** We write down the proof only for the case of spectral synthesis. Suppose first that \( \tilde{E} \) is of synthesis for \( A(G/K) \). If \( S \in VN(H/K) \) is supported in \( \tilde{E} \), then \( r_K^* S \) is also supported in \( \tilde{E} \). Thus, for \( \tilde{u} \in I_{A(G/K)}(\tilde{E}) \), we have \( \langle S, r_K^* \tilde{u} \rangle = \langle r_K^* S, \tilde{u} \rangle = 0 \). In view of Lemma 3.1, this shows that \( S \) annihilates \( I_{A(H/K)}(\tilde{E}) \) and so \( \tilde{E} \) is of synthesis for \( A(H/K) \).

Conversely, suppose that \( \tilde{E} \) is of synthesis for \( A(H/K) \). If \( S \in VN(G/K) \) is supported in \( \tilde{E} \subseteq H/K \), then \( S \) annihilates \( I_{A(G/K)}(H/K) \) since \( H/K \) is of synthesis. But \( I_{A(G/K)}(H/K) \supseteq I_{A(G/K)}(\tilde{E}) \) and so \( S \) annihilates the latter ideal, completing the proof.

We now pass on to the Varopoulos algebra and its relation to the Fourier algebra in the context of spectral synthesis. In the rest of the section, we assume that \( G \) is compact.

**Lemma 3.9.** Suppose \( G \) is compact. For \( \tilde{w} \in V(G, K\setminus G) \) define \( Q_K \tilde{w}(s, \tilde{t}) = \int_G w(sx, \tilde{t} . x) \, dx \). Then \( Q_K \) is a contractive projection from \( V(G, K\setminus G) \) onto \( V_{inv}(G, K\setminus G) \). In the same way we have a contractive projection \( Q_K^* \) from \( V(K\setminus G, G) \) onto \( V_{inv}(K\setminus G, G) \).

**Proof.** This is a simple verification.

The next theorem is the analogue, adapted to our setting, of the result that \( N \) is an isometry of \( A(G) \) onto \( V_{inv}(G) \), due to Varopoulos [28] and Spronk and Turowska [26].

For convenience, we write \( t^{-1} \) in place of \( t^{-1} \) for the left coset \( t^{-1} K \).

**Theorem 3.10.** If \( K \) is a closed subgroup of a compact group \( G \), for \( \tilde{u} \in A(G/K) \) define a function \( N_K \tilde{u} \) on \( G \times K\setminus G \) by \( N_K \tilde{u}(s, \tilde{t}) = \tilde{u}(s . t^{-1}) \). Then \( N_K \tilde{u} \in V_{inv}(G, K\setminus G) \) and \( N_K \) is an isometric isomorphism of \( A(G/K) \) onto \( V_{inv}(G, K\setminus G) \).
Proof. Spronk and Turowska [26] have shown that for \( u \in A(G) \) the function \( Nu \) defined on \( G \times G \) by \( Nu(s, t) = u(st^{-1}) \) belongs to \( V(G) \) and that \( N \) gives an isometric isomorphism of \( A(G) \) onto the invariant part \( V_{inv}(G) \) of \( V(G) \). For \( \tilde{u} \in A(G/K) \) consider the corresponding function \( u \in A(G) \) that is constant on left cosets of \( K \), and observe that \( Nu(s, t) = u(st^{-1}) = u(st^{-1}k) = Nu(s, kt) \) for \( k \in K \). In other words \( Nu \in V(G, K \setminus G) \) and so \( N_K \) as in the statement of the theorem is just the restriction of \( N \) to \( A(G/K) \subseteq A(G) \) if we identify \( A(G/K) \) as the subspace \( A(G : K) \) of \( A(G) \) and it maps \( A(G/K) \) onto \( V_{inv}(G, K \setminus G) \subseteq V_{inv}(G) \). This proves the theorem. \( \square \)

For \( E \subseteq G \) and \( \tilde{E} \subseteq G/K \), let \( E^* = \{(s, t) \in G \times G: st^{-1} \in E \} \) and \( \tilde{E}^\circ = \{(s, \tilde{t}) \in G \times K \setminus G: s \cdot \tilde{t}^{-1} \in \tilde{E} \} \). Let \( 1_G \) denote the identity map on \( G \).

Lemma 3.11. Let \( G \) be compact and consider a closed set \( \tilde{E} \subseteq G/K \). Then

i) \( q^{-1}(\tilde{E})^* = (1_G \times p)^{-1}(\tilde{E}^\circ) \).
ii) \( \tilde{u} \in I_{A(G/K)}(\tilde{E}) \Leftrightarrow N_K \tilde{u} \in I_{V(G, K \setminus G)}(\tilde{E}^\circ) \).
iii) \( \tilde{u} \in J_{A(G/K)}(\tilde{E}) \Leftrightarrow N_K \tilde{u} \in J_{V(G, K \setminus G)}(\tilde{E}^\circ) \).

Proof. i) is a simple, direct verification and ii) is immediate from the definitions of \( N_K \) and \( \tilde{E}^\circ \).

To prove iii), consider the map \( \theta_K: G \times K \setminus G \rightarrow G/K \) defined by \( \theta_K(s, \tilde{t}) = s \cdot \tilde{t}^{-1} \). Then \( \theta_K \) is an open, continuous map being the composition of the open, continuous map \( G \times K \setminus G \rightarrow K \setminus G \), \( (s, \tilde{t}) \mapsto \tilde{t} \cdot s \) and the homeomorphism \( K \setminus G \rightarrow G/K, \tilde{t} \mapsto \tilde{t}^{-1} \). Moreover, \( N_K \tilde{u} = \tilde{u} \circ \theta_K \) for \( \tilde{u} \in A(G/K) \). Observe also that \( \text{supp } N_K \tilde{u} \cap \tilde{E}^\circ = \theta_K^{-1}(\text{supp } \tilde{u} \cap \tilde{E}) \). This shows that \( \tilde{u} \in J_{A(G/K)}(\tilde{E}) \) if and only if \( N_K \tilde{u} \in J_{V(G, K \setminus G)}(\tilde{E}^\circ) \). The implication \( \tilde{u} \in J_{A(G/K)}(\tilde{E}) \Rightarrow N_K \tilde{u} \in J_{V(G, K \setminus G)}(\tilde{E}^\circ) \) is now an easy consequence of the continuity of \( N_K \). For the reverse implication, suppose that \( N_K \tilde{u} \in J_{V(G, K \setminus G)}(\tilde{E}^\circ) \). Let \( \{\tilde{w}_n\} \) be a sequence in \( J_{V(G, K \setminus G)}(\tilde{E}^\circ) \) converging to \( N_K \tilde{u} \). Then, since \( \text{supp } N_K Q_K(\tilde{w}_n) \cap \tilde{E}^\circ = \theta_K^{-1}(\text{supp } Q_K \tilde{w}_n \cap \tilde{E}) \) and \( \text{supp } Q_K \tilde{w}_n \subseteq \theta_K(\text{supp } \tilde{w}_n) \), it follows that \( Q_K \tilde{w}_n \in J_{A(G/K)}(\tilde{E}) \). But then \( \tilde{u} = Q_K N_K \tilde{u} = \lim Q_K \tilde{w}_n \in J_{A(G/K)}(\tilde{E}) \). \( \square \)

Here is the promised result on the relation between sets of spectral synthesis for \( A(G/K) \) and those for \( V(G, K \setminus G) \).

Theorem 3.12. Let \( G \) be a compact group with \( K \) a closed subgroup. A closed subset \( \tilde{E} \) of \( G/K \) is a set of (weak) spectral synthesis for \( A(G/K) \) if and only if \( \tilde{E}^\circ \) is a set of (weak) spectral synthesis for \( V(G, K \setminus G) \).

Proof. One part is now almost immediate. Suppose \( \tilde{E}^\circ \) is a set of weak spectral synthesis for \( V(G, K \setminus G) \) with \( \xi_{V(G, K \setminus G)}(\tilde{E}^\circ) = n \). If \( \tilde{u} \in I_{A(G/K)}(\tilde{E}) \) then, by Lemma 3.11 ii), \( N_K \tilde{u} \in I_{V(G, K \setminus G)}(\tilde{E}^\circ) \). By assumption, \( N_K \tilde{u} \in I_{V(G, K \setminus G)}(\tilde{E}^\circ) \). So Lemma 3.11 iii) implies that \( \tilde{u} \in J_{A(G/K)}(\tilde{E}) \) and hence \( \tilde{E} \) is of weak spectral synthesis for \( A(G/K) \) with \( \xi_{A(G/K)}(\tilde{E}) \leq n \).

For the converse part, we closely follow Spronk and Turowska [26, Theorem 3.1] (who consider synthesis in the case when \( K \) is the trivial one element subgroup). Assume that \( \tilde{E} \) is a set of weak spectral synthesis for \( A(G/K) \) with \( \xi_{A(G/K)}(\tilde{E}) = n \). Let \( \tilde{w}^l \in I_{V(G, K \setminus G)}(\tilde{E}^\circ), l = 1, \ldots, n \). It is enough to prove that \( \tilde{w}^1 \tilde{w}^2 \cdots \tilde{w}^n \in J_{V(G, K \setminus G)}(\tilde{E}^\circ) \). Let \( \pi \) be an irreducible unitary representation of \( G \) with representation space \( \mathcal{B}(H_\pi) \) defined by

\[ \tilde{w}^{l, \pi}: G \times K \setminus G \rightarrow \mathcal{B}(H_\pi) \text{ defined by} \]
\[ \tilde{w}^{l\pi}(s, \tilde{t}) = \int_G \tilde{w}^l(sx, \tilde{t} . x)\pi(x) \, dx \quad \text{and} \quad \tilde{w}^{l_{\ast\pi}}(s, \tilde{t}) = \pi(s) \tilde{w}^{l\pi}(s, \tilde{t}). \]

Observe that \( \tilde{w}^{l_{\ast\pi}}(s, \tilde{t}) \) for all \( s, x \in G, \tilde{t} \in K \setminus G \).

Fixing an orthonormal basis for \( H_\pi \), we write \( u_{ij}^{\pi} \) for the coefficient functions of \( \pi \). Let \( \tilde{w}_{ij}^{\pi} = u_{ij}^{\pi} \) and similarly \( \tilde{w}_{ij}^{l_{\ast\pi}} = u_{ij}^{\pi} \) \( \tilde{w}_{ij}^{l_{\ast\pi}} = \sum_k (u_{ik}^{\pi} \otimes 1) \tilde{w}_{kj}^{l\pi} \) belong to \( I_{V(G, K \setminus G)}(\hat{E}^5) \). As observed earlier, \( \tilde{w}^{l_{\ast\pi}} \) belongs to \( V_{inv}(G, K \setminus G) \) and so does \( \tilde{w}^{l_{\ast\pi}} \). Hence, by Lemma 3.11, \( N_{K}^{-1} \tilde{w}_{ij}^{l_{\ast\pi}} \) belongs to \( I_{A(G, K \setminus G)}(\hat{E}) \) and by the hypothesis on \( \hat{E} \), we have \( N_{K}^{-1}(\tilde{w}_{ij}^{l_{\ast\pi}}) \cdots N_{K}^{-1}(\tilde{w}_{ij}^{l_{\ast\pi}}) = N_{K}^{-1}(\tilde{w}_{ij}^{l_{\ast\pi}}) \cdots \tilde{w}_{ij}^{l_{\ast\pi}} \in J_{A(G, K \setminus G)}(\hat{E}) \). So, again by Lemma 3.11, \( \tilde{w}_{ij}^{l_{\ast\pi}} \cdots \tilde{w}_{ij}^{l_{\ast\pi}} \in J_{V(G, K \setminus G)}(\hat{E}^5) \). But \( w_{ij}^{l_{\ast\pi}}(s, \tilde{t}) = \pi(s^{-1})u_{ij}^{\pi}(s, \tilde{t}) \), and so \( w_{ij}^{l_{\ast\pi}} = \sum_k \tilde{u}^{\pi}_{ik} \otimes 1 \tilde{w}_{kj}^{l\pi} \) where \( \tilde{u}(s) = u(s^{-1}) \). Thus \( w_{ij}^{l_{\ast\pi}} \cdots \tilde{w}_{ij}^{l_{\ast\pi}} \in J_{V(G, K \setminus G)}(\hat{E}^5) \). Since \( V(G, K \setminus G) \) is an essential \( L^1(G) \)-module, the proof is now completed as in [26, Theorem 3.1] using an approximate identity argument to conclude that \( \tilde{w}^{l_{\ast\pi}} \tilde{w}^{l_{\ast\pi}} \cdots \tilde{w}^{l_{\ast\pi}} \in J_{V(G, K \setminus G)}(\hat{E}^5) \) and hence that \( \xi_{V(G, K \setminus G)}(\hat{E}^5) \leq n \).

The assertions about synthesis correspond, of course, to the case \( n = 1 \). \( \square \)

4. Completely bounded multipliers

In this section we consider completely bounded multipliers of \( A(G/K) \) and obtain a complete isometry from \( V_{inv}^{\infty}(G, K \setminus G) \) to \( M_{cb}(A(G/K)) \). We heavily borrow from the results and methods of Spronk [25] where the case when \( K \) is the trivial one element subgroup is discussed. We present this result here because we need it in the next section on operator synthesis. We begin with a few lemmas the first of which is found by Spronk [25, Lemma 5.1], although it is not explicitly stated there in this form.

Lemma 4.1 (Spronk). Let \( G \) be a nondiscrete locally compact group and let \( w \in V_{inv}^{\infty}(G) \). Consider \( w \) as a function on \( G \times G \) (as a representative of its equivalence class). For \( t \in G \), let \( F_t = \{ s \in G : w(s, t^{-1}s) \neq w(e, t^{-1}) \} \). Then the set \( E_w = \{ t \in G : F_t \text{ is not locally null} \} \) is locally null.

Lemma 4.2. Let the function \( w \) and the sets \( F_t, E_w \) be as in Spronk’s lemma. Suppose \( w \) satisfies the condition \( w(s, kt) = w(s, t) \) locally almost everywhere for each \( k \in K \). Then \( q^{-1}(q(E_w)) = E_w \).

Proof. If \( F_t \) is locally null for some \( t \in G \), then \( F_{tk} \) is also locally null for \( k \in K \). For, if \( w(s, t^{-1}s) = w(e, t^{-1}) \) for locally almost all \( s \), then \( w(s, (tk)^{-1}) = w(s, k^{-1}t^{-1}) = w(s, t^{-1}) = w(e, t^{-1}) \) for locally almost all \( s \). In other words, when \( t \in G \setminus E_w \) we have \( tK \subseteq G \setminus E_w \). Thus \( E_w \) is a union of cosets. This implies that if \( t \in q^{-1}(q(E_w)) \) then \( tK = sK \subseteq E_w \) for some \( s \in E_w \), so \( t \in E_w \). The lemma follows. \( \square \)

Lemma 4.3. Let \( K \) be a compact subgroup of a nondiscrete locally compact group \( G \) and let \( \tilde{w} \in V_{inv}^{\infty}(G, K \setminus G) \). Consider \( \tilde{w} \) as a function on \( G \times K \setminus G \). Then there is a locally null subset \( \tilde{E}_{\tilde{w}} \) of \( G/K \) such that \( \tilde{w}(s, \tilde{t}^{-1}s) = \tilde{w}(e, \tilde{t}^{-1}) \) for locally almost all \( s \), whenever \( \tilde{t} \in (G/K) \setminus \tilde{E}_{\tilde{w}} \).

Proof. Consider the function \( w \) on \( G \times G \) defined by \( w(s, t) = \tilde{w}(s, \tilde{t}) \). Let \( E_{\tilde{w}} \) be as in Spronk’s lemma, so that \( w(s, t^{-1}s) = w(e, t^{-1}) \) for locally almost all \( s \) when \( t \in G \setminus E_{\tilde{w}} \). Then, by
the preceding lemma, \( \tilde{E}_w := q(E_w) \) is locally null in \( G/K \) and \( \tilde{w}(s, i^{-1} \cdot s) = w(s, t^{-1}) = \tilde{w}(e, i^{-1}) \) for locally almost all \( s \) and for \( i \in (G/K) \setminus \tilde{E}_w \). \( \square \)

We need the following result of Spronk [25, Theorem 5.2].

**Theorem 4.4** (Spronk). For each \( w \in \text{Inv}^\infty(G) \) there is a bounded continuous function \( w_G \) on \( G \) such that \( w_G(t) = w(e, t^{-1}) \) for \( t \in G \setminus E_w \) and such that the map \( w \mapsto w_G \) is a complete contraction from \( \text{Inv}^\infty(G) \) to \( M_{cb}A(G) \).

**Lemma 4.5.** If \( \tilde{w} \in \text{Inv}^\infty(G, K \setminus G) \) is a bounded function, then there is a bounded continuous function \( \tilde{w}_{G/K} \) on \( G/K \) such that \( \tilde{w}_{G/K}(i) = \tilde{w}(e, i^{-1}) \) when \( i \in (G/K) \setminus \tilde{E}_w \).

**Proof.** Let \( \tilde{w} \in \text{Inv}^\infty(G, K \setminus G) \) be a bounded function and let \( w \) be the corresponding function in \( \text{Inv}^\infty(G) \): \( w(s, t) = \tilde{w}(s, i) \). Let \( w_G \) be the continuous function on \( G \) associated to \( w \) as in Spronk’s theorem. Then \( w_G(t) = w(e, t^{-1}) = w(s, t^{-1} s) \) for locally almost all \( s \) and \( t \) in the complement of the locally null set \( E_w \). Define, as usual, the resulting bounded continuous function on \( G/K \): \( \tilde{w}_{G/K}(i) = \int_{G} w_G(tk) \, dk \). Note that, for \( i \in (G/K) \setminus \tilde{E}_w \), we have \( \tilde{w}_{G/K}(i) = w_G(t) = w(e, t^{-1}) = \tilde{w}(s, i^{-1} \cdot s) \) for locally almost all \( s \). \( \square \)

Each \( \tilde{w} = \sum \psi_i \otimes \tilde{\psi}_i \in \text{Inv}^\infty(G, K \setminus G) \) gives a weakly continuous bounded operator \( T_{\tilde{w}} \) of \( \mathcal{B}(L^2(K \setminus G), L^2(G)) \) into itself given by the weak* convergent series \( \sum_i \langle \psi_i, \tilde{w}_{G/K} \rangle \psi_i \).

We have \( T_{\tilde{w}} = T_{\tilde{w}} T_{\tilde{w}}^* \) for \( \tilde{w}, \tilde{w} \in \text{Inv}^\infty(G, K \setminus G) \).

We are now ready for the main results of this section.

**Theorem 4.6.** Let \( K \) be a compact subgroup of a second countable locally compact group \( G \).

If \( \tilde{w} \in \text{Inv}^\infty(G, K \setminus G) \) then \( \tilde{w}_{G/K} \in M_{cb}(A(G/K)) \) and the map \( \tilde{w} \mapsto \tilde{w}_{G/K} \) gives a complete contraction from \( \text{Inv}^\infty(G, K \setminus G) \) to \( M_{cb}(A(G/K)) \).

**Proof.** Spronk [25] has shown that \( m_w \), for \( w \in \text{Inv}^\infty(G) \), is a multiplier of \( A(G) \) and \( P(m_w, \omega) = w_G P \omega \) for \( \omega \in T(G) \). Here \( P : T(G) \to A(G) \) is given by \( P(f \otimes g) = \langle \lambda(.) f, \tilde{g} \rangle \).

Observing that if \( f \) is constant on right cosets of \( K \), then the right side is constant on left cosets of \( K \), we can define \( \tilde{P} : T(K \setminus G, G) \to A(G/K) \) by \( \tilde{P}(f \otimes g)(i) = P(f \otimes g) \).

Then, for \( \tilde{w} \in \text{Inv}^\infty(G, K \setminus G) \) with associated \( w \in \text{Inv}^\infty(G) \) and \( \omega \in T(K \setminus G, G) \), we have \( \tilde{P} \omega = \tilde{w}_{G/K} \).

Thus \( \tilde{w}_{G/K} \) is a multiplier of \( A(G/K) \) and \( \tilde{P} \circ m_w = m_{\tilde{w}_{G/K}} \circ \tilde{P} \).

As in Spronk [25], taking adjoints we get \( \tilde{T}_{\tilde{w}} \mid_{\text{Inv}(G)} = M_{\tilde{w}_{G/K}} := m_{\tilde{w}_{G/K}}^* \), and then observe that \( \| \tilde{w}_{G/K} \|_{cbm} \leq \| \tilde{w} \|_{\text{Inv}} \) and \( \| [\tilde{w}_{ij}] \|_{cbm} \leq ||[\tilde{w}_{ij}]||_{\text{Inv}} \) for \( [\tilde{w}_{ij}] \in M_n(\text{Inv}^\infty(G, K \setminus G)) \) to finish the proof. \( \square \)

With the modifications needed, the proof of the next theorem faithfully follows Spronk [25]. If \( \tilde{u} \) is a function on \( G/K, S \subseteq G \) and \( p(S) = \tilde{S} \subseteq K \setminus G \), we write \( \tilde{u}^{\tilde{S}} \) for the function on \( G \times K \setminus G \) defined by \( \tilde{u}^{\tilde{S}}(s, i) = \tilde{u}(s \cdot i^{-1}) \) for \( s \in S, i \in \tilde{S} \). We let \( \text{b}_1(X) \) stand for the unit ball in a Banach space \( X \). For unexplained terminology and notation that we need in our next theorem, we refer to the paper of Spronk [25].

**Theorem 4.7.** Let \( G \) be a second countable locally compact group and let \( K \) be a compact subgroup. For continuous functions \( \tilde{u}_{ij} : G/K \to \mathbb{C}, i, j = 1, 2, \ldots, n \), the following are equivalent:
(i) \([\tilde{u}_{ij}] \in b_1(M_n(M_{cb}A(G/K))).\)
(ii) If \(F\) is a finite subset of \(G\) then \([\tilde{u}_{ij}^F] \in b_1(M_n(V^\infty(F, \widetilde{F}))).\)
(iii) \([\tilde{u}^{KG}_{ij}] \in b_1(M_n(V^\infty_{inv}(G_d, (K \setminus G)_d))).\)
(iv) \([\tilde{u}^{KG}_{ij}] \in b_1(M_n(V^\infty_{inv}(G, K \setminus G))).\)

In particular, the map \(\tilde{u} \mapsto \tilde{u}^{KG}\) is a complete isometry from \(M_{cb}(A(G/K))\) onto \(V^\infty_{inv}(G, K \setminus G)).\)

**Proof.** This proof is essentially an unsymmetrized version of the proof of Theorem 5.3 of Spronk [25] and so we omit some of the details. See Pisier’s theorem and remarks following it in Section 2.

Let \(F = \{s_1, s_2, \ldots, s_p\}, \widetilde{F} = \{\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_q\}\) and, for \(k = 1, 2, \ldots, p,\) and \(l = 1, 2, \ldots, q,\) let \(\{e_{kl}: k = 1, 2, \ldots, p, l = 1, 2, \ldots, q\}\) be the standard matrix units for \(M_{pq} = M_{pq}(\mathbb{C})\). Let

\[
E_{kl} = \lambda(s_k \tilde{s}_l^{-1}) \otimes M_{pq} \cong M_{pq}(VN(G/K)).
\]

Then span\(\{E_{kl}: k = 1, 2, \ldots, p, l = 1, 2, \ldots, q\}\)\(\cong M_{pq}\) via \(E_{kl} \mapsto e_{kl}\). Let us make the identification \(M_{pq} \cong \mathcal{B}(l^2(\widetilde{F}), l^2(F))\). Let, for \(w \in V^\infty(F, \widetilde{F})\), \(S_w e_{kl} = w(s_k, \tilde{s}_l)e_{kl}\), for each \(k, l\), where \(S_v\) is the Schur multiplier corresponding to \(w\). If \([w_{ij}] \in M_n(V^\infty(F, \widetilde{F}))\), let \(S_{[w_{ij}]} x = [S_{w_{ij}} x]\) for \(x \in \mathcal{B}(l^2(\widetilde{F}), l^2(F))\). Then, we have

\[
M_{[w_{ij}]} E_{kl} \cong [M_{w_{ij}} \lambda(s_k \tilde{s}_l^{-1}) \otimes e_{kl}]
\]

\[
= [u_{ij}(s_k \tilde{s}_l^{-1}) \lambda(s_k \tilde{s}_l^{-1}) \otimes e_{kl}]
\]

\[
\cong [u_{ij}(s_k \tilde{s}_l^{-1})]
\]

\[
\cong [u_{ij}^F(s_k, \tilde{s}_l)e_{kl}]
\]

\[
S_{[u_{ij}]} e_{kl}.
\]

So \([u_{ij}^F] \in M_n(V^\infty(F, \widetilde{F})).\) Moreover,

\[
\|u_{ij}^F\|_{V^\infty} = \|S_{u_{ij}^F}\|_{cb} = \|M_{[u_{ij}]}|\text{span}\{E_{kl}\}|_{cb} \leq \|u_{ij}\|_{cb} \leq 1
\]

proving that (i) implies (ii).

Next, let us prove that (iii) is a consequence of (ii). From the definition of \([\tilde{u}^{KG}_{ij}]\), it is clear that \([\tilde{u}^{KG}_{ij}] \in M_n(V^\infty(G_d, (K \setminus G)_d))).\) We have to show that it is in the unit ball. For each finite subset \(F \subseteq G\) we have a complete isometric embedding \(V^\infty(F, \widetilde{F}) \subseteq V^\infty(G_d, (K \setminus G)_d)).\) The net \((\{[\tilde{u}^{KG}_{ij}]\}): F \subseteq G\) is in the unit ball of \(M_n(V^\infty(G_d, (K \setminus G)_d))).\) A dual Banach space, has a weak* limit point. But since \(\bigcup F l^1(F) \otimes l^1(K \setminus G)_d\) is dense in \(l^1(G_d) \otimes l^1((K \setminus G)_d))\) it follows that \([\tilde{u}^{KG}_{ij}]\) is such a limit point and (iii) is proved.

Assuming (iii) we shall now prove (iv). As in Spronk, for some index set \(I,\) the given functions \([\tilde{u}^{KG}_{ij}]\) can be written as a matricial product \(A \otimes B,\) where \(A = [\varphi_{i}] \in M_n,I(l^\infty(G_d))\) and \(B = [\psi_{ij}] \in M_{I,n}(l^\infty((K \setminus G)_d))).\) With \(\|A\|\|B\| \leq 1\) so that

\[
\tilde{u}_{ij}^{KG} = \sum_{i \in I} \varphi_{i} \otimes \psi_{ij}.
\]
Then one constructs continuous functions $\tilde{\psi}_i$ and $\tilde{\psi}_{ij}$ such that

$$\left[\tilde{u}_{ij}^K\right] = \tilde{A} \circ \tilde{B}$$

where $\tilde{A} = [\tilde{\psi}_i]$, $\tilde{B} = [\tilde{\psi}_{ij}]$ and $\|\tilde{A}\| \leq \|A\|$ and $\|\tilde{B}\| \leq \|B\|$. It follows that $\left[\tilde{u}_{ij}^K\right] \in b_1(M_w(V_{inv}^\infty(G, K\setminus G)))$.

To conclude, note that if we write $\tilde{w} = \tilde{u}_{ij}^K$, then with the notation of the previous theorem, we have $\tilde{w}_{G/K} = \tilde{u}_{ij}$. Thus (i) follows from (iv) by Theorem 4.6. The proof is complete. \( \square \)

**Remark 4.8.** If, for a function $\tilde{u}$ on $G/K$, we write (as in Theorem 3.10) $N_{K} \tilde{u}$ for the function on $G \times K \setminus G$ given by $N_{K} \tilde{u}(s, \tilde{t}) = \tilde{u}(s \cdot \tilde{t}^{-1})$, then observe that $\tilde{u}^K_{G\setminus G} = N_{K} \tilde{u}$. Thus Theorem 4.7 says that $N_{K}$ is a complete isometry of $M_{cb}(A(G/K))$ onto $V_{inv}^\infty(G, K\setminus G)$. In the next section, it will be convenient to also consider the flip of $N_{K} \tilde{u}$. It is the function $N_{K}^{\dagger} \tilde{u}$ on $K \setminus G \times G$ defined by $N_{K}^{\dagger} \tilde{u}(\tilde{t}, s) = N_{K} \tilde{u}(s, \tilde{t})$. So $N_{K}^{\dagger}$ identifies $M_{cb}(A(G/K))$ with $V_{inv}^\infty(K\setminus G, G)$.

5. **Operator synthesis**

On noncompact groups, Ludwig and Turowska [14] have obtained relations between sets of spectral synthesis for $A(G)$ and sets of operator synthesis. We show how their methods can be extended to discuss these problems in the context of quotients. One of the fruits of our investigations is the inverse projection theorem for sets of local synthesis on nonabelian groups, apparently unknown in full generality hitherto. We begin by isolating some technical results in the form of lemmas.

For an operator $S \in \mathcal{B}(L^2(K\setminus G), L^2(G))$, the **operator support**, $\text{supp}_{op} S$, is defined as the set of ordered pairs $(\tilde{t}, s) \in K\setminus G \times G$ such that, for arbitrary neighbourhoods $\tilde{V}$ of $\tilde{t}$ and $U$ of $s$, there are $\tilde{f} \in L^2(K\setminus G)$ and $g \in L^2(G)$ with $\text{supp} \tilde{f} \subset \tilde{V}$, $\text{supp} g \subset U$, and $\langle S \tilde{f}, g \rangle \neq 0$. Here $\text{supp} g = \{x \in G : g(x) \neq 0\}$ and similarly for $\tilde{f}$.

If $\tilde{E} \subseteq G/K$, define $\tilde{E}^\sharp = \{(\tilde{t}, s) \in K\setminus G \times G : s \cdot \tilde{t}^{-1} \in \tilde{E}\}$.

**Lemma 5.1.** $\text{supp}_{op} S \subseteq (\text{supp} S)^\sharp$ for $S \in VN(G/K) \subseteq \mathcal{B}(L^2(K\setminus G), L^2(G))$.

**Proof.** Suppose $(\tilde{t}, s) \in \text{supp}_{op} S$, but $s \cdot \tilde{t}^{-1}$ does not belong to $\text{supp} S$. Choose neighbourhoods $U$ of $s$ in $G$ and $\tilde{V}$ of $\tilde{t}$ in $K\setminus G$ such that $U \tilde{V}^{-1} \cap W = \emptyset$ where $W$ is a neighbourhood of $\text{supp} S$. Now take $g \in L^2(G)$ and $\tilde{f} \in L^2(K\setminus G)$ such that $g$ is supported in $U$, $\tilde{f}$ is supported in $\tilde{V}$ and $\langle S \tilde{f}, g \rangle \neq 0$. On the other hand, letting $u = \langle \lambda(.) \tilde{f}, g \rangle$, we have that $\text{supp} u \subseteq U \tilde{V}^{-1}$ is disjoint from $\text{supp} S$, so $u \in j(\text{supp} S)$ and $\langle S \tilde{f}, g \rangle = \langle S, u \rangle = 0$. This contradiction proves the lemma. \( \square \)

**Lemma 5.2.** If $u \in I_{A(G/K)}^\epsilon(E)$, then i) $T_{N_{K}u}$ is weakly continuous and ii) $u \cdot S = T_{N_{K}u}(S)$ for $S \in VN(G/K)$.

**Proof.** To prove i), it is to be shown that if $S_k \rightarrow 0$ weakly, then so does $T_{N_{K}u}(S_k)$. So assume $S_k \rightarrow 0$ weakly and $\|S_k\| \leq C$ for all $k$. If $N_{K}u = \sum \varphi_i \otimes \tilde{\psi}_i$, then $T_{N_{K}u}(S_k) = \sum M_{\varphi_i} S_k M_{\tilde{\psi}_i}$ and

$$\left\|T_{N_{K}u}(S_k) \tilde{f}, g\right\| \leq \sum_{i=1}^{n} \left|\langle S_k \tilde{\psi}_i \tilde{f}, \varphi_i g \rangle\right| + \sum_{n+1}^{\infty} \left|\langle S_k \tilde{\psi}_i \tilde{f}, \varphi_i g \rangle\right|$$
\[
\begin{align*}
\| & S_k \| \leq n \left| \langle S_k \tilde{\psi}_i, \tilde{f}, \tilde{\phi}_i \rangle \right| + \| S_k \| \left( \sum_{n+1}^{\infty} \| \tilde{\psi}_i \tilde{f} \|_2^2 \right)^{\frac{1}{2}} \left( \sum_{n+1}^{\infty} \| \tilde{\phi}_i g \|_2^2 \right)^{\frac{1}{2}} \\
\| & S_k \| \leq n \left| \langle S_k \tilde{\psi}_i, \tilde{f}, \tilde{\phi}_i \rangle \right| + C \left( \sum_{n+1}^{\infty} \| \tilde{\psi}_i \tilde{f} \|_2^2 \right)^{\frac{1}{2}} \left( \sum_{n+1}^{\infty} \| \tilde{\phi}_i g \|_2^2 \right)^{\frac{1}{2}}
\end{align*}
\]

for \( \tilde{f} \in L^2(K \setminus G) \) and \( g \in L^2(G) \). For a fixed, sufficiently large \( n \), each sum in the second term, being a tail of a convergent series, can be made arbitrarily small and then, by weak convergence of \( S_k \), the first sum is small for large enough \( k \). In other words, \( T_{Ku}(S_k) \to 0 \) weakly and i) is proved.

To prove ii), observe that

\[
T_{Ku}(\lambda(\hat{s})) \tilde{f}(t) = \sum_{i=1}^{\infty} M_{\psi_i} \lambda(\hat{s}) M_{\tilde{\psi}_i} \tilde{f}(t) = N_{Ku}(t, \hat{s}^{-1} \cdot t) \tilde{f}(\hat{s}^{-1} \cdot t) = u(\hat{s}) \lambda(\hat{s}) \tilde{f}(t)
\]

for any \( \tilde{f} \in L^2(K \setminus G) \) and \( s \in G \). This suffices to prove ii) by i) since \( VN(G/K) \) is the weak closed span of the regular representation \( \lambda \).

Before we come to our results on the relations between spectral synthesis and operator synthesis, we state the following result of Shulman and Turowska for convenience of reference. \( \mathcal{B}(L^2(K \setminus G), L^2(G)) \) has a \( V^\infty(K \setminus G, G) \)-module structure given as follows: For \( \tilde{w} = \sum \tilde{\psi}_i \otimes \psi_i \in V^\infty(K \setminus G, G) \) and \( S \in \mathcal{B}(L^2(K \setminus G), L^2(G)) \) define \( \tilde{w} \cdot S = \sum M_{\psi_i} SM_{\tilde{\psi}_i} \).

**Theorem 5.3 (Shulman–Turowska).** For a closed set \( F \subseteq K \setminus G \times G \), the following are equivalent:

i) \( F \) is \( m_{K \setminus G \times G} \)-synthetic.

ii) \( \langle S, \omega \rangle = 0 \) for any \( S \in \mathcal{B}(L^2(K \setminus G), L^2(G)) \), \( \text{supp}_{op} S \subseteq F \) and \( \omega \in \Phi(F) \).

iii) \( \tilde{w} \cdot S = 0 \) for any \( S \in \mathcal{B}(L^2(K \setminus G), L^2(G)) \) with \( \text{supp}_{op} S \subseteq F \) and \( \tilde{w} \) in \( V^\infty(K \setminus G, G) \) that vanishes on \( F \).

**Proof.** This is immediate from the results of Shulman and Turowska [23, Theorem 4.6] and [24, Proposition 5.3].

**Theorem 5.4.** Let \( K \) be a compact subgroup of a second countable locally compact group \( G \) and let \( \tilde{E} \subseteq G / K \) be a closed subset. If \( \tilde{E}^z \) is a set of operator synthesis with respect to \( m_{K \setminus G} \times m_G \) then \( \tilde{E} \) is a set of local spectral synthesis.

**Proof.** Suppose that \( \tilde{E}^z \) is \( m_{K \setminus G} \times m_G \)-synthetic. Let \( \tilde{u} \in I^c_{A(G/K)}(\tilde{E}) \) and \( S \in VN(G/K) \), \( \text{supp} S \subseteq \tilde{E} \). Then \( N_K \tilde{u} \in V^\infty(G, K \setminus G) \) by Theorem 4.7 and \( N_K \tilde{u} \in V^\infty(K \setminus G, G) \) vanishes on \( \tilde{E}^z \). Lemma 5.1 shows that \( \text{supp}_{op} S \subseteq (\text{supp} S)^z \subseteq \tilde{E}^z \). Therefore, for each \( \tilde{w} \in T(K \setminus G, G) \),

\[
\langle T_{N_K \tilde{u}}(S), w \rangle = \langle S, N_K \tilde{u} \cdot w \rangle = 0
\]
and, by Lemma 5.2, $$\tilde{u} \cdot S = T_{\mathbb{C}K} \tilde{u}(S) = 0$$. Since $$A(G/K)$$ is regular, there exists a compactly supported function $$\tilde{v}$$ in $$A(G/K)$$, such that $$\tilde{v} = 1$$ on $$\text{supp} \tilde{u}$$. Hence

\[
(S, \tilde{u}) = (S, \tilde{v} \tilde{u}) = (\tilde{u} \cdot S, \tilde{v}) = 0.
\]

This proves the theorem. □

**Corollary 5.5.** Let $$K$$ be a compact subgroup of a second countable locally compact group $$G$$. Then singleton sets are of spectral synthesis for $$A(G/K)$$.

**Proof.** It suffices to consider the singleton set $$\{\dot{i}\} = \{K\}$$. A simple verification shows that

\[
\{K\}^1 = \{ (i, s) \in K \backslash G \times G : f(s) = g(i) \}
\]

where $$f : K \backslash G \to K \backslash G$$, $$f(i) = i$$ and $$g : G \to K \backslash G$$, $$g(s) = s$$. Hence $$\{K\}^1$$ is of operator synthesis with respect to $$m_{K \backslash G} \times m_G$$, by [23, Theorem 4.8]. Thus, by Theorem 5.3, singleton $$\{K\}$$ is of local synthesis and hence is of synthesis for $$A(G/K)$$. □

**Corollary 5.6.** Let $$K$$ be a compact subgroup of a second countable locally compact group $$G$$ such that $$A(G/K)$$ has an approximate identity (not necessarily bounded). Let $$\tilde{E} \subseteq G/K$$ be closed. If $$\tilde{E}^\sharp$$ is of operator synthesis with respect to $$m_{K \backslash G} \times m_G$$, then $$\tilde{E}$$ is of spectral synthesis for $$A(G/K)$$.

**Proof.** If $$A(G/K)$$ has an approximate identity, then any set of local spectral synthesis is a set of spectral synthesis. Hence the corollary is an immediate consequence of the theorem. □

**Lemma 5.7.** Suppose $$\tilde{E}$$ is a set of local spectral synthesis for $$A(G/K)$$. Then $$N_{\mathbb{C}K}^\dagger \tilde{u} \cdot T = 0$$ whenever $$\tilde{u} \in M_{cb}(A(G/K))$$ vanishes on $$\tilde{E}$$ and the operator $$T \in \mathcal{B}(L^2(K \backslash G), L^2(G))$$ is supported in $$\tilde{E}^\sharp$$.

**Proof.** Let $$\tilde{u} \in M_{cb}(A(G/K))$$ vanish on $$\tilde{E}$$. If $$\tilde{v} \in A(G/K)$$ has compact support, then $$\tilde{u} \tilde{v} \in I_{A(G/K)}^{\mathbb{C}}(\tilde{E})$$. Since $$\tilde{E}$$ is a set of local spectral synthesis, this implies $$\tilde{u} \tilde{v} = \lim \tilde{u}_n$$ with $$\tilde{u}_n \in j_{A(G/K)}(\tilde{E})$$ and therefore $$N_{\mathbb{C}K}^\dagger (\tilde{u} \tilde{v})$$ is a $$V^\infty(K \backslash G, G)$$ limit of $$N_{\mathbb{C}K}^\dagger \tilde{u} \in \Phi(\tilde{E}^\sharp)$$. Now [23, Theorem 4.3] implies that operators in $$\mathcal{B}(L^2(K \backslash G), L^2(G))$$ which are supported in $$\tilde{E}^\sharp$$ are precisely those which annihilate $$\Phi(\tilde{E}^\sharp)$$. Thus for any such operator $$T$$, we have $$0 = \langle T, N_{\mathbb{C}K}^\dagger \tilde{u}_n \cdot \omega \rangle = \langle N_{\mathbb{C}K}^\dagger \tilde{u} \cdot T, \omega \rangle = \langle N_{\mathbb{C}K}^\dagger (\tilde{u} \tilde{v}) \cdot T, \omega \rangle = 0$$ for all $$\omega$$ in $$T(K \backslash G, G)$$. This means that the operator $$N_{\mathbb{C}K}^\dagger \tilde{u} \cdot T$$ vanishes on the subspace generated by the set $$\{N_{\mathbb{C}K}^\dagger \tilde{v} \omega : \tilde{v} \in A^{\mathbb{C}}(G/K), \omega \in T(K \backslash G, G)\}$$. This subspace is clearly invariant in the sense of [23, Theorem 2.1] and has empty zero set and hence, by the Shulman–Turowska Wiener Tauberian theorem [23, Corollary 4.3], is dense. It follows that $$N_{\mathbb{C}K}^\dagger \tilde{u} \cdot T = 0$$ and the lemma is proved. □

Let $$\pi$$ be an irreducible unitary representation of $$G$$ and let $$u_{ij}^\pi$$ be the matrix coefficients of $$\pi$$ with respect to a fixed orthonormal basis $$\{e_i\}$$ for the representation space $$H_\pi$$. For $$\omega \in T(K \backslash G, G)$$, the operator valued function $$\omega^\pi(\tilde{i}, s) = \int_G \omega(\tilde{i} \cdot x, s x) \pi(x) dx$$ is well defined, as observed below. Let $$\tilde{\omega}^\pi(\tilde{i}, s) = \pi(s) \omega^\pi(\tilde{i}, s)$$. Define $$\omega_{ij}^\pi = u_{ij}^\pi \cdot \omega^\pi$$ and $$\tilde{\omega}_{ij}^\pi = u_{ij}^\pi \cdot \tilde{\omega}^\pi$$.

We indicate two proofs of the next lemma, the first adapting arguments from Ludwig and Turowska [14] and the second using results from the same paper.
Lemma 5.8. With notation as above, $\omega^\pi$ is well defined and both $\omega_{ij}^\pi$ and $\tilde{\omega}_{ij}^\pi$ belong to $V^\infty(K \setminus G, G)$.

Proof. The proof for the case when $K$ is the trivial one element subgroup contained in the proof of Theorem 4.11 of Ludwig and Turowska [14] can be adapted to the present case. That $\omega^\pi$ is well defined is a consequence of the Schwarz inequality. When $\omega = f \otimes g$ is an elementary tensor, computation shows that $\tilde{\omega}_{ij}^\pi = N^\dagger_K (f^\pi u_{ij}^\pi \ast \tilde{g})$, so $\tilde{\omega}_{ij}^\pi \in V^\infty(K \setminus G, G)$ by Theorem 4.7. Moreover, the linear map $\omega \mapsto \tilde{\omega}_{ij}^\pi$ defined on elementary tensors extends to a bounded linear operator from $T(K \setminus G, G)$ to $V^\infty(K \setminus G, G)$ and so $\tilde{\omega}_{ij}^{(\pi)} \in V^\infty(K \setminus G, G)$ for any $\omega \in T(K \setminus G, G)$.

Again, for $T \in \mathcal{B}(L^2(K \setminus G, L^2(G))$ and $\omega \in T(K \setminus G, G)$, one shows that the operator $\omega_{ij}^\pi \cdot T$ is well defined and satisfies $\|\omega_{ij}^\pi \cdot T\| \leq \|T\| \|\omega\|_{T(K \setminus G, G)}$. Further $T \mapsto w_{ij}^\pi \cdot T$ is a bounded left $L^\infty(G)$ and right $L^\infty(K \setminus G)$ module map on $\mathcal{B}(L^2(K \setminus G, L^2(G))$. Hence the theorem of Smith [22, Theorem 2.1] applies to yield that this map is completely bounded. Now the Blecher–Smith result [2, Theorem 4.2] can be applied to conclude that the map is of the form $w \cdot T$. For $w \in V^\infty(K \setminus G, G)$ and therefore $w_{ij}^\pi = w$ m.a.e. The lemma is proved.

Alternately, an easier way is to use the result of Ludwig–Turowska and argue as follows. Considering $\omega \in T(K \setminus G, G)$ as a function in $T(G)$ satisfying $\omega(kt, s) = \omega(t, s)$, $s, t \in G$, $k \in K$, it follows from the proof of Theorem 4.11 of Ludwig and Turowska [14], that $\omega^\pi$ is well defined and both $\omega_{ij}^\pi$ and $\tilde{\omega}_{ij}^\pi$ belong to $V^\infty(G)$. Thus it suffices to show that $\omega_{ij}^\pi$ and $\tilde{\omega}_{ij}^\pi$ are both right $G$-invariant in the first variable. But since $\omega$ satisfies this invariance, it is clear that $\omega^\pi$ and hence $\omega_{ij}^\pi$ and $\tilde{\omega}_{ij}^\pi$ also satisfy the same invariance. ⧫

Analogous to what is given in [14], we consider the $L^1(G)$ action on $T(K \setminus G, G)$ as follows: for $f \in L^1(G)$ and $\omega \in T(K \setminus G, G)$ $f \cdot \omega(\tilde{t}, s) = \int_G f(x) \Delta^{\frac{1}{2}}(x) \omega(\tilde{t} \cdot x, sx) \, dx$ where $\Delta$ is the modular function of the group. We have $\|f \cdot \omega\|_{T(K \setminus G, G)} \leq \|f\|_1 \|\omega\|_{T(K \setminus G, G)}$ and $T(K \setminus G, G)$ becomes a Banach $L^1(G)$-module. Thus, by Cohen’s factorisation theorem, any bounded approximate identity of $L^1(G)$ acts as an approximate identity for $T(K \setminus G, G)$. If $f$ is compactly supported, we also consider $f \cdot \omega(\tilde{t}, s) = \int_G f(x) \omega(\tilde{t} \cdot x, sx) \, dx$.

Here is a converse of Theorem 5.3. We again follow Ludwig and Turowska [14].

Theorem 5.9. Let $G$ be a second countable locally compact group and let $K$ be a compact subgroup. If a closed subset $\mathcal{E}$ of $G/K$ is a set of local synthesis for $A(G/K)$, then $\mathcal{E}^\pi$ is operator synthetic with respect to $m_{K \setminus G} \times m_G$.

Proof. It suffices, by Theorem 5.3, to prove that $w \cdot T = 0$ for $w \in V^\infty(K \setminus G, G)$ and $T \in \mathcal{B}(L^2(K \setminus G, L^2(G))$ with $w$ vanishing on $\mathcal{E}^{\pi}$ and $T$ supported in $\mathcal{E}^{\pi}$. Writing $G$ as the union of an increasing sequence $\{C_n\}$ of compact sets, we have $K \setminus G$ as the union of the increasing sequence $p(C_n)$ of compact sets. Then $M_{K \setminus G} \tau M_{p(C_n)}^{\pi}$ converges to $T$ strongly. Hence it is sufficient to consider the case when both $w$ and $T$ are compactly supported.

Thus let $w \in V^\infty(K \setminus G, G)$ have compact support and vanish on $\mathcal{E}^{\pi}$ and let $T \in \mathcal{B}(L^2(K \setminus G, L^2(G))$ have compact support.

With notation as earlier, it is easy to check, using the left invariance of the integral, that $\tilde{w}_{ij}^{\pi}(\tilde{t} \cdot x, sx) = \tilde{w}_{ij}^{\pi}(\tilde{t}, s)$ implying $\tilde{w}_{ij}^{\pi} \in V^\infty_{\text{inv}}(K \setminus G, G)$. Therefore, by Theorem 4.7,

$$\tilde{w}_{ij}^{\pi} = N^\dagger_{K \setminus G} \tilde{u}$$
for some $\tilde{u} \in M_{cb}A(G/K)$. Moreover, if $w$ vanishes on $\tilde{E}^\circ$, then $\tilde{u}$ vanishes on $E$. Lemma 5.7 now gives that $\tilde{w}_{ij}^\pi \cdot T = 0$ and consequently

$$w_{ij}^\pi \cdot T = \left( \sum_k u_{kj}^\pi \tilde{w}_{ik}^\pi \right) \cdot T = 0.$$ 

Choose a compact set $C$ of positive measure such that both $w$ and $T$ are supported in $p(C) \times C$. Let $g \in L^\infty(G)$, $\tilde{f} \in L^\infty(K\setminus G)$ with $\text{supp } g \subseteq C$, $\text{supp } \tilde{f} \subseteq p(C)$ and take $\omega = \tilde{f} \otimes g$. Write $h = u_{ij}^\pi \chi_{CC^{-1}}$. 

$$\chi_{p(C)}(\tilde{f}) \chi_C(s) \int_G w(\tilde{f} \cdot x, sx) h(x) \, dx = \int_G w(\tilde{f} \cdot x, sx) \chi_{CC^{-1}}(x) h(x) \, dx = h \chi_{CC^{-1}} \cdot w = h \cdot w.$$ 

Since $T = M_{\chi_C} T M_{\chi_{p(C)}}$ it follows that

$$\langle \omega \cdot T, h \cdot w \rangle = \langle \omega \cdot M_{\chi_C} T M_{\chi_{p(C)}}, h \cdot w \rangle = \langle \omega \cdot T, h \cdot w \rangle = \langle \omega \cdot T, u_{ij}^\pi \chi_{CC^{-1}} \cdot w \rangle = \langle T, w_{ij}^\pi \omega \rangle = \langle w_{ij}^\pi \cdot T, \omega \rangle = 0.$$ 

It follows that for any finite linear combination $\sum c_i u_i$ of matrix coefficients

$$\langle \omega \cdot T, \sum c_i u_i \chi_{CC^{-1}} \cdot w \rangle = 0.$$ 

Choose an approximate identity $\{e_\alpha\}$ for $L^1(G)$ consisting of nonnegative continuous functions with compact support contained in $CC^{-1}$. We know that any continuous function on $G$ can be approximated by linear combinations of pure positive definite functions [3, 13.6.5]. In particular, $\Delta_{1/2} e_\alpha$ can be approximated in $L^1(G)$ by finite linear combinations $\sum c_i u_i \chi_{CC^{-1}}$. It follows that $\langle \omega \cdot T, e_\alpha \bullet w \rangle = 0$ and hence $0 = \langle \omega \cdot T, w \rangle = \langle w \cdot T, \omega \rangle$. Therefore, we finally obtain $w \cdot T = 0$ and the proof is complete. 

Lemma 5.10. Let $\tilde{E} \subseteq G/K$ be closed. If $\tilde{E}^\circ$ is of operator synthesis with respect to $m_K \setminus G \times m_G$ then $q^{-1}(\tilde{E})^\circ$ is of operator synthesis with respect to $m_G \times m_G$. 

Proof. In view of Lemma 3.11 i), $q^{-1}(\tilde{E})^\circ = (p \times 1_G)^{-1}(\tilde{E})^\circ$. Hence the current lemma is an immediate consequence of the inverse image theorem of Shulman and Turowska [23, Theorem 4.7] applied to the maps $p : G \to K \setminus G$ and $1_G : G \to G$. 

In contrast to only partial results available, we have a complete inverse projection theorem for sets of local spectral synthesis. For the inverse projection theorem for sets of local synthesis for quotients by a normal subgroup, see Lohoué [13]; see also Derighetti [4].
Theorem 5.11 (Inverse projection theorem for sets of local synthesis). A closed set $\widehat{E} \subseteq G/K$ is a set of local synthesis for $A(G/K)$ if and only if $q^{-1}(\widehat{E})$ is a set of local synthesis for $A(G)$.

Proof. One part is easy. Suppose that $q^{-1}(\widehat{E})$ is of local synthesis for $A(G)$ and let $\tilde{u} \in I_{A(G/K)}(\widehat{E})$. Then $u = \tilde{u} \circ q \in I_{A(G)}(q^{-1}(\widehat{E}))$. Since $q^{-1}(\widehat{E})$ is of local synthesis, $u \in J_{A(G)}(q^{-1}(\widehat{E}))$. Let $(u_n)$ be a sequence in $J_{A(G)}(q^{-1}(\widehat{E}))$ converging to $u$ with each $u_n$ vanishing in an open set $U_n \supset q^{-1}(\widehat{E})$. The canonical map $q : G \to G/K$ is open. Thus the corresponding $\tilde{u}_n$ defined by $\tilde{u}_n(t) = \int_K u_n(kt) \, dk$ vanishes on the open set $q(U_n) \supset \widehat{E}$ and converges to $\tilde{u}$. In other words, $\tilde{u} \in J_{A(G/K)}(\widehat{E})$ and so $\widehat{E}$ is of local synthesis for $A(G/K)$.

Conversely, suppose $\widehat{E}$ is of local synthesis for $A(G/K)$. Then, by Theorem 5.9, $\widehat{E}^\sharp$ is operator synthetic with respect to $m_{K \setminus G} \times m_G$. By Lemma 5.10, $q^{-1}(\widehat{E})^*$ is of operator synthesis with respect to $m_G \times m_G$ and hence $q^{-1}(\widehat{E})$ is of local synthesis, by Theorem 5.4 (with $K$ the trivial subgroup) or Theorem 4.3 of [14].

Remark 5.12. The work of Forrest, Samei and Spronk [8] has been brought to our attention and it has been pointed out that our inverse projection theorem generalises a result of theirs.

The following special case of the Takesaki–Tatsuuma result on synthesis of closed subgroups is a consequence of our theorem.

Corollary 5.13. Any compact subgroup $K$ of $G$ is a set of spectral synthesis for $A(G)$.

Proof. The singleton set $\{K\}$ is of spectral synthesis for $A(G/K)$, by Corollary 5.5 and therefore Theorem 5.11 now gives that $q^{-1}(\{K\}) = K$ is of local spectral synthesis for $A(G)$. Since $K$ is compact, this implies that $K$ is of spectral synthesis for $A(G)$.

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References


