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MR1864396 (2003i:47001a)[Nikolski, Nikolai K.](#) [Nikol'skiĭ, Nikolai K.]★ **Operators, functions, and systems: an easy reading. Vol. 1. (English summary)**

Hardy, Hankel, and Toeplitz.

Translated from the French by Andreas Hartmann.

Mathematical Surveys and Monographs, 92.

American Mathematical Society, Providence, RI, 2002. xiv+461 pp. \$98.00. ISBN 0-8218-1083-9 47-02 (30D55 30E05 30H05 46Exx 47N70 93B28)[Journal](#)[Article](#)[Doc
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MR1892647 (2003i:47001b)[Nikolski, Nikolai K.](#) [Nikol'skiĭ, Nikolai K.]★ **Operators, functions, and systems: an easy reading. Vol. 2. (English summary)**

Model operators and systems.

Translated from the French by Andreas Hartmann and revised by the author.

Mathematical Surveys and Monographs, 93.

American Mathematical Society, Providence, RI, 2002. xiv+439 pp. \$98.00. ISBN 0-8218-2876-2 47-02 (30D55 30H05 46Exx 47N70 93B28)[Journal](#)[Article](#)[Doc
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FEATURED REVIEW.

The two-volume set *Operators, functions, and systems: an easy reading* is, to paraphrase the author, a treatise on the interplay between operator theory and complex function theory, as he understands it. In a nutshell, the “theme is developed as the theory of functions on the circle group and of operators acting on them, then passing to stationary filtering, and Hankel and Toeplitz operators as compressions of multiplication operators. . . , model theory for Hilbert space operators as (advanced) compressions of the same shift operator, and, finally all this machinery is brought together to control dynamical systems”. In more detail, an abridged list of the author’s favorite sub-

jects, in the order of their appearance in the books, includes: Hardy classes, the Hilbert transform, weighted polynomial approximation, cyclicity phenomena, maximal and Littlewood-Paley functions, Marcinkiewicz weak type interpolation, Wiener filtering theory, the Riemann ζ -function, Hankel operators (including spectral theory, Peller's theory, moment problems), reproducing kernel Hilbert spaces, Schatten-von Neumann operator ideals, Toeplitz operators, the operator corona problem, spectral theory of normal operators, the Sz.-Nagy-Foias functional model, von Neumann inequalities, Carleson and generalized free interpolation, the theory of spectral multiplicities, elements of semigroup theory, classical control of dynamical systems, bases of exponentials on intervals of the real line, and elements of H^∞ control theory.

The organization is somewhat unusual in that background material is introduced as needed to develop the main themes. Thus elementary and advanced topics are interwoven throughout the two books, rather than split into the earlier and later portions in the traditional fashion.

The two volumes are organized into four parts (two per volume): Part A, "An invitation to Hardy classes", is, as the title indicates, basically an introduction to the topic. The first six chapters cover much of the same ground as the well-known books of Hoffman, Duren and Koosis, plus Hardy space theory for vector-valued functions. The latter plays a major role in the sequel. Chapter 1 begins with invariant subspaces, in the style of Helson's book [H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964; [MR0171178 \(30 #1409\)](#)], though the latter, being less ambitious in scope, is friendlier and easier to access, especially for the beginner who does not know what he can skip. The focus on Hilbert spaces leads the author to introduce scalar inner functions as vectors $f \in H^2(\mathbb{D})$ such that $|f(\zeta)| = 1$ for a.e. point ζ on the boundary of the unit disk $\mathbb{D}$. True, but, at least to this reader, a little awkward. Moreover, the definition does not carry over in this form to inner functions defined over the half plane. There is a lot of interesting material in this part that a reader would be hard pressed to find elsewhere. The jewel in the crown is Chapter 5, which is devoted to harmonic analysis in weighted L^2 spaces on the circle. The main focus is on the Helson-Szegő theorem and related issues that center around the Hilbert transform, Carleson embedding, the Marcinkiewicz interpolation theorem, bases of exponentials and much more. This is a fine introduction to the subject that can almost be read as an independent monograph. Chapter 6 introduces H^p spaces on the upper half plane by conformal mapping. Chapters 7 and 8 cover applications to Wiener filtering and invariant subspaces related to the Riemann ζ -function.

Part B, "Hankel and Toeplitz operators", is a natural topic to pursue next, since the theory of these operators makes extensive use of Hardy space theory. The generic operators are defined in terms of a function φ that belongs to the space L^∞ of the boundary $\mathbb{T}$ of $\mathbb{D}$ by the rules:

$$T_\varphi: f \in H^2 \rightarrow P_+ \varphi f \quad \text{and} \quad H_\varphi: f \in H^2 \rightarrow P_- \varphi f,$$

where P_+ stands for the orthogonal projection of L^2 onto H^2 and $P_- = I - P_+$. The name of the game is to explore the properties of these operators in terms of the properties of the symbol φ . Thus, for example, the operator H_φ is compact if and only if $\varphi \in H^\infty + C$; H_φ belongs to the operator ideal of trace class operators if and only if the function $\psi(z) = \varphi_-(1/z)$ that is defined in terms of the "negative part" of φ meets the condition $\int_{\mathbb{D}} |\psi''(z)| dx dy < \infty$. The first of these results is due to P. Hartman, the second to V. Peller. The theory of Toeplitz and Hankel operators

based on matrix- and operator-valued symbols φ is also developed. The main topics include the Adamyan-Arov-Kreĭn study of the singular numbers of compact Hankel operators with operator-valued symbols, Fredholm theory, spectral theory, interpolation, the operator corona problem, stationary processes, the function spaces BMO, VMO, PC and much more.

Part C, “Model operators and free interpolation”, is another major consumer of Hardy space theory. Thus, for example, if T is a completely nonunitary contractive operator from a Hilbert space H into itself, then there exists an operator-valued function $\Theta(z)$ that is holomorphic and contractive in $\mathbb{D}$, a Hilbert space of functions $\mathcal{K}_\Theta$ and a unitary operator U on $\mathcal{K}_\Theta$ such that T is unitarily equivalent to the operator $M_\Theta = P_\Theta U|_{\mathcal{K}_\Theta}$, where P_Θ denotes the orthogonal projection onto $\mathcal{K}_\Theta$. In the special case that $\Theta(z)$ is an operator-valued inner function the model space simplifies to $\mathcal{K}_\Theta = H^2(E_*) \ominus \Theta H^2(E)$, where E_* and E are Hilbert spaces with $\dim E_* = \text{rank } \sqrt{I - TT^*}$ and $\dim E = \text{rank } \sqrt{I - T^*T}$ and U is just multiplication by z , i.e., the shift. Moreover, there is a one-to-one correspondence between the invariant subspaces of T and the regular factorizations of $\Theta(z)$ as the product of two contractive (operator-valued) holomorphic functions, and the spectrum of T is equal to the union of the points in $\mathbb{D}$ at which $\Theta(z)$ is not invertible and the points z on the boundary of $\mathbb{D}$ at which Θ either fails to have a holomorphic extension or fails to be unitary on a surrounding arc. The space $\mathcal{K}_\Theta$ is more complicated when $\Theta(z)$ is not inner. A sizeable chunk of Part C is devoted to developing this theme and variations thereof, when the starting operator is subject to different constraints from those furnished above. The discussion focuses largely on the model of Sz.-Nagy and Foias, starting from the Sz.-Nagy dilation theorem, which states that every contraction in a Hilbert space has a unitary dilation. Other models, including those of Pavlov, and de Branges-Rovnyak, are also touched upon. Connections are also made with the models introduced by Moshe Livšic, the founding father of the theory of characteristic functions, who worked with operators that are close to selfadjoint in the sense that the imaginary part is “small”. In the present text, the imaginary part is also taken to be definite and then linked to a contraction by an appropriately chosen Cayley transform. The more general theory of characteristic functions $\Theta(z) = D + zC(I - zA)^{-1}B$ corresponding to operator nodes with

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix},$$

in which J is a signature operator, is not discussed.

Part D, “Analytic problems in linear system theory”, presents “a glance at system theory from the operator and complex analysis point of view”. Roughly speaking, this means focusing on the kind of problems that analysts like to solve, which are not necessarily the important ones from the point of view of engineers who actually design control systems. This is not devoid of significance, because often translating a problem from one language to another raises new and interesting questions. The initial contact between Parts C and D rests on the observation that the characteristic function of certain classes of operators can also be interpreted as the transfer function of a linear system. There is a good mixture here of semigroup theory, an extensive chapter on bases of exponentials (on revient toujours à ses premières amours) and a quick introduction to H^∞ control. The latter ties in nicely with the theory of Hankel operators through the formula due

to Adamyan, Arov and Kreĭn:

$$s_k = \inf \{ \|\varphi - f\|_\infty : f \in H_k^\infty(\mathbb{C}^{p \times q}) \},$$

where s_k , $k = 0, 1, \dots$, denote the singular values of the Hankel operator H_φ with symbol $\varphi \in L^\infty(\mathbb{C}^{p \times q})$ and $H_k^\infty(\mathbb{C}^{p \times q})$ differs from $H^\infty(\mathbb{C}^{p \times q})$ by a rational $p \times q$ matrix-valued function in $L^\infty(\mathbb{C}^{p \times q})$ of degree at most k . The case $k = 0$ is the Nehari theorem for matrix-valued symbols. Methods based on this formula and the fact that all the solutions to this extremal problem admit a linear fractional representation made a big splash in the eighties, especially following the 1984 paper by K. Glover [Internat. J. Control **39** (1984), no. 6, 1115–1193; [MR0748558 \(86a:93029\)](#)] in which the AAK formula was redone in state space formalism for Hankel operators with rational matrix-valued symbols φ . Most of the sources that the author relies upon are the work of mathematicians rather than engineers. For additional perspective, see the discursive book reviews by J. A. Ball and C. Martin in the SIAM Review **41** (1999), no. 4, 809–825 and **43** (2001), no. 1, 207–213, respectively. The monograph *Robust and optimal control* by K. Zhou, J. C. Doyle and Glover [Prentice Hall, Upper Saddle River, NJ, 1996; Zbl 0999.49500] is a good supplementary reference.

Every chapter begins with a short introductory preface that indicates the contents of the chapter and often emphasizes the principal results or theorems contained therein. This is followed by a number of sections of more or less expository material. The final section in each chapter, called “Notes and remarks”, discusses sources and related literature. In all chapters but one, the next to last section, called “Exercises and further results”, is devoted to just that. The extensions of the presented theory are usually in the form of exercises with hints and supplementary comments that serve to place the result in question in context. (The chapters with the most pages dedicated to exercises and remarks are undoubtedly the ones closest to the author’s heart.) To make this work, the author has invested considerable effort in cross referencing (both backwards and forwards) and providing gentle reminders of concepts as they reoccur. A (rather long) list of symbols and a detailed subject and author index are also furnished.

The scope is impressive. There are on the order of 1000 references in the bibliography, and it appears that the author has seriously investigated most of them and abstracted those parts that are relevant to his main themes and recast them in his language. No small task! Anyone who has an interest in the main achievements in this branch of analysis over the last 50–60 years, and how this material hangs together, is sure to find much of interest in this exposition. It is encyclopedic in breadth and a valuable source book. The two volumes can be read on many levels, from that of a beginning graduate student to that of an expert. I suspect, however, that without guidance as to what could be skipped on a first pass, a beginner would be somewhat overwhelmed. Among the many excellent books on Hardy spaces, the friendliest introduction (at least in my opinion) is to be found in the first three chapters of Helson’s book *Invariant subspaces* [op. cit.], though it does not go nearly as far as the exposition under review. Another good starting point is Chapter 17 of Rudin’s book [W. Rudin, *Real and complex analysis*, Third edition, McGraw-Hill, New York, 1987; [MR0924157 \(88k:00002\)](#)]. The article by H. S. Shapiro [in *Twentieth century harmonic analysis—a celebration (Il Ciocco, 2000)*, 31–56, Kluwer Acad. Publ., Dordrecht, 2001;

[MR1858778 \(2002h:47001\)](#)] is a very readable (and mercifully short) introduction to the interplay between these two subjects. The uncited paper by Ball and N. Cohen [in *Topics in matrix and operator theory* (Rotterdam, 1989), 93–136, Birkhäuser, Basel, 1991; [MR1115026 \(93a:47011\)](#)] and the cited papers by T. Ando [“De Branges spaces and analytic operator spaces”, Hokkaido Univ., Sapporo, 1990; Zbl 0717.47003] and R. G. Douglas [in *Topics in operator theory*, 161–218. Math. Surveys, 13, Amer. Math. Soc., Providence, R.I., 1974; [MR0355647 \(50 #8121\)](#)] are also useful supplements.

There are a number of minor quibbles: The passage from Russian to French to English has produced some rather charming expressions as well as an occasional misuse: “shortly” instead of “is short for”; I:385 “in an emotional way” instead of “in a picturesque way”; II:337 “it is no matter” instead of “it is no small matter”; I:167 “[t] means the entire part of the integer t ”.

Somewhat less charming is the combination “60ies” in place of either sixties or 60’s. I personally also find it jarring to come across itemized lists (as on I:47) where the displacement of (1) from the left margin differs from the displacement of (2), (3), The extensive use of the word “also” as in “meromorphic, see *also* Nevanlinna function” in the subject index when no information is provided for the first entry is superfluous. It is akin to asking whether you are ready for seconds when you have not yet had firsts.

There are a few misprints and oversights: characteristic function is misprinted on II:28. (It appears correctly at the bottom of II:25.) A number of other misprints are listed on II:439. The list will be updated on the author’s home page <http://www.math.u-bordeaux.fr/~nikolski> in the near future. In item I:4.2.2, one should assume that $f(z)$ is not identically zero and, in the proof, treat the case that real part $f(z) \equiv 0$ separately.

In spite of the rather extensive bibliography there are some curious omissions. The contributions of the Leningrad school, naturally enough, seem to be very well represented. The Odessa school did not fare so well. There are a number of papers by V. M. Adamyan and D. Z. Arov that revealed the connections between Lax-Phillips scattering theory and the Sz. Nagy-Foias theory of characteristic functions for completely nonunitary contractions that would have fit very naturally into the Notes and Remarks to Chapter 2 of Part C.

The important paper by Arov and L. Z. Grossman [Math. Nachr. **157** (1992), 105–123; [MR1233051 \(94i:47019\)](#)] that underlies the abstract interpolation scheme of Katsnelson, Kheifets and Yuditskii is not mentioned. L. A. Sakhnovich is only referenced for one Doklady note. The monograph by M. L. Gorbachuk and V. I. Gorbachuk [*M. G. Krein’s lectures on entire operators*, Birkhäuser, Basel, 1997; [MR1466698 \(99f:47001\)](#)] is another surprising omission for a treatise that focuses on the interplay between complex function theory and operator theory.

The 1974 Doklady note by Arov [Dokl. Akad. Nauk SSSR **216** (1974), 713–716; [MR0361842 \(50 #14287\)](#)] (followed by a more detailed presentation [Sibirsk. Mat. Zh. **20** (1979), no. 2, 211–228, 457; [MR0530486 \(80g:93031\)](#)]) is another source for II:356.

The paper by D. Sarason [J. Math. Anal. Appl. **12** (1965), 244–246; [MR0192355 \(33 #580\)](#)] fits in perfectly with the discussion of the invariant subspaces of a Volterra operator in II:92.

The author reports that these two volumes are based on an extended version of four graduate

courses that he gave at the University of Bordeaux in the years 1992–1996. However, a more accurate description would seem to be the culmination of close to forty years of involvement at the front lines of many of the topics under discussion. The subtitle “An easy reading” is perhaps a little overly optimistic. As Ecclesiastes remarks, the acquisition of knowledge involves much vexation. Professor Nikolski has done a splendid job of minimizing that vexation. It will be the rare reader indeed who will not find something new and interesting in these two volumes; they are definitely worth three stars in the Michelin guide to the Literature of Mathematics.

Reviewed by [*Harry Dym*](#)

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