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# Invertible Composition Operators on $H^p$

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The weakly closed algebras generated by certain sets of composition operators are shown to be reflexive. A structure theorem for invertible composition operators on $H^2$ is obtained and used to show that such operators are reflexive. The structure theorem shows that invertible hyperbolic composition operators are similar to cosubnormal operators built up from bilateral weighted shifts. Another consequence of the structure theorem is that the composition operators induced by hyperbolic disc automorphisms are universal. Thus the general invariant subspace problem for Hilbert space operators is contained in the problem of determining the invariant subspace lattices of these operators. © 1987 Academic Press, Inc.

## 1. INTRODUCTION

Each analytic function $\phi$ that maps the unit disk into itself induces a composition operator $C_\phi$ on the Hardy space $H^p$ $(p \geq 1)$; $C_\phi$ is defined by $(C_\phi f)(z) = f(\phi(z))$ for $f \in H^P$ and $|z| < 1$. The study of composition operators, which began with the work of Ryff [28], Nordgren [20], and Schwartz [30], has generated an extensive literature (see [21] and [8]).

Our main results in this paper concern the structure of certain com-

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position operators and reflexivity of algebras generated by composition operators. Cima and Wogen [5] showed that the weakly closed algebra generated by the group of all invertible composition operators on $H^2$ is the set of all operators that leave the space of constant functions invariant. In Section 2 we give a simpler proof of the Cima–Wogen theorem and also give some answers to their question of which other groups of composition operators generate this algebra. As a consequence we obtain a strengthening of known results giving sufficient conditions that the linear span of a set of linear fractional transformations be uniformly dense in the disk algebra. In Section 3 we show that certain composition operators generate the algebra of upper triangular matrices relative to the standard basis of $H^p$. Section 4 contains a complete description of the common invariant subspaces of the backwards shift and a composition operator induced by an inner function. In Section 5 we derive a structure theorem for parabolic and hyperbolic composition operators and prove that the strongly closed algebra generated by a single invertible composition operator on $H^2$ is always reflexive, and in Section 6 we show that every hyperbolic composition operator on $H^2$ is cosubnormal and has universal translates. It follows that every operator on Hilbert space has an invariant subspace if and only if the minimal invariant subspaces of the operator $C_\phi$ for $\phi(z) = (2z-1)/(2-z)$ are one dimensional.

## 2. Algebras Generated by Invertible Composition Operators

The only nontrivial subspace of $H^p$ that is obviously invariant under all composition operators is the set of $\mathbb{C}$ of constant functions in $H^p$. Cima and Wogen proved in [5] that in fact $\mathbb{C}$ is the only nontrivial common invariant subspace of all the invertible composition operators. They also showed that the strongly closed unital algebra generated by the invertible composition operators on $H^2$ is $\operatorname{Alg}\{\{0\}, \mathbb{C}, H^2\}$, the algebra of all operators leaving $\{0\}$, $\mathbb{C}$ and $H^2$ invariant, and they raised the question of which subgroups of the group of invertible composition operators have this property. We will give a somewhat improved version of their theorem with a shorter proof and provide some information on their question.

Schwartz [30] showed that a composition operator $C_\phi$ is invertible if and only if $\phi$ is a disc automorphism, i.e., $\phi$ is a linear fractional transformation carrying the unit disc onto itself. A disc automorphism other than the identity either has one fixed point in the open unit disc, or one fixed point on the unit circle or two fixed points on the unit circle (see, e.g., [4, 12]). The three types are called elliptic, parabolic and hyperbolic respectively. We will apply the same labels to the composition operators induced by each of these types of disc automorphism. If $\phi$ is elliptic and there exists an $n$ such that the composite of $\phi$ with itself $n$ times is the identity, then $\phi$ is said to have finite order. Otherwise $\phi$ is said to have infinite order.

A basic fact we will use is that algebras which contain an infinite order elliptic composition operator also contain many projections. Define $P_n$ on $H^P$ for $1 \leqslant p < \infty$ as follows: if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $(P_n f)(z) = a_n z^n$.

LEMMA 2.1. *If a strongly closed algebra of operators on $H^P$ for $1 \leqslant p < \infty$ contains an infinite order elliptic disc automorphism, then it is similar to an algebra containing $P_n$ for $n = 0, 1, \ldots$.*

*Proof.* Let $\phi$ be an infinite order elliptic disc automorphism, and let $\mathfrak{A}$ be a strongly closed unital algebra of operators on $H^P$ that contains $C_\phi$. If $z_0$ is the fixed point of $\phi$ in the open unit disc and $\omega$ is a disc automorphism that moves $z_0$ to 0, then $C_\omega^{-1} C_\phi C_\omega = C_{\alpha z}$ where $|\alpha| = 1$ (see [20]). Since $\phi$ has infinite order, $\alpha$ is not a root of unity. Thus the similarity induced by $C_\omega$ transforms $\mathfrak{A}$ into an algebra containing $C_{\alpha z}$.

It will be shown that if a strongly closed algebra contains $C_{\alpha z}$, then it contains $P_n$ for $n = 0, 1, 2, \ldots$. This follows from a theorem of Wermer in the $H^2$ case (see [33]), and it follows from the mean ergodic theorem in the general case. The following elementary argument was shown to us by Don Hadwin. Let $C_n = \bar{\alpha}^n C_{\alpha z}$, and let $A_k = (1/k)(C_n + C_n^2 + \cdots + C_n^k)$. Then $\{A_k - P_n\}_{k=1}^{\infty}$ is a bounded sequence of operators on $H^P$ for $1 \leqslant p < \infty$. It is easy to see that $\{f \in H^P : \lim_{k \to \infty} (A_k - P_n) f = 0\}$ is a closed subspace of $H^P$ that contains $z^j$ for every $j$ and thus equals $H^P$. In other words, $P_n$ is the strong limit of $\{A_k\}$. Since $A_k$ is in the algebra for every $k$, it follows that $P_n$ is in the algebra.

THEOREM 2.2. *If $\phi$ is an infinite order elliptic disc automorphism, then, for $1 \leqslant p < \infty$, every strongly closed algebra of operators on $H^P$ containing $C_\phi$ is reflexive.*

*Proof.* By Lemma 2.1, if a strongly closed algebra contains $C_\phi$, then a similarity can be used to transform it to an algebra $\mathfrak{A}$ containing $P_n$ for $n = 0, 1, \ldots$. It suffices to prove that such an $\mathfrak{A}$ is reflexive.

The following argument is essentially the proof of Theorem 1 of [9]. Suppose $B$ is an operator such that every invariant subspace of $\mathfrak{A}$ is invariant under $B$. For arbitrary $n$, $Bz^n$ is in the cyclic subspace of $\mathfrak{A}$ determined by $z^n$, and thus there is a sequence $\{A_k\}$ in $\mathfrak{A}$ such that $Bz^n = \lim_{k \to \infty} A_k z^n$. It follows that $BP_n$ is the strong limit of $\{A_k P_n\}_{k=1}^{\infty}$, and hence $BP_n \in \mathfrak{A}$. Let $\sigma_k$ be the $k$th Cesaro mean of the series $\sum_{m=0}^{\infty} P_n$, i.e., $\sigma_k = \sum_{m=0}^{k} [(k+1-m)/(k+1)] P_n$. From the fact that $\{\sigma_k f\}$ converges to $f$ in $H^p$ for every $f$ in $H^p$ (see, e.g., [17]), it follows that $\{B\sigma_k\}$ converges strongly to $B$. Since $B\sigma_k \in \mathfrak{A}$, $B \in \mathfrak{A}$.

THEOREM 2.3. *Let $\phi$ be an elliptic disc automorphism of infinite order and let $\psi$ be any disc automorphism that does not commute with $\phi$ under composition. Then the only nontrivial common invariant subspace of $C_\phi$ and $C_\psi$ on $H^p$ ($1 \leqslant p < \infty$) is $\mathbb{C}$.*

*Proof.* Let $\mathfrak{A}$ be the strongly closed algebra generated by $C_\phi$ and $C_\psi$. As in the proof of Lemma 2.1, $\mathfrak{A}$ can be transformed by a similarity into an algebra containing $C_{\alpha z}$ and all the $P_n$. The transformed algebra has the constants as its only nontrivial invariant subspace if and only if the original one does, so from here on we let $\mathfrak{A}$ be the transformed algebra.

Let $\mathcal{M}$ be an invariant subspace of $\mathfrak{A}$ other than $\{0\}$ and $\mathbb{C}$. Thus $P_n \mathcal{M} \neq \{0\}$ for some $n > 0$, and since $\mathcal{M}$ is invariant under $P_n$, $\mathcal{M}$ contains $z^n$. Thus $P_1 C_\psi z^n$ is also in $\mathcal{M}$. But $P_1 C_\psi z^n = P_1 \psi^n = az$, where $a = (D\psi^n)(0)$, $D$ being the differentiation operator. Since neither $\psi(0)$ nor $\psi'(0)$ is 0, it follows from the chain rule that $a \neq 0$, and hence $z \in \mathcal{M}$. Thus $\psi = C_\psi z \in \mathcal{M}$ and $P_n \psi \in \mathcal{M}$ for every $n$, and, since all the Taylor coefficients of $\psi$ are nonzero, $z^n \in \mathcal{M}$ for every $n$. We have shown that $\mathcal{M} = H^p$.

We are grateful to M. D. Choi for showing us part of the above simplification of our earlier simplification of the proof.

COROLLARY 2.4. *If $\phi$ is an infinite order elliptic disc automorphism and $\psi$ is any disc automorphism that does not commute with $\phi$ under composition, then the weakly closed algebra generated by $C_\phi$ and $C_\psi$ is* $\mathrm{Alg}\{\{0\}, \mathbb{C}, H^p\}$.

*Proof.* This is an immediate consequence of Theorems 2.2 and 2.3. It strengthens the result of [5], which was obtained as a consequence of [24].

We remark that Cima and Wogen obtain the following corollary (an improvement of a result of Fisher [11]): the uniformly closed linear span of the set of all disc automorphisms is the disc algebra, the set of functions continuous on the closed unit disc and analytic on its interior. Our Corollary 2.4 implies another improvement.

COROLLARY 2.5. *Let $\phi$ be an infinite order elliptic disc automorphism and $\psi$ be any disc automorphism that does not commute with $\phi$. If $\mathcal{G}$ is the group of disc automorphisms generated by $\phi$ and $\psi$, then the uniform closure of the linear span of $\mathcal{G}$ is the disc algebra.*

*Proof.* We begin by noting that the proof of Cima and Wogen [5] applies to the case $\mathcal{G}$ contains all the disc automorphisms. In our case $\mathcal{G}$, being countable, is a proper set of disc automorphisms, but, as we show below, the uniform closure $\bar{\mathscr{G}}$ of $\mathscr{G}$ contains all disc automorphisms. (Thus Theorem 2.3 can be obtained as a consequence of the result of Cima and Wogen, but our proof is considerably shorter.) For this it suffices to consider the case where $\phi$ is an irrational rotation: $\phi(z) = \alpha z$ with $|\alpha| = 1$. Since the powers of $\alpha$ are uniformly dense in the unit circle, it follows that $\bar{\mathscr{G}}$ contains all rotations. Since $\psi$ does not commute with $\phi$, we have

$$\psi(z) = e^{it}(z - a)/(1 - \bar{a}z)$$

with $0 < |a| < 1$. Composing $\psi$ with $az/|a|$ on the right and $e^{-it}\bar{a}z/|a|$ on the left, we see that $\bar{\mathscr{G}}$ contains $\omega$ here

$$\omega(z) = (z - r)/(1 - rz)$$

and $r = |a|$. On composing $\omega$ with itself sufficiently many times, we see that $\bar{\mathscr{G}}$ contains transformations of the same form as $\omega$ but with $r$ arbitrarily close to 1. If

$$\omega_\rho(z) = (z - re^{i\rho})/(1 - re^{-i\rho}z),$$

then $\omega_\rho \in \bar{\mathscr{G}}$. A calculation shows that $\omega \circ \omega_\rho$ has the same form as $\psi$, with $|a|$ depending continuously on $\rho$ and covering the range from 0 to $2r/(1 + r^2)$. It follows that $\bar{\mathscr{G}}$ contains $(z - s)/(1 - sz)$ for all $s$ with $0 < s < 1$, and hence $\bar{\mathscr{G}}$ contains all disc automorphisms.

Now the proof given by Cima and Wogen [5, p. 1239] can be applied to show that the linear span of $\bar{\mathscr{G}}$ is dense in the disc algebra.

COROLLARY 2.6. *Let $\mathscr{S}$ be a nonabelian group of disc automorphisms. The composition operators induced by $\mathscr{S}$ generate* Alg$\{\{0\}, \mathbb{C}, H^p\}$ *as a strongly closed algebra if $\mathscr{S}$ contains an infinite order elliptic disc automorphism.*

*Proof.* This follows immediately from Corollary 2.4.

We remark that there exist nonabelian groups of composition operators having two dimensional invariant subspaces. Let

$$\phi(z) = -z$$

and

$$\psi(z) = \frac{z + r}{1 + rz}$$

for any fixed nonzero $r$ between $-1$ and 1. Then $\phi$ is elliptic of order two and $\psi$ is hyperbolic, so $\phi$ and $\psi$ generate a nonabelian group of disc automorphisms. For $a \in \mathbb{C}$ and $-\frac{1}{2} < \operatorname{Re} \alpha < \frac{1}{2}$ and $\beta(z) = i(1+z)/(1-z)$, put

$$f_{\pm}(z) = \exp[\pm \alpha \log \beta(z)].$$

Then $f_{\pm} \in H^2$ and $f_{\pm}$ are eigenvectors for $C_\psi$ corresponding to eigenvalues $((1+r)/(1-r))^{\pm\alpha}$ (see [20]). Note $\beta \circ \phi = -1/\beta$, so $f_{\pm} \circ \phi = e^{\pm \alpha \pi i} f_{\mp}$. Hence the two-dimensional subspace spanned by $f_+$ and $f_-$ is invariant under both $C_\phi$ and $C_\psi$ and under the group they generate.

The operators $C_\phi$ can be defined on $L^p(m)$, where $m$ is normalized Lebesque measure on the unit circle in $\mathbb{C}$, as well as on $H^p$ (see [20]), and minor modification of our proofs yield analogous results in this setting. We content ourselves with stating two results.

THEOREM 2.7. *Let $\phi$ be an infinite order elliptic disc automorphism, and let $\psi$ be any disc automorphism that does not commute with $\phi$ under composition. The only nontrivial common invariant subspaces of $C_\phi$ and $C_\psi$ on $L^p(m)$ $(1 \leqslant p < \infty)$ are $\mathbb{C}$, $H^p$, and $H^{p*}$, where $H^{p*}$ is the set of complex conjugates of functions in $H^p$.*

COROLLARY 2.8. *If $\phi$ and $\psi$ are as in Theorem 2.7, then the strongly closed algebra generated by $C_\phi$ and $C_\psi$ is* $\operatorname{Alg}\{\{0\}, \mathbb{C}, H^p, H^{p*}, L^p(m)\}$.

## 3. Other Reflexive Algebras

Lemma 2.1 and Theorem 2.2 also yield the following two results.

THEOREM 3.1. *Let $\lambda$ be a nonzero complex number in the closed unit disc that is not a root of unity. Also let $a$ and $b$ be nonzero complex numbers that satisfy $|a| + |b| \leqslant 1$. Then for $1 \leqslant p < \infty$ the strongly closed algebra generated by $\{1, C_{\lambda z}, C_{az+b}\}$ is the algebra of all operators in* $\mathfrak{B}(H^p)$ that leave the subspaces $\mathcal{M}_k = \bigvee_{n=0}^{k} \{z^n\}$ *invariant for* $k = 0, 1, 2, \ldots$.

*Proof.* If $\mathfrak{A}$ is the strongly closed algebra generated by $\{1, C_{\lambda z}, C_{az+b}\}$, then $\mathfrak{A}$ contains all the diagonal operators, by Lemma 2.1. (The case where $|\lambda| < 1$ is elementary.) Consequently, the invariant subspaces of $\mathfrak{A}$ are all spanned by basis vectors $z^k$. If $\mathcal{M}$ is invariant under $\mathfrak{A}$ and $z^k \in \mathcal{M}$, then $\mathcal{M}$ contains $C_{az+b}\, z^k = (az+b)^k$. Since neither $a$ nor $b$ is zero, $C_{az+b} z^k$ is not orthogonal to any basis vector $z^j$ with $j \leqslant k$. Hence $\mathcal{M}$ contains $\mathcal{M}_k$ whenever it contains $z^k$, and thus the invariant subspace lattice of $\mathfrak{A}$ consists of $\{0\}$, $H^p$ and $\mathcal{M}_k$ for $k = 0, 1, 2, \ldots$. By Theorem 2.2 $\mathfrak{A}$ is reflexive, and thus $\mathfrak{A}$ contains all upper triangular matrices in $\mathfrak{B}(H^p)$.

THEOREM 3.2. *Suppose* $\lambda$, $a$, $b$ *are as above and let* $C_\psi$ *be a composition operator with* $\psi$ *not linear. Then the weakly closed algebra generated by* $\{1, C_{\lambda z}, C_{az+b}, C_\psi\}$ *is* $\mathrm{Alg}\{\{0\}, \mathbb{C}, H^p\}$.

*Proof.* As above, we need only show that the only nontrivial common invariant subspace of $C_{\lambda z}$, $C_{az+b}$, and $C_\psi$ is $\mathbb{C}$. Suppose $\mathcal{M}$ is invariant under $C_{\lambda z}$, $C_{az+b}$, and $C_\psi$, so $\mathcal{M} = \mathcal{M}_k$ for some $k$, as above. If $k \geqslant 1$, then we have that $\psi = C_\psi e_1 \in \mathcal{M}_k$, and thus $\psi$ is a polynomial. It is not linear by hypothesis, so $k > 1$. Thus $\psi^k = C_\psi z^k \in \mathcal{M}_k$. But $\psi^k$ has degree greater than $k$, which is impossible, and hence the only common invariant subspace is $\mathbb{C}$.

## 4. Common Invariant Subspaces of Certain Operators and the Backward Shift

In this section we will consider an inner $\phi$ and determine the invariant subspaces that $C_\phi$ has in common with the adjoint of the unilateral shift $S$ ($Sf(z) = zf(z)$ on $H^2$). If $\phi$ has a fixed point in the open unit disc, then $C_\phi$ is similar to an isometry [20] and all the invariant subspaces of $C_\phi$ are known via the Beurling–Lax–Halmos theory. We will write $\phi_0$ for $\phi(0) = (\phi, 1)$.

THEOREM 4.1. *Suppose* $\phi$ *is inner and* $\phi_0 \neq 0$. *Then the common invariant subspaces of* $C_\phi$ *and* $S^*$ *are the subspaces* $H^2 \ominus zgH^2$ *where* $g$ *is inner and* $g \circ \phi$ *is a divisor of* $g$.

*Proof.* Let $\mathcal{M}$ be a proper subspace of $H^2$ that is invariant under both $C_\phi$ and $S^*$. By Beurling's theorem, $\mathcal{M} = H^2 \ominus \psi H^2$ for some nonconstant inner function $\psi$. Since $S^*\phi \in \mathcal{M}$ it follows $C_\phi S^*\psi \in \mathcal{M}$. We have

$$C_\phi S^*\psi = C_\phi \bar{z}(\psi - \psi_0) = \bar{\phi}(\psi \circ \phi - \psi_0),$$

where $\psi_0 = (\psi, 1)$. Consequently

$$\bar{\phi}(\psi \circ \phi - \psi_0) \perp \psi H^2:$$

thus

$$\bar{\psi}\bar{\phi}(\psi \circ \phi - \psi_0) \perp H^2,$$

and hence

$$\psi\phi(\overline{\psi \circ \phi} - \overline{\psi_0}) = zh$$

for some $h$ in $H^2$. This implies

$$\frac{\psi\phi}{\psi \circ \phi} = \bar{\psi}_0 \psi\phi + zh, \qquad (*)$$

and taking inner products of both sides of the above with the constant functions 1 yields

$$\frac{\psi_0 \phi_0}{\psi(\phi_0)} = |\psi_0|^2 \phi_0.$$

If $\psi_0 \neq 0$, then, since $\phi_0 \neq 0$, we would conclude on cancelling $\psi_0$ and $\phi_0$ that $1 = \bar{\psi}_0 \psi(\phi_0)$, which by the maximum principle would imply $\psi$ is constant. Thus $\psi_0 = 0$, and $\psi = zg$ for inner $g$. By $(*)$,

$$zg\phi = zh\phi(g \circ \phi),$$

which implies $g \circ \phi$ is a divisor of $g$.

Conversely, suppose $\mathcal{M} = H^2 \ominus zgH^2$, where $g$ is inner and

$$g = \omega(g \circ \phi)$$

for some inner $\omega$. If $f \perp zgH^2$, then $\bar{g}f \perp zH^2$ and hence $g\bar{f} \in H^2$. Consequently, $g \circ \phi \overline{f \circ \phi} \in H^2$, which implies $f \circ \phi \perp z(g \circ \phi) H^2$. Since $z(g \circ \phi) H^2$ includes $\omega z(g \circ \phi) H^2$, $f \circ \phi$ is orthogonal to $\omega z(g \circ \phi) H^2 = zgH^2$. Hence $\mathcal{M}$ is $C_\phi$-invariant.

Corollary 4.2. *Let $C_\phi$ be invertible and let $g$ be an inner function. Then the following are equivalent*:

(1) *$gH^2$ is doubly invariant under $C_\phi$, i.e., is invariant under both $C_\phi$ and $C_\phi^{-1}$;*

(2) *$H^2 \ominus zgH^2$ is doubly invariant under $C_\phi$;*

(3) *$g$ is an eigenvector of $C_\phi$.*

*Proof.* Given (1), $C_\phi g = gh$ and $C_\phi^{-1} g = gh_1$. This implies $g \circ \phi$ and $g \circ \phi^{-1}$ both divide $g$, which by the theorem implies $H^2 \ominus zgH^2$ is invariant under both $C_\phi$ and $C_\phi^{-1}$, and conversely. Thus (1) and (2) are equivalent. Also, given (1), $h$ and $h_1$ are inner and $g = (g \circ \phi)(h_1 \circ \phi) = gh(h_1 \circ \phi)$, which implies $h(h_1 \circ \phi) = 1$, and thus $h$ is constant. Hence $g$ is an eigenvector. Thus (1) implies (3), and the converse is trivial.

## 5. Structure and Reflexivity of Invertible Composition Operators

In this section we will examine composition operators on $H^2$ induced by hyperbolic and parabolic disc automorphisms and show that they are reflexive. The only other invertible composition operators are those induced by elliptic disc automorphisms, and it is easy to see that they are reflexive. For if $\phi$ is a disc automorphism with a fixed point $z_0$ in the open unit disc, then $C_\phi$ is similar to a composition operator of the form $C_{\alpha z}$ where $|\alpha| = 1$ (see [20]). The operator $C_{\alpha z}$ is reflexive because it is unitary, and all normal operators are reflexive [29].

Let us describe canonical hyperbolic and parabolic disc automorphisms (see [4, 12]). Define a linear fractional transformation $\beta$ that carries the unit disc to the upper half plane by

$$\beta(z) = i(1+z)/(1-z).$$

The hyperbolic disc automorphism with fixed points $-1$ and 1 such that 1 is attracting may be obtained by choosing $a > 1$ and defining $\phi$ by $\phi = \beta^{-1}(a\beta)$; thus $\phi(z) = (z+r)/(1+rz)$ where $r = (a-1)/(a+1)$. A parabolic disc automorphism with fixed point 1 is obtained by choosing real $s \neq 0$ and defining $\phi$ by $\phi = \beta^{-1}(\beta + s)$; thus $\phi(z) = [(s-2i)z - s]/(sz - s - 2i)$. Given an arbitrary hyperbolic or parabolic $\phi$, there exists a disc automorphism $\omega$ that moves the fixed points of $\phi$ to the special ones. Then $C_\omega^{-1} C_\phi C_\omega = C_{\omega \circ \phi \circ \omega^{-1}}$, i.e., $C_\phi$ is similar to a composition operator of the special type.

For any disc automorphism $\phi$, define $\phi^{(n)}$ as follows:

$$\phi^{(0)}(z) = z$$

and for $n = 1, 2, \ldots$,

$$\phi^{(n)} = \phi \circ \phi^{(n-1)} \qquad \text{and} \qquad \phi^{(-n)} = \phi^{-1} \circ \phi^{(-n+1)}.$$

Let $z_n = \phi^{(n)}(0)$, so in the hyperbolic case

$$z_n = (a^n - 1)/(a^n + 1), \tag{1}$$

and in the parabolic case

$$z_n = ns/(ns + 2i). \tag{2}$$

We record two facts for later use. From (1) and (2), $|z_n| = |z_{-n}|$, and hence for $n \neq 0$

$$\phi^{(n)}(z) = -\frac{z_n}{z_{-n}} \frac{z - z_{-n}}{1 - \bar{z}_{-n} z}. \tag{3}$$

Consequently

$$\phi^{(n)}(z)-z_n=-\frac{z_n}{z_{-n}}(1-|z_n|^2)\frac{z}{1-\bar{z}_{-n}z}. \tag{4}$$

LEMMA 5.1. *Let $\{z_n\}_{n=-\infty}^{\infty}$ be the orbit of zero under the iterates of a hyperbolic or parabolic disc automorphism. Then $\{z_n\}$ is an interpolating sequence.*

*Proof.* It suffices to consider the special cases where the fixed points are $-1$ and $1$ for hyperbolic $\phi$ and $1$ for parabolic $\phi$. By Carleson's theorem [3], it suffices to show that $\prod_{n=-\infty, n\neq k}^{\infty} |(z_n-z_k)/(1-\bar{z}_k z_n)|$ is bounded away from zero independently of $k$. Note that by (3)

$$\left|\frac{z_n-z_k}{1-\bar{z}_k z_n}\right|=|\phi^{-k}(z_n)|=|z_{n-k}|,$$

and thus all that needs to be shown is that $\prod_{n=1}^{\infty} |z_n|>0$, or, equivalently $\{z_n\}$ is a Blaschke sequence. In the hyperbolic case

$$1-|z_n|=2/(a^{|n|}+1)$$

and in the parabolic case

$$\begin{aligned}1-|z_n|&=(|ns+2i|-|ns|)/|ns+2i|\\&=4/[|ns+2i|\,(|ns+2i|+|ns|)]\\&\leqslant 2/n^2s^2.\end{aligned}$$

Hence $\{z_n\}$ is a Blaschke sequence in either case. This establishes the lemma.

Before proceeding to the reflexivity results, we will examine the structure of hyperbolic and parabolic composition operators. Let $B$ be the Blaschke product with $\{z_n\}$ as its sequence of zeros. Thus $B=\prod_{n\in\mathbb{Z}}\lambda_n\phi^{(n)}$, where $\lambda_0=1$ and $\lambda_n=\bar{z}_n/|z_n|$ if $n\neq 0$. It follows that

$$B\circ\phi=\tau B, \tag{5}$$

where $\tau=\prod_{n\in\mathbb{Z}}\lambda_n/\lambda_{n+1}=(\lim_{n\to-\infty}\lambda_n)/(\lim_{n\to\infty}\lambda_n)$. Thus $\tau=-1$ in the hyperbolic case and $\tau=1$ in the parabolic case. Because of the $-1$, the hyperbolic case contains some complications that can be avoided in the parabolic case, but we will treat both cases in the same way as much as possible.

Let $\mathscr{K}_0=H^2\ominus zBH^2$. By Theorem 4.1, $\mathscr{K}_0$ is invariant under both $C_\phi$ and $C_\phi^{-1}$. This also follows from the fact, which we now demonstate, that

$\mathscr{K}_0$ is the subspace spanned by $\{\phi^{(n)}: n \in \mathbb{Z}\}$. Since taking the inner product of an $H^2$ function with $1/(1-\bar{w}z)$ is the same as evaluating the function at $w$, it follows that $z/(1-\bar{z}_{-n}z)$ is in $\mathscr{K}_0$. Hence by (4), every $\phi^{(n)}$ is in $\mathscr{K}_0$. Since $\|1/(1-\bar{w}z)\| = (1-|w|^2)^{-1/2}$, it also follows from (4) that $\lim_{n\to\infty}(\phi^{(n)} - z_n) = 0$. Thus $\lim_{n\to\infty}\phi^{(n)} = 1$, and hence the subspace spanned by $\{\phi^{(n)}: n \in \mathbb{Z}\}$ contains the constant functions. Suppose $f$ is orthogonal to every $\phi^{(n)}$. Then $f$ is orthogonal to both the constants and to $\phi^{(0)}(z) = z$, so $f$ has a zero of order two at the origin. Further, it follows from (4) that $f$ is orthogonal to $z/(1-\bar{z}_n z)$ for every $n$. Thus

$$f(z_n)/z_n = (f/z, 1/(1-\bar{z}_n z)) = (f, z/(1-\bar{z}_n z)) = 0$$

for every $n \neq 0$, and it follows that $f$ has a zero at every $z_n$. Hence $f$ is a multiple of $zB$; i.e., $f$ is orthogonal of $\mathscr{K}_0$. We have shown that $\mathscr{K}_0$ is spanned by the $\phi^{(n)}$. Understanding how $C_\phi$ behaves on $\mathscr{K}_0$ is the key to the other results. Note that $\mathscr{K}_0$ includes the constants, so we may write $\mathscr{K}_0 = \mathbb{C} \oplus \mathscr{L}$, where $\mathscr{L}$ is semi-invariant for $C_\phi$.

THEOREM 5.2. *Suppose $\phi$ is a disc automorphism, and $\phi$ is parabolic with fixed point 1 or $\phi$ is hyperbolic with fixed points 1 and $-1$. Let $B$ be the Blaschke product with zeros $\phi^{(n)}(0)$, $n \in \mathbb{Z}$, and let $\mathscr{K}_0$ be the subspace spanned by $\{\phi^{(n)}: n \in \mathbb{Z}\}$. Then $\mathscr{K}_0 = (zBH^2)^\perp = \mathbb{C} \oplus \mathscr{L}$, and the compression of $C_\phi$ to $\mathscr{L}$ is similar to a bilateral weighted shift.*

*Proof.* The fact that $(zBH^2)^\perp$ is equal to the span of $\{\phi^{(n)}: n \in \mathbb{Z}\}$ was shown above. Let $W$ be the compression of $C_\phi$ to $\mathscr{L}$, i.e., if $Q$ is the orthogonal projection of $\mathscr{K}_0$ onto $\mathscr{L}$, then $W = QC_\phi | \mathscr{L}$. Obviously $C_\phi \phi^{(n)} = \phi^{(n+1)}$, and by the $C_\phi$-invariance of the projection $1-Q$, we have

$$WQ\phi^{(n)} = QC_\phi Q\phi^{(n)} = QC_\phi \phi^{(n)} = Q\phi^{(n+1)}.$$

Thus if $f_n = (1/\|Q\phi^{(n)}\|)\, Q\phi^{(n)}$, and if

$$w_n = \|Q\phi^{(n+1)}\| / \|Q\phi^{(n)}\|, \tag{6}$$

then $Wf_n = w_n f_{n+1}$. Since $Q\phi^{(n)} = \phi^{(n)} - z_n$, it follows from (4) that

$$\|Q\phi^{(n)}\| = (1-|z_{-n}|^2)^{-1/2} \tag{7}$$

and

$$f_n = (-z_n/z_{-n})(1-|z_{-n}|^2)^{1/2} z/(1-\bar{z}_{-n}z).$$

By Lemma 5.1, $\{z_n\}$ is an interpolating sequence, and hence a result of Shapiro and Shields [31] implies that $\{(1-|z_{-n}|^2)^{1/2}/(1-\bar{z}_{-n}z)\}$ is similar to an orthonormal set (see also Cowen [6, p. 23]). Since multiplication by $z$ is an isometry and $|-z_n/z_{-n}|=1$, it follows that $\{f_n\}$ is similar to an orthonormal set. Hence $W$ is similar to the bilateral weighted shift with weight sequence $\{w_n\}$.

We remark that one could avoid an appeal to interpolation theory to obtain the similarity of $\{f_n\}$ to an orthonormal set if one could show the Grammian of $\{f_n\}$ is a boundedly invertible matrix. Calculation shows that this Grammian is a Laurent matrix in the hyperbolic case and unitarily equivalent to such a matrix in the parabolic case. The Laurent matrix in the hyperbolic case is induced by the function whose Fourier series is $\sum_{n=-\infty}^{\infty}[2/(a^{n/2}+a^{-n/2})]\,e^{in\theta}$ and in the parabolic case by the function $-4\pi e^{-(2/s)\theta}/s(1-e^{-4\pi/s})$ for $0\leqslant\theta<2\pi$. It is easy to see that both these functions are in $L^\infty(0, 2\pi)$ and that the latter is invertible. Unfortunately, it is not immediate that the former is nonvanishing and hence invertible (since it is continuous) for every positive $a\,(\neq 1)$, but of course the fact established above that $\{f_n\}$ is similar to an orthonormal set and hence its Grammian is invertible implies that it is. It would be desirable to have a direct proof.

In the following corollaries we identify $W$ and draw some conclusions concerning $C_\phi|\mathscr{K}_0$.

COROLLARY 5.3. *Let $\phi$ be a hyperbolic disc automorphism with fixed points* 1 *and* $-1$ *and $\phi(0)=(a-1)/(a+1)$ for $a>1$. The compression $W$ of $C_\phi$ to $\mathscr{L}$ is similar to the bilateral weighted shift with weight sequence $\{w_n\}$ where $w_n=\sqrt{a}\,(a^n+1)/(a^{n+1}+1)$.*

*Proof.* The only thing that still needs to be verified is the formula for $w_n$, and this follows easily from (6), (7), and (1).

COROLLARY 5.4. *If $\phi$ satisfies the hypotheses of Corollary* 5.3, *then $C_\phi|\mathscr{K}_0$ has a spanning set of eigenvectors, and $(C_\phi-\lambda)\,\mathscr{K}_0=\mathscr{K}_0$ for every $\lambda$ satisfying $1\sqrt{a}<|\lambda|<\sqrt{a}$.*

*Proof.* Since $C_\phi$ is the identity on $\mathbb{C}$, we have $C_\phi|\mathscr{K}_0=\left[\begin{smallmatrix}1 & X\\ 0 & W\end{smallmatrix}\right]$ relative to the decomposition $\mathscr{K}_0=\mathbb{C}\oplus\mathscr{L}$. The sequence $\{w_n\}$ is decreasing and has limits $\sqrt{a}$ at $-\infty$ and $1/\sqrt{a}$ at $\infty$. In what follows we will make use of results on weighted shifts due to Gellar [13, 14], Kelley [18], and Ridge [26], but for convenience we will cite references to Shield's paper [32]. By Theorem 9, p. 71, and Proposition 15, p. 72 of [32], the point spectrum of $W$ includes the interior of the zero centered annulus with inner radius $1/\sqrt{a}$ and outer radius $\sqrt{a}$. Let $T^*$ be the weighted shift similar to $W$. Then $T$ is also a weighted shift, and $T$ may be represented as multiplication by $z$ on a weighted sequence space $L^2(\beta)$ (see [32, Sect. 3]). By Theorem 10, p. 79 of [32], all eigenvalues of $T^*$ are bounded point evaluations on $L^2(\beta)$, the reproducing kernels associated with these eigenvalues being the eigenvectors of $T^*$ (see [32, Sect. 6]). Since every vector in $L^2(\beta)$ can be represented as an analytic function on the annulus, the only vector in $L^2(\beta)$ that can be orthogonal to all the eigenvectors, or to any "large" set of eigenvectors, is 0. Thus the eigenvectors of $W$ corresponding to eigenvalues $\lambda \neq 1$ span $\mathscr{L}$. If $\lambda = 1$, then $1 \otimes 0$ is a corresponding eigenvalue for $C_\phi$ in $\mathscr{K}_0$. If $\lambda$ is an eigenvalue for $W$, $\lambda \neq 1$, and $f$ is a corresponding eigenvalue, then $\mu \oplus f$ is a corresponding eigenvector for $C_\phi$ in $\mathscr{K}_0$ provided $\mu = Xf/(\lambda - 1)$. By the preceding remarks, the eigenvectors $1 \oplus 0$ and all the $\mu \oplus f$ span $\mathscr{K}_0$.

To see that $1/\sqrt{a} < |\lambda| < \sqrt{a}$ implies $(C_\phi - \lambda) \mid \mathscr{K}_0$ is onto we first observe that $W - \lambda$ is onto. For by Proposition 15, p. 72, and Theorem 7, p. 70 of [32], the approximate point spectrum of $W^*$ consists of the circles with center 0 and radii $\sqrt{a}$ and $1/\sqrt{a}$. Thus $W^* - \bar{\lambda}$ has a left inverse, which implies $W - \lambda$ has a right inverse, and hence $W - \lambda$ is onto. If $\lambda \neq 1$, then clearly the constants are in $(C_\phi - \lambda)\mathscr{K}_0$, and thus $(C_\phi - \lambda)\mathscr{K}_0 \supset \mathbb{C} \oplus (W - \lambda)\mathscr{L} = \mathscr{K}_0$.

The case $\lambda = 1$ requires a slightly different argument, for although the above shows that $W - 1$ is onto, it is not immediate that the constants are in $(C_\phi - 1)\mathscr{K}_0$. For this we need only show that if $h$ is an eigenvector of $W$ corresponding to eigenvalue 1, then $Xh \neq 0$. Suppose $h$ is in $\mathscr{L}$, $Wh = h$ and $Xh = 0$. This implies $h \circ \phi = h$, and hence $h(z_n) = h(0) = 0$ for every $n$. It follows that $B$ divides $h$, and thus $zh \perp \mathscr{K}_0$. But $H^2 = \mathbb{C} \oplus \mathscr{L} \oplus zB\mathbb{C} \oplus z^2BH^2$, and thus $z\mathscr{K}_0 = \mathscr{L} \oplus zB\mathbb{C}$. It follows that $z\mathscr{K}_0 \cap \mathscr{K}_0^\perp = (\mathscr{L} \oplus zB\mathbb{C}) \cap (zBH^2)^\perp = zB\mathbb{C}$, i.e., $zh = \alpha zB$ for some $\alpha$ in $\mathbb{C}$. Hence $h = \alpha B$, which implies $\alpha = 0$ since $h = h \circ \phi = \alpha B \circ \phi = -\alpha B$. Thus $h = 0$.

COROLLARY 5.5. *Let $\phi$ be a parabolic disc automorphism with fixed point 1 and $\phi(0) = s/(s + 2i)$ with real $s \neq 0$. The compression $W$ of $C_\phi$ to $\mathscr{L}$ is similar to the bilateral weighted shift with weight sequence $\{w_n\}$ where $w_n = [(n^2s^2 + 4)/((n+1)^2 s^2 + 4)]^{1/2}$.*

*Proof.* This follows from Theorem 5.2 and a simple calculation based on (6), (7), and (2).

COROLLARY 5.6. *If $\phi$ satisfies the hypotheses of Corollary 5.5, then $C_\phi \mid \mathscr{K}_0$ has a spanning set of eigenvectors.*

*Proof.* The bilateral shift with weight sequence $\{w_n\}$ has the unit circle for its spectrum (see [32, Theorem 9, p. 71 and Proposition 15, p. 72]). Corresponding to each $\lambda$ on the unit circle one obtains an eigenvector

$f_\lambda = \{a_n\}_{n\in\mathbb{Z}}$, where $a_n = \lambda^{-n}(n^2s^2+4)^{-1/2}$. The set of $f_\lambda$ with $\lambda \neq 1$ form a spanning set. For if $g = \{b_n\}$ is orthogonal to this set, then

$$\sum_{n=-\infty}^{\infty} b_n(n^2s^2+4)^{-1/2}\lambda^n = (g, f_\lambda) = 0.$$

Writing $\lambda = e^{i\theta}$, we see that the series is an absolutely convergent Fourier series, and hence it can vanish on a dense subset of the unit circle only if all its coefficients vanish, i.e., only if $g = 0$. As in the hyperbolic case, it now follows that $C_\phi | \mathscr{K}_0$ has a spanning set of eigenvectors. This completes the proof.

THEOREM 5.7. *Let $\phi$ be a parabolic or hyperbolic disc automorphism with fixed points in $\{-1, 1\}$, let $\mathscr{K}_0$ be the span of $\{\phi^{(n)}: n \in \mathbb{Z}\}$, and let $B$ be the Blaschke product with zeros $\phi^{(n)}(0)$. Define $\mathscr{K} = \mathscr{K}_0 + B\mathscr{K}_0$. Then $\mathscr{K} = (zB^2H^2)^\perp$, $B^n\mathscr{K}$ is invariant under $C_\phi$ for $n = 0, 1, 2, \ldots$, and $B^{2n}\mathscr{K} \perp B^{2m}\mathscr{K}$ whenever $|n-m| \geq 2$. If $\mathscr{M} = \sum_{n=0}^{\infty} \oplus B^{4n}\mathscr{K}$ and $\mathscr{N} = \sum_{n=0}^{\infty} \oplus B^{4n+2}\mathscr{K}$, then $\mathscr{M}$ and $\mathscr{N}$ are invariant under $C_\phi$, $C_\phi | \mathscr{M}$ and $C_\phi | \mathscr{N}$ are unitarily equivalent to inflations of $C_\phi | \mathscr{K}$, and $\mathscr{M} + \mathscr{N} = H^2$.*

*Proof.* Since $\mathscr{K}_0 = (zBH^2)^\perp$ (see the remarks preceding Theorem 5.2), it follows that $\mathscr{K}_0 \perp zB^2H^2$ and $B\mathscr{K}_0 \perp zB^2H^2$. Thus $\mathscr{K} \subset (zB^2H^2)^\perp$. To obtain the reverse inclusion observe first that

$$H^2 = \mathscr{K}_0 \oplus zBH^2 = (BH^2)^\perp \oplus B\mathscr{K}_0 \oplus zB^2H^2.$$

Also observe that $BH^2 \supset zBH^2$ implies $(BH^2)^\perp \subset \mathscr{K}_0$. Thus

$$(zB^2H^2)^\perp = (BH^2)^\perp \oplus B\mathscr{K}_0 \subset \mathscr{K}_0 + B\mathscr{K}_0,$$

which is the desired inclusion. Hence $\mathscr{K} = (zB^2H^2)^\perp$.

It is a consequence of (5) that if $M_B$ is the analytic Toeplitz operator of multiplication by $B$, then

$$C_\phi M_B = \tau M_B C_\phi,$$

and hence $C_\phi$ commutes with $M_B^2$. The last equality also implies that $B^n\mathscr{K}$ is invariant under $C_\phi$ for every $n$. Since $M_B$ is an isometry, the commutativity of $C_\phi$ with $M_B^2$ implies that $C_\phi | \mathscr{K}$ is unitarily equivalent to $C_\phi | B^{2n}\mathscr{K}$ for every $n$. If $k \geq 3$, then $B^k\mathscr{K} \subset zB^2H^2$, and it follows that $B^{2n}\mathscr{K} \perp B^{2m}\mathscr{K}$ whenever $|n-m| \geq 2$.

On putting $\mathscr{M} = \sum_{n=0}^{\infty} \oplus B^{4n}\mathscr{K}$ and $\mathscr{N} = \sum_{n=0}^{\infty} \oplus B^{4n+2}\mathscr{K}$, we obtain invariant subspaces for $C_\phi$ such that the restriction of $C_\phi$ to any summand of $\mathscr{M}$ or $\mathscr{N}$ is unitarily equivalent to $C_\phi | \mathscr{K}$. Hence $C_\phi | \mathscr{M}$ and $C_\phi | \mathscr{N}$ are unitarily equivalent to inflations, and thus $C_\phi | \mathscr{M}$ and $C_\phi | \mathscr{N}$ are reflexive (see [25, p. 179]). Finally, to see that $\mathcal{M}+\mathcal{N}=H^2$ observe that $(B^2H^2)^\perp \subset \mathcal{K}$, and hence

$$H^2=\sum_{n=0}^{\infty} \oplus B^{2n}(B^2H^2)^\perp \subset \sum_{n=0}^{\infty} B^{2n}\mathcal{K} \subset \mathcal{M}+\mathcal{N}.$$

COROLLARY 5.8. *If $\phi$ is a disc automorphism, then $C_\phi$ has a spanning set of eigenvectors.*

*Proof.* An operator has the asserted property if and only if it is similar to an operator with the property. Thus it suffices to consider elliptic $\phi$ with fixed point 0 and parabolic or hyperbolic $\phi$ with fixed points in $\{-1, 1\}$. In the elliptic case all the basis vectors $z^n$ are eigenvectors. Thus we may suppose $\phi$ is parabolic or hyperbolic, and Theorem 5.7 applies.

To see that $H^2$ is spanned by eigenvectors of $C_\phi$ it suffices to show that $\mathcal{M}$ and $\mathcal{N}$ are spanned by eigenvectors of $C_\phi$. Since $C_\phi|\mathcal{M}$ and $C_\phi|\mathcal{N}$ are unitarily equivalent to inflations of $C_\phi|\mathcal{K}$, it is enough to show that $\mathcal{K}$ is spanned by eigenvectors of $C_\phi$. Because of the relation $C_\phi M_B=\tau M_B C_\phi$, it will follow that $B\mathcal{K}_0$ is spanned by eigenvectors of $C_\phi$ if $\mathcal{K}_0$ is. Thus the problem reduces to showing that $\mathcal{K}_0$ is spanned by eigenvectors of $C_\phi$. But this is the content of Corollaries 5.4 and 5.6.

COROLLARY 5.9. *If $\phi$ is hyperbolic and $\lambda$ is in the interior of $\sigma(C_\phi)$, then $C_\phi-\lambda$ is onto.*

*Proof.* It suffices to consider the case where $\phi$ has fixed points $-1$ and 1, since every hyperbolic $C_\phi$ is similar to such a special one. In the notation of Corollary 5.3 the spectrum of $C_\phi$ is the zero centered annulus with inner radius $1/\sqrt{a}$ and outer radius $\sqrt{a}$ (see [20]). Hence for every $\lambda$ in the interior of $\sigma(C_\phi)$, Corollary 5.4 may be applied to obtain $(C_\phi-\lambda)\mathcal{K}_0=\mathcal{K}_0$ and also $(C_\phi+\lambda)\mathcal{K}_0=\mathcal{K}_0$. Thus by using $M_BC_\phi=-C_\phi M_B$, we see that

$$(C_\phi-\lambda)B\mathcal{K}_0=-M_B(C_\phi+\lambda)\mathcal{K}_0=B\mathcal{K}_0,$$

and it follows that $(C_\phi-\lambda)\mathcal{K}=\mathcal{K}$. Since $C_\phi|\mathcal{M}$ and $C_\phi|\mathcal{N}$ are unitarily equivalent to inflations of $C_\phi|\mathcal{K}$, and $H^2=\mathcal{M}+\mathcal{N}$, we obtain that $C_\phi-\lambda$ is onto, which completes the proof.

For $\phi$ parabolic or hyperbolic with fixed points in $\{-1, 1\}$ we know by Theorem 5.7 that $C_\phi|\mathcal{M}$ is unitarily equivalent to an inflation. Thus $C_\phi|\mathcal{M}$ is reflexive (see [25, p. 179]). These observations almost put in a position to use the following theorem to show that $C_\phi$ is reflexive in both the hyperbolic and parabolic cases.

THEOREM 5.10. *Let $\mathfrak{A}$ be a weakly closed unital algebra in $\mathfrak{B}(\mathcal{H})$ with invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that*

(a) *there exists an invertible operator* $T: \mathcal{M} \to \mathcal{N}$ *satisfying* $T(A \mid \mathcal{M}) = (A \mid \mathcal{N}) T$ *for all* $A$ *in* $\mathfrak{A}$,

(b) *the weak closure of* $\mathfrak{A} \mid \mathcal{M}$ *is reflexive,*

(c) $\mathcal{M}$ *is spanned by one dimensional invariant subspaces of* $\mathfrak{A}$, *and*

(d) $\mathcal{M} + \mathcal{N} = \mathcal{H}$.

*Then* $\mathfrak{A}$ *is reflexive.*

*Proof.* Suppose $B$ is an operator such that every invariant subspace of $\mathfrak{A}$ is $B$-invariant. Then by (b), there exists a net $\{A_\alpha\}$ in $A$ such that $\lim_\alpha A_\alpha \mid \mathcal{M} = B \mid \mathcal{M}$. Hence $\lim_\alpha T(A_\alpha \mid \mathcal{M}) T^{-1} = T(B \mid \mathcal{M}) T^{-1}$. It will suffice to show that $T(B \mid \mathcal{M}) T^{-1} = B \mid \mathcal{N}$. For then $\lim_\alpha A_\alpha \mid \mathcal{M} = B \mid \mathcal{M}$ and by (a), $\lim_\alpha A_\alpha \mid \mathcal{N} = B \mid \mathcal{N}$, which implies $\lim_\alpha A_\alpha = B$, by (d).

Consider an $f$ in $\mathcal{M}$ such that $\mathbb{C}f$ is invariant under $\mathfrak{A}$. If $A \in \mathfrak{A}$ and $Af = \lambda f$, then $ATf = TAf = \lambda Tf$, and consequently the restriction of $\mathfrak{A}$ to the subspace spanned by $f$ and $Tf$ consists only of scalar operators. Hence the restriction of $B$ to this subspace is also a scalar. Thus

$$BTf = BTf = (TBT^{-1})\, Tf.$$

By (c), the one-dimensional invariant subspaces of $\mathfrak{A}$ span $\mathcal{M}$, and hence $B \mid \mathcal{N} = T(B \mid \mathcal{M}) T^{-1}$. This completes the proof.

Our goal is to use the above theorem to prove that hyperbolic and parabolic disc automorphisms induce reflexive composition operators. We will in fact obtain a stronger result. An algebra $\mathfrak{A}$ of operators is called superflexive in case every weakly closed unital subalgebra of $\mathfrak{A}$ is reflexive, and an operator is called superreflexive in case the weakly closed unital algebra that it generates is superreflexive. Sarason [29] showed that normal operators and the unilateral shift are superreflexive. (See [15] and [19] for more on this concept.) Every inflation is in fact superreflexive (see [15]). Thus if $\mathcal{M}$ is the invariant subspace of $C_\phi$ described in Theorem 5.7, then $C_\phi \mid \mathcal{M}$ is superreflexive. It is easy to see that if hypothesis (b) of Theorem 5.10 is strengthened by changing reflexive to superreflexive, then the conclusion may be corresponding strengthened. Therefore the following holds.

THEOREM 5.11. *Let* $\mathfrak{A}$ *we weakly closed unital algebra in* $\mathfrak{B}(\mathcal{H})$ *with invariant subspaces* $\mathcal{M}$ *and* $\mathcal{N}$ *such that*

(a) *there exists an invertible operator* $T: \mathcal{M} \to \mathcal{N}$ *satisfying* $T(A \mid \mathcal{M}) = (A \mid \mathcal{N}) T$ *for all* $A$ *in* $\mathfrak{A}$,

(b) *the weak closure of* $\mathfrak{A} \mid \mathcal{M}$ *is superreflexive,*

(c) *$\mathscr{M}$ is spanned by one-dimensional invariant subspaces of $\mathfrak{A}$, and*

(d) $\mathscr{M} + \mathscr{N} = \mathscr{H}$.

*Then $\mathfrak{A}$ is superreflexive.*

THEOREM 5.12. *Every composition operator $C_\phi$ induced by a hyperbolic or parabolic disc automorphism $\phi$ is superreflexive.*

*Proof.* Let $\mathfrak{A}$ be the weakly closed unital algebra generated by a hyperbolic or parabolic composition operator. If $\mathscr{M}$ and $\mathscr{N}$ are the subspaces defined in Theorem 5.7, then condition (d) of Theorem 5.11 is satisfied. The restriction of $M_B^2$ to $\mathscr{M}$ is an invertible operator from $\mathscr{M}$ onto $\mathscr{N}$ satisfying $M_B^2(C_\phi \mid \mathscr{M}) = (C_\phi \mid \mathscr{N})(M_B^2 \mid \mathscr{M})$, so if we take $T = M_B^2 \mid \mathscr{M}$, then part (a) of the hypothesis of Theorem 5.11 is satisfied. As noted before the statement of Theorem 5.11, the weak closure of $\mathfrak{A} \mid \mathscr{M}$ is superreflexive, so (b) is satisfied. Hypothesis (c) follows from the proof of Corollary 5.8. Hence $C_\phi$ is superreflexive.

COROLLARY 5.3. *Every invertible composition operator is superreflexive.*

*Proof.* If $\phi$ is an elliptic disc automorphism, then $C_\phi$ is superreflexive by the discussion at the beginning of this section. The theorem covers the remaining cases.

## 6. SUBNORMALITY AND UNIVERSALITY OF HYPERBOLIC OPERATORS

We conclude this paper with two additional properties of composition operators induced by hyperbolic disc automorphisms. It will be shown they are similar to cosubnormal operators, and hence, by the result of Olin and Thompson [22] that all subnormal operators are reflexive, we obtain a second proof of reflexivity in this case. We also show that they have translates that are universal, so the invariant subspace problem can be reformulated as a problem about composition operators. Carl Cowen has obtained Theorem 6.1 by different methods.

THEOREM 6.1. *The adjoint of a composition operator induced by a hyperbolic disc automorphism is similar to a subnormal operator.*

*Proof.* If $\phi$ is a hyperbolic disc automorphism, then, as before, there is no loss of generality in assuming that the fixed points of $\phi$ are 1 and $-1$ and that 1 is attracting. Hence

$$\phi(z) = (z + r)/(1 + rz),$$

where $0<r<1$. The space $H^2$ may be regarded as a subspace of $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle with normalized Lebesgue measure, and $C_\phi$ may be extended to $L^2(\mathbb{T})$ in the obvious way. We will also denote the extension by $C_\phi$. The proof of subnormality depends on transforming $L^2(\mathbb{T})$ to $L^2(\mathbb{R})$, where $\mathbb{R}$ is the real line, and applying the Halmos–Bram criterion [16, 1].

We obtain a unitarity operator $\Omega$ from $L^2(\mathbb{T})$ to $L^2(\mathbb{R})$ as follows: for $f\in L^2(\mathbb{T})$ put

$$\Omega f(t)=f\circ\beta^{-1}(t)/\pi^{1/2}(t+i),$$

where $\beta$ is the mapping defined at the beginning of Section 5 (see [17]). If $\beta\circ\phi=a\beta$, then define $\psi$ on $\mathbb{R}$ by $\psi(t)=at$, and let $C_\psi$ be the operator on $L^2(\mathbb{R})$ defined by $C_\psi F=F\circ\psi$. Thus $\phi\circ\beta^{-1}=\beta^{-1}\circ\psi$, and for $f\in L^2(\mathbb{T})$ we obtain

$$\begin{aligned}\Omega C_\phi f(t)&=f\circ\phi\circ\beta^{-1}(t)/\pi^{1/2}(t+i)\\&=[(\psi(t)+i)/(t+i)]\,f\circ\beta^{-1}\circ\psi(t)/\pi^{1/2}(\psi(t)+i).\end{aligned}$$

Let $M$ be the operator "multiplication by $(at+i)/(t+i)$" on $L^2(\mathbb{R})$. We have $\Omega C_\phi=MC_\psi\Omega$, and thus it suffices to show that $C_\psi^*M^*$ is subnormal. To do this we employ the Halmos–Bram criterion: $T$ is subnormal if and only if $(T^{*m}T^l)_{l,m=0}^n\geqslant 0$ for every $n$ [16, 1].

Let $M_k$ be "multiplication by $(a^kt+i)/(t+i)$" on $L^2(\mathbb{R})$. We claim that if $T^*=MC_\psi$, then $T^{*m}=M_mC_\psi^m$. The case $m=1$ is obvious, so suppose the asserted formula is true for a given $m$. Note that if $\omega\in L^\infty(\mathbb{R})$ and $M_\omega$ is "multiplication by $\omega$" on $L^2(\mathbb{R})$, then $C_\psi M_\omega=M_{\omega\circ\psi}C_\psi$. If $\omega=(a^mt+i)/(t+i)$, then we have

$$\begin{aligned}T^{*(m+1)}&=MC_\psi M_\omega C_\psi^m\\&=MM_{\omega\circ\psi}C_\psi^{m+1}\\&=M_{m+1}C_\psi^{m+1},\end{aligned}$$

as required.

To compute $T^{*m}T^l$ it is convenient to introduce some additional notation. Write $M(\omega)$ for $M_\omega$, let $\psi_l(t)=a^lt$, and let $\omega_l(t)=(a^lt+i)/(t+i)$. Then

$$\begin{aligned}T^{*m}T^l&=M(\omega_m)\,C_\psi^mC_\psi^{*l}M(\omega_l)^*\\&=M(\omega_m)\,C_\psi^{*l}C_\psi^mM(\bar\omega_l)\\&=C_\psi^{*l}M(\omega_m\circ\psi_l)\,M(\bar\omega_l\circ\psi_m)\,C_\psi^m.\end{aligned}$$

It follows that if $D = \mathrm{diag}(1, C_\psi, \ldots, C_\psi^n)$, then

$$(T^{*m}T^l)_{l,m=0}^n = D^*(M(\omega_m \circ \psi_l)\, M(\bar{\omega}_l \circ \psi_m))_{l,m=0}^n D,$$

and hence verifying that $C_\phi$ satisfies the Halmos–Bram criterion amounts to showing that the matrix $(\omega_m \circ \psi_l(t)\, \bar{\omega}_l \circ \psi_m(t))_{l,m=0}^n$ is positive for each real $t$. Let $\psi_{lm}$ be the $l$, $m$ entry of this matrix, so

$$\psi_{lm} = \frac{(a^{l+m}t + i)}{a^l t + i} \frac{a^{l+m}t - i}{a^m t - i}.$$

Note $\psi_0(t) = t$, so $\psi_{l0} = \overline{\omega_l(t)}$, and $\psi_{00} = 1$. Hence if $d = \mathrm{diag}((t-i)^{-1}, (at-i)^{-1} \cdots (a^n t - i)^{-1})$, then

$$(\psi_{lm})_{l,m=0}^n = d^*(a^{2(l+m)}t^2 + 1)_{l,m=0}^n d,$$

and the problem reduces to proving positivity of $(a^{2(l+m)}t^2 + 1)_{l,m=0}^n$. The latter matrix is just $t^2(a^{2(l+m)})_{l,m=0}^n$ plus the identity, and $(a^{2(l+m)})_{l,m=0}^n = V \otimes V$, where $V$ is the vector with components $1, a, \ldots, a^n$. The asserted positivity is now obvious, and the proof is complete.

We remark that the similarity of hyperbolic composition operators to cosubnormal operators distinguishes them from the other types of invertible composition operators. Elliptic composition operators are similar to unitary operators, whereas a parabolic composition operator can not be similar even to a seminormal operator. For if $C_\phi$ is parabolic and similar to a seminormal operator $T$, then since $\sigma(C_\phi)$ is the unit circle [20], Putnam's theorem [23] implies that $T$ is unitary. Further, every point of the unit circle is an eigenvalue for $C_\phi$ [20], and hence the same is true of $T$. But a unitary operator on a separable space cannot have uncountably many eigenvalues.

An operator $U$ is called universal in case for every operator $T$, some multiple of $T$ is similar to the restriction of $U$ to some invariant subspace. Rota showed in [27] that the adjoint of the unilateral shift of infinite multiplicity is universal, and Caradus [2] showed that an operator $U$ is universal whenever $U$ is onto and has an infinite dimensional kernel. Recall that the spectrum of a hyperbolic composition operator is an annulus with interior [20].

THEOREM 6.2. *If $\phi$ is a hyperbolic disc automorphism and $\lambda$ is in the interior of the spectrum of $C_\phi$, then $C_\phi - \lambda$ is universal.*

*Proof.* Since every operator similar to a universal operator is universal, it suffices to consider the case where $\phi$ has fixed points $-1$ and $1$ as before. Let $\lambda$ be in the interior of the spectrum of $C_\phi$. Then $\lambda$ is in the point spectrum of $C_\phi$ see [20]). If $f$ is an eigenvector of $C_\phi$ corresponding to $\lambda$ and $B$ is the Blaschke product defined earlier, then $B^{2^n}f$ is also an eigenvector of $C_\phi$ corresponding to $\lambda$ for every integer $n \geqslant 0$. Thus $C_\phi - \lambda$ has an infinite dimensional kernel. It is onto by Corollary 5.9, and hence $C_\phi - \lambda$ is universal by Caradus' theorem.

COROLLARY 6.3. *Let $C_\phi$ be an invertible composition operator that is hyperbolic. Every operator has an invariant subspace if and only if the minimal nontrivial invariant subspaces of $C_\phi$ are all one dimensional.*

*Proof.* Choose any $\lambda$ in the interior of $\sigma(C_\phi)$. Let $T$ be any operator. By Theorem 6.2, $T$ is similar to a restriction of $C_\phi - \lambda$ to an invariant subspace $\mathcal{M}$. If $(C_\phi - \lambda) | \mathcal{M}$ has a proper invariant subspace, then so does $T$; if not, then $\mathcal{M}$ is minimal invariant subspace of $C_\phi$.

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