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Completely Bounded Module Maps and the Haagerup Tensor Product

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We obtain a representation theorem for module maps defined on the algebra $K(H)$ of compact operators. This is used to show that the Haagerup tensor product of two operator algebras enjoys strong versions of Tomita's commutant theorem and the slice map property. We also give a general theorem concerning the automatic complete boundedness of module maps. © 1991 Academic Press, Inc.

1. INTRODUCTION

If $\mathscr{A}$ and $\mathscr{B}$ are algebras of operators in $B(H)$ and $\mathscr{E}$ is a left-$\mathscr{A}$ right-$\mathscr{B}$ submodule of $B(H)$ then $\phi\colon \mathscr{E} \to B(H)$ is called a module map if $\phi(aeb) = a\phi(e)b$ for $a \in \mathscr{A}$, $b \in \mathscr{B}$, $e \in \mathscr{E}$. Such maps arise naturally in various contexts and so it is important to determine their structure. The most satisfactory answers have been obtained when restricting to the class of completely bounded module maps, and so we seek conditions on $\mathscr{A}$ and $\mathscr{B}$ which imply that module maps are automatically completely bounded. This has been featured in several recent papers and we summarize the results below in chronological order.

A module map $\phi\colon \mathscr{E} \to B(H)$ is completely bounded and $\|\phi\| = \|\phi\|_{cb}$ in the following circumstances:

(i) $\mathscr{E} = B(H)$, ϕ is normal, $\mathscr{A} = \mathscr{B} = \mathscr{M}$, where $\mathscr{M}$ is a von Neumann algebra whose commutant $\mathscr{M}'$ has the property that all normal states are vector states (Haagerup [8]).

(ii) Same conditions as (i) except that ϕ need not be normal (Effros and Kishimoto [7]).

(iii) H is finite dimensional and $\mathscr{A} = \mathscr{B} = \mathscr{D}$, the algebra of diagonal operators in $B(H)$ (Paulsen, Power, and Smith [14]).

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(iv) $\mathcal{A}$ and $\mathcal{B}$ are maximal abelian C^*-subalgebras of $B(H)$ (Davidson and Power [4]).

If H is finite dimensional then, of course, all linear maps on $B(H)$ are completely bounded. The object of (iii) was to give a short proof of $\|\phi\| = \|\phi\|_{cb}$ for a special case of (i).

The aim of the first part of this paper is to present a theorem which includes the cases cited above, and which has a simpler proof, based on a technique developed by Christensen [2]. The second part focuses on a structure theorem for completely bounded module maps. Haagerup [8] has obtained the representation

$$\phi(x) = \sum a_i' x b_i'$$

in the case where $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, ϕ is normal, and the elements a_i' and b_i' lie in the commutants of $\mathcal{A}$ and $\mathcal{B}$. The extension to non-normal maps was discussed in [7]. In the third section we obtain Haagerup's representation theorem, but with no restrictions on $\mathcal{A}$ and $\mathcal{B}$. This more general version is needed for applications to cohomology theory which we outline at the end of the section. In the last part of the paper we investigate the Haagerup tensor product $B(H) \otimes_h B(H)$. The main results are Corollaries 4.6–4.8 where we show that this algebra satisfies strong versions of Tomita's commutant theorem for von Neumann algebras and the slice map property. These results usually fail in other tensor products, making the Haagerup tensor product special in this regard.

For background material the reader should consult the book by Paulsen [13] and the survey paper by Christensen and Sinclair [3]. Taken together they provide a comprehensive overview of the theory of complete boundedness.

2. Automatic Complete Boundedness

Throughout this section $\mathcal{A}$ and $\mathcal{B}$ will be unital C^*-subalgebras of $B(H)$ and $\mathcal{E}$ will be a norm closed subspace of $B(H)$ which is both a left-$\mathcal{A}$ module and a right-$\mathcal{B}$ module. We consider module maps $\phi: \mathcal{E} \to B(H)$ which satisfy

$$\phi(aeb) = a\phi(e)b$$

for $a \in \mathcal{A}$, $e \in \mathcal{E}$, $b \in \mathcal{B}$. The main result is the following:

THEOREM 2.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ have cyclic vectors η and ξ, respectively, and let $\phi: \mathcal{E} \to B(H)$ be a bounded module map. Then ϕ is completely bounded and $\|\phi\|_{cb} = \|\phi\|$.*

Proof. We assume without loss of generality that $\|\phi\| = 1$ and, to reach a contradiction, we assume that for some integer n the norm of $\phi_n\colon M_n(\mathscr{E}) \to M_n(B(H))$ exceeds one. Then there exists an element $(e_{ij}) \in M_n(\mathscr{E})$ of unit norm such that $\|(\phi(e_{ij}))\| > 1$. Then vectors

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

may be chosen from the open unit ball of $H \oplus \cdots \oplus H$ such that

$$\left| \left\langle (\phi(e_{ij})) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \right\rangle \right| > 1. \tag{2.1}$$

Since $\mathscr{A}$ and $\mathscr{B}$ have cyclic vectors we may choose elements $a_i \in \mathscr{A}$, $b_i \in \mathscr{B}$ such that $\|a_i\eta - \eta_i\|$ and $\|b_i\xi - \xi_i\|$ are so small that

$$\left\| \begin{pmatrix} a_1\eta \\ \vdots \\ a_n\eta \end{pmatrix} \right\|, \left\| \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix} \right\| < 1$$

and

$$\left| \left\langle (\phi(e_{ij})) \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix}, \begin{pmatrix} a_1\eta \\ \vdots \\ a_n\eta \end{pmatrix} \right\rangle \right| > 1. \tag{2.2}$$

We will assume temporarily that $a = \sum a_i^* a_i$ and $b = \sum b_i^* b_i$ are invertible elements, and remove this restriction at the end of the proof.

Let $\tilde{\eta} = a^{1/2}\eta$, $\tilde{\xi} = b^{1/2}\xi$, $c_i = a_i a^{-1/2}$, and $d_i = b_i b^{-1/2}$. Then, by definition,

$$c_i\tilde{\eta} = a_i\eta \qquad \text{and} \qquad d_i\tilde{\xi} = b_i\xi,$$

and (2.2) may be rewritten as

$$\left| \left\langle \sum_{ij} \phi(c_i^* e_{ij} d_j)\tilde{\xi}, \tilde{\eta} \right\rangle \right| > 1, \tag{2.3}$$

using the module properties of ϕ. Now

$$\|\tilde{\xi}\|^2 = \langle b^{1/2}\xi, b^{1/2}\xi \rangle = \langle b\xi, \xi \rangle = \sum_i \langle b_i\xi, b_i\xi \rangle = \left\| \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix} \right\|^2 < 1,$$

and a similar calculation shows that $\|\tilde{\eta}\| < 1$. The element $\sum_{ij} c_i^* e_{ij} d_j$ may be expressed as the matrix product

$$(c_1^*, ..., c_n^*)(e_{ij}) \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

and so has norm at most one since $\sum c_i^* c_i = \sum d_i^* d_i = 1$. It follows from (2.3) that $\|\phi\| > 1$ and the desired contradiction is reached.

A modification is necessary if either $\sum a_i^* a_i$ or $\sum b_i^* b_i$ fails to be invertible. Replace $(e_{ij}) \in M_n(\mathscr{E})$ by $(e_{ij}) \oplus 0 \in M_{n+1}(\mathscr{E})$ and replace the vectors

$$\begin{pmatrix} a_1\eta \\ \vdots \\ a_n\eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix}$$

by

$$\begin{pmatrix} a_1\eta \\ \vdots \\ a_n\eta \\ \varepsilon\eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \\ \varepsilon\xi \end{pmatrix},$$

respectively, for some sufficiently small $\varepsilon > 0$. The new vectors will still have norms less than one, and the proof proceeds exactly as above.

Remark 2.2. The most convenient formulation of Theorem 2.1 is in terms of cyclic vectors, but it is clear from the proof that the result remains valid if the following weaker hypothesis is substituted:

> Given two finite dimensional subspaces K_1 and K_2 of H, there exist vectors $\eta, \xi \in H$ such that $K_1 \subseteq \overline{\mathscr{A}\eta}$ and $K_2 \subseteq \overline{\mathscr{B}\xi}$ (norm closures). (2.4)

In order to relate Theorem 2.1 to previous results on module maps we require the following lemma, which is close to [5, p. 223].

LEMMA 2.3. *Let $\mathscr{M} \subseteq B(H)$ be a von Neumann algebra with commutant $\mathscr{M}'$, and consider the following conditions:*

(i) *Every normal state of $\mathscr{M}'$ is a vector state,*

(ii) *Given a finite dimensional subspace K of H, there exists a vector $\xi \in H$ such that $K \subseteq \overline{\mathscr{M}\xi}$.*

Then (i) *implies* (ii).

Proof. Suppose that (i) is satisfied and let K be a finite dimensional subspace of H with an orthonormal basis $\{\xi_1, ..., \xi_n\}$. Define a normal state by

$$\omega(m') = n^{-1} \sum \langle m'\xi_i, \xi_i \rangle, \qquad m' \in \mathcal{M}',$$

and let ξ be a unit vector which implements ω. The space $\overline{\mathcal{M}\xi}$ is an invariant subspace for $\mathcal{M}$ and so the associated projection p lies in $\mathcal{M}'$. Then

$$1 = \langle p\xi, \xi \rangle = \omega(p) = n^{-1} \sum \langle p\xi_i, \xi_i \rangle$$

and so $\langle p\xi_i, \xi_i \rangle = 1$ for $1 \leqslant i \leqslant n$. Thus $\xi_i \in \overline{\mathcal{M}\xi}$ and so $K \subseteq \overline{\mathcal{M}\xi}$.

In view of Lemma 2.3 and Remark 2.2 it is clear that Theorem 2.1 includes the previously known cases. We conclude this section by showing that this C^*-algebra theorem cannot be deduced from von Neumann algebra results. The details are easy and are left to the reader.

EXAMPLE 2.4. Let $H = L^2[0, 1]$ and let $\mathcal{A} = \mathcal{B} = C[0, 1]$, acting as an algebra of multiplication operators on H. Let μ denote Lebesgue measure on $[0, 1]$. There exists a measurable set Ω such that for any open interval I,

$$0 < \mu(\Omega \cap I) < \mu(I)$$

[16]. Let $u \in L^\infty[0, 1]$ be the self-adjoint unitary $1 - 2\chi_\Omega$. Then $\mathcal{A} \cap \mathcal{A}u = 0$ and $\mathcal{E} = \mathcal{A} + \mathcal{A}u$ is a closed $\mathcal{A}$-bimodule in $B(H)$. The map $\phi: \mathcal{E} \to B(H)$ defined by $\phi(f + gu) = f$ is bounded. Then ϕ is a bounded $\mathcal{A}$-bimodule map and, since $\mathcal{A}$ has a cyclic vector, is also completely bounded by Theorem 2.1. It thus extends to an $\mathcal{A}$-bimodule map $\psi: B(H) \to B(H)$ [17, 20]. Since $\psi(u) = \phi(u) = 0$, while $u\psi(1) = u\phi(1) = u$, it is clear that ψ is a $C[0, 1]$-bimodule map, but not an $L^\infty[0, 1]$-bimodule map.

3. THE STRUCTURE OF MODULE MAPS

We begin this section by considering the structure of left-$\mathcal{A}$ right-$\mathcal{B}$ completely bounded module maps $\phi: K(H) \to B(H)$ where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary norm closed unital subalgebras of $B(H)$. For notational simplicity we will restrict to separable Hilbert spaces although the results remain true in general with arbitrary index sets replacing the integers. We will have many occasions to consider infinite sums of the form $\sum s_i k t_i$, and we remark that for any fixed $k \in K(H)$ the convergence is always in the strong operator topology. When referring to closed subspaces or the closures of subspaces we will always mean in the norm topology. The theorem to be proved is the following:

THEOREM 3.1. *If* $\phi: K(H) \to B(H)$ *is a completely bounded left-*$\mathscr{A}$ *right-*$\mathscr{B}$ *module map then there exist sequences* $\{s_i\}$ *and* $\{t_i\}$ *in the commutants* $\mathscr{A}'$ *and* $\mathscr{B}'$, *respectively, such that* $\sum_i s_i s_i^*$, $\sum_i t_i^* t_i \in B(H)$, $\|\sum_i s_i s_i^*\| \, \|\sum_i t_i^* t_i\| = \|\phi\|_{cb}^2$ *and, for all* $k \in K(H)$,

$$\phi(k) = \sum_i s_i k t_i.$$

In the case where $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras this result was established by Haagerup in [8]. There the theorem was formulated in terms of normal maps on $B(H)$, which is equivalent to our situation. The von Neumann algebra case was further refined in [6, 7] but these proofs do not appear to generalize, and we need the full result for the applications we have in mind.

If H is a Hilbert space then H^n, $1 \leqslant n \leqslant \infty$, will denote the n-fold direct sum of copies of H, while if $x \in B(H)$ then x^n will denote the n-fold direct sum of copies of x, acting as an operator on H^n. We will have no occasion to consider powers of an operator and so no confusion should arise. An operator $t \in B(H, H^\infty)$ may be viewed as a column matrix of operators $t_i \in B(H)$ where $\sum_i t_i^* t_i \in B(H)$, and if $\lambda = \{\lambda_1, \lambda_2, ...\} \in l_2$ then $\lambda \cdot t$ will mean the operator $\sum_i \lambda_i t_i \in B(H)$, where the sum converges in the norm topology. In the same way an operator $s \in B(H^\infty, H)$ may be viewed as a row matrix of operators $s_i \in B(H)$ where $\sum_i s_i s_i^* \in B(H)$. It will be convenient to make the following definition.

DEFINITION 3.2. *Let* $\mathscr{W}$ *be a norm closed subspace of* $B(H)$. *A set of operators* $\{t_i\}$, $t_i \in B(H)$, $\sum_i t_i^* t_i \in B(H)$, *is said to be strongly independent over* $\mathscr{W}$ *if* $\lambda = 0$ *whenever* $\lambda \in l_2$ *and* $\lambda \cdot t \in \mathscr{W}$. *If* $\mathscr{W}$ *is the zero subspace then we will say simply that the set* $\{t_i\}$ *is strongly independent.*

LEMMA 3.3. *Suppose that* $s \in B(H^\infty, H)$, $t \in B(H, H^\infty)$, $\{t_i\}$ *is strongly independent over* $\mathscr{B}'$, *and*

$$sk^\infty tb - sk^\infty b^\infty t = 0 \tag{3.1}$$

for $k \in K(H)$, $b \in \mathscr{B}$. *Then* $s = 0$.

Proof. Let $\{\xi_i\}_{i=1}^4$ be arbitrary vectors in H. If k is the rank one operator $\xi_1 \otimes \xi_2$ then (3.1) becomes

$$\sum_i \langle (t_i b - b t_i)\xi_3, \xi_2 \rangle \langle s_i \xi_1, \xi_4 \rangle = 0 \tag{3.2}$$

for all $b \in \mathscr{B}$. Thus $\sum_i \langle s_i \xi_1, \xi_4 \rangle t_i \in \mathscr{B}'$ (letting ξ_2 and ξ_3 vary over H), and so $\langle s_i \xi_1, \xi_4 \rangle = 0$ by the strong independence of $\{t_i\}$ over $\mathscr{B}'$. The two vectors ξ_1 and ξ_4 were arbitrary and so $s = 0$.

Proof of Theorem 3.1. The representation theorem for completely bounded maps, obtained independently by Haagerup [8] and Paulsen [12], allows us to write such a map $\phi: K(H) \to B(H)$ in the form

$$\phi(k) = x\pi(k)y, \qquad k \in K(H)$$

where π is a representation of $K(H)$ on some Hilbert space H_1, $y: H \to H_1$, $x: H_1 \to H$, and $\|x\| \, \|y\| = \|\phi\|_{cb}$. All irreducible representations of $K(H)$ are unitarily equivalent to the identity representation and so in this case ϕ has the form

$$\phi(k) = xk^{\infty}y, \qquad k \in K(H),$$

where $x \in B(H^{\infty}, H)$ and $y \in B(H, H^{\infty})$ [8]. Under the assumption that ϕ is a left-$\mathscr{A}$ right-$\mathscr{B}$ module map, we first show that y may be chosen to have components in $\mathscr{B}'$.

Decompose l_2 as the orthogonal sum $L_1 \oplus L_2 \oplus L_3$ where $L_1 = \{\boldsymbol{\lambda} \in l_2 : \boldsymbol{\lambda} \cdot y = 0\}$, L_2 is the orthogonal complement of L_1 in $\{\boldsymbol{\lambda} \in l_2 : \boldsymbol{\lambda} \cdot y \in \mathscr{B}'\}$, and L_3 is the orthogonal complement of $L_1 \oplus L_2$ in l_2. The case in which all three subspaces are infinite dimensional is typical. Form an orthonormal basis $\{\boldsymbol{\alpha}_i\}$ for l_2 so that $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_7, ...\}$, $\{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_5, \boldsymbol{\alpha}_8, ...\}$, and $\{\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_6, \boldsymbol{\alpha}_9, ...\}$ are bases for L_1, L_2, and L_3, respectively. Let u be the unitary matrix whose ith row is $\boldsymbol{\alpha}_i$ and observe that u^* commutes with k^{∞}. Thus

$$\begin{aligned} \phi(k) &= xk^{\infty}y = xk^{\infty}u^*uy = xu^*k^{\infty}uy \\ &= \tilde{x}k^{\infty}\tilde{y}, \end{aligned}$$

where $\tilde{x} = xu^*$ and $\tilde{y} = uy$.

Since $\tilde{y}_i = \boldsymbol{\alpha}_i \cdot y$ it is clear from the construction that $\tilde{y}_{3i-2} = 0$ and $\tilde{y}_{3i-1} \in \mathscr{B}'$ for $i \geq 1$. We claim that $\{\tilde{y}_{3i}\}$ is strongly independent over $\mathscr{B}'$ and that $\{\tilde{y}_{3i-1}\}$ is strongly independent. The methods are the same in both cases and so we look only at the first. Suppose that $\sum_i |\lambda_i|^2 < \infty$ and $\sum_i \lambda_i \tilde{y}_{3i} \in \mathscr{B}'$. Then $(\sum_i \lambda_i \boldsymbol{\alpha}_{3i}) \cdot y \in \mathscr{B}'$ and so $\sum_i \lambda_i \boldsymbol{\alpha}_{3i} \in L_1 \oplus L_2$, forcing $\lambda_i = 0$ since $\{\boldsymbol{\alpha}_{3i}\}$ is a basis for L_3.

Now ϕ is a right-$\mathscr{B}$ module map, and so

$$\phi(k)b - \phi(kb) = 0, \qquad \text{for} \quad k \in K(H), \quad b \in \mathscr{B}.$$

It follows that

$$\sum_i \tilde{x}_i k(\tilde{y}_i b - b\tilde{y}_i) = 0 \tag{3.3}$$

which reduces to

$$\sum_i \tilde{x}_{3i} k(\tilde{y}_{3i} b - b\tilde{y}_{3i}) = 0 \tag{3.4}$$

since $\tilde{y}_{3i-2} = 0$ and $\tilde{y}_{3i-1} \in \mathscr{B}'$. We have already shown that $\{\tilde{y}_{3i}\}$ is strongly independent over $\mathscr{B}'$ and so Lemma 3.3 allows us to conclude that $\tilde{x}_{3i} = 0$. Thus $\phi(k) = \sum_i \tilde{x}_{3i-1} k \tilde{y}_{3i-1}$, which has the form

$$\phi(k) = sk^{\infty} t, \tag{3.5}$$

where the components of t lie in $\mathscr{B}'$ and are strongly independent.

We now wish to show that the components of s lie in $\mathscr{A}'$. We claim that the span of vectors of the form $k^{\infty} t\xi$, $k \in K(H)$, $\xi \in H$, is dense in H^{∞}. If not, there exists a non-zero vector $(\xi_i) \in H^{\infty}$ such that

$$\sum_i \langle kt_i \xi, \xi_i \rangle = 0, \qquad k \in K(H), \quad \xi \in H. \tag{3.6}$$

Without loss of generality suppose that $\xi_1 \neq 0$, and let $k = \xi_1 \otimes \eta$ where $\eta \in H$ is arbitrary. Then (3.6) becomes

$$\sum_i \langle t_i \xi, \eta \rangle \langle \xi_1, \xi_i \rangle = 0, \qquad \xi, \eta \in H \tag{3.7}$$

and so

$$\sum_i \langle \xi_1, \xi_i \rangle t_i = 0.$$

Since $\{\langle \xi_1, \xi_i \rangle\}$ is a non-zero element of l_2, we have contradicted the strong independence of $\{t_i\}$.

The left-$\mathscr{A}$ module property of ϕ may be expressed as

$$(as - sa^{\infty}) k^{\infty} t\xi = 0, \qquad a \in \mathscr{A}, \quad k \in K(H), \quad \xi \in H$$

and so, from above, the operator $as - sa^{\infty}$ annihilates H^{∞}. It follows from this that $s_i \in \mathscr{A}'$. Since s and t are submatrices of xu^* and uy, respectively, it is clear that $\|s\| \, \|t\| = \|\phi\|_{cb}$ and the proof is complete.

Remark 3.4. As a consequence of Theorem 3.1, a completely bounded left-$\mathscr{A}$ right-$\mathscr{B}$ module map $\phi: K(H) \to B(H)$ is automatically a left-$\mathscr{A}''$ right-$\mathscr{B}''$ module map. If $\mathscr{A}$ and $\mathscr{B}$ are not C^*-algebras then this is perhaps surprising, since in general $\mathscr{A}''$ and $\mathscr{B}''$ can be much larger than $\mathscr{A}$ and $\mathscr{B}$ even if $\mathscr{A}$ and $\mathscr{B}$ are σ-weakly closed (see Example 3.5 below).

It was shown in [17, 19, 20] that if $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras, $\mathscr{E}$ is a left-$\mathscr{A}$ right-$\mathscr{B}$ module in $B(H)$, and $\phi: \mathscr{E} \to B(H)$ is a completely bounded module map then ϕ has a completely bounded module extension on $B(H)$. As we now show, no such extension theorem is possible in general.

EXAMPLE 3.5. Let $\mathscr{A}_6 \subseteq M_6$ be the algebra of matrices

$$\begin{pmatrix} * & * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

where stars denote arbitrary entries. Let u denote the unitary matrix $\left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right)$ where I_3 is the identity on $\mathbb{C}^3$. It is easy to check that $\mathscr{A}_6$ and $\mathscr{A}_6 u$ are disjoint subspaces of M_6. Let $\mathscr{E} = \mathscr{A}_6 \oplus \mathscr{A}_6 u$ and define $\phi\colon \mathscr{E} \to M_6$ by

$$\phi(a+bu) = a, \qquad a, b \in \mathscr{A}_6.$$

Then ϕ is a left-$\mathscr{A}_6$ right-$\mathbb{C}$ module map, and it of course completely bounded since we are working on a finite dimensional Hilbert space. If ϕ had a module map extension ψ then it would necessarily have the form

$$\psi(x) = xt,$$

by Theorem 3.1, since $\mathscr{A}_6' = \mathbb{C}$. But then

$$0 = \psi(u) = ut,$$

and so $t = 0$, an impossibility.

Our interest in module maps stems from some joint work with F. L. Gilfeather on the cohomology of operator algebras. This will appear elsewhere and so we only briefly indicate the connection. If $\mathscr{A}$ and $\mathscr{B}$ are algebras of operators in $B(H)$ then a new algebra $\{\left(\begin{smallmatrix} b & 0 \\ t & a \end{smallmatrix}\right) : a \in \mathscr{A},\ b \in \mathscr{B},\ t \in B(H)\}$ may be formed. If $\phi\colon B(H) \to B(H)$ is a left-$\mathscr{A}$ right-$\mathscr{B}$ module map then

$$\begin{pmatrix} b & 0 \\ t & a \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \phi(t) & 0 \end{pmatrix}$$

is a derivation. It very often happens that such maps are automatically normal and completely bounded using Theorem 2.1, and so the first cohomology group may be determined using Theorem 3.1.

4. Properties of the Haagerup Tensor Product

If $\mathscr{E}$ and $\mathscr{F}$ are subspaces of $B(H)$ then the algebraic tensor product $\mathscr{E} \odot \mathscr{F}$ may be given a norm in the following manner. If $v = \sum_{i=1}^{n} e_i \otimes f_i$ is an element of $\mathscr{E} \odot \mathscr{F}$ then

$$\|v\|_h = \inf \left\| \sum_i e_i e_i^* \right\|^{1/2} \left\| \sum_i f_i^* f_i \right\|^{1/2},$$

where the infimum is taken over all representations of v as a finite sum of elementary tensors. This norm was introduced in [7] where it was called the Haagerup norm. It was shown in [15] that if $\mathscr{E} \subseteq \mathscr{E}_1$ and $\mathscr{F} \subseteq \mathscr{F}_1$ then the Haagerup norm of an element in $\mathscr{E} \odot \mathscr{F}$ coincides with its norm in $\mathscr{E}_1 \odot \mathscr{F}_1$. Thus $\mathscr{E} \otimes_h \mathscr{F}$, the completion of $\mathscr{E} \odot \mathscr{F}$, may be regarded as a subspace of $B(H) \otimes_h B(H)$. It was observed in [1] that $B(H) \otimes_h B(H)$ is a Banach algebra under any one of the four multiplications $(a \otimes b)(c \otimes d) = ac \otimes bd$, $ac \otimes db$, $ca \otimes bd$, or $ca \otimes db$, and our results are valid in all cases, although we choose to work with the first. Thus if $\mathscr{A}$ and $\mathscr{B}$ are subalgebras of $B(H)$ then $\mathscr{A} \otimes_h \mathscr{B}$ is a Banach subalgebra of $B(H) \otimes_h B(H)$. We wish to investigate some properties of this algebra, but must first establish some notation and four technical results which lead to Theorem 4.5.

If $\{e_i\}$ is a set of operators in $B(H)$ then $[e_i]$ will denote the closed linear span of these elements. If $\psi \in B(H)^*$ then the right slice map $R_\psi : B(H) \otimes_h B(H) \to B(H)$ is defined on sums of elementary tensors by

$$R_\psi \left(\sum_i a_i \otimes b_i \right) = \sum_i \psi(a_i) b_i$$

and the left slice map L_ψ is defined similarly by applying ψ to the elements b_i. Since linear functionals are completely bounded [13], it follows from the definition of the Haagerup norm that R_ψ and L_ψ are bounded maps and so extend uniquely to $B(H) \otimes_h B(H)$. Given subspaces $\mathscr{E}_1 \subseteq \mathscr{E}_2$ and $\mathscr{F}_1 \subseteq \mathscr{F}_2$ the Fubini product $F(\mathscr{E}_1, \mathscr{F}_1; \mathscr{E}_2 \otimes_h \mathscr{F}_2)$ of $\mathscr{E}_1$ and $\mathscr{F}_1$ relative to $\mathscr{E}_2 \otimes_h \mathscr{F}_2$ is defined to be

$$\{v \in \mathscr{E}_2 \otimes_h \mathscr{F}_2 : R_\psi(v) \in \mathscr{F}_1,\ L_\psi(v) \in \mathscr{E}_1 \text{ for all } \psi \in B(H)^*\}.$$

In the case $\mathscr{E}_2 = \mathscr{F}_2 = B(H)$ we will write $F(\mathscr{E}_1, \mathscr{F}_1)$ for $F(\mathscr{E}_1, \mathscr{F}_1; B(H) \otimes_h B(H))$. Such Fubini products have been considered for other tensor norms [10, 18]. It is clear from the definition that $F(\mathscr{E}_1, \mathscr{F}_1; \mathscr{E}_2 \otimes_h \mathscr{F}_2)$ must contain $\mathscr{E}_1 \otimes_h \mathscr{F}_1$, and we will establish equality below, a property not enjoyed by other tensor products.

If $v \in \mathscr{E} \odot \mathscr{F}$ then we may associate a map $\phi_v \in CB(K(H))$, the space of completely bounded maps on $K(H)$ in the cb-norm, by

$$\phi_v(k) = \sum_{i=1}^{n} e_i k f_i, \qquad k \in K(H),$$

where $v = \sum_{i=1}^{n} e_i \otimes f_i$. When v has norm less than one there is a representation $v = \sum_{i=1}^{m} \tilde{e}_i \otimes \tilde{f}_i$ where $\|\sum \tilde{e}_i \tilde{e}_i^*\| < 1$ and $\|\sum \tilde{f}_i^* \tilde{f}_i\| < 1$, by defintion. Writing $\tilde{e}$ and $\tilde{f}$ for the row and column matrices $(\tilde{e}_1, ..., \tilde{e}_m)$ and

$$\begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_m \end{pmatrix},$$

we may represent ϕ_v as the map $k \to \tilde{e}k^m\tilde{f}$, from which it is clear that $\|\phi_v\|_{cb} < 1$. Thus $v \to \phi_v$ is a contractive map on $\mathscr{E} \odot \mathscr{F}$ and so has a unique extension to a contractive map of $\mathscr{E} \otimes_h \mathscr{F}$ into $CB(K(H))$. If e and f are respectively row and column matrices with components e_i and f_i then maps in $CB(K(H))$ will be written $ek^\infty f$ or $\sum e_i k f_i$ as convenient. As in Section 3 we will assume that H is separable for notational simplicity, although the theorems are valid in general.

All results in this section depend on the following:

LEMMA 4.1. *Let $s \in B(H^\infty, H)$ and $t \in B(H, H^\infty)$ have components $s_i, t_i \in B(H)$, respectively, and let $\mathscr{W}$ be a closed subspace of $B(H)$. Then there exist unitaries u_1, $u_2 \in B(l_2)$ and disjoint decompositions $M_1 \cup M_2 \cup M_3$, $N_1 \cup N_2 \cup N_3$ of $\mathbb{N}$ such that the components $\tilde{s}_i$ and $\tilde{t}_i$ of $\tilde{s} = su_2$ and $\tilde{t} = u_1 t$ satisfy*

(i) *$\tilde{s}_i = 0$ for $i \in M_1$, $\tilde{t}_i = 0$ for $i \in N_1$,*

(ii) *$\tilde{s}_i \in \mathscr{W} \cap [s_i]$ for $i \in M_2$, $\tilde{t}_i \in \mathscr{W} \cap [t_i]$ for $i \in N_2$, and $\{\tilde{s}_i\}_{i \in M_2}$, $\{\tilde{t}_i\}_{i \in N_2}$ are strongly independent,*

(iii) *$\tilde{s}_i \in [s_i]$ for $i \in M_3$, $\tilde{t}_i \in [t_i]$ for $i \in N_3$, and $\{\tilde{s}_i\}_{i \in M_3}$, $\{\tilde{t}_i\}_{i \in N_3}$ are strongly independent over $\mathscr{W}$,*

(iv) *$\|\tilde{s}\| = \|s\|$ and $\|\tilde{t}\| = \|t\|$,*

(v) *If $\mathscr{W}$ is finite dimensional then M_2 and N_2 are finite sets.*

Proof. The fourth part is clear since u_1 and u_2 are unitaries. We will only consider the case of column matrices. Once this is established we may apply it to $s^* \in B(H, H^\infty)$ and the subspace $\mathscr{W}^*$ to obtain the result for row matrices.

Decompose l_2 as an orthogonal sum $L_1 \oplus L_2 \oplus L_3$ where $L_1 = \{\lambda \in l_2 : \lambda \cdot t = 0\}$, L_2 is the orthogonal complement of L_1 in

$\{\boldsymbol{\lambda} \in l_2 : \boldsymbol{\lambda} \cdot t \in \mathscr{W}\}$, and L_3 is the orthogonal complement of $L_1 \oplus L_2$ in l_2. Amalgamate orthonormal bases for L_1, L_2, and L_3 into an orthonormal basis $\{\boldsymbol{\alpha}_i\}$ for l_2. Amalgamate orthonormal bases for L_1, L_2, and L_3 into an orthonormal basis $\{\boldsymbol{\alpha}_i\}$ for l_2. Then there exists a decomposition $\mathbb{N} = N_1 \cup N_2 \cup N_3$ so that $\{\alpha_i\}_{i \in N_r}$ is a basis for L_r, $1 \leqslant r \leqslant 3$. Let u_1 be the unitary matrix whose ith row is $\boldsymbol{\alpha}_i$ and write $\tilde{t} = u_1 t$. Then it is clear that the components $\tilde{t}_i$ lie in $[t_i]$ and their remaining properties (i)–(iii) are verified by following the proof of Theorem 3.1.

Let $\mathscr{W}$ be finite dimensional with dimension j. Suppose that the cardinality of N_2 exceeds j, and choose integers $i_1, ..., i_{j+1}$ in N_2. Then $\boldsymbol{\alpha}_{i_r} \cdot t \in \mathscr{W}$ for $1 \leqslant r \leqslant j+1$ and so there must exist a non-trivial linear dependence

$$\sum_{r=1}^{j+1} \lambda_r \boldsymbol{\alpha}_{i_r} \cdot t = 0.$$

By definition $\sum_{r=1}^{j+1} \lambda_r \boldsymbol{\alpha}_{i_r} \in L_1$, contradicting the disjointness of L_1 and L_2. Thus N_2 has at most j elements and (v) is verified.

COROLLARY 4.2. *If* $s \in B(H^\infty, H)$, $t \in B(H, H^\infty)$, $e \in (H^n, H)$, *and* $f \in B(H, H^n)$ *satisfy*

$$\sum_{i=1}^{\infty} s_i k t_i - \sum_{i=1}^{n} e_i k f_i = 0, \qquad k \in K(H)$$

and $\|s\|$, $\|t\| \leqslant 1$, *then there exist* $\tilde{s} \in B(H^m, H)$ *and* $\tilde{t} \in B(H, H^m)$ *such that*

$$\sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i - \sum_{i=1}^{n} e_i k f_i = 0,$$

$\|\tilde{s}\|$, $\|\tilde{t}\| \leqslant 1$ *and* $\tilde{s}_i \in [e_i] \cap [s_i]$, $\tilde{t}_i \in [f_i] \cap [t_i]$.

Proof. Apply Lemma 4.1 to t with $\mathscr{W}$ chosen to be the finite dimensional space $[f_i]$. Then there exists a unitary matrix u and three sets of integers N_1, N_2, and N_3 such that the components of $t' = ut$ satisfy

(a) $t'_i = 0$, $i \in N_1$

(b) $t'_i \in [f_i] \cap [t_i]$, $i \in N_2$, a finite set of cardinality m,

(c) $\{t'_i\}_{i \in N_2}$ are linearly independent and $\{t'_i\}_{i \in N_3}$ are strongly independent over $[f_i]$.

Put $s' = su^*$ and observe that

$$sk^\infty t = sk^\infty u^* ut = su^* k^\infty ut = s' k^\infty t'$$

from which it follows that

$$\sum_{i=1}^{\infty} s'_i k t'_i - \sum_{i=1}^{n} e_i k f_i = 0. \tag{4.1}$$

This reduces, by (a), to

$$\sum_{i\in N_2} s_i' k t_i' + \sum_{i\in N_3} s_i' k t_i' - \sum_{i=1}^{n} e_i k f_i = 0. \tag{4.2}$$

Let $\xi_1, ..., \xi_4$ be arbitrary vectors in H, put $k = \xi_1 \otimes \xi_2$ in (4.2), and take the inner product to obtain

$$\sum_{i\in N_2\cup N_3} \langle t_i'\xi_3, \xi_2\rangle\langle s_i'\xi_1, \xi_4\rangle - \sum_{i=1}^{n} \langle f_i\xi_3, \xi_2\rangle\langle e_i\xi_1, \xi_4\rangle = 0. \tag{4.3}$$

Letting ξ_3 and ξ_2 vary, this becomes

$$\sum_{i\in N_3} \langle s_i'\xi_1, \xi_4\rangle t_i' = \sum_{i=1}^{n} \langle e_i\xi_1, \xi_4\rangle f_i - \sum_{i\in N_2} \langle s_i'\xi_1, \xi_4\rangle t_i' \in [f_i] \tag{4.4}$$

and so $\langle s_i'\xi_1, \xi_4\rangle = 0$ for $i \in N_3$ by the strong independence of $\{t_i'\}_{i\in N_3}$ over $[f_i]$. The two vectors ξ_1 and ξ_4 were arbitrary and so $s_i' = 0$ for $i \in N_3$. Thus (4.2) becomes

$$\sum_{i\in N_2} s_i' k t_i' - \sum_{i=1}^{n} e_i k f_i = 0. \tag{4.5}$$

Let $\{i_1, ..., i_m\}$ be the integers in N_2 and define $\tilde{s}_j = s_{i_j}'$ and $\tilde{t}_j = s_{i_j}'$. Then $\tilde{s} \in B(H^m, H)$, $\tilde{t} \in B(H, H^m)$, and

$$\sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i - \sum_{i=1}^{n} e_i k f_i = 0. \tag{4.6}$$

It is clear that $\|\tilde{s}\| \leqslant \|su^*\| = \|s\| \leqslant 1$ and a similar inequality holds for $\tilde{t}$. It only remains to show that $\tilde{s}_i \in [e_i]$ since clearly $\tilde{s}_i \in [s_i]$. The set $\{\tilde{t}_i\}$ is linearly independent in $[f_i]$ by (c), and so extends to a basis $\{\tilde{t}_i\}_{i=1}^{r}$ for $[f_i]$. After expressing each f_i in terms of this basis (4.6) may be rewritten

$$\sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i - \sum_{i=1}^{r} \tilde{e}_i k \tilde{t}_i = 0, \tag{4.7}$$

where $\tilde{e}_i \in [e_i]$. Thus

$$\sum_{i=1}^{m} (\tilde{s}_i - \tilde{e}_i) k \tilde{t}_i - \sum_{i=m+1}^{r} \tilde{e}_i k \tilde{t}_i = 0$$

from which it follows that $\tilde{s}_i = \tilde{e}_i \in [e_i]$, as above. This completes the proof.

The following theorem allows us to study the Haagerup tensor product in terms of completely bounded maps. We have been unable to provide a reference which includes a proof, although the result is mentioned in [11]. (We thank Professor Vern Paulsen for drawing our attention to Mathieu's work on elementary operators and their generalizations.) Christensen and Sinclair [3] point out that the case of a finite dimensional Hilbert space may be deduced from a result in [9]. A closely related characterization of the Haagerup norm was given in [15], and Blecher [1] has also obtained an isometric representation of $\mathscr{A} \otimes_h \mathscr{B}$, but on a different Hilbert space. None of these seem to imply the general case and, since it is an easy consequence of our previous work, we include a proof.

THEOREM 4.3. *The map $v \to \phi_v$ from $B(H) \otimes_h B(H)$ into $CB(K(H))$ is an isometry.*

Proof. We remarked in the introduction to this section that $\|\phi_v\|_{cb} \leqslant \|v\|_h$ and so we need only establish the reverse inequality. Let $v = \sum_{i=1}^n e_i \otimes f_i \in B(H) \odot B(H)$ and suppose that $\|\phi_v\|_{cb} = 1$. Then, as in the third section, there exist $s \in B(H^\infty, H)$, $t \in B(H, H^\infty)$, both of unit norm, such that

$$\phi_v(k) = sk^\infty t, \qquad k \in K(H)$$

or equivalently

$$\sum_{i=1}^{\infty} s_i k t_i - \sum_{i=1}^{n} e_i k f_i = 0.$$

By Corollary 4.2, there exist $\tilde{s} \in B(H^m, H)$ and $\tilde{t} \in B(H, H^m)$ satisfying $\|\tilde{s}\|, \|\tilde{t}\| \leqslant 1$ such that

$$\sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i - \sum_{i=1}^{n} e_i k f_i = 0.$$

From this it is clear that $v = \sum_{i=1}^m \tilde{s}_i \otimes \tilde{t}_i$ and that $\|v\|_h \leqslant \|\tilde{s}\| \, \|\tilde{t}\| \leqslant 1$.

We will also require a result which may be viewed as an asymptotic version of Corollary 4.2. We identify $H^\infty \oplus H^\infty$ and H^∞, and regard H^n as a subspace of H^∞.

LEMMA 4.4. *Suppose that operators $s, c \in B(H^\infty, H)$, $t, d \in B(H, H^\infty)$, $e \in B(H^n, H)$, and $f \in B(H, H^n)$ satisfy*

$$sk^\infty t + ck^\infty d - ek^\infty f = 0, \qquad k \in K(H) \tag{4.8}$$

and the norm inequalities

$$\|s\|, \|t\| \leqslant 1, \qquad \|c\|, \|d\| \leqslant \varepsilon < 1.$$

Then there exist operators $\tilde{s}, \tilde{c} \in B(H^m, H)$ *and* $\tilde{t}, \tilde{d} \in B(H, H^m)$ *with the following properties*:

(i) $\tilde{s}_i \in [s_i]$ *and* $\tilde{t}_i \in [t_i]$,

(ii) $\|\tilde{s}\|$, $\|\tilde{t}\| \leqslant 1$ *and* $\|\tilde{c}\|$ $\|\tilde{d}\| \leqslant (3\varepsilon)^{1/2}$,

(iii) $\tilde{s}k^\infty \tilde{t} + \tilde{c}k^\infty \tilde{d} - ek^\infty f = 0$, $k \in K(H)$.

Proof. Equation (4.8) may be written

$$(s \oplus c)k^\infty \ (t \oplus d) - ek^\infty f = 0.$$

By Lemma 4.1 and the proof of Corollary 4.2 with $\mathscr{W} = [f_i]$ there exists a unitary matrix u such that $(s \oplus c)u^*$ and $u(t \oplus d)$ have only finitely many simultaneously non-zero components. Thus there exists a finite rank diagonal projection $p \in B(l_2)$ such that

$$\begin{aligned}(s \oplus c)k^\infty \ (t \oplus d) &= (s \oplus c)u^* k^\infty u(t \oplus d)\\ &= (s \oplus c)u^* p k^\infty p u(t \oplus d).\end{aligned}$$

Write $\tilde{s} = (s \oplus 0)u^* p$, $\tilde{t} = pu(t \oplus 0)$. These matrices have only finitely many non-zero entries, the components lie in $[s_i]$ and $[t_i]$, respectively, and $\|\tilde{s}\|$, $\|\tilde{t}\| \leqslant 1$. In addition the map

$$\begin{aligned}\phi(k) &= (s \oplus c)u^* p k^\infty p u(t \oplus d) - \tilde{s}k^\infty \tilde{t}\\ &= (0 \oplus c)u^* p k^\infty p u(t \oplus 0) + (s \oplus 0)u^* p k^\infty p u(0 \oplus d)\\ &\quad + (0 \oplus c)u^* p k^\infty p u(0 \oplus d)\end{aligned}$$

has cb-norm at most $\varepsilon + \varepsilon + \varepsilon^2 \leqslant 3\varepsilon$ and so may be represented as $\tilde{c}k^\infty \tilde{d}$ where $\|\tilde{c}\|$, $\|\tilde{d}\| \leqslant (3\varepsilon)^{1/2}$. Thus

$$\tilde{s}k^\infty \tilde{t} + \tilde{c}k^\infty \tilde{d} = (s \oplus c)k^\infty \ (t \oplus d)$$

and the result follows from making this substitution in (4.8).

If ξ_1 and ξ_2 are vectors in H then L_{12} and R_{12} will denote respectively the left and right slice maps on $B(H) \otimes_h B(H)$ with respect to the vector functional $\langle \cdot \xi_1, \xi_2 \rangle$. If $v = \sum_{i=1}^n a_i \otimes b_i \in B(H) \odot B(H)$ then

$$\begin{aligned}\langle L_{12}(v)\xi_3, \xi_4 \rangle &= \left\langle \sum a_i \langle b_i \xi_1, \xi_2 \rangle \xi_3, \xi_4 \right\rangle\\ &= \left\langle \sum a_i \xi_3 \otimes \xi_2 b_i \xi_1, \xi_4 \right\rangle\\ &= \langle \phi_v(\xi_3 \otimes \xi_2)\xi_1, \xi_4 \rangle, \qquad (4.9)\end{aligned}$$

and by continuity this also holds for $v \in B(H) \otimes_h B(H)$. In the same way

$$\langle R_{12}(v)\xi_3, \xi_4\rangle = \langle \phi_v(\xi_1 \otimes \xi_4)\xi_3, \xi_2\rangle. \tag{4.10}$$

THEOREM 4.5. *Let $v \in B(H) \otimes_h B(H)$, and let $\mathscr{E}$ and $\mathscr{F}$ be closed subspaces of $B(H)$. Then the following statements are equivalent.*

(i) $v \in \mathscr{E} \otimes_h \mathscr{F}$,

(ii) *$R_\psi(v) \in \mathscr{F}$ and $L_\psi(v) \in \mathscr{E}$ for all $\psi \in B(H)^*$,*

(iii) *ϕ_v has a representation*

$$\phi_v(k) = ek^\infty f, \qquad k \in K(H),$$

where $e \in B(H^\infty, H)$ with components in $\mathscr{E}$, $f \in B(H, H^\infty)$ with components in $\mathscr{F}$, and $\|e\| = \|f\| = \|v\|_h^{1/2}$.

Proof. (i) $\Rightarrow$ (ii) is obvious. Suppose now that (ii) is true, and assume without loss of generality that $\|v\|_h = 1$. Then $\|\phi_v\|_{cb} = 1$ by Theorem 4.3, and so ϕ_v has a representation

$$\phi_v(k) = ek^\infty f, \qquad k \in K(H),$$

where $\|e\| = \|f\| = 1$. By Lemma 4.1 there exists a unitary matrix u and a decomposition $\mathbb{N} = N_1 \cup N_2 \cup N_3$ so that, writing $\tilde{e} = eu^*$ and $\tilde{f} = uf$,

$$\phi_v(k) = \tilde{e}k^\infty \tilde{f}, \qquad k \in K(H),$$

$\tilde{f}_i = 0$ for $i \in N_1$, $\tilde{f}_i \in \mathscr{F}$ for $i \in N_2$, and $\{\tilde{f}_i\}_{i \in N_3}$ is a strongly independent set over $\mathscr{F}$. Then, using (4.10),

$$\begin{aligned}\langle R_{12}(v)\xi_3, \xi_4\rangle &= \langle \phi_v(\xi_1 \otimes \xi_4)\xi_3, \xi_2\rangle \\ &= \sum_{i \in N_2} \langle \tilde{e}_i\xi_1, \xi_2\rangle\langle \tilde{f}_i\xi_3, \xi_4\rangle + \sum_{i \in N_3} \langle \tilde{e}_i\xi_1, \xi_2\rangle\langle \tilde{f}_i\xi_3, \xi_4\rangle.\end{aligned} \tag{4.11}$$

Since $R_{12}(v) \in \mathscr{F}$ by hypothesis and $\tilde{f}_i \in \mathscr{F}$ for $i \in N_2$ by construction, (4.11) implies that

$$\sum_{i \in N_3} \langle \tilde{e}_i\xi_1, \xi_2\rangle \tilde{f}_i \in \mathscr{F}$$

which forces $\tilde{e}_i = 0$ for $i \in N_3$ since $\{\tilde{f}_i\}_{i \in N_3}$ is strongly independent over $\mathscr{F}$. Thus

$$\phi_v(k) = \sum_{i \in N_2} \tilde{e}_i k \tilde{f}_i$$

which has the form

$$\phi_v(k) = ek^\infty f, \qquad k \in K(H),$$

where the components of f lie in $\mathscr{F}$. The proof is completed by applying the same argument on the left, using Lemma 4.1 and the hypothesis that $L_{12}(v) \in \mathscr{E}$, and observing that if f has components in $\mathscr{F}$ then so does uf for any unitary matrix $u \in B(l_2)$. The norm estimates are clear from the construction. Thus (ii) implies (iii).

Now suppose that $\|v\|_h = 1$ and

$$\phi_v(k) = ek^\infty f, \qquad k \in K(H),$$

where the components of e and f lie in $\mathscr{E}$ and $\mathscr{F}$, respectively. Let $0 < \varepsilon < 1$ and choose $v_0 = \sum_{i=1}^n a_i \otimes b_i \in B(H) \odot B(H)$ so that $\|v - v_o\|_h \leqslant \varepsilon^2$. Put $v_1 = v_0 - v$. Then $\|v_1\|_h \leqslant \varepsilon^2$ so $\|\phi_{v_1}\|_{cb} \leqslant \varepsilon^2$, by Theorem 4.3, and thus ϕ_{v_1} may be represented by

$$\phi_{v_1}(k) = ck^\infty d, \qquad k \in K(H),$$

where $\|c\|$, $\|d\| \leqslant \varepsilon$. Since $v + v_1 - v_0 = 0$, we have

$$\sum_{i=1}^\infty e_i k f_i + \sum_{i=1}^\infty c_i k d_i - \sum_{i=1}^n a_i k b_i = 0.$$

By Lemma 4.4 there exist $\tilde{e} \in B(H^m, H)$, $\tilde{f} \in B(H, H^m)$, $\tilde{c} \in B(H^\infty, H)$, $\tilde{d} \in B(H, H^\infty)$ such that $\|\tilde{e}\|$, $\|\tilde{f}\| \leqslant 1$, $\|\tilde{c}\|$, $\|\tilde{d}\| \leqslant (3\varepsilon)^{1/2}$,

$$\tilde{e}k^m\tilde{f} + \tilde{c}k^\infty\tilde{d} - ak^n b = 0 \tag{4.12}$$

and the components of $\tilde{e}$ and $\tilde{f}$ lie in $\mathscr{E}$ and $\mathscr{F}$, respectively.

Let $v_2 = \sum_{i=1}^m \tilde{e}_i \otimes \tilde{f}_i \in \mathscr{E} \odot \mathscr{F}$. Then by (4.12)

$$\phi_{v_2 - v_0}(k) = -\tilde{c}k^\infty\tilde{d}$$

so

$$\|v_2 - v_0\|_h = \|\phi_{v_2 - v_0}\|_{cb} \leqslant \|\tilde{c}\| \ \|\tilde{d}\| \leqslant 3\varepsilon$$

from above. Thus

$$\begin{aligned} \|v - v_2\|_h = \|v - v_0 + v_0 - v_2\|_h &\leqslant \|v - v_0\|_h + \|v_0 - v_2\|_h \\ &\leqslant \varepsilon^2 + 3\varepsilon \leqslant 4\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $v \in \mathscr{E} \otimes_h \mathscr{F}$, and we have proved that (iii) $\Rightarrow$ (i).

We are now able to prove the main results of this section which we state as corollaries of Theorem 4.5. We remark that they are all false for other tensor product norms, which highlights the special nature of the Haagerup tensor product.

COROLLARY 4.6. *If* $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{F}_1$, $\mathscr{F}_2$ *are closed subspaces of* $B(H)$ *then*

$$(\mathscr{E}_1 \otimes_h \mathscr{F}_1) \cap (\mathscr{E}_2 \otimes_h \mathscr{F}_2) = (\mathscr{E}_1 \cap \mathscr{E}_2) \otimes_h (\mathscr{F}_1 \cap \mathscr{F}_2).$$

Proof. If $v \in (\mathscr{E}_1 \otimes_h \mathscr{F}_1) \cap (\mathscr{E}_2 \otimes_h \mathscr{F}_2)$ and $\psi \in B(H)^*$ then clearly $R_\psi(v) \in \mathscr{F}_1 \cap \mathscr{F}_2$ and $L_\psi(v) \in \mathscr{E}_1 \cap \mathscr{E}_2$. It follows from Theorem 4.5 that $v \in (\mathscr{E}_1 \cap \mathscr{E}_2) \otimes_h (\mathscr{F}_1 \cap \mathscr{F}_2)$, and he reverse inclusion is trivial.

The next result may be viewed as a strong version in the Haagerup tensor product of Tomita's commutant theorem for von Neumann algebras. Below, $\mathscr{Z}(\mathscr{A})$ will denote the center of an algebra $\mathscr{A}$.

COROLLARY 4.7. *Let* $\mathscr{A} \subseteq \mathscr{A}_1$ *and* $\mathscr{B} \subseteq \mathscr{B}_1$ *be unital subalgebras of* $B(H)$. *Then*

(i) $(\mathscr{A} \otimes_h \mathscr{B})' = \mathscr{A}' \otimes_h \mathscr{B}'$,

(ii) *the relative commutant of* $\mathscr{A} \otimes_h \mathscr{B}$ *in* $\mathscr{A}_1 \otimes_h \mathscr{B}_1$ *is* $(\mathscr{A}' \cap \mathscr{A}_1) \otimes_h (\mathscr{B}' \cap \mathscr{B}_1)$,

(iii) $\mathscr{Z}(\mathscr{A} \otimes_h \mathscr{B}) = \mathscr{Z}(\mathscr{A}) \otimes_h \mathscr{Z}(\mathscr{B})$.

Proof. (i) If $v \in (\mathscr{A} \otimes_h \mathscr{B})'$ then v commutes with $a \otimes 1$ and $1 \otimes b$ for $a \in \mathscr{A}$, $b \in \mathscr{B}$. If $w = \sum_{i=1}^n e_i \otimes f_i \in B(H) \odot B(H)$ and $k \in K(H)$ then

$$\phi_{w(a \otimes 1)}(k) = \sum_{i=1}^n e_i a k f_i = \phi_w(ak)$$

while

$$\phi_{(a \otimes 1)w}(k) = \sum_{i=1}^n a e_i k f_i = a\phi_w(k).$$

By continuity these relations hold for $w \in B(H) \otimes_h B(H)$ and so, replacing w by v, ϕ_v is a left-$\mathscr{A}$ module map. A similar calculation shows that it is also a right-$\mathscr{B}$ module map and so, by Theorem 3.1, ϕ_v may be represented as

$$\phi_v(k) = \sum s_i k t_i, \qquad k \in K(H),$$

where $s_i \in \mathscr{A}'$ and $t_i \in \mathscr{B}'$. It follows from Theorem 4.5 that $v \in \mathscr{A}' \otimes_h \mathscr{B}'$ and so $(\mathscr{A} \otimes_h \mathscr{B})'$ is contained in $\mathscr{A}' \otimes_h \mathscr{B}'$. The reverse inclusion is trivial.

(ii) The relative commutant of $\mathscr{A} \otimes_h \mathscr{B}$ is $(\mathscr{A} \otimes_h \mathscr{B})' \cap (\mathscr{A}_1 \otimes_h \mathscr{B}_1)$ which, by part (i) and Corollary 4.6, is $(\mathscr{A}' \cap \mathscr{A}_1) \otimes_h (\mathscr{B}' \cap \mathscr{B}_1)$.

(iii) This is a special case of (ii) with $\mathscr{A}_1 = \mathscr{A}$ and $\mathscr{B}_1 = \mathscr{B}$, since $\mathscr{Z}(\mathscr{A}) = \mathscr{A} \cap \mathscr{A}'$.

The last result concerns a strong slice map property of the Haagerup tensor product. Many papers have been written on slice maps, the most recent being [10] where the idea of synthesis is discussed (the author thanks Professor David Larson for drawing his attention to this). This may be roughly stated as the property that elements of $B(H) \otimes_h B(H)$ may be reconstituted from their left and right slices (see (ii) below). The Fubini product was introduced at the beginning of the section and we will also need the following definition. If $v \in B(H) \otimes_h B(H)$, let $\mathscr{R}_v$ be the closed linear span of the images of all right slice maps applied to v, with a similar definition for $\mathscr{L}_v$ in terms of left slices.

COROLLARY 4.8. (i) *Suppose that $\mathscr{E}_1 \subseteq \mathscr{E}_2$ and $\mathscr{F}_1 \subseteq \mathscr{F}_2$ are closed subspaces of $B(H)$. Then*

$$\mathscr{E}_1 \otimes_h \mathscr{F}_1 = F(\mathscr{E}_1, \mathscr{F}_1; \mathscr{E}_2 \otimes_h \mathscr{F}_2).$$

(ii) *If $v \in B(H) \otimes_h B(H)$ then $v \in \mathscr{L}_v \otimes_h \mathscr{R}_v$.*

Proof. Part (i) is immediate from the equivalence of (i) and (ii) in Theorem 4.5. By definition $v \in F(\mathscr{L}_v, \mathscr{R}_v)$, so by (i) $v \in \mathscr{L}_v \otimes_h \mathscr{R}_v$, proving the second assertion.

Remark 4.9. The previous three corollaries are valid without change in $B(H_1) \otimes_h B(H_2)$ where H_1 and H_2 are possibly distinct Hilbert spaces, and this may be seen by working within the algebra $B(H_1 \oplus H_2) \otimes_h B(H_1 \oplus H_2)$. Elements of $B(H_1 \oplus H_2)$ may be viewed as square matrices of operators relative to H_1 and H_2. Subspaces $\mathscr{E} \subseteq B(H_1)$ and $\mathscr{F} \subseteq B(H_2)$ may be identified respectively with the subspaces

$$\left\{ \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} : e \in \mathscr{E} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} : f \in \mathscr{F} \right\}$$

of $B(H_1 \oplus H_2)$, while unital subalgebras $\mathscr{A} \subseteq B(H_1)$ and $\mathscr{B} \subseteq B(H_2)$ may be embedded respectively in the unital subalgebras

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} : a \in \mathscr{A}, \lambda \in \mathbb{C} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & b \end{pmatrix} : b \in \mathscr{B}, \lambda \in \mathbb{C} \right\}$$

of $B(H_1 \oplus H_2)$.

Note added in proof. An unpublished manuscript by U. Haagerup entitled "The α-Tensor Product of C^*-Algebras" contains an earlier proof of Theorem 4.3.

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