QUASI-MULTIPLIERS AND ALGEBRIZATIONS OF AN OPERATOR SPACE

MASAYOSHI KANEDA*

Abstract. Let $X$ be an operator space, let $\varphi$ be a product on $X$, and let $(X, \varphi)$ denote the algebra that one obtains. We give necessary and sufficient conditions on the bilinear mapping $\varphi$ for the algebra $(X, \varphi)$ to have a completely isometric representation as an algebra of operators on some Hilbert space. In particular, we give an elegant geometrical characterization of such products by using the Haagerup tensor product. Our result makes no assumptions about identities or approximate identities. Our proof is independent of the earlier result of Blecher-Ruan-Sinclair (6) that solved the case when the algebra has an identity of norm one, and our result is used to give a simple direct proof of this earlier result. We also develop further the connections between quasi-multipliers of operator spaces, and shows that the quasi-multipliers of operator spaces defined in 12 coincide with their $C^*$-algebraic counterparts.

1. Introduction.

One of the most interesting questions in the operator space theory was: what are the possible operator algebra products which a given operator space can be equipped with? I was investigating many types of multipliers of operator spaces for their own interests. Meantime, V. I. Paulsen defined quasi-multipliers of operator spaces (12 Definition 2.2), and suggested to me to study them. Then, accidentally, I found that the quasi-multipliers happened to answer the question above. That is, the possible operator algebra products which a given operator space can be equipped with are precisely the bilinear mappings implemented by the contractive quasi-multipliers of the operator space (12 Theorem 2.6). In this paper, we give a striking geometrical characterization of operator algebra products (Theorem 4.1).

In Section 3, we study quasi-multipliers in a special case in which an operator space is an operator algebra with a two-sided contractive approximate identity (we will abbreviate as "c.a.i."). In this case, the quasi-multiplier space is quite manageable like $C^*$-algebra case, and equivalent to other definitions using representation on a Hilbert space, or considering in the second dual. It is also equivalent to the set of quasi-centralizers, as a result, we obtain

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that the definition of quasi-multipliers of operator spaces coincide with the existing ones in
the C*-algebra case which were defined by L. G. Brown ([7]).

In Section 4 we present the main result (Theorem 4.1) of this paper. There, we give
a beautiful geometrical characterization of operator algebra products under no assumption
about identities or approximate identities. That is, the possible operator algebra products
which a given operator space can be equipped with are completely determined by matrix
norm structure of the operator space by using the Haagerup tensor product (Theorem 4.1).
This can be considered as the quasi-version of the Blecher-Effros-Zarikian theorem ([3].
In which they characterized left multiplier mappings in terms of the matrix norms ([3].
As a simple corollary, we obtain a generalized version of Blecher-Ruan-Sinclair theorem ([6].
The reader who hurries for the main result may skip Section 3 and directly move on to
Section 4 after reading Section 2 for a background if necessary.

2. Preliminaries.

We begin by recalling a construction of an injective envelope of an operator space. See,
e.g., [5], [13] Chapter 15 for more details. Let \( X \subset B(K, H) \) be an operator space, and
consider the Paulsen operator system

\[ S_X := \left[ \begin{array}{cc} C_1 H & X \\ X^* & C_1 K \end{array} \right] \subset B(H \oplus K). \]

One then takes a minimal (with respect to a certain ordering) completely positive \( S_X \)-
projection \( \Phi \) on \( B(H \oplus K) \) whose image \( \text{Im} \Phi \) turned out to be an injective envelope \( I(S_X) \) of
\( S_X \). By a well-known result of M.-D. Choi and E. G. Effros ([8]), \( \text{Im} \Phi \) is a unital C*-algebra
with the product \( \odot \) defined by \( \xi \odot \eta := \Phi(\xi \eta) \) for \( \xi, \eta \in \text{Im} \Phi \) and other algebraic operations
and norm are the original ones in \( B(H \oplus K) \). One may write

\[ \text{Im} \Phi = I(S_X) = \left[ \begin{array}{cc} I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) \end{array} \right] \subset B(H \oplus K), \]

where \( I(X) \) is an injective envelope of \( X \), and \( I_{11}(X) \) and \( I_{22}(X) \) are injective unital C*-algebras.

By well-known trick one may decompose

\[ \Phi = \left[ \begin{array}{cc} \psi_1 & \phi \\ \phi^* & \psi_2 \end{array} \right]. \]

The new product \( \odot \) induces new products \( \bullet \) between elements of \( I_{11}(X), I_{22}(X), I(X) \)
and \( I(X)^* \). For example, \( x \bullet a = \phi(xa) \) for \( x \in I(X), a \in I_{22}(X) \). Note that the associativity
of \( \bullet \) is guaranteed by that of \( \odot \).

We call the embedding \( i : X \hookrightarrow \left[ \begin{array}{cc} O & X \\ X & O \end{array} \right] \subset S_X; \ x \rightarrow \left[ \begin{array}{cc} 0 & x \\ 0 & 0 \end{array} \right] \) the Šilov embedding.

Now we recall the definitions of quasi-multipliers for operator spaces, and also define some
related notions.
Definition 2.1. (1) \([\mathbb{12} \text{ Section 2}]\) Let \(X\) be an operator space, and let \(\pi\) be a complete isometry from \(X\) into an operator algebra \(A\). Then \((A, \pi)\)-relative quasi-multiplier space of \(X\) is the set
\[
QM^\pi(X) := \{a \in A; \pi(X) a \pi(X) \subset \pi(X)\}.
\]

(2) Let \(QM^\pi(X)\) and \(QM^{\pi'}(X)\) be, respectively, \((A, \pi)\)-relative and \((A', \pi')\)-relative quasi-multiplier spaces for \(X\). Then a linear mapping \(\sigma : QM^\pi(X) \to QM^{\pi'}(X)\) is a quasi-homomorphism\(^1\) if \(\pi^{-1}(\pi(x_1)y\pi(x_2)) = \pi'^{-1}(\pi'(x_1)\sigma(y)\pi'(x_2)), \forall x_1, x_2 \in X, y \in QM^\pi(X)\). Furthermore, if \(\sigma\) is one-to-one and onto, then we call \(\sigma\) a quasi-isomorphism.

(3) \([\mathbb{12} \text{ Definition 2.6}]\) The quasi-multiplier space for \(X\) is the set
\[
QM(X) := \{z \in I(X)^*; X \bullet z \bullet X \subset X\}.
\]
We call an element of \(QM(X)\) a quasi-multiplier of \(X\).

Note that \(QM(X)\) is a subspace of \(QM'(X)\) under the identification
\[
QM(X) = \begin{bmatrix}
O & O \\
QM(X) & O
\end{bmatrix}, \text{where } i \text{ is the Šilov embedding } X \to \begin{bmatrix}
O & X \\
O & O
\end{bmatrix} \text{ defined above.}
\]

The following theorem shows the universal property of quasi-multipliers.

Theorem 2.2. \([\mathbb{12} \text{ Theorem 2.3}]\) Let \(X\) be an operator space and \(A\) be an operator algebra, and suppose that \(\pi : X \to A\) is a complete isometry. Then there exists a unique completely contractive quasi-homomorphism \(\sigma : QM^\pi(X) \to QM(X)\), i.e., \(\pi(x_1)y\pi(x_2) = \pi(x_1 \bullet \sigma(y) \bullet x_2)\), \(\forall x_1, x_2 \in X, y \in QM^\pi(X)\), where \(X\) is regarded as a subset of \(I(S_X)\). In particular, if \(X\) itself is an operator algebra with product \(\cdot\), then there exists a unique \(z \in QM(X)\) such that \(x_1 \cdot x_2 = x_1 \bullet z \bullet x_2, \forall x_1, x_2 \in X\).

In this paper, we discuss bilinear mappings on operator spaces. We would like to make sure of our terminology. Let \(\varphi\) be a bilinear mapping on an operator space \(X\), and let \(\varphi : X \otimes_h X \to X\) be the linear mapping corresponding to \(\varphi\), where \(X \otimes_h X\) is the Haagerup tensor product which plays the central role in Section 4. For the Haagerup tensor product, see \([\mathbb{13}, \mathbb{10}, \mathbb{16}]\). We define the completely bounded norm of \(\varphi\) by \(\|\varphi\|_{cb} := \|\varphi\|_{cb}\), and we say that \(\varphi\) is completely bounded (respectively, completely contractive) if \(\|\varphi\|_{cb} < \infty\) (respectively, \(\|\varphi\|_{cb} \leq 1\)). Note that the term “completely bounded” for a bilinear mapping defined here is in the sense of Christensen-Sinclair \([\mathbb{9}]\), and it is called multiplicatively bounded in \([\mathbb{10}]\).

Throughout this paper, a product means an associative bilinear mapping.

3. QUASI-MULTIPLIERS OF AN OPERATOR ALGEBRA WITH A TWO-SIDED C.A.I.

In this section, we study some equivalent notions of quasi-multipliers of an operator algebra with a two-sided c.a.i. in a similar manner to \([\mathbb{2}, \mathbb{1}]\). We have already studied the left multipliers \((\mathbb{4})\) and the right multipliers \((\mathbb{2})\) of an operator algebra with a “right” c.a.i.,

\(^1\)This is different from a left (right) quasihomomorphism defined in \([\mathbb{12} \text{ Definition 5.1}]\).
So the reader may think that we should study here the quasi-multipliers of an operator algebra with a right c.a.i.. However, this seems very messy and not worth writing here, so we study only the “two-sided c.a.i.” case. This is very satisfactory, and as a result, we show that “quasi-multipliers” coincide with the existing ones in the C*-algebra case ([6], also see [14] §3.12).

Definition 3.1. Let $\mathcal{A}$ be an operator algebra with a two-sided c.a.i.. Then the quasi-centralizer space of $\mathcal{A}$ is the set

$$QC(\mathcal{A}) := \{\varphi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}; \, \varphi(\cdot, a) \in CB(\mathcal{A}), \, \varphi(a, \cdot) \in CB_{\mathcal{A}}(\mathcal{A}) \, \forall a \in \mathcal{A}\}.$$ 

We call an element of $QC(\mathcal{A})$ a quasi-centralizer. In Theorem 3.2, we prove that elements of $QC(\mathcal{A})$ are completely bounded in the sense of Christensen-Sinclair which is explained at the end of the previous section. Hence, for each $n \in \mathbb{N}$ and each $(\tilde{\varphi}_{i,j}) \in M_n(QC(\mathcal{A}))$, there is a corresponding matrix of linear mappings $(\tilde{\varphi}_{i,j}) \in M_n(CB(\mathcal{A} \otimes_h \mathcal{A}, \mathcal{A}))$, and we define a matrix norm of $QC(\mathcal{A})$ by $\|\tilde{\varphi}_{i,j}\|_n := \|\tilde{\varphi}_{i,j}\|_n$.

We now define a quasi-multiplier extension of $\mathcal{A}$ to be a pair $(X, \pi)$ consisting of an operator space $X$ which is a subspace of some operator algebra $\mathcal{B}$ and a completely isometric homomorphism $\pi : \mathcal{A} \rightarrow X$ such that $\pi(\mathcal{A})X \pi(\mathcal{A}) \subset \pi(\mathcal{A})$, where the product is taken in $\mathcal{B}$. We say that $(X, \pi)$ is an essential quasi-multiplier extension of $\mathcal{A}$ if in addition the canonical completely contractive mapping $X \rightarrow QC(\mathcal{A})$ is one-to-one. For two quasi-multiplier extensions $(X, \pi)$ and $(X', \pi')$ of $\mathcal{A}$, we write $(X, \pi) \leq (X', \pi')$ if there exists a completely contractive homomorphism $\theta : X \rightarrow X'$ such that $\theta \circ \pi = \pi'$. We say that two quasi-multiplier extensions $(X, \pi)$ and $(X', \pi')$ are $\mathcal{A}$-equivalent if there exists a completely isometric quasi-isomorphism $\theta : X \rightarrow X'$ with $\theta \circ \pi = \pi'$. This is an equivalence relation, and “$\leq$” induces a well-defined ordering on the equivalence classes.

It follows that if there exists a maximum essential quasi-multiplier extension of $\mathcal{A}$, then it is unique up to $\mathcal{A}$-equivalence. Also if two quasi-multiplier extensions are $\mathcal{A}$-equivalent, and if one is essential, then so is the other.

Now let us recall the following facts from [1].

Lemma 3.2. ([1] Theorem 2.3) Let $\mathcal{A}$ be an operator algebra with a right c.a.i.. Then there exists a $v^* \in Ball(QM(\mathcal{A}))$ such that

1. $v \cdot v^*$ is an orthogonal projection in $I_{11}(\mathcal{A})$,
2. $v^* \cdot v$ is the identity of $I_{22}(\mathcal{A})$,
3. $a \cdot v^* \cdot b = ab, \, \forall a, b \in \mathcal{A}$, and hence $\hat{\psi} : I(\mathcal{A}) \rightarrow I_{11}(\mathcal{A})$ defined by $\hat{\psi}(a) := a \cdot v^*, \, \forall a \in I(\mathcal{A})$ is a complete isometry that restricts to a homomorphism on $\mathcal{A}$.

The following corollary immediately follows from the lemma above.

Corollary 3.3. If $\mathcal{A}$ has a two-sided contractive approximate identity, then $v \cdot v^*$ is the identity of $I_{11}(\mathcal{A})$.

Proof. By symmetry, there exists a $w^*Ball(QM(\mathcal{A}))$ such that $w \cdot w^*$ is the identity of $I_{11}(\mathcal{A})$. But $v^* = w^*$ by uniqueness of a quasi-multiplier (Theorem 2.2). \qed
The following lemma tells us that if an operator algebra $\mathcal{A}$ has a two-sided c.a.i., then the quasi-multiplier space $QM(\mathcal{A})$ can be replaced by a better one, $QM^{\psi}(\mathcal{A})$, which contains a copy of $\mathcal{A}$ preserving the product.

**Lemma 3.4.** Let $\mathcal{A}$ be an operator algebra with a two-sided c.a.i., and we regard $\mathcal{A}$ as $A = \begin{bmatrix} O & A \\ O & O \end{bmatrix} \subset I(S_{\mathcal{A}})$. Let $\tilde{\psi} : \mathcal{A} \rightarrow I_{11}(\mathcal{A})$; $\tilde{\psi} := \tilde{\psi}|_{\mathcal{A}}$ and $\rho : QM(\mathcal{A}) \rightarrow I_{11}(\mathcal{A})$; $\rho(z) := v \circ z$, where $\tilde{\psi}$ and $v$ are as in Lemma 3.3. Then $\rho$ is a completely isometric quasi-isomorphism from $QM(\mathcal{A})$ onto $QM^{\psi}(\mathcal{A})$, and $\tilde{\psi}(\mathcal{A}) \subset QM^{\psi}(\mathcal{A}) (= \rho(QM(\mathcal{A})))$.

**Proof.** For $a_1, a_2 \in \mathcal{A}$, $z \in QM(\mathcal{A})$, $\tilde{\psi}(a_1) \circ \rho(z) \circ \tilde{\psi}(a_2) = a_1 \circ v^* \circ z \circ a_2 \circ v^* = a_1 \circ z \circ a_2 \circ v^* = \tilde{\psi}(a_1 \circ z \circ a_2)$. This shows that $\rho$ is a quasi-homomorphism and $\rho(QM(\mathcal{A})) \subset QM^{\psi}(\mathcal{A})$. On the contrary, let $x \in I_{11}(\mathcal{A})$ be such that $\tilde{\psi}(a_1) \circ x \circ \tilde{\psi}(a_2) = \tilde{\psi}(a_3)$ for some $a_3 \in \mathcal{A}$. Then $a_1 \circ v^* \circ x \circ a_2 \circ v^* = a_3 \circ v^*$. By multiplying the both sides by $v^*$ on the right, we obtain $a_1 \circ v^* \circ x \circ a_2 = a_3$. This implies that $v^* \circ x \in QM(\mathcal{A})$. Hence, $x = v \circ v^* \circ x \in \rho(QM(\mathcal{A}))$ since $v \circ v^*$ is the identity of $I_{11}(\mathcal{A})$ (Corollary 3.3). That $\rho$ is a complete isometry easily follows from the facts that $\rho$ is a left multiplication by $v$ and that $v^* \circ v$ is the identity of $I_{22}(\mathcal{A})$. That $\tilde{\psi}(\mathcal{A}) \subset QM^{\psi}(\mathcal{A})$ follows from the fact that $\tilde{\psi}$ is a homomorphism. □

Let $\mathcal{A}$ be an operator algebra with a two-sided c.a.i.. We consider the following three notions:

1. $QM^{**}(\mathcal{A}) := \{ z \in \mathcal{A}^{**}; \hat{\mathcal{A}}z\hat{\mathcal{A}} \subset \hat{\mathcal{A}} \}$,
2. $QM^{\pi}(\mathcal{A}) := \{ T \in \mathbb{B}(\mathcal{H}); \pi(\mathcal{A})T\pi(\mathcal{A}) \subset \pi(\mathcal{A}) \}$, where $\pi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ is a completely isometric nondegenerate representation,
3. $QM^{\bar{\psi}}(\mathcal{A})$,

where $\hat{\mathcal{A}}$ is the canonical image of $\mathcal{A}$ in $\mathcal{A}^{**}$. Also hereafter, we denote by $\hat{a} \in \hat{\mathcal{A}}$ the canonical image of $a \in \mathcal{A}$.

**Theorem 3.5.** Let $\mathcal{A}$ be an operator algebra with a two-sided c.a.i.. Then (I), (II), and (III) as above are quasi-multiplier extensions of $\mathcal{A}$, and are all $\mathcal{A}$-equivalent. Moreover, these are maximum essential quasi-multiplier extensions of $\mathcal{A}$.


Let $\pi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a completely isometric homomorphism, and consider the following canonical completely contractive homomorphisms:

$$\mathcal{A} \hookrightarrow \mathcal{A}^{**} \xrightarrow{\pi^{**}} \mathbb{B}(\mathcal{H})^{**} \rightarrow \mathbb{B}(\mathcal{H})$$

Let $\hat{\pi}$ be the composition of the last two mappings. Then $\hat{\pi}$ is completely isometric on $\hat{\mathcal{A}}$, and also $\hat{\pi}(\hat{a}) = \pi(a)$, $\forall a \in \mathcal{A}$. Let $\tilde{\pi} := \hat{\pi}|_{QM^{**}(\mathcal{A})}$.

Let us consider the canonical complete contractions:

$$QM^{**}(\mathcal{A}) \xrightarrow{\tilde{\pi}} QM^{\pi}(\mathcal{A}) \xrightarrow{\phi} QM^{\bar{\psi}}(\mathcal{A}) \xrightarrow{\theta} QC(\mathcal{A}),$$

where, $\tilde{\pi}$ and $\phi$ are quasi-homomorphisms. Explicitly, $\phi = \rho \circ \sigma$, where $\sigma$ is as in Theorem 2.2 and $\rho$ is as in Lemma 3.3 $\theta(x)(a_1, a_2) = a_1 \circ v^* \circ x \circ a_2$ for $x \in QM^{\bar{\psi}}(\mathcal{A})$, $a_1, a_2 \in \mathcal{A}$ with $v^*$ as in Lemma 3.2.
To check that \( \tilde{\pi} \) maps \( \mathcal{QM}^*(A) \) into \( \mathcal{QM}^*(A) \), take \( z \in \mathcal{QM}^*(A), a, b \in A \). Then \( \pi(a)\tilde{\pi}(z)\pi(b) = \tilde{\pi}(azb) \in \tilde{\pi}(A) = \pi(A) \). That \( \tilde{\pi} \) is an isometry follows from the facts that \( \forall z \in \mathcal{QM}^*(A), \|\tilde{\pi}(z)\| \geq \|\pi(a)\tilde{\pi}(z)\pi(e_b)\| = \|\tilde{\pi}(az\tilde{e}_b)\| = \|\tilde{e}_az\tilde{e}_b\| \), and that the identity of \( A \) is a weak* limit point of \( \{e_a\}_{a \in A} \), using the separate weak* continuity of the product on \( A^* \). In fact, for each \( \epsilon > 0 \) and each \( f \in \text{Ball}(A^*) \), \( \sup_a \sup_{\beta} \|\tilde{\pi}(e_a\tilde{e}_\beta)\| \geq |z(f)| - \epsilon \), so that \( \sup_{\beta} \|\tilde{\pi}(e_a\tilde{e}_\beta)\| = \sup_{f \in \text{Ball}(A^*)} \|\tilde{\pi}(e_a\tilde{e}_\beta)\| = \sup_{f \in \text{Ball}(A^*)} \|\tilde{\pi}(e_a\tilde{e}_\beta)\| \geq \sup_{f \in \text{Ball}(A^*)} |z(f)| - \epsilon = \|z\| - \epsilon \). Since \( \epsilon > 0 \) is arbitrary, \( \sup_{\beta} \|\tilde{\pi}(e_a\tilde{e}_\beta)\| \geq \|z\| \). A similar calculation at the matrix level shows that \( \tilde{\pi} \) is a complete isometry.

Let \( x \in \mathcal{QM}^\theta(A) \), and write \( \varphi_x := \theta(x) \). Then for \( a, b, c \in A \), \( \varphi_x(a, b) = a \cdot v^* \cdot x \cdot b \), and \( \varphi_x(a, c) \in CB(A) \) and \( \varphi_x(a, c) \in CB(A) \). Also \( \varphi_x(ab, c) = (ab) \cdot v^* \cdot x \cdot c = a \cdot v^* \cdot b \cdot v^* \cdot x \cdot c = a(\varphi_x(a, b)). \) Similarly, \( \varphi_x(a, bc) = \varphi_x(a, b)c \). Hence \( \theta(z) = \varphi_x \in \mathcal{QM}^\theta(A) \). That \( \theta \) is one-to-one easily follows from \( [5] \) Corollary 1.3. In fact, let \( \theta(x) = 0 \), i.e., \( \varphi_x(a, b) = 0 \), \( \forall a, b \in A \). Then \( a \cdot v^* \cdot x \cdot b = 0 \), \( \forall a, b \in A \). By \( [5] \) Corollary 1.3, \( v^* \cdot x \cdot b = 0 \), so that \( v \cdot v^* \cdot x \cdot b = 0 \), and hence \( x \cdot b = 0 \), \( \forall b \in A \) since \( v \cdot v^* \) is the identity of \( I_{11}(A) \). Thus again by \( [5] \) Corollary 1.3, \( x = 0 \).

Let \( \varphi \in \mathcal{QC}(A) \), and let \( F \) be a weak* accumulation point of \( \varphi(e_{\alpha}, e_{\beta}) \) in \( A^* \). Clearly, \( \|F\| \leq \|\varphi\|_{cb} \). For \( a, b \in A \), we have \( \varphi(a, b) = \lim_{\alpha, \beta} \varphi(a e_{\alpha}, e_{\beta}b) = \lim_{\alpha, \beta} a \varphi(e_{\alpha}, e_{\beta})b = a F b \). Hence, \( F \in \mathcal{QM}^\theta(A) \) and \( \theta \circ \phi \circ \pi(F) = \varphi \) and \( \|\varphi\|_{cb} \leq \|F\| \), and thus \( \theta \circ \phi \circ \pi \) is an onto isometry. Here, to see that \( \theta \circ \phi \circ \pi(F) = \varphi \), first note that \( \pi(\bar{\alpha}) \tilde{\pi}(F) \pi(\bar{b}) = \pi(\alpha \sigma(\tilde{\pi}(F)))b \) by Theorem \( [2, 2] \). But the left hand side is \( \tilde{\pi}(\alpha) \tilde{\pi}(F) \tilde{\pi}(\bar{b}) = \tilde{\pi}(\alpha) \tilde{\pi}(\bar{b}) = \tilde{\pi}(\varphi(a, b)) = \pi(\varphi(a, b)) \). On the other hand, \( \pi(a \cdot \sigma(\tilde{\pi}(F)))b = \pi(a \cdot v^* \cdot v \cdot \sigma(\tilde{\pi}(F)))b = \pi(\alpha \cdot v^* \cdot v \cdot \sigma(\tilde{\pi}(F)))b = \pi(\alpha \cdot v^* \cdot \phi(\tilde{\pi}(F)))b = \pi(\theta(\phi(\tilde{\pi}(F))))(a, b) \) follows. Now since we know that elements of \( \mathcal{QC}(A) \) are completely bounded, we can equip \( \mathcal{QC}(A) \) with a matrix norm as mentioned after the definition of \( \mathcal{QC}(A) \). Since the operation of \( \varphi \) is given by the multiplication by \( F \), a similar calculation works at the matrix level, and \( \theta \circ \phi \circ \pi \) is a complete isometry. It is easy to check that \( \phi(= \rho \circ \sigma) \) is one-to-one by using the fact that \( \pi(e_{\alpha}) \overset{SOT}{\rightarrow} 1_H \). In fact, since \( \rho \) is a complete isometry (Lemma \( [3, 4] \), it suffices to show that \( \sigma \) is one-to-one. Let \( \sigma(y) = 0 \), \( \forall y \in \mathcal{QM}(A) \). Then \( \pi(e_{\alpha})y \pi(e_{\beta}) = \pi(e_{\alpha})y \pi(e_{\beta}) = 0 \), thus \( \pi(e_{\alpha})y \pi(e_{\beta})\xi = 0 \), \( \forall \xi \in H \). By taking the limits \( \alpha, \beta \to +\infty \), \( y \xi = 0 \), \( \forall \xi \in H \), so that \( y = 0 \).

Thus, we have proved that \( \tilde{\pi} \) is a complete isometry; \( \phi \) and \( \theta \) are one-to-one complete contractions; \( \theta \circ \phi \circ \pi \) is an onto complete isometry. All these facts force that each of \( \tilde{\pi} \), \( \phi \), and \( \theta \) is an onto complete isometry. Hence (I)-(III) are all \( A \)-equivalent, and they are essential quasi-multiplier extensions of \( A \), and \( \mathcal{QC}(A) \) is an operator space.

Finally, we prove that (III) is a maximum essential quasi-multiplier extension of \( A \). Then by Theorem \( [2, 2] \) and Lemma \( [3, 4] \) there exists a completely contractive quasi-homomorphism \( \phi : X \to \mathcal{QM}^\theta(A) \). To see that \( \phi \circ \pi = \tilde{\psi} \), take any \( a, b, c \in A \). By Theorem \( [2, 2] \), \( \pi(ab)(\pi(b)\pi(c) = \pi(a \cdot \sigma(\pi(b)) \cdot c) \), so that \( \pi(abc) = \pi(a \cdot \sigma(\pi(b)) \cdot c) \), and thus \( abc = a \cdot v^* \cdot b \cdot v^* \cdot c \) by Lemma \( [3, 2] \) (3), so that \( a \cdot \sigma(\pi(b)) \cdot c = a \cdot v^* \cdot b \cdot v^* \cdot c \). Since \( a \in A \) is arbitrary, by \( [4] \) Corollary 1.3, \( \pi(b) \cdot c = v^* \cdot b \cdot v^* \cdot c \). Hence \( \phi(\pi(b)) \cdot c = \rho(\pi(b)) \cdot c = \rho(\sigma(\pi(b))) \cdot c = v \cdot \sigma(\pi(b)) \cdot c = v \cdot v^* \cdot b \cdot v^* \cdot c = b \cdot v^* \cdot c \) since \( v \cdot v^* \) is the identity of \( I_{11}(A) \). Since \( c \in A \)
is arbitrary, again by [5] Corollary 1.3, \( \phi(\pi(b)) = b \bullet v^* = \tilde{\psi}(b) \). Since \( b \in A \) is arbitrary, \( \phi \circ \pi = \tilde{\psi} \). Thus \( (X, \pi) \leq (\mathcal{QM}^{\tilde{\psi}}(A), \tilde{\psi}) \), so that \( (\mathcal{QM}^{\tilde{\psi}}(A), \tilde{\psi}) \) is a maximum essential quasi-multiplier extension.

**Corollary 3.6.** In the C*-algebra case, our definition of the quasi-multipliers coincides with the existing one ([2], also see [14] §3.12) in the sense that these are completely isometrically quasi-isomorphic.

**Corollary 3.7.** If \( A \) is an operator algebra with a two-sided c.a.i., then \( \mathcal{QC}(A) \) is an operator space, and it can be taken as a maximum essential quasi-multiplier extension of \( A \).

### 4. Quasi-multipliers and Algebrizations of an Operator Space

In this section we present the main theorem (Theorem 4.1) of this paper. We consider possible functors from the category of operator spaces together with complete isometries into the category of operator algebras together with completely isometric homomorphisms. Hence comes the name algebrization. But we do not have to express this “algebrization functor” explicitly, the name “algebrization” shows up only in the titles of this paper and this section.

Theorem 4.1 completely characterizes an operator algebra without any assumption on identities or approximate identities. The item (iii) of the following theorem is regarded as the “quasi” version of the Blecher-Effros-Zarikian Theorem (so-called “\( \tau \)-trick”) ([3] Theorem 4.6), that is, operator algebra products are characterized only in terms of the matrix norm. To use the Haagerup tensor norm is essential. Also a generalization of the Blecher-Ruan-Sinclair theorem is obtained as a simple corollary.

**Theorem 4.1.** Let \( X \) be a non-zero operator space with a bilinear mapping \( \phi : X \times X \to X \), and let \( I(S_X) \) be as in Section 2 and \( 1 \) be its identity. We regard \( X \) as a subspace of \( I(S_X) \) by the Šilov embedding as explained in Section 2. Let

\[
\Gamma_{\phi} : \begin{bmatrix} X \otimes_h C1 & X \otimes_h X \\ O & C1 \otimes_h X \end{bmatrix} \to \begin{bmatrix} X & X \\ O & X \end{bmatrix}
\]

be defined by

\[
\Gamma_{\phi} \left( \begin{bmatrix} x_1 \otimes 1 & x \otimes y \\ 0 & 1 \otimes x_2 \end{bmatrix} \right) := \begin{bmatrix} x_1 & \phi(x, y) \\ 0 & x_2 \end{bmatrix}
\]

and their linear extensions. Then, the following are equivalent:

(i) \( (X, \phi) \) is an abstract operator algebra (i.e., there is a completely isometric homomorphism from \( X \) into a concrete operator algebra, hence, in particular, \( \phi \) is associative),

(ii) there exists a \( z \in \mathcal{QM}(X) \) with \( \|z\| \leq 1 \) such that \( \forall x, y \in X, \phi(x, y) = x \bullet z \bullet y \),

(iii) \( \Gamma_{\phi} \) is completely contractive.

Moreover, such a \( z \) is unique.

When these conditions hold, we say that \( \phi \) is an operator algebra product (OAP) on \( X \) and denote the set of all OAP’s on \( X \) by \( \text{OAP}(X) \).
Proof. 2 Uniqueness of $z$ easily follows from [5] Corollary 1.3. In fact, let $z_1, z_2 \in \mathcal{Q}M(X)$ be such that $\|z_1\| \leq 1$, $\|z_2\| \leq 1$, $x \cdot z_1 \cdot y = \varphi(x, y) = x \cdot z_2 \cdot y$, $\forall x, y \in X$. Then $x \cdot (z_1 - z_2) \cdot y = 0$, $\forall x, y \in X$, so that $(z_1 - z_2) \cdot y = 0$, $\forall y \in X$ by [5] Corollary 1.3, and thus $(z_1 - z_2)^* (z_1 - z_2) = 0$, and $z_1 = z_2$.

(ii) $\implies$ (i): This direction follows from the same way as Remark 2 on page 194 of [5].

Define $\rho : X \to I(S_X)$ by

$$\rho(x) := \begin{bmatrix} x \cdot z & x \cdot \sqrt{1_{22} - z \cdot z^*} \\ 0 & 0 \end{bmatrix}, \ \forall x \in X.$$  

Since

$$\rho(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 \\ z & \sqrt{1_{22} - z \cdot z^*} \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} 0 & 0 \\ z & \sqrt{1_{22} - z \cdot z^*} \end{bmatrix} \odot \begin{bmatrix} 0 & z^* \\ 0 & \sqrt{1_{22} - z \cdot z^*} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1_{22} \end{bmatrix},$$

it follows that $\rho$ is a completely isometric homomorphism.

(i) $\implies$ (iii): We may assume that $(X, \varphi) \subset (\mathcal{B}(\mathcal{H}), \cdot)$ as a subalgebra of operators. By the construction of $I(S_X)$ in Section 2, $1_{11} = \begin{bmatrix} 1_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1\mathcal{H} & 0 \\ 0 & 0 \end{bmatrix}$, $1_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1\mathcal{H} \end{bmatrix}$.

We write $1_{21} := \begin{bmatrix} 0 & 0 \\ 1\mathcal{H} & 0 \end{bmatrix} \in \mathcal{M}_2(\mathcal{B}(\mathcal{H}))$ and define a linear mapping $\phi : I(S_X) \otimes_h I(S_X) \to \mathcal{M}_2(\mathcal{B}(\mathcal{H}))$ by $\phi(\eta \otimes \zeta) := \eta \cdot 1_{21} \cdot \zeta \ \forall \eta, \zeta \in I(S_X)$, and their linear extensions, where the products between elements of $\mathcal{M}_2(\mathcal{B}(\mathcal{H}))$ are induced from the original products $\cdot$ of $\mathcal{B}(\mathcal{H})$, and still denoted by $\cdot$.” Then, obviously $\phi$ is completely contractive. Let

$$\xi = \begin{bmatrix} x \otimes 1 & \sum_{i=1}^n x_i \otimes y_i \\ 0 & 1 \otimes y \end{bmatrix} \in \begin{bmatrix} X \otimes_h \mathcal{C} 1 & X \otimes_h X \\ \mathcal{C} 1 \otimes_h X \end{bmatrix},$$

then

$$\|\Gamma_\varphi(\xi)\| = \|\Gamma_\varphi\left(\begin{bmatrix} x \otimes 1 & \sum_{i=1}^n x_i \otimes y_i \\ 0 & 1 \otimes y \end{bmatrix}\right)\| = \left\|\begin{bmatrix} x & \sum_{i=1}^n \varphi(x_i, y_i) \\ 0 & y \end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix} x \cdot 1_{21} & \sum_{i=1}^n x_i \cdot 1_{21} \cdot y_i \\ 0 & 1 \cdot 1_{21} \cdot y_i \end{bmatrix}\right\| = \left\|\begin{bmatrix} \phi(x \otimes 1) & \phi(\sum_{i=1}^n x_i \otimes y_i) \\ \phi(0 \otimes 0) & \phi(1 \otimes y) \end{bmatrix}\right\|$$

$$= \left\|\phi_2\left(\begin{bmatrix} x \otimes 1 & \sum_{i=1}^n x_i \otimes y_i \\ 0 & 1 \otimes y \end{bmatrix}\right)\right\| = \|\phi_2(\xi)\|,$$

where $\phi_2 : \mathcal{M}_2(I(S_X)) \to \mathcal{M}_2(\mathcal{M}_2(\mathcal{B}(\mathcal{H})))$; $\phi_2((x_{i,j})) := (\phi(x_{i,j}))$. Since $\phi$ is completely contractive, we obtain that $\|\Gamma_\varphi(\xi)\| \leq \|\xi\|$, so that $\Gamma_\varphi$ is contractive. A similar calculation at the matrix level shows that $\Gamma_\varphi$ is completely contractive.

Now, we show the hardest direction.

Historically, first I proved the equivalence of (i) $\iff$ (ii) separately, [12] Theorem 2.6.
(iii) \(\implies\) (ii): Let \(I(\hat{S}_X)\) be a copy of \(I(S_X)\) which shares \(\mathbb{C}1\) with \(I(S_X)\), and let \(\sim:\)
\(I(S_X) \to I(\hat{S}_X)\) be the canonical mapping. Note that \(\hat{1} = 1\). \(I(S_X) \ast_1 I(\hat{S}_X)\) be the completion of the free product of \(I(S_X)\) and \(I(\hat{S}_X)\) amalgamated over \(\mathbb{C}1\), which is a \(C^*\)-algebra. We embed \(I(S_X) \otimes h I(S_X)\) into \(I(S_X) \ast_1 I(\hat{S}_X)\) by the complete isometry \(\gamma\) defined by setting \(\gamma(\xi \otimes \eta) := \xi \eta\) and extending linearly. The reader unfamiliar with this embedding is recommended to refer to \([15], [16],\) and also \([13]\) Chapter 17. The important properties of this free product employed in the proof below are that \(I(S_X) \ast_1 I(\hat{S}_X)\) contains both \(I(S_X)\) and \(I(\hat{S}_X)\) as \(C^*\)-subalgebras, and that these three \(C^*\)-algebras share the common identity \(1 = \hat{1} = 1 \ast \hat{1}\). Let us define
\[
S := \begin{bmatrix}
\mathbb{C}1_{11} & \gamma(X \otimes_h \mathbb{C}1) & O & \gamma(X \otimes_h X) \\
\gamma(X \otimes_h \mathbb{C}1)^* & \mathbb{C}1_{12} & O & O \\
O & O & \mathbb{C}1_{11} & \gamma(\mathbb{C}1 \otimes_h X) \\
\gamma(X \otimes_h X)^* & O & \gamma(\mathbb{C}1 \otimes_h X)^* & \mathbb{C}1_{12}
\end{bmatrix},
\]
which is a subset of \(\mathbb{M}_4(I(S_X) \ast_1 I(\hat{S}_X))\). Let \(C^*(S)\) be the \(C^*\)-algebra generated by \(S\) in \(\mathbb{M}_4(I(S_X) \ast_1 I(\hat{S}_X))\). The elements \([\xi_{i,j}]_{1 \leq i,j \leq 4} \in C^*(S)\) of the form \(\xi_{i,j} = \xi_{i,j}^{1} \cdots \xi_{i,j}^{n_{i,j}}, n_{i,j} \in \mathbb{N}\) with
\[
\xi_{i,1}^{1} \in \begin{cases} X, & \text{if } i = 1; \\
X^*, & \text{if } i = 2; \\
\bar{X}, & \text{if } i = 3; \\
\bar{X}^*, & \text{if } i = 4,
\end{cases}
\] and
\[
\xi_{i,j}^{n_{i,j}} \in \begin{cases} X^*, & \text{if } j = 1; \\
X, & \text{if } j = 2; \\
\bar{X}^*, & \text{if } j = 3; \\
\bar{X}, & \text{if } j = 4
\end{cases}
\]
span a dense subset of \(C^*(S)\). Hence \(\text{diag}\{1_{11}, 1_{22}, \bar{1}_{11}, \bar{1}_{22}\}\) is the identity of \(C^*(S)\), so that \(S\) is an operator system in \(C^*(S)\). Define a linear mapping
\[
\Phi : S \to \begin{bmatrix}
\mathbb{C}1_{11} & X & O & X \\
X^* & \mathbb{C}1_{12} & O & O \\
O & O & \mathbb{C}1_{11} & X \\
O & X^* & \mathbb{C}1_{12}
\end{bmatrix} \subset \mathbb{M}_2(I(S_X))
\]
by
\[
\Phi := \begin{bmatrix}
\lambda_{11} & \gamma(x_1 \otimes 1) & 0 & \gamma(x_5 \otimes x_6) \\
\gamma(x_2 \otimes 1)^* & \mu_{12} & 0 & 0 \\
0 & 0 & \alpha_{11} & \gamma(1 \otimes x_3) \\
\gamma(x_7 \otimes x_8)^* & 0 & \gamma(1 \otimes x_4)^* & \beta_{122}
\end{bmatrix}
\]
and their linear extensions. By the canonical shuffle, and the fact that \(\gamma\) is a complete isometry, and our assumption that \(\Gamma_{\phi}\) is completely contractive, we know that \(\Phi\) is completely
positive. We extend $\Phi$ to a linear mapping $\Phi'$ from

$$S' := \text{span}\{S \cup \text{diag}\{C1, C1, C1, C1\}\}$$

onto the same range such that

$$\Phi'(\text{diag}\{\lambda1, \mu1, \alpha1, \beta1\}) = \text{diag}\{\lambda1_{11}, \mu1_{22}, \alpha1_{11}, \beta1_{22}\}.$$

This uniquely well defines $\Phi'$. In particular,

$$\text{Ker}\Phi' = \{C1_{22}, C1_{11}, \tilde{C}1_{22}, \tilde{C}1_{11}\}.$$

To see that $\Phi'$ is completely positive, simply observe that for $\xi \in S'$,

$$\Phi' (\xi) = \Phi(\text{diag}\{1_{11}, 1_{22}, \tilde{1}_{11}, \tilde{1}_{22}\} \xi \text{diag}\{1_{11}, 1_{22}, \tilde{1}_{11}, \tilde{1}_{22}\}).$$

Since $S'$ is an operator system containing the identity of $M_4(I(S_X) \ast 1 \widehat{I(S_X)})$, we can extend $\Phi'$ to a completely positive map

$$\tilde{\Phi} : M_4(I(S_X) \ast 1 \widehat{I(S_X)}) \rightarrow \begin{bmatrix} I_{11}(X) & I(X) & I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) & I(X)^* & I_{22}(X) \\ I_{11}(X) & I(X) & I_{11}(X) & I(X) \\ I(X)^* & I_{22}(X) & I(X)^* & I_{22}(X) \end{bmatrix}$$

by the injectivity of the right hand side. Since $\tilde{\Phi}$ “fixes” each scalar element on the diagonal, $\tilde{\Phi}$ is factored to $[\phi_{i,j}]_{1 \leq i,j \leq 4}$ by a common argument. $\tilde{\Phi}$ also “fixes” the C*-subalgebra

$$\begin{bmatrix} I_{11}(X) & I(X) & 0 & 0 \\ I(X)^* & I_{22}(X) & 0 & 0 \\ 0 & 0 & \widehat{I_{11}(X)} & \widehat{I(X)} \\ 0 & 0 & \widehat{I(X)^*} & \widehat{I_{22}(X)} \end{bmatrix}$$
by the rigidity. Hence, \( \tilde{\Phi} \) is a “module map” over it in the sense of \( [5] \) Lemma 1.6. Let \( x, y \in X \), then
\[
\begin{bmatrix}
0 & 0 & 0 & \varphi(x, y) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
= \tilde{\Phi}
\begin{bmatrix}
0 & 0 & 0 & x \ast \tilde{y} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
= \tilde{\Phi}
\begin{bmatrix}
0 & x & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & x \ast \tilde{y} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
where \( z := \phi_{2,3}(1) \).
\]

The new point in the following corollary is that we do not assume that \((X, \varphi)\) has a “two-sided” c.a.i.. This also could follow from \([1]\).

**Corollary 4.2.** (A generalization of the BRS Theorem \([5, 17]\)) Let \( X \) be a non-zero operator space, and \( \varphi \) be a completely contractive bilinear mapping on \( X \). If there exists \( \{e_\alpha\} \) and \( \{f_\beta\} \) with \( \|e_\alpha\| \leq 1, \|f_\beta\| \leq 1 \) such that \( \lim_\alpha \varphi(x, e_\alpha) = \lim_\beta \varphi(f_\beta, x) = x, \forall x \in X \), then \((X, \varphi)\) is an operator algebra.

**Proof.** For any element \( \xi = \begin{bmatrix} x \otimes 1 & \sum_{i=1}^{n} x_i \otimes y_i \\ 0 & 1 \otimes y \end{bmatrix} \in \begin{bmatrix} X \otimes_h \text{Cl} & X \otimes_h X \\ O & \text{Cl} \otimes_h X \end{bmatrix} \),
\[
\Gamma_\varphi \left( \begin{bmatrix} x \otimes 1 & \sum_{i=1}^{n} x_i \otimes y_i \\ 0 & 1 \otimes y \end{bmatrix} \right) = \begin{bmatrix} x & \sum_{i=1}^{n} \varphi(x_i, y_i) \\ 0 & \varphi(f_\beta, y) \end{bmatrix}
\]
\[
= \lim_\alpha \lim_\beta \left( \begin{bmatrix} \varphi(x, e_\alpha) & \sum_{i=1}^{n} \varphi(x_i, y_i) \\ 0 & \varphi(f_\beta, y) \end{bmatrix} \right)
= \lim_\alpha \lim_\beta \tilde{\varphi}_2 \left( \begin{bmatrix} 1 \otimes 1 & 0 \\ 0 & f_\beta \otimes 1 \end{bmatrix} \begin{bmatrix} x \otimes 1 & \sum_{i=1}^{n} x_i \otimes y_i \\ 0 & 1 \otimes y \end{bmatrix} \begin{bmatrix} 1 \otimes e_\alpha & 0 \\ 0 & 1 \otimes 1 \end{bmatrix} \right),
\]
where \( \tilde{\varphi} \) is the linear mapping \( X \otimes_h X \rightarrow X \) corresponding to the bilinear mapping \( \varphi : X \times X \rightarrow X \), and \( \tilde{\varphi}_2 : \text{M}_2(X \otimes_h X) \rightarrow \text{M}_2(X); \tilde{\varphi}_2((\zeta_{i,j})) := (\varphi(\zeta_{i,j})) \). Also we abused notation, and regarded \( I(S_X) \otimes_h I(S_X) \) as a subset of \( I(S_X) \ast_1 I(S_X) \) by the canonical complete isometry as in the proof of Theorem 4.1 (iii) \( \Rightarrow \) (ii), and the product between elements of
\[\text{[3]In this case, (X, \varphi) has a two-sided contractive approximate identity.} \]


Masayoshi Kaneda: Department of Mathematics, University of California, Irvine, CA 92697-3875 U.S.A.

E-mail address: mkaneda@math.uci.edu, [http://www.math.uci.edu/~mkaneda/](http://www.math.uci.edu/~mkaneda/)