Analyticity and Flows in Von Neumann Algebras

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Communicated by the Editors

Received December 11, 1975

Let \( \mathfrak{B} \) be a von Neumann algebra, let \( \{ \alpha_t \}_{t \in \mathbb{R}} \) be an ultraweakly continuous one-parameter group of \(*\)-automorphisms of \( \mathfrak{B} \), and let \( \mathfrak{A} \) be the set of all \( A \) such that for each \( \rho \) in \( \mathfrak{B}_\ast \), the function \( t \mapsto \rho(\alpha_t(A)) \) lies in \( H^\infty(\mathbb{R}) \). Then \( \mathfrak{A} \) is an ultraweakly closed subalgebra of \( \mathfrak{B} \) containing the identity which is proper and non-self-adjoint if \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is not trivial. In this paper, a systematic investigation into the structure theory of \( \mathfrak{A} \) is begun. Two of the more noteworthy developments are these. First of all, conditions under which \( \mathfrak{A} \) is a subdiagonal algebra in \( \mathfrak{B} \), in the sense of Arveson, are determined. The analysis provides a common perspective from which to view a large number of hitherto unrelated algebras. Second, the invariant subspace structure of \( \mathfrak{A} \) is determined and conditions under which \( \mathfrak{A} \) is a reductive subalgebra of \( \mathfrak{B} \) are found. These results are then used to produce examples where \( \mathfrak{A} \) is a proper, non-self-adjoint, reductive subalgebra of \( \mathfrak{B} \). The examples do not answer the reductive algebra question, however, because although ultraweakly closed, the subalgebras are weakly dense in \( \mathfrak{B} \).

1. Introduction

Suppose that \( \mathfrak{B} \) is a von Neumann algebra and that \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is an ultraweakly continuous representation of the real line \( \mathbb{R} \) as a group of \(*\)-automorphisms of \( \mathfrak{B} \). Consider the space \( \mathfrak{A} \) consisting of those operators \( A \) in \( \mathfrak{B} \) with the property that for each \( \rho \) in \( \mathfrak{B}_\ast \), the predual of \( \mathfrak{B} \), the function of \( t \), \( \rho(\alpha_t(A)) \), lies in the classical Hardy space \( H^\infty(\mathbb{R}) \). Such operators will be called \emph{analytic} (with respect to \( \{ \alpha_t \}_{t \in \mathbb{R}} \)). As we shall see, \( \mathfrak{A} \) is an ultraweakly closed subalgebra of \( \mathfrak{B} \),

* Supported in part by a Wayne State University Faculty Research award.
† Supported in part by a grant from the National Science Foundation.
containing the identity operator, such that $\mathcal{U} + \mathcal{U}^* = \{A + B^* \mid A, B \in \mathcal{U}\}$ is ultraweakly dense in $\mathcal{B}$ and such that $\mathcal{U} \cap \mathcal{U}^* = \{A \in \mathcal{B} \mid \alpha_t(A) = A \text{ for all } t\}$. Thus, unless $\{\alpha_t\}_{t \in \mathbb{R}}$ is trivial, $\mathcal{U}$ is a non-self-adjoint subalgebra of $\mathcal{B}$ and provides one example of an extension to a noncommutative setting of certain well-known classes of function algebras. In particular, $\mathcal{U}$ may well be regarded as a noncommutative, weak-* Dirichlet algebra [52]. Our primary objective in this paper is to initiate a systematic investigation into the structure of $\mathcal{U}$. Before doing so, we wish to provide some background and history and, because of the length of this paper, a section-by-section survey of the results.

To fix ideas and to provide some initial examples (more will be presented later), consider the following. Suppose first that $\mathcal{B}$ is the Lebesgue space $L^\infty(\mathbb{R})$ and that $\{\alpha_t\}_{t \in \mathbb{R}}$ is given by translation; i.e., $(\alpha_t(A))(x) = A(x + t)$, $A \in L^\infty(\mathbb{R})$. In this case it is easy to see that $\mathcal{U}$ is simply $H^\infty(\mathbb{R})$ itself. At an opposite extreme, suppose $\mathcal{B}$ is the von Neumann algebra of all $n \times n$ complex matrices and suppose $A$ is a self-adjoint, diagonal matrix whose eigenvalues are distinct and labeled in decreasing order. If $\{\alpha_t\}_{t \in \mathbb{R}}$ is defined on $\mathcal{B}$ by the formula $\alpha_t(B) = e^{itA}Be^{-itA}$, $B \in \mathcal{B}$, $t \in \mathbb{R}$, then an easy calculation reveals that $\mathcal{U}$ is simply the algebra of upper triangular matrices. We shall see later in Section 4.1 that when $\mathcal{B}$ is type I, $\mathcal{U}$ is a synthesis of these two extreme cases.

The notion of analyticity we are investigating may be traced back to Mackey [33], although it was undeservedly by-passed until the paper [9] of deLeeuw and Glicksberg appeared. This paper, in turn, was motivated by the studies of Arens and Singer [1] and Helson and Lowdenslager [21-23] concerning generalized analytic functions on compact abelian groups. Later, independently, and from a somewhat different perspective, the notion of analyticity re-emerged and was developed in the work of Calderon [6], Fife [13], and Weiss [58]. However, it was really Forelli [14] who supplied the basic tools necessary for a systematic study of analyticity and who showed in several papers [15-17] that the theory is a rich and fruitful one. In a series of papers [38-43], the first of which overlaps with Weiss' [58], the theme was taken up by the second author and may of the basic function algebraic properties of the notion were established. At about the same time that Forelli was beginning his contributions to the theory, Arveson began to develop a general theory of noncommutative function algebras [2]. Subsequently, in a very important paper [4], he transferred much of Forelli's work [14] to the setting of noncommutative operator algebras and showed the importance of the theory of spectra in the sense of spectral synthesis for the purpose of analyzing algebras of analytic operators. In a sense, the present paper arose as a response to his suggestion that algebras of analytic operators in von Neumann and $C^*$-algebras should be systematically explored, and that the results of the second author, particularly [38], should be extendable to the noncommutative setting. This program began in the first author's dissertation [31] and continues in the present paper.

We turn now to a summary of the contents of this paper. In the next section
we establish notation and discuss aspects of the theory of spectra in the sense of spectral synthesis which we will need for our analysis. Also, we generalize the notion of analyticity somewhat. Specifically, we shall deal with ultraweakly continuous representations \( \{\alpha_g\}_{g \in G} \) of an arbitrarily locally compact Abelian group \( G \) on the von Neumann algebra \( \mathcal{B} \), we shall assume that there is a distinguished subsemigroup \( \Sigma \) of the dual group \( \hat{G} \) satisfying certain very general properties, and we shall let \( \mathcal{A} \), the space of analytic operators, be the subspace of all those operators \( A \) in \( \mathcal{B} \) such that the (distributional) Fourier transform of the \( \mathcal{B} \)-valued function on \( G \), \( \alpha_g(A) \), is supported in \( \Sigma \). Although we shall be interested primarily in the case when \( G = \mathbb{R} \), we proceed, when possible, at this level of generality for three reasons: first, it requires absolutely no extra effort; second, the more general setting helps to clarify the obstacles present when one tries to extend the results which are valid when \( G = \mathbb{R} \); and third, such generality may prove useful later.

In Section 3, we determine conditions under which \( \mathcal{A} \) is a subdiagonal algebra in \( \mathcal{B} \) in the sense of Arveson [2]. Recall that the diagonal \( \mathcal{D} \) of \( \mathcal{A} \) is simply \( \mathcal{A} \cap \mathcal{A}^* \). By a theorem of Kovacs and Szücs [29] there is a faithful, normal \( \alpha \)-invariant expectation \( \Phi \) from \( \mathcal{B} \) onto \( \mathcal{D} \) precisely when there are sufficiently many invariant normal states of \( \mathcal{B} \) to separate the points of \( \mathcal{B}^+ \). We show that when this happens, \( \Phi \) is multiplicative on \( \mathcal{A} \) so that by definition, \( \mathcal{A} \) is subdiagonal with respect to \( \Phi \). We show too that in fact \( \mathcal{A} \) is a maximal subdiagonal subalgebra of \( \mathcal{B} \) in the sense that \( \mathcal{A} \) is not contained in any larger subalgebra on which \( \Phi \) is multiplicative. As was shown in [2], maximality is a very important property to establish for any given subdiagonal algebra.

Section 4 is devoted to examples. In Section 4.1, we consider the case when \( \mathcal{B} \) is type I, concentrating mainly on the homogeneous case. Here we use a recent result of Brown [5] (cf. the appendix of [7] also) to show that all one-parameter automorphism groups of a homogeneous type I von Neumann algebra are spatially implemented. This enables us, then, to put into evidence all the ingredients necessary for constructing the most general algebra of analytic operators in a type I von Neumann algebra. In Section 4.2, we show that if \( \{\alpha_t\}_{t \in \mathbb{R}} \) is inner, then \( \mathcal{A} \) is a nest algebra in \( \mathcal{B} \) in the sense of Ringrose [47] and conversely, if \( \mathcal{A} \) is a nest algebra in \( \mathcal{B} \), then it is the algebra of analytic operators with respect to an inner automorphism group. Finally, in Section 4.2, we consider crossed products and show how “duality theory” leads to numerous examples to which our results apply. In particular, we obtain several simplifications and extensions of some of Arveson’s results in [2].

The fifth, and final, section contains what are, perhaps, our most important contributions to operator theory in general. We begin by analyzing the invariant subspace structure of \( \mathcal{A} \), showing that “most” of the nonreducing subspaces invariant under \( \mathcal{A} \) are determined by strongly continuous unitary representations of \( \mathbb{R} \) which implement \( \{\alpha_t\}_{t \in \mathbb{R}} \). We then use this result to exhibit examples where \( \mathcal{A} \) has no nonreducing invariant subspaces by showing that in these
examples $\{\mathcal{A}_t\}_{t \in \mathbb{R}}$ is not spatially implementable. Thus, on the surface, it would appear that we have found an answer to the reductive algebra question: "Are there any weakly closed, non-self-adjoint algebras of operators on Hilbert space all whose invariant subspaces are reducing? This unfortunately (fortunately?) is not the case because our examples, although ultraweakly closed, are *weakly* dense in the von Neumann algebras which they generate. This discovery, of course, makes the reductive algebra question all the more piquant and exhibits as well hitherto unsuspected differences between the weak and ultraweak topologies. In addition, it suggests the question: "Are there any ultraweakly closed algebras of operators on $\mathcal{H}$, whose only invariant subspaces are $\{0\}$ and $\mathcal{H}$?"—an "ultraweak" version of the transitive algebra question. It is clear from the results of Section 4.1 that our methods will not lead directly to an affirmative answer, at least not when the candidate is determined by a representation of $\mathbb{R}$.

2. Preliminaries

In this section we establish notation and describe the theory of spectra (in the sense of spectral synthesis) as needed for this paper. In [3], Arveson developed powerful machinery to handle the problems facing us, so rather than reproduce his results here, we shall merely describe the general setting and present just enough details so that the reader may pass easily between this paper and Arveson's. As a rule, we shall state the results of this section in more generality than is needed for our immediate purposes in the belief that they will prove useful later.

Throughout this paper, $G$ will denote a locally compact Abelian group with the operation written additively. Elements of $G$ will be denoted by lowercase Roman letters and Haar measure on $G$ will be denoted by $m$. The dual of $G$ will be written $\hat{G}$ and the elements of $\hat{G}$ will be distinguished from those of $G$ by a caret. The pairing between $G$ and $\hat{G}$ will be written $\langle t, \hat{s} \rangle$, $t \in G$, $s \in \hat{G}$, and the Fourier transform will take this form: $\hat{f}(\hat{s}) = \int_G \langle t, \hat{s} \rangle f(t) \, dm(t)$, $f \in L^1(G)$.

We are especially interested in representations of $G$, $\{V_t\}_{t \in G}$, acting as isometries on a Banach space $\mathcal{X}$. In case $\{V_t\}_{t \in G}$ is strongly continuous, the theory evolves without any difficulty. In this paper, however, the representations under consideration frequently fail to be strongly continuous and are continuous only in some weaker topology. Consequently, the development of the theory is impeded by numerous technical difficulties. Fortunately, these have been analyzed in considerable detail by Arveson [3], and the state of the art is such that we may proceed formally, as we now do, to describe the various constructs under consideration.
In each of the cases we consider, the topology on $\mathcal{A}$ is sufficiently strong for us to assert that if $f \in L^1(G)$ and $X \in \mathcal{A}$, then the integral

$$\int_G V_t(X) f(t) \, dm(t),$$

(2.1)
defined in the usual manner as a limit of integrals of simple functions (cf. [10]), exists and determines an element of $\mathcal{A}$. This element is denoted by $X * f$, or by $X * f$ if it is necessary to keep track of the representation. This process of convolution makes $\mathcal{A}$ into a module over $L^1(G)$ with the property that $\| X * f \|_{\mathcal{A}} \leq \| X \|_{\mathcal{A}} \| f \|_{L^1(G)}$ for all $X \in \mathcal{A}$ and $f \in L^1(G)$. From this inequality we see that the annihilator $\mathcal{J}(X)$ of an element $X$ in $\mathcal{A}$, which is by definition $\{ f \in L^1(G) \mid X * f = 0 \}$, is a closed ideal of $L^1(G)$.

**Definitions** 2.1. (a) The hull of $\mathcal{J}(X)$, defined to be $\bigcap_{t \in \mathcal{J}(X)} \{ \hat{f} \in \hat{G} \mid \hat{f}(t) = 0 \}$, is called the *spectrum* of $X$ (in the sense of spectral synthesis) and will be written $sp(X)$ or $sp_e(X)$.

(b) if $E$ is a closed subset of $\hat{G}$, then $\mathcal{S}(E)$ or $\mathcal{S}_e(E)$ will denote the set of $\{ X \in \mathcal{A} \mid sp(X) \subseteq E \}$, and will be referred to as the *spectral subspace* of $\mathcal{A}$ associated with $E$.

It is easy to see that $\mathcal{S}(E)$ is, in fact, a norm-closed, linear space invariant under $\{ \hat{V}_t \}_{t \in G}$. In the cases of interest to us, $\mathcal{S}(E)$ is also closed in the strongest topology with respect to which $\{ \hat{V}_t \}_{t \in G}$ is continuous. Significantly, however, as we shall see in Section 5.8 *et seq.*, $\mathcal{S}(E)$ may fail to be closed in some important weaker topologies, even though the representation happens to be continuous in those topologies.

To justify our terminology, and to illustrate these ideas in a way which will be useful subsequently, we note that the spaces $\mathcal{S}(E)$ play the role of what are customarily called the spectral subspaces associated with a unitary representation of $G$. Indeed, if $\mathcal{A}$ is a Hilbert space, if $\{ V_t \}_{t \in G}$ is a unitary representation of $G$ in $\mathcal{H}$, and if $P$ is the spectral measure on $\hat{G}$ associated with $\{ V_t \}_{t \in G}$. via Stone's theorem, then for each closed set $E$ in $\hat{G}$, $\mathcal{S}(E) = P(E) \mathcal{A}$.

At this point we pause momentarily to caution the reader about some confusion which may develop because of our choice of notation—a choice which is made somewhat arbitrarily but which is, nevertheless, one possible resolution of an annoying conflict of notation which exists in the literature. The conflict basically is due to three things: First, most people write the Fourier transform with a minus sign, $\int \langle t, -s \rangle f(t) \, dm(t)$; second, most people write the abstract process of convolution as we have in formula (2.1)—note the lack of minus sign; and third, when using Stone's theorem, most people express the spectral resolution of a unitary group $\{ U_t \}_{t \in G}$ on a Hilbert space $\mathcal{H}$ as $U_t = \int \langle t, s \rangle \, dP(s)$; again, note the lack of minus sign. When these three notational conventions are adopted, certain familiar formulas must be altered. For example, the spectral
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subspaces \( \mathcal{X}(M) \), \( M \) a closed subset of \( \mathcal{G} \), defined in Definition 2.1 are no longer the ranges of \( P(M) \), but instead, \( \mathcal{X}(M) = P(-M)\mathcal{X} \) where \( -M = \{ -i \mid i \in M \} \). To preserve what are, for our purposes, the most important formulas, we have opted to suppress the minus sign in the Fourier transform. This choice forces us to alter other well-known formulae, of course, but their number in this paper is fairly small. To be sure, ours is not the only way to resolve the conflict, but we have found it to be the most serviceable.

By way of further illustration, we state the following lemma, which will be used frequently and which is easily proven.

**Lemma 2.2.** For \( \hat{t} \in \mathcal{G} \), \( \mathcal{X}([\hat{t}]) = \{ X \in \mathcal{X} \mid V_s(X) = \langle s, \hat{t} \rangle X, \text{ for all } s \in G \} \); in particular, \( \mathcal{X}([\hat{0}]) \) is the space of vectors in \( \mathcal{X} \) left fixed by every \( V_t \).

The first context in which we consider representations which are not necessarily strongly continuous is that in which the Banach space is a von Neumann algebra \( \mathfrak{B} \) endowed with the ultraweak topology (cf. [11, Chap. I, Sect. 3.1]) and \( G \) is represented as an ultraweakly continuous group of \( * \)-automorphisms \( \{ \alpha_t \}_{t \in G} \) of \( \mathfrak{B} \). (We use lowercase Greek letters from the beginning of the alphabet to denote such representations on von Neumann algebras.) That is, each \( \alpha_t \) is a \( * \)-automorphism of \( \mathfrak{B} \), \( \alpha_{t+s} = \alpha_t \alpha_s \) for \( t, s \in G \), and the map \( (t, A) \rightarrow \alpha_t(A) \) from \( G \times \mathfrak{B} \) to \( \mathfrak{B} \) is continuous when \( \mathfrak{B} \) is given the ultra-weak topology.

As a consequence of Propositions 1.4 and 3.0 of [3], for each \( A \in \mathfrak{B} \) and \( f \) in \( L^1(G) \) the integral (2.1) (with \( X \) replaced by \( A \) and \( V_t \) by \( \alpha_t \)) converges and defines an element in \( \mathfrak{B} \). Furthermore, by [3, Proposition 1.4 and Definition 2.1 et seq.], the spectral subspaces \( \mathfrak{B}(E) \), \( E \) closed in \( \mathcal{G} \), are ultraweakly closed subspaces of \( \mathfrak{B} \). However, as we shall see in Section 5.8 et seq., even though \( \{ \alpha_t \}_{t \in G} \) is automatically continuous in the weak operator topology, it can happen that \( \mathfrak{B}(E) \) fails to be closed in that topology. Such an anomaly can occur even when \( \mathfrak{B}(E) \) is a subalgebra of \( \mathfrak{B} \).

The second setting in which we consider representations which are not necessarily strongly continuous is as follows. Suppose that \( \mathfrak{B} \) and \( \mathfrak{R} \) are two von Neumann algebras endowed with the ultraweak topologies, and that \( \{ \alpha_t \}_{t \in G} \) and \( \{ \beta_t \}_{t \in G} \) are ultraweakly continuous representations of \( G \) as groups of \( * \)-automorphisms of \( \mathfrak{B} \) and \( \mathfrak{R} \), respectively. We let \( \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \) denote the collection of all ultraweakly continuous linear maps from \( \mathfrak{B} \) to \( \mathfrak{R} \) and we give it the topology of pointwise convergence. The representations \( \{ \alpha_t \}_{t \in G} \) and \( \{ \beta_t \}_{t \in G} \) induce a representation \( \{ (\beta \alpha^*)_t \}_{t \in G} \) on \( \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \) which is continuous with respect to the topology on \( \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \) and which is defined by the formula \( (\beta \alpha^*)_t(\Phi) = \beta_t \circ \Phi \circ \alpha_{-t}, \Phi \in \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \). By virtue of 1.6 and 3.0 of [3], the integral (2.1), with \( X \) replaced by \( \Phi \in \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \), and \( V_t \) replaced by \( (\beta \alpha^*)_t \), converges and determines an element \( \Phi * f \) in \( \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \). Moreover, by [3, 1.6 and 2.1 et seq.], the spectral subspaces \( \{ \mathcal{L}_w(\mathfrak{B}, \mathfrak{R})(E) \} \) are closed in the topology on \( \mathcal{L}_w(\mathfrak{B}, \mathfrak{R}) \).
We do not exclude the possibility that $\mathcal{R}$ consists solely of scalars. In this case $\mathcal{L}_w(\mathcal{B}, \mathcal{R})$ is simply the predual $\mathcal{B}^*$ of $\mathcal{B}$ endowed with the weak-* topology. Likewise, we do not exclude the possibility that $\{\beta_t\}_{t \in G}$ is trivial, i.e., that $\beta_t \equiv \beta_0$ for all $t$. In this case, we write $\{\alpha_t^*\}_{t \in G}$ for $\{((\beta_t^*)_t)_{t \in G}\}$.

The final setting in which we consider representations of $G$ which are not necessarily strongly continuous arises as follows. Let $\mathcal{R}$ be a von Neumann algebra and let $C_u(G, \mathcal{R})$ denote the collection of all bounded, uniformly continuous functions from $G$ to $\mathcal{R}$ where $\mathcal{R}$ is again given the ultraweak topology. We give the space $C_u(G, \mathcal{R})$ the topology of uniform convergence; i.e., a net $(F_t)$ converges to zero in $C_u(G, \mathcal{R})$ precisely when for each ultraweak neighborhood $\mathcal{V}$ of 0 in $\mathcal{R}$, there is an index $A$, such that for all $h > A$, and $t \in G$, $F_t(t)$ lies in $\mathcal{V}$. We write $\{T_t\}_{t \in G}$ for the representation of $G$ on $C_u(G, \mathcal{R})$ defined by (backward) translation; i.e., $(T_tF)(s) = F(s - t)$. $F \in C_u(G, \mathcal{R})$. It is a simple matter to verify that for each $F \in C_u(G, \mathcal{R})$ and $f \in L^1(G)$, the integral (2.1), with the appropriate change in notation, converges and determines an element $F * f$ in $C_u(G, \mathcal{R})$. Moreover, it is easy to see that the spectral subspaces $(C_u(G, \mathcal{R}))(E)$ are closed in $C_u(G, \mathcal{R})$ for each closed set $E$ in $G$.

Before proceeding with more definitions, notation, and terminology, we illustrate some of the ideas just introduced in the following useful lemma.

**Lemma 2.3.** Let $\mathcal{B}$ be a von Neumann algebra with an ultra-weakly continuous representation $\{\alpha_t\}_{t \in G}$ of $G$ as a group of *-automorphisms of $\mathcal{B}$. Let $\mathcal{R}$ be another von Neumann algebra and assume that $G$ acts trivially on $\mathcal{R}$. For $\Phi$ in $\mathcal{L}_w(\mathcal{B}, \mathcal{R})$ and $A$ in $\mathcal{B}$, let $(\Phi \circ A)$ denote the element in $\mathcal{L}_w(\mathcal{B}, \mathcal{R})$ defined by the formula $(\Phi \circ A)(F) = \Phi(\alpha_t(A)(F))$, where $\{T_t\}_{t \in G}$ is the translation group on $C_u(G, \mathcal{R})$ discussed above.

(i) If $\text{sp}_a(\Phi) \subseteq \{0\}$, then $\text{sp}_a(\Phi \circ A) \subseteq \text{sp}_a(A)$.

(ii) If $A \in \mathcal{B}$, $\Phi \in \mathcal{L}_w(\mathcal{B}, \mathcal{R})$, and if $F$ is the function in $C_u(G, \mathcal{R})$ defined by the formula $F(t) = \Phi(\alpha_t(A))$, then $\text{sp}_r(F) \subseteq -\text{sp}_a(A) \cap \text{sp}_a(\Phi)$, where $\{T_t\}_{t \in G}$ is the translation group on $C_u(G, \mathcal{R})$ discussed above.

**Proof.** (i) By Lemma 2.2, $\alpha_t^*(\Phi) = \Phi$ for all $t \in G$, and consequently $(\Phi \circ A) * f = \Phi \circ (A * f)$ for all $f$ in $L^1(G)$. Assertion (i) follows from this.

(ii) The argument we present is essentially Forelli's proof of formula (30) [15. p. 50]. It suffices to show that if $\check{r}$ lies in the complement of $\text{sp}_a(\Phi)$ or in the complement of $-\text{sp}_a(A)$, then $\check{r}$ is not in $\text{sp}_r(F)$. Suppose first that $\check{r}$ is not in $\text{sp}_a(\Phi)$ and select an $f$ in $\mathcal{J}(\Phi)$ with $\check{f}(\check{r}) = 1$. Then $\int F(-t)f(t)dm(t) - \int \Phi(\alpha_t(A))f(t)dm(t) = (\Phi * f)(\check{r}) = 0$. Since $\mathcal{J}(\Phi)$ is a closed ideal in $L^1(G)$, $\mathcal{J}(\Phi)$ is closed under translation. Hence $\int F(s - t)f(t) dm(t) = 0$ for all $s \in G$. This means that $\check{f} \in \mathcal{J}(F)$ so that $\check{r}$ is not in $\text{sp}_r(F)$. If, on the other hand, $\check{r}$ is not in $-\text{sp}_a(A)$, then there is an $f$ in $\mathcal{J}(A)$ with $\check{f}(\check{r}) = 1$. But then $\int F(t)f(t)dm(t) = \int \Phi(\alpha_t(A)) \times f(t) dm(t) = \Phi(A * f) = 0$, and since, once again, $\mathcal{J}(A)$ is closed under trans-
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lation, we find that \( 0 = \int G \, F(t) \, f(t - s) \, dm(t) = \int F(s - t) \, \tilde{f}(t) \, dm(t) = (F * \tilde{f})(s) \) where \( \tilde{f}(t) = f(-t) \). Thus \( \tilde{f} \) lies in \( \mathcal{J}(F) \), and since \( (\tilde{f})^*(\tilde{r}) = 1 \), we find that \( \tilde{r} \) is not in \( \text{sp}_r(F) \). This completes the proof.

DEFINITION 2.4. Throughout this paper, \( \Sigma \) will denote a closed subsemigroup of \( \hat{G} \) which satisfies these two conditions: (i) \( \Sigma \cap (-\Sigma) = \{0\} \), and (ii) \( \Sigma \) is the closure of its interior.

Condition (i) is an "antisymmetry" condition and implies that certain types of generalized analytic functions associated with \( \Sigma \) are constant whenever they are real-valued (cf. 3.13 below). The second condition implies that \( \Sigma \) has positive Haar measure and that it is a set of spectral synthesis [49, Theorem 7.5.6]. This turns out to be of considerable importance in the theory.

Examples of the sort of semigroups we have in mind are plentiful. Most important, particularly for our present purposes, are the cases when \( G = \hat{G} = \mathbb{R} \) and \( \Sigma = [0, \infty) \) and when \( G = \mathbb{T}, \hat{G} = \mathbb{Z}, \) and \( \Sigma = \{0, 1, 2, \ldots \} \). However, we emphasize that many of our results apply to semigroups \( \Sigma \) which do not totally order \( \hat{G} \) in the sense that \( \Sigma \cup (-\Sigma) = \hat{G} \). For example, let \( G = \mathbb{G} = \mathbb{R}^n, \ n > 1, \) and let \( \Sigma \) be the closure of an arbitrary regular cone in \( \hat{G} \) (cf. [53, Chap. III]). Then \( \Sigma \) satisfies our requirements, but certainly \( \Sigma \cup (-\Sigma) \neq \hat{G} \).

We note in passing that if, in this case, \( \mathcal{B} = L^\infty(\mathbb{R}^n) \) and if \( \mathbb{R}^n \) acts on \( \mathcal{B} \) via (forward) translation, i.e., if \( \alpha_t(\varphi)(x) = \varphi(x + t), \varphi \in L^\infty(\mathbb{R}^n) \), then \( \mathcal{B}(\Sigma) \), the principal object of study in this paper, is precisely the algebra of (boundary values of) bounded holomorphic functions in the tube domain determined by \( \Sigma \).

For the remainder of this chapter, \( \mathcal{B} \) will be a fixed von Neumann algebra, \( \{\alpha_t\}_{t \in \mathbb{G}} \) will be an ultraweakly continuous representation of \( G \) on \( \mathcal{B} \) as a group of \( * \)-automorphisms of \( \mathcal{B} \), and \( \Sigma \) will be a fixed subsemigrup of \( \hat{G} \) satisfying the conditions of Definition 2.4.

DEFINITION 2.5. A covariant representation of the pair \( (\mathcal{B}, \alpha) \) is a pair \( (\pi, U) \) consisting of an ultraweakly continuous \( * \)-representation \( \pi \) of \( \mathcal{B} \) on a Hilbert space and a strongly continuous unitary representation \( U \) of \( G \) on the space of \( \pi \) such that \( U_t \pi(A) = U_t \pi(\alpha_t(A)) \) for all \( A \) in \( \mathcal{B} \) and \( t \) in \( \mathcal{G} \).

PROPOSITION 2.6. There is at least one faithful covariant representation of \( (\mathcal{B}, \alpha) \).

Proof. A proof may be found in [19, 57]. Since, however, we need to refer to it later, we present an outline. Suppose the space upon which \( \mathcal{B} \) acts is \( \mathcal{K} \) and let \( \mathcal{K} = L^\infty(G, \mathcal{K}) \), the Hilbert space of all Bochner measurable, norm-square integrable, \( \mathcal{K} \)-valued functions on \( G \). We define \( \pi, \) representing \( \mathcal{B} \) on \( \mathcal{K} \), by the formula \( (\pi(A)f)(x) = \alpha_t(A)f(x), \ A \in \mathcal{B}, \ f \in \mathcal{K}, \) and \( x \in G \). It is easy to see that \( \pi \) is a faithful, ultraweakly continuous representation of \( G \) on \( \mathcal{K} \).
The representation $U = \{U_t\}_{t \in G}$ of $G$ is taken to be the regular representation of $G$ on $\mathcal{H}$; i.e., $(U_t f)(s) = f(s + t), \quad f \in \mathcal{H}, \quad s, t \in G$. A straightforward calculation shows that $(\pi, U)$ is indeed a covariant representation of $(\mathcal{B}, \alpha)$.

**Definition 2.7.** The representation $(\pi, U)$ of $(\mathcal{B}, \alpha)$ constructed in the preceding proposition will be called the canonical covariant representation of $(\mathcal{B}, \alpha)$ and the von Neumann algebra generated by $(\pi(|\mathcal{B}|)$ and $(U_t)_{t \in G}$ will be called the crossed product of $\mathcal{B}$ by $G$ (determined by $\alpha$).

To understand better the next result, which lies at the heart of our analysis, consider first the following observations on the Weyl commutation relation. Let $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ be two strongly continuous unitary representations of $\mathbb{R}$ on a Hilbert space $\mathcal{H}$. The Weyl commutation relation is simply the equation

$$U_t V_s = e^{ist} V_s U_t, \quad t, s \in \mathbb{R}. $$

There are two equivalent formulations of this equation, or rather, of the assertion that two unitary groups satisfy this equation, which make sense in much more general contexts when no pair of unitary groups are in sight. For the first, let $\{\varphi_t\}_{t \in \mathbb{R}}$ be the representation of $\mathbb{R}$ as a group of automorphisms of $L(\mathcal{H})$ defined by the formula $\varphi_t(A) = U_t A U_t^*$, i.e., $\{\varphi_t\}_{t \in \mathbb{R}}$ is the adjoint representation of $\mathbb{R}$ on $L(\mathcal{H})$ determined by $\{U_t\}_{t \in \mathbb{R}}$. Then the Weyl commutation relation is the same as the equation

$$\varphi_t(V_s) = e^{ist} V_s, \quad t, s \in \mathbb{R}. $$

But by Lemma 2.2, this equation is satisfied precisely when $sp_s(V_t) = \{t\}$, $t \in \mathbb{R}$. Thus $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ satisfy the Weyl commutation relation if and only if $sp_s(V_t) = \{t\}$ for all $t$. For the second formulation, write $\{U_t\}_{t \in \mathbb{R}}$ in its spectral form: $U_t = \int e^{ist} dP(t)$. Then an easy computation reveals that $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ satisfy the Weyl commutation relation if and only if $V_t P(E) V_t^* = P(U + t)$ for all $t$ and all Borel sets $E \subseteq \mathbb{R}$. By regularity, this equation is satisfied for all Borel sets if and only if it is satisfied for all intervals of the form $(-\infty, s]$ and $[s, \infty)$, $s \in \mathbb{R}$. Replacing the projections by their ranges, we may conclude that $\{U_t\}_{t \in \mathbb{R}}$ and $\{V_t\}_{t \in \mathbb{R}}$ satisfy the Weyl commutation relation if and only if $V_t \mathcal{H}^U((-\infty, s]) \subseteq \mathcal{H}^U((-\infty, s + t])$ and $V_t \mathcal{H}^U([s, \infty)) \subseteq \mathcal{H}^U([s + t, \infty))$ for all $s$ and $t$ in $\mathbb{R}$. Combining the two formulations, we arrive at the following assertion in which no explicit mention of the Weyl commutation relation is made; in addition, it may be proved easily and directly without the intervention of the Weyl commutation relation. For any operator $A$, $sp_s(A) \subseteq \{t\}$ if and only if $A \mathcal{H}^U((-\infty, s]) \subseteq \mathcal{H}^U((-\infty, s + t])$ and $A \mathcal{H}^U([s, \infty)) \subseteq \mathcal{H}^U([s + t, \infty))$ for all $s \in \mathbb{R}$. This assertion, together with the analysis which led to it, suggests that in general there may be an intimate relationship between

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1 We are indebted to the referee for these observations.
the distribution of the spectrum of $A$ with respect to $\varphi$ and the way $A$ acts on the spectral subspaces of $U$.

Forelli [14] showed that indeed there is such a relationship, a relationship which we shall call Forelli's Spectral-Commutation Principle. Subsequently, Arveson [3] refined this principle to cover technically more general situations than those considered by Forelli. Since it constitutes the basic tool for our analysis, we present it here for the sake of clarity, but without the technical hypotheses. In those places where we apply it, the technical hypotheses are satisfied, and the proofs may all be found in Arveson's paper.

Scholium 2.8 (Forelli's Spectral-Commutation Principle). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\{U_t\}_{t \in G}$ and $\{V_t\}_{t \in G}$ be isometric representations of $G$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively which need not be strongly continuous but which are continuous in some weaker topologies. Let $\mathcal{L}_w(\mathcal{X}, \mathcal{Y})$ be the space of weakly continuous linear maps from $\mathcal{X}$ to $\mathcal{Y}$, and let it be endowed with a topology which makes continuous the representation $\{\varphi_t\}_{t \in G}$ of $G$ defined by the formula $\varphi_t(A) = V_t \circ A \circ U_{-t}$, $t \in G$, $A \in \mathcal{L}_w(\mathcal{X}, \mathcal{Y})$. Then under suitable hypotheses on the topologies involved, one may assert that the following conditions that $A \in \mathcal{L}_w(\mathcal{X}, \mathcal{Y})$ may satisfy are equivalent:

(a) $\text{sp}_\varphi(A) \subseteq \Sigma + i$; and

(b) $\mathcal{A} \mathcal{F} U(\Sigma + i) \subseteq \mathcal{Y} V(\Sigma + i + i)$, for all $i \in \hat{G}$.

The following is a frequently used special case of Forelli's principle which appears as Corollary 2 to Theorem 2.3 in [3]. Note that it directly extends the above discussion.

Theorem 2.9. Let $\{U_t\}_{t \in G}$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$, let $P$ denote its spectral measure defined on $\hat{G}$, and let $\{\beta_t\}_{t \in G}$ be defined on $\mathcal{L}(\mathcal{H})$ by the formula $\beta_t(A) = U_t A U_t^*$, $A \in \mathcal{L}(\mathcal{H})$. Then for $i \in \hat{G}$ and $A \in \mathcal{L}(\mathcal{H})$, we have $\text{sp}_h(A) \subseteq \Sigma + i$ if and only if $AP(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + i + i) \mathcal{H}$ for all $i \in \hat{G}$.

The following is another application of Forelli's principle which we shall use and, although the proof may be dug out of [3], it does not appear there all in one place. We therefore present an outline.

Proposition 2.10. Let $\mathcal{R}$ be another von Neumann algebra with an ultra-weakly continuous representation $\{\beta_t\}_{t \in G}$ of $G$ as a group of *-automorphisms of $\mathcal{R}$.

\[ \text{Because of the dual use of the notion of spectrum which is contained in this sentence, we have had difficulty refraining from adopting the following proposal for a change in terminology: Use the term "spectrum" without modification to refer to the spectrum of an operator in the sense of invertibility and call the quantity sp.}(X) \text{ defined in Definition II.1a the energy spectrum or the energy distribution of } X \text{ with respect to } \{V_t\}_{t \in G}. \]

This would not only avoid the dual use of "spectrum" but would also be more in keeping with Wiener's intent when he first applied the term "spectrum" in harmonic analysis.
and let $\Phi$ be in $L_\alpha(\mathcal{B}, \mathbb{R})$. Then for $i \in \mathcal{G}$, $\text{sp}(\Phi_\alpha)(\Phi) \subseteq \Sigma + i$ if and only if $\Phi(\mathcal{B}(\Sigma + \delta)) \subseteq \mathcal{R}(\Sigma + \delta + i)$, for all $\delta \in \mathcal{G}$.

**Proof.** By multiplying $\beta$, by $\langle t, -i \rangle$, it suffices to prove the assertion with $i = 0$. By [3, Proposition 3.01, (8, ) 01 and ('%, /3) satisfy the hypotheses (1.5) of [3]. On the other hand, the topology on $L_\alpha(\mathcal{B}, \mathbb{R})$ satisfies the hypotheses (1.1) of [3]. Thus the conditions of Theorem 2.3 of [3] are met, and the proposition is proved.

**Corollary 2.11.** Suppose $\Phi$ in Proposition 2.10 is self-adjoint in the sense that $\Phi(A^*) = \Phi(A)^*$ for all $A$ in $\mathcal{B}$. Then $\Phi(\mathcal{B}(\Sigma + \delta)) \subseteq \mathcal{R}(\Sigma + \delta)$ for all $\delta \in \mathcal{G}$ if and only if $\text{sp}(\Phi_\alpha)(\Phi) \subseteq \{0\}$; i.e., if and only if $\beta_t \circ \Phi = \Phi_\alpha$ for all $t \in \mathcal{G}$.

For the proof we need a simple observation which we state for frequent use later as

**Lemma 2.12.** For each $A \in \mathcal{B}$, $\text{sp}_\alpha(A^*) = -\text{sp}_\alpha(A)$.

**Proof of 2.11.** From Lemma 2.12 we see that $\text{sp}(\Phi_\alpha)(\Phi)$ is symmetric and from Proposition 2.10 we see that $\Phi(\mathcal{B}(\Sigma + \delta)) \subseteq \mathcal{R}(\Sigma + \delta)$ for all $\delta \in \mathcal{G}$ if and only if $\text{sp}(\Phi_\alpha)(\Phi) \subseteq \Sigma$. Thus, since $\Sigma \cap (-\Sigma) = \{0\}$, the result follows.

We are now prepared for a result of the utmost importance for our subsequent applications.

**Theorem 2.13.** Let $\pi$ be an ultraweakly continuous *-representation of the von Neumann algebra $\mathcal{B}$ on a Hilbert space $\mathcal{H}$, let $U = \{U_t\}_{t \in \mathcal{G}}$ be a strongly continuous unitary representation of $\mathcal{G}$ on $\mathcal{H}$, and let $P$ be the spectral measure for $U$ on $\mathcal{G}$. Then $(\pi, U)$ is a covariant representation of $(\mathcal{B}, \alpha)$ if and only if

$$
\pi(\mathcal{B}(\Sigma + i)) P(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + \delta + i) \mathcal{H}, \quad \text{for all } \delta, i \in \mathcal{G}.
$$

**Proof.** Let $\{\beta_t\}_{t \in \mathcal{G}}$ be the representation of $G$ on $L(\mathcal{H})$ defined by the formula $\beta(A) = U_t A U_t^*$, $A \in L(\mathcal{H})$. If equation (2.2) is satisfied, then by Theorem 2.9, $\pi(\mathcal{B}(\Sigma + i)) \subseteq (L(\mathcal{H}))^\alpha(\Sigma + i)$ for all $i \in \mathcal{G}$. By Corollary 2.11, then, $\beta_t \circ \pi = \pi \circ \alpha_t$ for all $t \in \mathcal{G}$; i.e., $(\pi, U)$ is a covariant representation of $(\mathcal{B}, \alpha)$. Since the steps are clearly reversible, the proof is complete.

**Corollary 2.14.** Let $(\pi, U)$ be a covariant representation of $(\mathcal{B}, \alpha)$ on a Hilbert space $\mathcal{H}$, let $P$ be the central support of $U$, and let $A$ lie in $\mathcal{B}$. Then $A F$ belongs to $\mathcal{B}(\Sigma + i)$ for some $i$ in $\mathcal{G}$ if and only if

$$
\pi(A) P(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + \delta + i) \mathcal{H}, \quad \text{for all } \delta \in \mathcal{G}.
$$

**Proof.** Since $(\pi, U)$ is a covariant representation of $(\mathcal{B}, \alpha)$ Theorem 2.13 implies that $\pi(\mathcal{B}(\Sigma + i)) P(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + \delta + i) \mathcal{H}$ for all $\delta \in \mathcal{G}$. Thus,
if $AF$ lies in $\mathcal{B}(\Sigma + i)$, $\pi(A) = \pi(AF)$ satisfies equation (2.3). The converse follows from the facts that $\pi$ is an isomorphism of the reduced algebra $\mathcal{B}F$ and that $F$ is invariant.

**Corollary 2.15.** For $s, t \in G$, $\mathcal{B}(\Sigma + i) \mathcal{B}(\Sigma + i) \subseteq \mathcal{B}(\Sigma + 2i + i)$.

**Proof.** Arveson [3] proves this when $G = \mathbb{R}$ and $\Sigma = [0, \infty)$ and his proof works equally well here—with the obvious modifications. However, we offer a different proof which also serves to illustrate the preceding developments. Let $(\pi, U)$ be a faithful covariant representation of $(\mathcal{B}, \alpha)$ acting on a Hilbert space $\mathcal{H}$ and let $P$ be the spectral measure on $\hat{G}$ associated with $U$. Since $\pi$ is faithful, $A$ lies in $\mathcal{B}(\Sigma + i)$ precisely when $\pi(A) P(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + i + i) \mathcal{H}$ for all $i \in \hat{G}$ by Corollary 2.14. Similarly $B$ belongs to $\mathcal{B}(\Sigma + i)$ if and only if $\pi(B) P(\Sigma + i + i) \mathcal{H} \subseteq P(\Sigma + i + i) \mathcal{H}$ for all $i \in \hat{G}$. But then, for $A \in \mathcal{B}(\Sigma + i)$ and $B \in \mathcal{B}(\Sigma + i)$, we have $\pi(BA) P(\Sigma + i) \mathcal{H} = \pi(B) \pi(A) \times P(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + i + i) \mathcal{H}$ for all $i \in \hat{G}$, and so by Corollary 2.14, once more, $BA$ is in $\mathcal{B}(\Sigma + i + i)$.

**Corollary 2.16.** The subspace $\mathcal{B}(\Sigma)$ is actually a subalgebra of $\mathcal{B}$.

**Definition 2.17.** Henceforth, we write $\mathcal{A}$ for $\mathcal{B}(\Sigma)$ and refer to it as the algebra of analytic operators in $\mathcal{B}$ (relative to $\alpha$).

**Remark 2.18.** Since $\Sigma$ is a set of spectral synthesis, $\mathcal{A} (= \mathcal{B}(\Sigma))$ is characterized as the set of $A$ in $\mathcal{B}$ such that $A * f = 0$ for all $f$ in $L^1(G)$ such that $f(i) = 0$, $i \in \Sigma$ [44, Lemma 2.3.8]. When $G = \mathbb{R}$ and $\Sigma = [0, \infty)$, this subspace of $L^1(\mathbb{R})$ is usually denoted by $K^1(\mathbb{R})$. When $\mathcal{B} = L^1(\mathbb{R})$ and $\mathbb{R}$ acts via translation, we have the familiar fact that $F$ lies in $H^\infty(\mathbb{R}) (= \mathcal{B}(0, \infty))$ precisely when $\int_{-\infty}^{\infty} F(t) \overline{f(t)} dt = 0$ for all $f \in K^1(\mathbb{R})$. We will have occasion to use this in the discussion preceding Theorem 4.1.6 as well as in its proof.

**Remark 2.19.** Suppose $(\pi, U)$ is a faithful covariant representation of $(\mathcal{B}, \alpha)$ acting on a Hilbert space $\mathcal{H}$ and let $P$ be the spectral measure on $\hat{G}$ associated with $U$. Then by Corollary 2.14, $\mathcal{A}$ is precisely the collection of operators $A \in \mathcal{B}$ with the property that $\pi(A) P(\Sigma + i) \mathcal{H} \subseteq P(\Sigma + i) \mathcal{H}$ for all $i \in \hat{G}$. It is natural to ask if this condition may be replaced by the weaker condition $\pi(A) P(\Sigma) \mathcal{H} \subseteq P(\Sigma) \mathcal{H}$. Examples show that the answer depends on the algebra and on the representation, of course, but the determination of exactly when this occurs seems to be a difficult problem.

### 3. The Algebra of Analytic Operators

In this section we investigate the general structure of algebras of analytic operators, paying particular attention to their relation to the subdiagonal algebras.
of Arveson [2]. We fix, once and for all, the following ingredients: a locally compact abelian group \( G \); a subsemigroup \( \Sigma \) of \( G \) satisfying the conditions of Definition 2.4; a von Neumann algebra \( \mathfrak{B} \); and an ultraweakly continuous representation \( \{ \alpha_t \}_{t \in G} \) of \( G \) on \( \mathfrak{B} \) as a group of \( \ast \)-automorphisms. As in Section 2, we shall write \( \mathfrak{A} \) for \( \mathfrak{B}^\gamma(\Sigma) \). After some preliminary results, we shall specialize to the case when \( \Sigma \) totally orders \( G \). This assumption yields the sharpest results.

**Definition 3.1.** A normal expectation from \( \mathfrak{B} \) onto a von Neumann subalgebra \( \mathfrak{D} \) is an ultraweakly continuous linear map \( \Phi \) from \( \mathfrak{B} \) onto \( \mathfrak{D} \) such that (i) \( \| \Phi \| = 1 \), and (ii) the restriction of \( \Phi \) to \( \mathfrak{D} \) is the identity map.

Such a mapping \( \Phi \) also has the following properties: (iii) it is positive (and hence self-adjoint), i.e., \( \Phi(A) \geq 0 \) if \( A \geq 0 \); (iv) it is idempotent, i.e., \( \Phi \circ \Phi = \Phi \); and (v) it is \( \mathfrak{D} \)-homogeneous, i.e., \( \Phi(AXB) = A\Phi(X)B \) for all \( A, B \in \mathfrak{D} \). A nice exposition of the basic properties of expectations may be found in [2, Appendix]. Since we shall have no occasion to consider nonnormal expectations, we shall henceforth assume that all expectations under consideration are normal.

We shall abuse the terminology somewhat and refer to the zero map on \( \mathfrak{B} \) as the zero expectation. Finally, we shall say that the expectation \( \Phi \) is faithful in case \( \Phi(A^*A) = 0 \) only when \( A = 0 \).

**Definition 3.2.** Let \( \mathfrak{D} \) denote the subalgebra of fixed points of \( \{ \alpha_t \}_{t \in G} \). Then \( \mathfrak{B} \) is called \( G \)-finite (relative to \( \alpha \)) in case there is a faithful expectation \( \Phi \) from \( \mathfrak{B} \) onto \( \mathfrak{D} \) such that \( \Phi \circ \alpha_t = \Phi \) for all \( t \in G \). At the opposite extreme, we say that \( \mathfrak{B} \) is completely non-\( G \)-finite (relative to \( \alpha \)) in case the zero expectation is the only expectation of \( \mathfrak{B} \) into \( \mathfrak{D} \) left invariant by \( \{ \alpha_t \}_{t \in G} \).

**Remark 3.3.** (See [29].) The algebra \( \mathfrak{B} \) is \( G \)-finite if and only if there is a separating family of \( \alpha \)-invariant normal states on \( \mathfrak{B} \). (It suffices even to assume that the states separate the points of \( \mathfrak{D} \).) On the other hand, \( \mathfrak{B} \) is completely non-\( G \)-finite if and only if there are no invariant normal states.

**Remark 3.4.** (See [8].) There is a projection \( E \) in the center of \( \mathfrak{D} \) such that \( E\mathfrak{B}E \) is \( G \)-finite while \( (I - E)\mathfrak{B}(I - E) \) is completely non-\( G \)-finite. In particular, if \( \mathfrak{D} \) is a factor, then \( \mathfrak{B} \) is either \( G \)-finite or completely non-\( G \)-finite.

**Remark 3.5.** There is at most one faithful invariant expectation from \( \mathfrak{B} \) onto \( \mathfrak{D} \), so that if \( \mathfrak{B} \) is \( G \)-finite, the expectation is unique. This is proved in [29] and also follows from the general developments in [2, Appendix].

**Remark 3.6.** If \( \mathfrak{B} \) is \( G \)-finite with respect to \( \{ \alpha_t \}_{t \in G} \), and if \( \Phi \) is the unique invariant, faithful expectation from \( \mathfrak{B} \) onto \( \mathfrak{D} \), then there is a net \( \{ \varphi_i \}_{i \in I} \) of convex combinations of the \( \alpha_t \) (i.e., \( \varphi_i = \sum_{k=1}^{n_i} \lambda_{k}^{(i)} \alpha_t \), \( \sum_{k=1}^{n_i} \lambda_{k}^{(i)} = 1 \)) such that for each \( A \) in \( \mathfrak{B} \), \( \Phi(A) \) is the limit in the ultrastrong topology of \( \{ \varphi_i(A) \}_{i \in I} \) [12].

**Remark 3.7.** If \( \mathfrak{L}(\mathfrak{B}) \) is \( G \)-finite, then \( \mathfrak{D} \) is a type I von Neumann algebra and its center is discrete [55].
Before proceeding, we pause to remind the reader of an analogy. If $G$ were the full group of unitary operators in $\mathcal{B}$ acting on $\mathcal{B}$ via conjugation, then to say that $\mathcal{B}$ is $G$-finite is simply to say that $\mathcal{B}$ is finite as a von Neumann algebra. In this instance $\Phi$ becomes the center-valued trace, Remark 3.5 asserts its uniqueness—a familiar fact—and Remark 3.6 is the approximation theorem [11, Théorème 1, Sect. 5, Chap. III]—also a familiar fact. On the other hand, in this analogy, to say that $\mathcal{B}$ is completely non-$G$-finite is simply to say that $\mathcal{B}$ is properly infinite and Remark 3.4 affirms the decomposition of $\mathcal{B}$ into its finite and properly infinite parts. This way of viewing $G$-finitude is not due to us but has been exploited with good effect in the literature (cf. [8, 27, 56]).

For the sequel recall that by Lemma 2.2, $\mathcal{D}$ is $\mathcal{B}^\omega(\{0\})$.

**Theorem 3.8.** If $\mathcal{B}$ is $G$-finite, then the unique faithful invariant expectation $\Phi$ from $\mathcal{B}$ onto $\mathcal{B}^\omega(\{0\})$ is multiplicative on $\mathcal{A}$; i.e., $\Phi(AB) = \Phi(A) \Phi(B)$, for all $A$ and $B$ in $\mathcal{A}$.

**Proof.** Following Remark 3.6 we fix a net $\{\varphi_i\}_{i\in I}$ of convex combinations of the $\alpha_i$ such that $\lim_{i} \varphi_i(X) = \Phi(X)$ in the ultraweak topology for all $X$ in $\mathcal{B}$. Now fix $A$ and $B$ in $\mathcal{A}$ and let $F(t) = \Phi(\alpha_i(A)B) = (\Phi \circ \alpha_i)(B)$. Then by Lemma 2.3, $\text{sp}(F) \subseteq \text{sp}(B) \cap \text{sp}(\Phi \circ \alpha_i)(-\Sigma) \cap \Sigma = \{0\}$, and so by Lemma 2.2, $F(t) = F(0)$. But then, for each $i$ in $I$, $\Phi(AB) = F(0) = (\Phi \circ \alpha_i)(\varphi_i(B))$ and so $\Phi(AB) = \lim_{i} (\Phi \circ \alpha_i)(\varphi_i(B)) = (\Phi \circ \alpha_i)(\Phi(B)) = \Phi(\alpha_i(\Phi(B))) = \Phi(A) \Phi(B)$ since $\Phi$ is $\mathcal{B}^\omega(\{0\})$-homogeneous (property (v) of expectations).

**Definition 3.9.** Let $\mathcal{B}$ be a von Neumann algebra and let $\Phi$ be a faithful expectation of $\mathcal{B}$ onto a von Neumann algebra $\mathcal{D}$. Then a subalgebra $\mathcal{A}$ of $\mathcal{B}$, containing $\mathcal{D}$, is called a subdiagonal algebra in $\mathcal{B}$ with respect to $\Phi$ if

(i) $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$;

(ii) $\Phi$ is multiplicative on $\mathcal{A}$; and

(iii) $\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}$.

The subalgebra $\mathcal{D}$ is called the diagonal of $\mathcal{A}$. We say that $\mathcal{A}$ is a maximal subdiagonal algebra in $\mathcal{B}$ with respect to $\Phi$ in case $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{B}$ which is subdiagonal with respect to $\Phi$.

** Remarks 3.10.** (a) Subdiagonal algebras should be regarded as the non-commutative analogue of weak-* Dirichlet algebras [52].

(b) As we have given it, our definition of subdiagonal algebras is not the same as the original definition of Arveson [2]. However, by his Proposition 2.1.4, they agree.

(c) If $\mathcal{A}$ is a subdiagonal algebra in $\mathcal{B}$ and if $\mathcal{A}$ is also a maximal ultraweakly closed subalgebra $\mathcal{B}$, then $\mathcal{A}$ is certainly a maximal subdiagonal algebra.
of $\mathcal{B}$. (This is because maximal subdiagonal algebras are ultraweakly closed [2, Theorem 2.2.1].) However, the converse is false. We note in passing that the problem of deciding when a subdiagonal algebra is a maximal ultraweakly closed subalgebra of its containing von Neumann algebra appears to be quite difficult. It was solved for commutative algebras by the second author in [37, 38]. We note too that this problem is intimately tied up with the problem in Remark 2.19.

Recall that $\Sigma$ totally orders $\hat{G}$ in case $\Sigma \cup \{-\Sigma\} = \hat{G}$. We shall show in Theorem 3.15 below that if $\Sigma$ totally orders $\hat{G}$ and if $\mathfrak{B}$ is $G$-finite, then $\mathfrak{A}$ is a maximal subdiagonal algebra in $\mathfrak{B}$ with respect to the unique faithful, $\alpha$-invariant expectation on $\mathfrak{B}$. Before doing this we pursue a number of technical results (some will be used in the proof) which provide greater insight into the structure of $\mathfrak{A}$ and the role that total ordering plays.

We shall write $\Sigma'$ for $\Sigma \setminus \{0\}$, and for $t^\prime$ in $\Sigma'$, we shall write $\Sigma_{t^\prime}$ for $\Sigma + t^\prime$. Thus each $\Sigma_{t^\prime}$ is contained in $\Sigma'$, and $\Sigma' = \bigcup_{t^\prime \in \Sigma'} \Sigma_{t^\prime}$. In addition, we shall write $\mathfrak{A}_{t^\prime}$ for the ultraweak closure of $\{A \in \mathfrak{A} \mid \text{sp}(A) \subseteq \Sigma_{t^\prime}, \ t^\prime \in \Sigma'\}$. The following lemma and proposition constitute a mild generalization of Forelli's Proposition 2 [14].

**Lemma 3.11.** The space $\mathfrak{A}_{0}$ is a two-sided ideal in $\mathfrak{A}$.

**Proof.** First observe that $\Sigma'$ is a subsemigroup of $\Sigma$. Indeed, if $s$ and $t$ are in $\Sigma'$ and if $s + t^\prime = 0$, then $-s = t^\prime \in \Sigma$, so that $s$ lies in $\Sigma \cap (-\Sigma) = \{0\}$—a contradiction. Next observe that $\Sigma' + \Sigma - \{s + t^\prime \mid s \in \Sigma', \ t^\prime \in \Sigma\}$ is contained in $\Sigma'$. Thus, on the basis of Corollary 2.15 and the fact that $\text{sp}(A + B) \subseteq \text{sp}(A) \cup \text{sp}(B)$, we find that $\{A \in \mathfrak{A} \mid \text{sp}(A) \subseteq \Sigma'\}$ is an ideal in $\mathfrak{A}$ and so, therefore, is its ultraweak closure $\mathfrak{A}_{0}$.

**Proposition 3.12.** Suppose $\Sigma$ totally orders $\hat{G}$ and that $G$ acts trivially on another von Neumann algebra $\mathfrak{N}$. Then $(\mathcal{L}_{w}(\mathfrak{B}, \mathfrak{N}))(\Sigma) = \{\Phi \in \mathcal{L}_{w}(\mathfrak{B}, \mathfrak{N}) \mid \Phi(\mathfrak{A}_{0}) = 0\}$.

**Proof.** This result follows easily from Proposition 2.10 once it is noted that if $\{\beta_{t}\}_{t \in \Sigma}$ denotes the trivial action of $G$ on $\mathfrak{N}$, then $\mathfrak{N}^{\beta}(E) = \mathfrak{N}$ if $0 \in E$ and is $\{0\}$ otherwise. However, for the sake of variation, we supply an alternate proof.

Suppose $\text{sp}(A) \subseteq \Sigma_{t^\prime}$ for some $t^\prime \in \Sigma'$, and let $\Phi$ be an element in $\mathcal{L}_{w}(\mathfrak{B}, \mathfrak{N})(\Sigma)$. If $\mathcal{Y}(t) = \Phi(\mathfrak{N}_{t^\prime}(A))$, $\text{sp}(\mathcal{Y}) \subseteq \text{sp}(A) \cap \text{sp}(\Phi) \subseteq (-\Sigma_{t^\prime}) \cap \Sigma$ which is empty. Hence, by a familiar theorem, $\mathcal{Y} = 0$. Thus $\Phi(A) = 0$ and it follows that $\Phi$ annihilates $\mathfrak{A}_{0}$.

Suppose conversely that $\Phi$ annihilates $\mathfrak{A}_{0}$. To show that $\Phi$ belongs to $\mathcal{L}_{w}(\mathfrak{B}, \mathfrak{N})(\Sigma)$ we need to show that for each $t^\prime$ in the complement of $\Sigma$ we can produce a function $f$ in $L^{1}(G)$ such that $\hat{f}(t^\prime) \neq 0$ while $\Phi * f = 0$. Fix such a $t^\prime$. Then because $\Sigma$ totally orders $\hat{G}$ and because $\Sigma$ is the closure of its interior, we can find an $s$ in $\Sigma'$ such that $t^\prime$ lies in interior of $-\Sigma_{s}$. Now choose an $f$ in $L^{1}(G)$ such that $\hat{f}$ is supported in $-\Sigma_{s}$ with $\hat{f}(t^\prime) \neq 0$. Then for $A$ in $\mathfrak{B}$ we
find that by definition of convolution on $L^\infty(G)$, $(\Phi * f)(A) = \Phi(A * f^\ast)$ where $f^\ast(t) = f(-t)$. Since the Fourier transform of $f^\ast$ is supported in $\Sigma'$, $A * f$ lies in $\mathfrak{A}(\Sigma')$ which is contained in $\mathfrak{A}_0$. Thus $\Phi(A * f) = 0$, and since $A$ is arbitrary, $\Phi * f = 0$. This completes the proof.

When $\Sigma$ totally orders $G$, one may write $\Sigma = G\langle -\Sigma' \rangle$. One might well suspect, therefore, that if $\Sigma$ does not totally order $G$, then the annihilator of $\mathfrak{A}_0$ in $L^\infty(G)$ is the closure of $\{ \Phi \mid \text{sp}(\Phi) \subseteq \hat{G}\langle -\Sigma' \rangle \}$. This suspicion is borne out by numerous examples, but we are not able to decide its validity in general.

The following corollary generalizes Proposition 5.1 in [3] (cf. [31] also). It makes explicit an important relation between analytic operators and scalar-valued generalized analytic functions (cf. [1])—a relation which is exploited implicitly at least in other parts of the paper. We shall call a $\rho$ in $\mathcal{B}_*$ *analytic* in case $\text{sp}(\rho) \subseteq \Sigma'$. Also, when $G$ is totally ordered by $\Sigma$, we shall write $H^\infty(G)$ for the space of functions $\varphi$ in $L^\infty(G)$ such that $\int \varphi(t) f(t) \, dt = 0$ for all $f$ in $L^1(G)$ with $f$ supported in some $-\Sigma_i, \ i \in \Sigma'$. Evidently, $H^\infty(G)$ is the space of all functions in $L^\infty(G)$ whose spectra, computed with respect to the representation of $G$ by (forward) translation on $L^\infty(G)$, are contained in $\Sigma$.

**Corollary 3.13.** If $\Sigma$ totally orders $G$, then the following assertions about a functional $\rho$ in $\mathcal{B}_*$ are equivalent:

(i) $\rho$ is analytic;

(ii) $\rho$ annihilates $\mathfrak{A}_0$; and

(iii) for all $A$ in $\mathcal{B}$, the function $\varphi(t) = \rho(\alpha_{\Sigma_i}(A))$ belongs to $H^\infty(G)$.

**Proof.** The equivalence of (i) and (ii) is the content of Proposition 3.12. Also, the fact that (i) implies (iii) is simply a consequence of Lemma 2.3. If (iii) holds, if $A$ is an arbitrary operator in $\mathcal{B}$, and if $f$ is a function with Fourier transform supported in $-\Sigma_i$ for some $i \in \Sigma'$, then $0 = \int \varphi(t) f(t) \, dt = \int \rho(\alpha_{\Sigma_i}(A)) f(t) \, dt = (\rho * f)(A)$. Since $f$ is supported in $-\Sigma_i$, the argument in Proposition 3.12 now applies and allows us to conclude that since $A$ is arbitrary, $\text{sp}(\rho) \subseteq \Sigma$; i.e., $\rho$ is analytic.

In the commutative setting, the next proposition appears in [38, proof of Theorem 1].

**Proposition 3.14.** Suppose $G$ is totally ordered by $\Sigma$ and that $\mathcal{B}$ is $G$-finite. Then $\mathfrak{A}_0$ coincides with the kernel of $\Phi \mid \mathfrak{A}$, where $\Phi$ is the unique, faithful, $\infty$-invariant expectation from $\mathcal{B}$ onto $\mathcal{B}(\{0\})$.

**Proof.** Since $\Phi$ is invariant, $\text{sp}(\Phi) = \{0\} \subseteq \Sigma$. So by Proposition 3.12, $\mathfrak{A}_0 \subseteq \ker(\Phi)$. On the other hand, suppose $A$ is in $(\ker \Phi) \cap \mathfrak{A}$ and let $\rho \in \mathcal{B}_*$ annihilate $\mathfrak{A}_0$. Then, by Corollary 3.13 and Lemma 2.3, we find that the spectrum of the function $\psi(t) = \rho(\alpha_{\Sigma_i}(A))$ (computed with respect to backward translation) is contained in $-\text{sp}(A) \cap \text{sp}(\rho) \subseteq (-\Sigma) \cap \Sigma = \{0\}$. Thus $\psi$ is a constant.
But then if \( \{ \varphi_i \}_{i \in I} \) is a net of convex combinations of the \( \alpha_i \) converging ultra-
strongly to \( \Phi(A) \) (cf. Remark 3.6), we find that \( 0 = \rho(0) = \rho(\Phi(A)) = \lim_i \rho(\varphi_i(A)) = \rho(A) \). Since \( \rho \) is an arbitrary element of the annihilator of \( \mathcal{U}_0 \),
this and the Hahn–Banach theorem yield the result.

We arrive, finally, at the principal result of this section.

**Theorem 3.15.** If \( \Sigma \) totally orders \( G \), then \( \mathfrak{A} + \mathfrak{A}^* \) is ultraweakly dense
in \( \mathfrak{B} \). If, moreover, \( \mathfrak{B} \) is \( G \)-finite, and if \( \Phi \) is the unique faithful \( \alpha \)-invariant expectation
from \( \mathfrak{B} \) onto \( \mathfrak{B}((0)) \), then \( \mathfrak{A} \) is a maximal subdiagonal algebra in \( \mathfrak{B} \) with
respect to \( \Phi \).

**Proof.** Suppose \( \rho \in \mathfrak{B}_* \) annihilates \( \mathfrak{A} + \mathfrak{A}^* \). Then \( \rho \) annihilates \( \mathfrak{A}_0 \) and
so by Corollary 3.13, \( \text{sp}(\rho) \subseteq \Sigma \). On the other hand, since \( \text{sp}(A^*) = -\text{sp}(A) \)
(cf. Lemma 2.12), \( \text{sp}(\rho) \subseteq -\Sigma \) as well. Hence \( \text{sp}(\rho) \subseteq (-\Sigma) \cap (\Sigma) = \{0\} \) and
\( \rho \) is invariant. But \( \rho \) annihilates \( \mathfrak{A} \supseteq \mathfrak{B}((0)) \), and so \( \rho = 0 \) [29]. Thus \( \mathfrak{A} + \mathfrak{A}^* \) is
ultraweakly dense in \( \mathfrak{B} \).

This, with Theorem 3.8, shows that \( \mathfrak{A} \) is a subdiagonal algebra in \( \mathfrak{B} \) with
respect to \( \Phi \). To establish its maximality, it suffices, by [2, Theorem 2.2.1],
to show that any \( X \) in \( \mathfrak{B} \) satisfying the equation
\[
\Phi(AXT) = 0
\]  
(3.1)
for all \( A \) in \( \mathfrak{A} \) and all \( T \) in \( (\ker \Phi) \cap \mathfrak{A} \), is already in \( \mathfrak{A} \). In fact, Arveson showed
that the collection \( \mathfrak{A}_{\text{max}} \) of all such \( X \) is the maximal subdiagonal algebra in \( \mathfrak{B} \)
with respect to \( \Phi \) containing \( \mathfrak{A} \). It follows, therefore, that since \( \mathfrak{A}, \Phi, \), and
\( \ker \Phi \cap \mathfrak{A} \) are all invariant under \( \{ \alpha_t \}_{t \in \mathcal{G}} \), so is \( \mathfrak{A}_{\text{max}} \). Hence, if \( X \) is in \( \mathfrak{A}_{\text{max}} \),
so is \( X \ast f \) for all \( f \in L^1(G) \). But, if there were an \( X \in \mathfrak{A}_{\text{max}} \setminus \mathfrak{A} \), then because \( \Sigma \)
totally orders \( \mathcal{G} \), we could find a \( t \in \Sigma' \) and a function \( f \in L^1(G) \) with Fourier
transform supported in \( -\Sigma_t \) such that \( X \ast f \neq 0 \). This new element \( Y \) would
then lie in \( \mathfrak{A}_{\text{max}} \) and \( Y^* \) would lie in \( \mathfrak{A}_0 \), which is \( \Phi \cap \mathfrak{A} \) by Proposition 3.14.
Thus, taking \( A = I, X = Y, \) and \( T = Y^* \) in Eq. (3.1), we would arrive at
the equation \( \Phi(YY^*) = 0 \). Since \( \Phi \) is faithful we would have to conclude that
\( Y = 0 \), contrary to hypothesis. Thus \( \mathfrak{A} = \mathfrak{A}_{\text{max}} \) and the proof is complete.

We conclude this section with an observation which is a partial converse
to some of the preceding developments.

**Proposition 3.16.** Suppose \( \Sigma \) totally orders \( \mathcal{G} \) and suppose there is a faithful
expectation \( \Psi \) from \( \mathfrak{B} \) onto \( \mathfrak{B}((0)) \) with respect to which \( \mathfrak{A} \) is a subdiagonal algebra
in \( \mathfrak{B} \). If \( \mathfrak{A}_0 \subseteq \ker(\Psi) \), then \( \mathfrak{B} \) is \( G \)-finite and \( \Psi \) is the unique faithful, \( \alpha \)-invariant
expectation from \( \mathfrak{B} \) onto \( \mathfrak{B}((0)) \).

**Proof.** By Remark 3.5, it clearly suffices to show that \( \Psi \) is invariant. But
if we take \( \Phi \) to be \( \mathfrak{B}((0)) \) with the trivial action of \( G \) in Corollary 2.11 and
Proposition 3.12 then the assertion is immediate.
4. Examples

In this section we survey a number of concrete examples of the abstract algebras we have introduced in the previous chapters. We want to emphasize at the outset that we do not intend to present a complete analysis of these examples here. Indeed, each example merits considerable study in its own right, and we intend to pursue such studies in the future. Our goal, then, is to exhibit the examples, putting into view the essential constructs which enter into their structure, and to show how they relate to various classes of non-self-adjoint operator algebras which appear elsewhere in the literature. The examples fall naturally into three relatively disjoint classes. In the first, the von Neumann algebras are type I, and the subalgebras of analytic operators are clearly seen to be the natural, noncommutative generalization of the function algebras associated with flows studied by the second author [38]. In the second class, the von Neumann algebras are arbitrary, but the automorphism groups are assumed to be inner. It is shown that this class coincides with the class of nest algebras introduced by Ringrose [47] as a generalization of the hyperreducible, triangular algebras of Kadison and Singer [29]. In the last class, the von Neumann algebras are crossed products and we show how to analyze a number of algebras studied by Arveson [2] via "spectral theory."

4.1. Type I von Neumann Algebras

To begin with, we need the fact that if the homogeneous summands of a type I von Neumann algebra are preserved by an action of \( R \), then the action is spatially implemented. This fact rests on a deep theorem of Brown [5] which generalizes the well-known result of Bargmann [4] to the effect that every strongly continuous projective representation of \( L^1(\mathbb{R}) \) is in fact a unitary representation. Although the analysis which relates Brown's theorem to the problem of spatial implementation is fairly well known, at least to the cognoscenti, we present a discussion of it for the reader's convenience and also because much of it is necessary for our ultimate goal of analyzing all the possibilities for algebras of analytic operators in type I von Neumann algebras.

Throughout this subsection, \( \mathcal{B} \) will denote a type I von Neumann algebra acting on a separable space \( \mathcal{H} \) and \( \{\alpha_t\}_{t \in \mathbb{R}} \) will be an ultraweakly continuous representation of \( \mathbb{R} \) as a group of \(*\)-automorphisms of \( \mathcal{B} \). Recall that in general a group of \(*\)-automorphisms of a von Neumann algebra is said to act ergodically if and only if the only operators in the algebra fixed by all the automorphisms in the group are the scalar multiples of the identity. Whenever convenient we will assume that \( \{\alpha_t\}_{t \in \mathbb{R}} \) acts ergodically on the center \( \mathcal{Z} \) of \( \mathcal{B} \). There is no real loss of generality in doing this because, by a result of Guichardet and Kastler [18, Théorème 6], we may always express \( \mathcal{B} \) and \( \{\alpha_t\}_{t \in \mathbb{R}} \) as direct integrals of objects of the same kind, \( \mathcal{B} = \int \mathcal{B}(\lambda) d\nu(\lambda) \), and \( \{\alpha_t\}_{t \in \mathbb{R}} = \{\int \alpha_t(\lambda) d\nu(\lambda)\}_{t \in \mathbb{R}} \).
where for almost all \( \lambda, \{ \alpha_t(\lambda) \}_{t \in \mathbb{R}} \) acts ergodically on the center of \( \mathcal{B}(\lambda) \). We note too that if \( \mathcal{A} \) (resp. \( \mathcal{A}(\lambda) \)) denotes the algebra of analytic operators with respect to \( \{ \alpha_t(\lambda) \}_{t \in \mathbb{R}} \), then \( \mathcal{A} = \int_{\mathbb{R}} \mathcal{A}(\lambda) d\nu(\lambda) \).

Recall that if \( \mathcal{M} \) is a type I factor then it is called type I, \( m = 1, 2, \ldots, \infty \), in case it is spatially isomorphic to the von Neumann algebra \( \mathcal{M}_{n,m} \) which is the tensor product of the matrix algebra \( \mathcal{M}_{n,1} \) of \( n \times n \) matrices over \( \mathbb{C} \) (all bounded operators on a separable space if \( n = \infty \)) with the scalar multiples of the identity on an \( m \)-dimensional Hilbert space [51, Theorem 1.1.16]. This "fine" classification of type I factors provides a complete set of spatial isomorphism invariants for such algebras in the sense that if \( \mathcal{M} \) (resp. \( \mathcal{M}_1 \)) is of type I, \( \{n, m\} \) then \( \mathcal{M} \) and \( \mathcal{M}_1 \) are spatially isomorphic if and only if \( n = n_1 \) and \( m = m_1 \). The following proposition is an immediate consequence of these remarks and [51, Corollary III.1.3].

**Proposition 4.1.1.** There is a doubly indexed sequence \( E_{n,m}, n, m = 1, 2, \ldots, \infty \), of orthogonal projections in \( \mathcal{Z} \) (some of which may be zero) whose sum is \( \mathcal{I} \) such that \( E_{n,m} \mathcal{B} E_{n,m} \) is spatially isomorphic to \( \mathcal{M}_{n,m} \otimes \mathcal{E}_{n,m} \) where \( \mathcal{E}_{n,m} \) is a maximal Abelian von Neumann algebra \(*\)-isomorphic to \( \mathcal{B}_{n,m} \mathcal{Z}_{n,m} \). If, in addition, \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is spatially implemented, then each \( E_{n,m} \) is invariant under \( \{ \alpha_t \}_{t \in \mathbb{R}} \).

**Corollary 4.1.2.** If \( \{ \alpha_t \}_{t \in \mathbb{R}} \) acts ergodically on \( \mathcal{Z} \) and is spatially implemented, then \( \mathcal{B} \) is spatially isomorphic to \( \mathcal{M}_{n,m} \otimes \mathcal{E} \) for suitable \( n \) and \( m \) and maximal abelian von Neumann algebra \( \mathcal{E} \).

Our objective now is to show that conversely if \( \mathcal{B} \) is spatially isomorphic to some \( \mathcal{M}_{n,m} \otimes \mathcal{E} \), then \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is spatially implementable. We then provide an "internal" description of \( \mathcal{M} (= \mathcal{B}^\ast([0, \infty))) \) in this case.

The subscripts \( n \) and \( m \) will be of no use to us in the sequel and so will be dropped; we shall simply use \( \mathcal{M} \) to denote a type I factor. The reader is urged to keep in mind, however, that \( \mathcal{M} \) need not be spatially isomorphic to the algebra of operators on some Hilbert space.

From basic reduction theory one knows that an algebra of the form \( \mathcal{M} \otimes \mathcal{E} \) with \( \mathcal{E} \) maximal Abelian and acting on a separable space may be identified as the space \( \mathcal{L}^\infty(X, \mathcal{M}) \) of all essentially bounded, weakly measurable, \( \mathcal{M} \)-valued functions on a convenient standard Borel space \( (X, \mu) \) of finite measure. We shall henceforth assume that \( \mathcal{B} \) has such a representation and we shall write \( \mathcal{H}_0 \) as \( \mathcal{L}^2(X, \mathcal{H}_0) \) where \( \mathcal{H}_0 \) is the space on which \( \mathcal{M} \) acts and where \( \mathcal{L}^2(X, \mathcal{H}_0) \) denotes the space of measurable \( \mathcal{H}_0 \)-valued functions on \( X \) such that \( \int_X \| f(x) \|_{\mathcal{H}_0}^2 d\mu(x) < \infty \). Of course, \( \mathcal{L}^2(X, \mathcal{H}_0) \) is a Hilbert space with the usual operations and \( \mathcal{L}^\infty(X, \mathcal{M}) \) acts on it in the obvious way. Since \( \{ \alpha_t \}_{t \in \mathbb{R}} \), regarded now as acting on \( \mathcal{L}^\infty(X, \mathcal{M}) \), carries the center of \( \mathcal{L}^\infty(X, \mathcal{M}) \) onto itself, and since this in turn is isomorphic to \( \mathcal{L}^\infty(X) \) in the obvious way, a deep theorem of Mackey [34] may be applied to allow us to assert that there is a measurable action of \( \mathbb{R} \).
on $X$ such that $(\alpha_t(A))(x) = A(x + t)$ a.e. $(\mu)$ for all $A$ in the center of $L^\infty(X, \mathcal{M})$.

It follows from this that the measure $\mu$ is quasi-invariant, meaning that for each null set $E$ for $\mu$ and each $t \in \mathbb{R}$, $E + t$ is also null. Thus, for each $t \in \mathbb{R}$ we may form the Radon–Nikodym derivative $J(t, x) = (d\mu_t/d\mu)(x)$ where $\mu_t$ is measure determined by the equation $\mu_t(E) = \mu(E + t)$. It is important to note that although for each $t \in \mathbb{R}$, $J(t, x)$ is unique only up to a set of $\mu$-measure zero, it is possible to choose $J$ so that it is a Borel function on $\mathbb{R} \times X$ [34]. We assume that such a choice has been made, and we define a unitary representation $\{T_t\}_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $L^2(X, \mathcal{M}_0)$ by the formula

$$(T_t f)(x) = f(x + t) J^{1/2}(t, x), \quad f \in L^2(X, \mathcal{M}_0). \quad (4.1.1)$$

Then since the action of $\mathbb{R}$ on $X$ is measurable, and since $J$ is a Borel function, $\{T_t\}_{t \in \mathbb{R}}$ is measurable and hence a strongly continuous unitary representation of $\mathbb{R}$ on $L^2(X, \mathcal{M}_0)$. Let $\{\beta_t\}_{t \in \mathbb{R}}$ be the ultraweakly continuous representation of $\mathbb{R}$ as a group of $\ast$-automorphisms of $L^\infty(X, \mathcal{M}_0)$ implemented by $\{T_t\}_{t \in \mathbb{R}}$ and observe that $\{\beta_t\}_{t \in \mathbb{R}}$ normalizes $L^\infty(X, \mathcal{M})$ while on the center of $L^\infty(X, \mathcal{M})$, $\{\beta_t\}_{t \in \mathbb{R}}$ coincides with $\{\alpha_t\}_{t \in \mathbb{R}}$. Thus we find that $\{\gamma_t\}_{t \in \mathbb{R}}, \gamma_t = \alpha_t \circ \beta_t^{-1}$, is a family (not necessarily a group) of automorphisms of $L^\infty(X, \mathcal{M})$ which fixes the center elementwise and which has the property that for each $A$ in $L^\infty(X, \mathcal{M})$ the function of $t$, $\gamma_t(A)$, is continuous in the ultraweak topology on $L^\infty(X, \mathcal{M})$. By a well-known theorem [11, Chap. III, Sect. 3, Corollary 2], there is, for each $t \in \mathbb{R}$, a unitary operator $\Theta(t)$ in $L^\infty(X, \mathcal{M})$ such that $\gamma_t(A) = \Theta(t) A \Theta^*(t)$ for all $A \in L^\infty(X, \mathcal{M})$. Moreover, the continuity condition on $\{\gamma_t\}_{t \in \mathbb{R}}$ coupled with an argument of Kallman [28], allows us to choose $\Theta(t)$ so that the map $t \rightarrow \Theta(t)$ is a Borel map from $\mathbb{R}$ to the group $M_n(X)$ of unitary operators in $L^\infty(X, \mathcal{M})$ endowed with the Borel structure generated by the weak operator topology. We have thus proved the first half of the following proposition.

**Proposition 4.1.3.** There is a Borel map $\Theta$ from $\mathbb{R}$ to $M_n(X)$ such that

$$\alpha_t(A) = \Theta(t) T_t A T_t^* \Theta^*(t) \quad (4.1.2)$$

for all $t \in \mathbb{R}$. If $\Psi$ is another map with the same properties, then there is a Borel map $\varphi$ from $\mathbb{R}$ to the group of unitary operators $U(X)$ in the center of $L^\infty(X, \mathcal{M})$ such that $\Psi = \varphi \Theta; \text{ and conversely, given such a map } \varphi, \text{ the map } \Psi = \varphi \Theta \text{ satisfies Eq. (4.1.2).}$

**Proof.** We need only attend to the uniqueness statement. Suppose, therefore, that $\Psi$ is a Borel map from $\mathbb{R}$ to $M_n(X)$ satisfying Eq. (4.1.2). Then for all $A$ in $L^\infty(X, \mathcal{M})$, $\Psi(t) T_t A T_t^* \Psi^*(t) = \Theta(t) T_t A T_t^* \Theta^*(t)$ for all $t \in \mathbb{R}$ and so $(\Theta^*(t) \Psi(t))(T_t A T_t^*) = (T_t A T_t^*)(\Theta^*(t) \Psi(t))$. Since $T_t A T_t^*$ lies in $L^\infty(X, \mathcal{M})$.
precisely when $A$ does, and since $\Theta^*(t) \Psi(t)$ lies in $L^\infty(X, \mathcal{M})$ for each $t$, it follows from the last equation that $\Theta^*(t) \Psi(t)$ belongs to the center of $L^\infty(X, \mathcal{M})$. Since the map $t \mapsto \Theta^*(t) \Psi(t)$ is clearly Borel, the first half of the uniqueness part is proved. Since the second is trivial, we omit it.

Continuing with our discussion, suppose $\Theta$ is a Borel map from $\mathbb{R}$ to $\mathcal{M}_u(X)$ satisfying Eq. (4.1.2), let $s, t \in \mathbb{R}$ be fixed, choose $A$ in $L^\infty(X, \mathcal{M})$ and observe that

$$\Theta(s + t)(T_{s+t}A T^*_r, t) \Theta^*(s + t)$$

$$= \alpha_{s+t}(A) \alpha_s(\alpha_t(A))$$

$$= (\Theta(s) T_r)(\Theta(t) T_s) A(T^*_r \Theta^*(t))(T^*_s \Theta^*(s))$$

$$= (\Theta(s) T_s \Theta(t) T^*_s)(T_{s+t} A T^*_r, t) (\Theta(s) T_s \Theta(t) T^*_s)^*.$$  

From this it follows that there is a Borel map $\omega$ from $\mathbb{R} \times \mathbb{R}$ into $\mathcal{U}(X)$ such that

$$\omega(s, t) \Theta(s + t) = \Theta(s) T_s \Theta(t) T^*_s$$  (4.1.3)

for all $s, t \in \mathbb{R}$. Straightforward calculations which we omit show that $\omega$ satisfies the following identities:

$$\omega(s, 0) = \omega(0, s) = I, \quad \text{for all } s \in \mathbb{R};$$

and

$$\omega(r + s, t) \omega(r, s) = \omega(r, s + t) T_r \omega(s, t) T^*_r, \quad \text{for all } r, s, t \in \mathbb{R}.$$  

These identities characterize what is known as a 2-cocycle for $\mathbb{R}$ with values in $\mathcal{U}(X)$ (cf. [36]). Suppose $b$ is an arbitrary Borel function from $\mathbb{R}$ to $\mathcal{U}(X)$ such that $b(0) = I$. Then a calculation reveals that

$$\omega(s, t) = b(s + t)(b(s) T_s b(t) T^*_s)^*$$  (4.1.4)

is also a 2 cocycle for $\mathbb{R}$ with values in $\mathcal{U}(X)$. Such 2-cocycles are called 2-co-boundaries. The following important theorem was proved by L. Brown [5] and, independently, by Connes and Takesaki [7, Appendix].

**Theorem 4.1.4.** Every 2-cocycle for $\mathbb{R}$ with values in $\mathcal{U}(X)$ is a 2-coboundary.

As a consequence of this theorem and our discussion to this point we arrive at the following result which we promised at the beginning of this section.

**Theorem 4.1.5.** There exists a strongly continuous unitary representation $\{V_t\}_{t \in \mathbb{R}}$ on $L^2(X, \mathcal{H})$ such that $\omega_t(A) = V_t AV_t^*$ for all $t \in \mathbb{R}$ and $A$ in $L^\infty(X, \mathcal{M})$.

**Proof.** We know that there is a Borel map $\Theta$ from $\mathbb{R}$ into $\mathcal{M}_u(X)$ and there is a 2-cocycle $\omega$ for $\mathbb{R}$ with values in $\mathcal{U}(X)$ such that equations (4.1.2) and (4.1.3)
hold. By Theorem 4.1.4 there is a Borel function \( b \) from \( \mathbb{R} \) to \( \mathcal{B}(X) \) satisfying equation (4.4). If \( \Psi = b \Theta \), then by Proposition 4.1.3, \( \Psi \) also satisfies equation (4.1.2) and by virtue of equations (4.1.3) and (4.1.4), \( \Psi \) also satisfies the equation

\[
\Psi(s + t) = \Psi(s) T_s \Psi(t) T^*_t
\]

for all \( s, t \in \mathbb{R} \). Consequently, if \( V_t = \Psi(t) T_t \), then \( V_t \) is a unitary operator and \( V_{s+t} = \Psi(s + t) T_{s+t} = \Psi(s) T_s \Psi(t) T^*_s T^*_t T_{s+t} = (\Psi(s) T_s)(\Psi(t) T_t) = V_s V_t \); i.e., \( \{V_t\}_{t \in \mathbb{R}} \) is a unitary representation of \( \mathbb{R} \) on \( L^2(X, \mathcal{M}_0) \) which implements \( \{\alpha_t\}_{t \in \mathbb{R}} \) by Proposition 4.1.3. Finally, since \( \Psi \) is a Borel map, and \( \{T_t\}_{t \in \mathbb{R}} \) is strongly continuous, \( \{V_t\}_{t \in \mathbb{R}} \) is measurable, and so is strongly continuous. This completes the proof.

The proof actually shows a bit more than is stated. First it shows that it is possible to implement \( \{\alpha_t\}_{t \in \mathbb{R}} \) with a unitary representation \( \{V_t\}_{t \in \mathbb{R}} \) so that \( V_t T^*_t \) is inner for all \( t \), and second, it shows that any Borel function \( \Psi \) from \( \mathbb{R} \) to \( \mathcal{M}_u(X) \) satisfying equation (4.1.5) already is continuous with respect to the strong operator topology.

Continuous functions \( \Psi \) from \( \mathbb{R} \) to \( \mathcal{M}_u(X) \) with the strong operator topology satisfying equation (4.1.5) are called 1-cocycles, or simply cocycles. Two cocycles \( \Psi \) and \( \Theta \) are called cohomologous in case there is an operator \( R \in \mathcal{M}_u(X) \) such that \( \Psi(t) = R \Theta(t) T_t B^* T^*_t \) for all \( t \in \mathbb{R} \). Equivalently, two cocycles are cohomologous in case the unitary groups they determine are unitarily equivalent within \( \mathcal{M}_u(X) \). If a cocycle \( \Psi \) itself can be written as \( \Psi(t) = B T_t B^* T^*_t, B \in \mathcal{M}_u(X) \), then it is called a coboundary. Unless \( \mathcal{M} \) is the algebra of scalar multiples of the identity on \( \mathcal{H}_0 \), so that \( L^\infty(X, \mathcal{M}) \) is Abelian, the cocycles do not form a group under pointwise multiplication. Nonetheless, the relation of being cohomologous is an equivalence relation and we shall refer to the equivalence class of a cocycle as its cohomology class.

Since the elements of \( L^\infty(X, \mathcal{M}) \) are themselves functions on \( X \), a cocycle with values in \( \mathcal{M}_u(X) \) may be regarded as a Borel function on \( \mathbb{R} \times X \) with values in the group of unitary operators in \( \mathcal{M} \) (cf. [36]). When this is done, the cocycle identity (4.1.5) becomes

\[
\Psi(s + t, x) = \Psi(s, x) \Psi(t, x + s) \quad \text{a.e. } \mu
\]

for each \( s, t \in \mathbb{R} \). As with \( f(t, x) \), which is itself a kind of cocycle, the exceptional null set in this equation depends upon \( s \) and \( t \). Although we do not need this fact in the sequel, we note that Mathew [35] has recently proved that for each \( t \), \( \Psi \) may be modified on a null set of \( x \), depending on \( t \), so that the resulting function \( \Psi' \) is still Borel and so that Eq. (4.1.6) is satisfied for all \( x, s, \) and \( t \) without exception. Of course, when regarded as functions from \( \mathbb{R} \) to \( \mathcal{M}_u(X) \), \( \Psi \) and \( \Psi' \) are the same.

The next theorem shows how to express \( \Psi \), which, recall, is the collection of all \( A \) in \( L^\infty(X, \mathcal{M}) \) with \( \text{sp}_u(A) \subseteq [0, \infty) \), in terms of \( \Psi \) and the action of \( \mathbb{R} \) on \( X \).
and exhibits another sense in which the operators in \( \mathcal{U} \) are analytic. We denote
the algebra of bounded measurable \( \mathbb{M} \)-valued functions on \( \mathbb{R} \) which admit
bounded analytic extension to the upper half-plane by \( \mathcal{H}^\infty(\mathbb{R}, \mathbb{M}) \). Equivalently,
a bounded measurable \( \mathbb{M} \)-valued function \( A \) in \( \mathbb{R} \) lies in \( \mathcal{H}^\infty(\mathbb{R}, \mathbb{M}) \) if and only if
the matrix entries of \( A(t) \) with respect to any orthonormal basis in \( \mathcal{H}_0 \) lie in
\( H^\infty(\mathbb{R}) \). It follows from this that \( A \) belongs to \( \mathcal{H}^\infty(\mathbb{R}, \mathbb{M}) \) if and only if
\[ \int_{-\infty}^{\infty} A(t) f(t) \, dt = 0 \quad \text{for all } f \in \mathcal{K}(\mathbb{R}) \] (cf. Remark 2.18).

**Theorem 4.1.6.** Let \( \Psi \) be a cocycle on \( \mathbb{R} \) with values in \( \mathcal{M}_a(X) \) which deter-
mines a unitary representation of \( \mathbb{R} \) implementing \( \{\alpha_t\}_{t \in \mathbb{R}} \). Then \( \mathcal{U} \) is the algebra
of all \( A \) in \( \mathcal{L}(X, \mathbb{M}) \) such that for almost all \( x \) (the exceptional set depends on \( A \),
the function of \( t ) \),
\[ \Psi(t, x) A(x + t) \Psi^*(t, x) \]
lies in \( \mathcal{H}^\infty(\mathbb{R}, \mathbb{M}) \).

**Proof.** The proof is an adaptation of a well-known argument due to Helson [20]. Since \( \{\Psi(t) T_t\}_{t \in \mathbb{R}} \) is a unitary group implementing \( \{\alpha_t\}_{t \in \mathbb{R}} \), it follows that
for each \( A \) in \( \mathcal{L}(X, \mathbb{M}) \), \( (\alpha_t(A))(x) = (\Psi(t) T_t A T_t^* \Psi^*(t))(x) = \Psi(t, x) \times A(x + t) \Psi^*(t, x) \) a.e. \( \mu \). Thus for \( f \in \mathcal{L}^1(\mathbb{R}) \),
\[ (A \ast_a f)(x) = \int \Psi(t, x) A(x + t) \Psi^*(t, x) f(t) \, dt \quad \text{a.e. } \mu. \] (4.1.7)
If \( A \) lies in \( \mathcal{U} \) and if \( f \in \mathcal{K}(\mathbb{R}) \), the right hand side of this equation vanishes
except on a null set of \( x \) depending on \( f \). But since \( \mathcal{K}(\mathbb{R}) \) is separable, and since
the matrix entries of \( \Psi(t, x) A(x + t) \Psi^*(t, x) \) with respect to any orthonormal
basis for \( \mathcal{H}_0 \) are bounded Borel functions on \( \mathbb{R} \times X \), there is one set of measure
zero off of which the right hand side of Eq. (4.1.7) vanishes for all \( f \in \mathcal{K}(\mathbb{R}) \). The
discussion preceding the theorem implies that for such \( x \), \( \Psi(t, x) \times A(x + t) \Psi^*(t, x) \) lies in \( \mathcal{H}^\infty(\mathbb{R}, \mathbb{M}) \). Since the steps are easily reversible, we
omit the converse part of the proof.

We mention in passing that by Mathew's theorem cited above, one may
suppose \( \Psi \) satisfies Eq. (4.1.6) for all \( x, s, \) and \( t \). When this is done, the exceptional
null set in Theorem 4.1.6 is actually invariant.

At the present, the principal problem appears to be that of determining the
extent to which the various parameters we have associated with the algebra \( \mathcal{U} \)
of analytic operators determined by \( \{\alpha_t\}_{t \in \mathbb{R}} \) acting on \( \mathcal{L}^\infty(X, \mathbb{M}) \) distinguish
and are distinguished by \( \mathcal{U} \). Certainly the cohomology class of the cocycle associated
with \( \{\alpha_t\}_{t \in \mathbb{R}} \) and the conjugacy class of the action of \( \mathbb{R} \) on \( X \) are both isomorphism
invariants, in fact, unitary invariants, for \( \mathcal{U} \); but more than just these are
necessary to classify \( \mathcal{U} \) completely. In the commutative case it would appear at
first glance that the conjugacy class of the action of \( \mathbb{R} \) on \( X \) characterizes \( \mathcal{U} \) up
to isomorphism, but the results of [42] show that this is not the case. However,
the condition presented there, which is a slight extension of the notion of conjugacy, may be sufficient to classify \( \mathfrak{A} \), at least in the commutative case. At the other extreme, when \( X \) is a point so that \( L^\infty(X, \mathcal{M}) \) is merely the factor \( \mathcal{M} \) itself, then as we shall see in the next section (cf. Theorem 4.2.3), \( \mathfrak{A} \) may be determined by entirely different automorphism groups. So in this case, the problem of classifying \( \mathfrak{A} \) in terms of the automorphism group alone is hopeless. One reason for the very weak bond between \( \mathfrak{A}^1 \) and \( \{\alpha_t\}_{t \in \mathbb{R}} \) in the case of a type I factor seems to be due to the fact that \( \{\alpha_t\}_{t \in \mathbb{R}} \) is inner. We conjecture that if one hypothesizes the opposite extreme, namely, if one assumes that \( \{\alpha_t\}_{t \in \mathbb{R}} \) acts ergodically on \( L^\infty(X, \mathcal{M}) \), not just on the center, then the bond between \( \mathfrak{A} \) and \( \{\alpha_t\}_{t \in \mathbb{R}} \) is very strong.

4.2. Nest Algebras

Recall that a nest of subspaces of a Hilbert space \( \mathcal{H} \) is simply a totally ordered (by inclusion) family of subspaces of \( \mathcal{H} \). Given a nest \( \mathfrak{N} \), \( \mathcal{P}_\mathfrak{N} \) will denote the collection of projections onto the subspaces in \( \mathfrak{N} \). We say that a nest \( \mathfrak{N} \) is affiliated with a von Neumann algebra \( \mathcal{B} \) in case \( \mathcal{P}_\mathfrak{N} \subseteq \mathcal{B} \). Given a von Neumann algebra \( \mathcal{B} \) and a nest \( \mathfrak{N} \) affiliated with \( \mathcal{B} \), the nest algebra \( \mathfrak{N}_\mathfrak{R} \) determined by \( \mathfrak{N} \) and \( \mathcal{B} \) is the algebra \( \{A \in \mathcal{B} \mid AN \subseteq N \text{ for all } N \in \mathfrak{N}\} \).

Nest algebras were introduced by Ringrose [47] as a generalization of hyper-reducible triangular algebras studied by Kadison and Singer [26]. It should be noted, however, that they were studied implicitly, at least, somewhat earlier in the Russian work on triangular representations of operators. It should also be noted that Ringrose dealt only with the case when \( \mathcal{B} \) is \( \mathcal{L}(\mathcal{H}) \). Our objective is to show that every nest algebra in a von Neumann algebra \( \mathcal{B} \) (on a separable space) is the algebra of analytic operators with respect to a representation of \( \mathfrak{N} \) as a group of inner \( * \)-automorphisms of \( \mathcal{B} \); and conversely, given such a representation, the algebra of analytic operators which it determines is a nest algebra.

Observe that a nest algebra \( \mathfrak{N}_\mathfrak{R} \) is unaffected if the nest \( \mathfrak{N} \) is enlarged (if necessary) to contain \( \{0\} \) and \( \mathcal{H} \) and all intersections and spans of subsets of \( \mathfrak{N} \). A nest with these properties is called complete. We may thus assume, whenever convenient, that the nest determining a nest algebra is complete. The following proposition is well known; one proof of it may be found in [26].

**Proposition 4.2.1.** For each complete nest \( \mathfrak{N} \) of subspaces of a separable Hilbert space \( \mathcal{H} \), there is a spectral measure \( P \) on \( \mathbb{R} \) whose support is a closed subset \( \Omega \) of \([0, 1]\), containing 0 and 1, and there is an order reversing bijection \( \lambda \to N_\lambda \) between \( \Omega \) and \( \mathfrak{N} \) such that \( N_\lambda = P([\lambda, \infty)) \mathcal{H} \) for each \( \lambda \in \Omega \).

**Remark 4.2.2.** It should be noted that in general \( P \) is highly nonunique; simply consider a change of variables in \( \Omega \).

The following theorem has some overlap Theorem 3.1.1 of [2] and should also be compared with the paper of Schue [50].
4.2.3. Let \( \{\alpha_t\}_{t \in \mathbb{R}} \) be an ultraweakly continuous representation of \( \mathbb{R} \) as a group of inner \(*\)-automorphisms of a von Neumann algebra \( \mathcal{B} \) acting on a separable Hilbert space \( \mathcal{H} \), and let \( \mathfrak{A} \) be the algebra of analytic operators with respect to \( \{\alpha_t\}_{t \in \mathbb{R}} \). Then there is a nest \( \mathfrak{N} \) affiliated with \( \mathfrak{A} \) such that \( \mathfrak{A}_{\mathfrak{N}} = \mathfrak{A}_{\mathfrak{M}} \). Conversely, given a nest \( \mathfrak{N} \) affiliated with \( \mathfrak{A} \), there is a uniformly continuous representation of \( \mathbb{R} \), \( \{\alpha_t\}_{t \in \mathbb{R}} \), as a group of inner \(*\)-automorphisms of \( \mathcal{B} \) such that \( \mathfrak{A}_{\mathfrak{N}} \) is the algebra of analytic operators with respect to \( \{\alpha_t\}_{t \in \mathbb{R}} \).

Proof. Since \( \{\alpha_t\}_{t \in \mathbb{R}} \) is inner by hypothesis, there is a unitary representation of \( \mathbb{R} \), \( \{U_t\}_{t \in \mathbb{R}} \) contained in \( \mathcal{B} \) such that \( \alpha_t(B) = U_t B U_t^* \) for all \( t \in \mathbb{R} \). Thus \( (\text{id}, \{U_t\}_{t \in \mathbb{R}}) \) is a faithful covariant representation of \( (\mathcal{B}, \{\alpha_t\}_{t \in \mathbb{R}}) \), and if \( E \) is the spectral measure of \( \{U_t\}_{t \in \mathbb{R}} \) so that \( U_t = \int_{-\infty}^{\infty} e^{i\lambda t} dE(\lambda) \), then \( \mathfrak{N} = \{E([\lambda, \infty)) \mathcal{H}\}_{\mathfrak{N}} \) is a nest affiliated with \( \mathfrak{A} \) and \( \mathfrak{A}_{\mathfrak{N}} = \mathfrak{A}_{\mathfrak{M}} \) by Corollary 2.14. For the converse, simply enlarge \( \mathfrak{N} \), if necessary, to a complete nest \( \mathfrak{N}' \), apply Proposition IV.2.1 to obtain a spectral measure \( \mathcal{P} \) with compact support such that \( \mathfrak{P}_{\mathfrak{N}'} \) coincides with \( \{\mathcal{P}([\lambda, \infty))\}_{\mathfrak{N}'} \) and let \( \{\alpha_t\}_{t \in \mathbb{R}} \) be the uniformly continuous representation of \( \mathbb{R} \) on \( \mathfrak{A} \) implemented by the unitary representation of \( \mathbb{R} \) which is the Fourier–Stieltjes transform of \( \mathcal{P} \). Then \( \{\alpha_t\}_{t \in \mathbb{R}} \) is inner, and by Corollary 2.14 once more, \( \mathfrak{A}_{\mathfrak{N}} \) is the algebra of analytic operators determined by \( \{\alpha_t\}_{t \in \mathbb{R}} \).

We state the following corollary for the sake of completeness.

4.2.4. Let \( \{\alpha_t\}_{t \in \mathbb{R}} \) be an ultraweakly continuous representation of \( \mathbb{R} \) as a group of \(*\)-automorphisms of \( \mathcal{L}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \). Then there is a nest \( \mathfrak{N} \) of subspaces of \( \mathcal{H} \) such that the algebra of analytic operators determined by \( \{\alpha_t\}_{t \in \mathbb{R}} \) is \( \mathfrak{A}_{\mathfrak{N}} \).

Proof. This follows immediately from Theorem 4.2.1 and Bargmann’s result that \( \{\alpha_t\}_{t \in \mathbb{R}} \) is spatially implemented, i.e., is a group of inner \(*\)-automorphisms (see Theorem 4.1.5).

Our final result in this section was proved by Arveson [2, Corollary 3.1.2] using different methods.

4.2.5. Let \( \mathfrak{A} \) be a nest affiliated with a von Neumann algebra \( \mathcal{B} \) acting on a separable space and let \( \mathfrak{A}_{\mathfrak{N}} \) be the nest algebra determined by \( \mathcal{B} \) and \( \mathfrak{A} \). Suppose there is a faithful normal expectation \( \Phi \) from \( \mathcal{B} \) onto the relative commutant of \( \mathfrak{N} \) in \( \mathfrak{A} \). Then \( \mathfrak{A}_{\mathfrak{N}} \) is a maximal subdiagonal subalgebra of \( \mathcal{B} \) with respect to \( \Phi \).

Proof. Let \( \{\alpha_t\}_{t \in \mathbb{R}} \) be a uniformly continuous representation of \( \mathbb{R} \) on \( \mathcal{B} \) determined by \( \mathfrak{A} \) as in the proof of Theorem 4.2.2 so that \( \mathfrak{A}_{\mathfrak{N}} = \mathcal{B}^\prime([0, \infty)) \). By construction, the relative commutant of \( \mathfrak{N} \) in \( \mathcal{B} \) is \( \mathcal{B}^\prime([0]) \) and so the hypothesis implies that \( \mathcal{B} \) is \( \mathbb{R} \)-finite relative to \( \{\alpha_t\}_{t \in \mathbb{R}} \). The proof is now completed by appeal to Theorem 3.15.
4.3. Crossed Products

A large number of the examples studied by Arveson in [2] are crossed products. Likewise, Kadison and Singer [26] used these algebras to construct examples of irreducible triangular algebras. In this final section we show how all of these examples fit within our general scheme of things.

We suppose for the remainder of this section that $G$ is a compact abelian group so that $G$ is discrete. Although it seems that this restriction is unduly severe for what we are about to present, there is a problem of a spectral theoretic nature which we are unable to avoid except by making this extreme hypothesis—see the proof of Corollary 4.3.2.

We suppose, too, that $\mathfrak{B}_0$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and that $\{\beta_\lambda\}_{\lambda \in G}$ is a representation of $\hat{G}$ on $\mathfrak{B}_0$ as a group of *-automorphisms. We let $\mathfrak{B}$ denote the crossed product of $\mathfrak{B}_0$ by $\{\beta_\lambda\}_{\lambda \in G}$. Recall that $\mathfrak{B}$ is the von Neumann algebra on the Hilbert space $L^2(G, \mathcal{H}_0) = \{f: G \to \mathcal{H}_0 | \sum_{\lambda \in G} \|f(\lambda)\|_{\mathcal{H}_0}^2 < \infty\}$ generated by the operators $\pi(B), B \in \mathfrak{B}_0$, and $\{V_\beta\}_{\beta \in G}$ defined by the formulas

$$\pi(B)f(\beta) = \beta(B)f(\beta)$$

and

$$(V_\beta f)(\hat{h}) = f(\hat{h} + \beta),$$

$f \in L^2(G, \mathcal{H}_0), \hat{h}, \beta, \in \hat{G}$. The group $G$ acts on $L^2(G, \mathcal{H}_0)$ and on $\mathfrak{B}$ in a canonical fashion. For $g \in G$, $U_g$ is the unitary operator on $L^2(G, \mathcal{H}_0)$ defined by the formula

$$(U_g f)(\beta) = \langle -g, \beta \rangle f(\beta), f \in L^2(G, \mathcal{H}_0).$$

Clearly $\{U_g\}_{g \in G}$ is a strongly continuous unitary representation of $G$ on $L^2(G, \mathcal{H}_0)$ satisfying the equations

$$U_g \pi(B) U_g^* = \pi(B), \quad B \in \mathfrak{B}_0,$$

$$U_g V_\beta U_g^* = \langle g, \beta \rangle V_\beta, \quad g \in G, \; \beta \in \hat{G}.$$}

If $\{\alpha_\mu\}_{\mu \in G}$ denotes the automorphism group implemented by $\{U_g\}_{g \in G}$, then $\{\alpha_\mu\}_{\mu \in G}$ normalizes $\mathfrak{B}$ and we find that $\pi(\mathfrak{B}_0) \subseteq \mathfrak{B}^\alpha(\{0\})$ and $\text{sp}_a(V_\beta) = \{\beta\}$ for all $\beta \in \hat{G}$. Takesaki [57] calls $\{\alpha_\mu\}_{\mu \in G}$ the dual action of $G$ on $\mathfrak{B}$.

**Proposition 4.3.1.** $\pi(\mathfrak{B}_0) = \mathfrak{B}^\alpha(\{0\})$.

**Proof.** We begin by noting that the equation is invariant under isomorphisms in the following sense. Suppose $\mathfrak{B}_{1,0}$ is another von Neumann algebra and that $\{\beta_\lambda\}_{\lambda \in G}$ is a representation of $\hat{G}$ as a group of *-automorphisms of $\mathfrak{B}_{1,0}$. Let $\mathfrak{B}_1$ be the crossed product of $\mathfrak{B}_{1,0}$ by $\{\beta_\lambda\}_{\lambda \in G}$ and let $\{\alpha_\mu\}_{\mu \in G}$ be the dual action of $G$ on $\mathfrak{B}_1$. If $\Phi_0$ is an isomorphism from $\mathfrak{B}_0$ onto $\mathfrak{B}_{1,0}$ such that $\beta_\lambda \circ \Phi_0 = \Phi_0 \circ \beta_\lambda$ for all $\lambda \in G$, then there is a canonical lifting of $\Phi_0$ to an isomorphism
\[240\]

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Thus appealing to Proposition 2.6 we may replace \( G_0 \) by an isomorphic copy and assume, without loss of generality, that \( \{ \beta_{\hat{g}} \}_{\hat{g} \in \hat{G}} \) is spatially implemented.

Now observe that \( \mathcal{B}_0(\{0\}) = \mathcal{B} \cap \{ U_{\hat{g}} \}_{\hat{g} \in \hat{G}} \) and that any operator \( B \) in \( \{ U_{\hat{g}} \}_{\hat{g} \in \hat{G}} \) is determined by a bounded function \( \tau \) from \( G \) into \( \mathbb{C}(\mathfrak{a}) \) by the equation

\[ \tau(\hat{g}) f(\hat{g}) \quad \text{for all} \quad g \in G. \]

We want to show that if \( B \) is also in \( \mathcal{B} \), then there is an \( \Lambda \in \mathcal{B}_0 \) such that \( \tau(\hat{g}) = \beta_{\hat{g}}(\Lambda) \) for all \( \hat{g} \in \hat{G} \). To this end, let \( \tau' \) be the representation of \( \mathcal{B}_0 \), on \( L^2(\hat{G}, \mathfrak{H}_0) \) defined by the formula

\[ \tau'(\hat{g}) \cdot \langle f, \hat{g} \rangle = \langle f, \hat{g} \rangle \quad \text{for all} \quad f \in L^2(\hat{G}, \mathfrak{H}_0), \]

and define \( \{ V_{\hat{g}} \}_{\hat{g} \in \hat{G}} \) by the formula \( V_{\hat{g}} f = V_{\hat{g}} f_{\hat{g}} \). It so happens that the commutant of \( \mathcal{B} \) is generated by \( \tau'(\mathcal{B}) \) and \( \{ V_{\hat{g}} \}_{\hat{g} \in \hat{G}} \), but we only need the fact that these are contained in the commutant, a fact which is easy enough to verify. Since \( \mathcal{B} \), which is supposed to be represented by \( \tau \), is contained in \( \mathcal{B} \), it commutes with \( \tau'(\mathcal{B}) \) and so, as a calculation reveals, \( \tau(\hat{g}) \) lies in \( \mathcal{B}_0 \) for each \( \hat{g} \in \hat{G} \). Since \( B \) also commutes with \( \{ V_{\hat{g}} \}_{\hat{g} \in \hat{G}} \), another calculation reveals that for all \( f \in L^2(\hat{G}, \mathfrak{H}_0) \) and \( \hat{g} \in \hat{G} \), \( \tau'(\hat{g}) \cdot f(\hat{g}) = \langle B f, \hat{g} \rangle = \langle V_{\hat{g}} B( V_{\hat{g}}^* )^*( f, \hat{g} ) \rangle \) so that \( \tau(\hat{g}) = \beta_{\hat{g}}(\tau(\hat{g})) \) for all \( \hat{g} \in \hat{G} \). Setting \( A = \tau(\hat{0}) \), we find from this that \( \tau(\hat{g}) = \beta_{\hat{g}}(\tau(\hat{0})) = \beta_{\hat{g}}(\Lambda) \) and the proof is complete.

\textbf{Corollary 4.3.2.} For any subset \( E \) of \( \hat{G} \), \( \mathfrak{B}_0(E) \) is the ultraweakly closed span of the operators \( \{ \tau(B) V_{\hat{g}} \mid B \in \mathcal{B}_0, \hat{g} \in E \} \).

\textit{Proof.} Let \( \mathcal{E} \) be the span in question and note that since \( \text{sp}_\alpha(V_{\hat{g}}) = \{ \hat{g} \} \) and \( \text{sp}_\alpha(\tau(B)) \subseteq \{ 0 \} \) for all \( B \in \mathcal{B}_0 \), it follows that \( \text{sp}_\alpha(\tau(B) V_{\hat{g}}) \subseteq \{ \hat{g} \} \) for all \( B \in \mathcal{B}_0 \). Hence \( \mathcal{E} \subseteq \mathfrak{B}_0(E) \). Since \( L^1(G) \) has approximate identities consisting of trigonometric polynomials, \( \mathfrak{B}_0(E) \) is the ultraweak closure of the linear manifold in \( \mathfrak{B}_0(E) \) consisting of those elements whose spectrum with respect to \( \alpha \) is finite. But by the regularity of \( L^1(G) \), any such element is itself a sum of elements in \( \mathfrak{B}_0(E) \) whose spectra are singletons. Thus it suffices to note that if \( \text{sp}_\alpha(B) \subseteq \{ \hat{g} \} \), then \( B = \text{tr} V_{\hat{g}} B \) with \( B \in \mathcal{B}_0 \). Indeed, if \( B_0 - B V_{\hat{g}}^* \), then \( B = B_0 V_{\hat{g}} \) and \( \text{sp}_\alpha(B_0) \subseteq \{ 0 \} \) by 2.12 and 2.15 so that by Proposition 4.3.1, \( B_0 \in \mathcal{B}_0 \). This completes the proof.

Observe that if a compact group \( G \) acts as group of \( * \)-automorphisms of a von Neumann algebra, then the algebra is \( G \)-finite. Indeed, the integral provides the required expectation (cf. [31]). Thus, in our setting, \( \mathfrak{B} \) is \( G \)-finite and we call the expectation determined by the integral on \( G \) the \textit{canonical} expectation.
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**Theorem 4.3.3.** If $\Sigma$ totally orders $G$, then the ultraweakly closed subalgebra generated by $\pi(\mathcal{B}_0)$ and $\{V_g\}_{g \in \Sigma}$ is a maximal subdiagonal algebra in $\mathcal{B}$ with respect to the canonical expectation on $\mathcal{B}$.

*Proof.* By Corollary 4.3.2, this subalgebra is $\mathcal{B}^\Sigma(\Sigma)$ so the result follows from Theorem 3.15.

We conclude this section by noting if $\mathcal{B}_0$ is a finite factor and if $G$ acts trivially on $\mathcal{B}_0$, then Theorem 4.3.3 yields Theorem 5.3.2 of [2]. On the other hand, if $\mathcal{B}_0$ is $L^\infty(X)$ for some measure space $X$ and if $\{\rho_g\}_{g \in G}$ is determined by a measurable action of $G$ on $X$, then the analysis of this subsection provides an alternate approach to the results of [2, Sect. 3.3], at least if the groups there are assumed to be abelian. In particular, our Theorem 4.3.3 shows, in the notation of [2, Sect. 3.3], that $\mathcal{C}$ is the ultraweak closure of $\mathcal{A}_0$ (see Theorem 5.5.5(i) as well).

5. **Invariant Subspaces**

In this section we investigate the invariant subspaces of the algebra of analytic operators in a von Neumann algebra for the case when the group $G$ is $\mathbb{R}$. Our ultimate goal is to use our analysis to exhibit ultraweakly closed reductive algebras which generate type II von Neumann algebras.

As usual, $\mathcal{B}$ will denote our basic von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\{\alpha_t\}_{t \in \mathbb{R}}$ will denote an ultraweakly continuous representation of $\mathbb{R}$ as a group of $*$-automorphisms of $\mathcal{B}$, and $\mathcal{A}$ will denote the algebra of analytic operators with respect to $\{\alpha_t\}_{t \in \mathbb{R}}$, $\mathcal{B}^\Sigma([0, \infty))$.

**Definition 5.1.** Let $\mathcal{M}$ be an invariant subspace for $\mathcal{A}$. Then $\mathcal{M}$ is said to be left- (right-) normalized in case

$$\bigwedge_{t < 0} \{\mathcal{B}^\Sigma([t, \infty))\mathcal{M}\}^{\text{cl}} = \mathcal{M} \left(\bigvee_{t > 0} \{\mathcal{B}^\Sigma([t, \infty))\mathcal{M}\}^{\text{cl}} = \mathcal{M}\right).$$

If $\mathcal{M}$ is both left- and right-normalized, $\mathcal{M}$ is said to be completely normalized.

Since $G = \mathbb{R}$, Theorem 3.15 implies that $\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}$. Consequently $\mathcal{A}$ and $\mathcal{B}$ have the same reducing subspaces, namely, the ranges of the projections in $\mathcal{B}'$. Recall that if $E$ is a projection in $\mathcal{B}'$, then $E$ determines an ultraweakly continuous representation of $\mathcal{B}$ on the Hilbert space $E\mathcal{H}$. This representation, denoted by $\pi_E$, is called the *induction* determined by $E$ and is defined by the formula $\pi_E(B) = EB | E\mathcal{H}$, $B \in \mathcal{B}$. The following theorem exhibits the structure of certain nonreducing invariant subspaces for $\mathcal{A}$. It is a direct descendant of the fundamental study of Helson and Lowdenslager [23].
**Theorem 5.2.** Let $\mathcal{M}$ be a nonreducing invariant subspace for $\mathfrak{A}$ and assume that $\mathcal{M}$ is left-normalized (resp. right-normalized). Then there are projections $E_1$ and $E_2$ in $\mathcal{B}$, $E_1 \preceq E_2$, and there is a strongly continuous unitary representation $U = \{U_t\}_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $E\mathcal{H}$, $E = E_2 \ominus E_1$, such that $(\pi_E, U)$ is a covariant representation of $(\mathfrak{B}, \alpha)$ and such that

$$\mathcal{M} = \{F([0, \infty)) E\mathcal{H}\} \oplus E_1 \mathcal{H},$$

(resp. $\mathcal{M} = \{F((0, \infty)) E\mathcal{H}\} \oplus E_1 \mathcal{H}$),

(5.1) (5.2)

where $F$ is the spectral measure for $U$ on $E\mathcal{H}$. Conversely, given orthogonal projections $E$ and $E_1$ in $\mathcal{B}$ and a strongly continuous unitary representation $U = \{U_t\}_{t \in \mathbb{R}}$ on $E\mathcal{H}$, with spectral measure $F$, such that $(\pi_E, U)$ is a covariant representation of $(\mathfrak{B}, \alpha)$, the subspace $\mathcal{M}$ of $\mathcal{H}$ defined by Eq. (5.1) is a left-normalized invariant subspace for $\mathfrak{A}$.

**Proof.** We shall proceed under the assumption that $\mathcal{M}$ is left-normalized; the proof when $\mathcal{M}$ is right-normalized is similar. For $t \in \mathbb{R}$, let $\bar{P}_t$ be the projection of $\mathcal{H}$ onto $\Lambda_{s > t} \{\mathbb{B}^\circ([s, \infty)) \mathcal{M}\}$, let $E_1 = \bigwedge_{t \in \mathbb{R}} \bar{P}_t$, and let $E_2 = \bigvee_{t \in \mathbb{R}} \bar{P}_t$.

By Corollary 2.15, $\mathbb{B}^\circ([t, \infty)) (\mathbb{B}^\circ([s, \infty)) \mathcal{M}) \subseteq \{\mathbb{B}^\circ([s + t, \infty)) \mathcal{M}\}$ for all $s, t \in \mathbb{R}$, and so $\mathbb{B}^\circ([t, \infty)) \bar{P}_s \mathcal{H} \subseteq \bar{P}_{s+t} \mathcal{H}$ for all $s, t \in \mathbb{R}$. It follows that the ranges of $E_1$ and $E_2$ are invariant under every operator in $\mathfrak{B}$ whose spectrum with respect to $\alpha$ is compact. Since the family of all such operators is ultraweakly dense in $\mathfrak{B}$ [44], it follows that $E_1$ and $E_2$ both belong to $\mathfrak{B}$. By construction, $\bar{P}_s \preceq \bar{P}_t$, when $t < s$, and $\bar{P}_s = \bigwedge_{t < s} \bar{P}_t$. Hence there is a spectral measure with values in the projections on $E\mathcal{H}$, $E = E_2 \ominus E_1$, such that $F([t, \infty)) = \bar{P}_t \ominus E_1$ for all $t \in \mathbb{R}$. By construction and the hypothesis that $\mathcal{M}$ is left-normalized, $\mathcal{M} = \bar{P}_0 \mathcal{H} = \{F([0, \infty)) E\mathcal{H}\} \oplus E_1 \mathcal{H}$. Also by construction,

$$\pi_E(\mathbb{B}^\circ([t, \infty)) (\mathcal{H}^\circ([s, \infty)) E\mathcal{H}) \subseteq F([s + t, \infty)) E\mathcal{H}$$

for all $s, t \in \mathbb{R}$. Consequently, if $U = \{U_t\}_{t \in \mathbb{R}}$ is the strongly continuous unitary representation of $\mathbb{R}$ on $E\mathcal{H}$ which is the Fourier–Stieltjes transform of $F$, then $(\pi_E, U)$ is a covariant representation of $(\mathfrak{B}, \alpha)$ by Theorem 2.13.

The converse is an easier application of Theorem 2.13. It implies that a subspace $\mathcal{M}$ of the form in Eq. (5.1) is invariant under $\mathfrak{A}$ and is left-normalized because $\mathcal{M} \subseteq \Lambda_{s < 0} \mathbb{B}^\circ([s, \infty)) \mathcal{M} \subseteq \Lambda_{s < 0} \pi_E(\mathbb{B}^\circ([s, \infty)) E\mathcal{H}) \oplus E_1 \mathcal{H} \subseteq \Lambda_{s < 0} \{F([s, \infty)) E\mathcal{H}\} \oplus E_1 \mathcal{H} = \{F([0, \infty)) E\mathcal{H}\} \oplus E_1 \mathcal{H} = \mathcal{M}$.

**Remark 5.3.** The assumption that $\mathcal{M}$ does not reduce $\mathfrak{A}$ forces $E$ to be nonzero; however, a subspace $\mathcal{M}$ of the form in Eq. (5.1) may reduce $\mathfrak{A}$ even if $E \neq 0$. Indeed as Arveson’s proof of Borcher’s theorem [3, Theorem 3.1] shows, $\mathcal{M}$ will reduce $\mathfrak{A}$ if and only if $\{U_t\}_{t \in \mathbb{R}}$ is inner and $F$ is supported in $[0, \infty)$. We note too that the unitary representation associated with an invariant subspace is not generally unique as the examples consisting of nest algebras
(Section 4.2) demonstrate. However, if $\mathcal{B}$ is Abelian and if the action of $\mathbb{R}$ on $\mathcal{B}$ is ergodic and not periodic, then the unitary representation associated with an invariant subspace is unique \cite{43}. Finally, we note that a subspace of the form in Eq. (5.2) may not be right-normalized. Indeed, if $E_1$ is a non-zero, invariant central projection in $\mathcal{B}$ such that $\{E_1\}_{t \in \mathbb{R}}$ acts trivially on $E_1 \mathcal{B} E_1$, then $E_1 \mathcal{H}$ is an invariant subspace which is not right-normalized because $\mathcal{B}^\epsilon([s, \infty)) E_1 \mathcal{H} = \{0\}$ for all $s > 0$.

The decomposition of a left-normalized invariant subspace (Eq. (5.1)) afforded by Theorem 5.2 is reminiscent of the World decomposition of a stationary process. Consequently we propose the following definition.

**Definition 5.4.** Let $\mathcal{M}$ be a nonreducing invariant subspace for $\mathcal{A}$ which is left-normalized. Then the decomposition $\mathcal{M} = \{F([0, \infty)) \mathcal{E} \} \oplus E_1 \mathcal{H}$ given in Theorem 5.2 will be called the *Wold decomposition* of $\mathcal{M}$. The subspace $\{F([0, \infty)) \mathcal{E} \}$ will be called the *pure part* of $\mathcal{M}$, and the subspace $E_1 \mathcal{H}$ will be called the *self-adjoint or deterministic part* of $\mathcal{M}$.

Theorem 5.2 deals only with invariant subspaces which are normalized on the left or on the right. A general invariant subspace may not be so normalized. Indeed, an analysis of such invariant subspaces is one of the important, albeit technical, questions which should be investigated. The following proposition and corollaries allow us to circumvent this problem in the present investigation.

**Proposition 5.5.** If $\mathcal{M}$ is an invariant subspace for $\mathcal{A}$, then $\mathcal{M}$ is contained in a unique, minimal, left-normalized, invariant subspace $\mathcal{M}(+)\subseteq \mathcal{M}$ and contains a unique, maximal, right-normalized, invariant subspace $\mathcal{M}(-)\subseteq \mathcal{M}$. Moreover, the vector state determined by any unit vector in $\mathcal{M}(+) \cap \mathcal{M}(-)$ is invariant.

**Proof.** Let

$$\mathcal{M}(+) = \bigwedge_{s < 0} (\mathcal{B}^\epsilon([s, \infty))) \mathcal{M}$$

and let

$$\mathcal{M}(-) = \bigvee_{s > 0} (\mathcal{B}^\epsilon([s, \infty))] \mathcal{M}.$$  

Then by Corollary 2.15, $\mathcal{M}(+)$ and $\mathcal{M}(-)$ are invariant subspaces for $\mathcal{A}$ and $\mathcal{M}(+) \supseteq \mathcal{M} \supseteq \mathcal{M}(-)$. Next observe that for all $t < 0$ and arbitrary $s$, the same corollary implies that $\mathcal{B}^\epsilon([s, \infty)) \mathcal{M}(+) \subseteq \mathcal{B}^\epsilon([s, \infty))(\mathcal{B}^\epsilon([t, \infty])) \mathcal{M} \subseteq \mathcal{B}^\epsilon([s + t, \infty)) \mathcal{M}$. From this we obtain the following inclusion, valid for all $s \in \mathbb{R}$.

$$\mathcal{B}^\epsilon([s, \infty)) \mathcal{M}(+) \subseteq \bigwedge_{t < 0} [\mathcal{B}^\epsilon([s + t, \infty)) \mathcal{M}]. (5.3)$$
Hence,
\[ \bigcap_{s<0} [\mathcal{V}([s, \infty)), \mathcal{M}^{(+)}]_{cl} \subseteq \bigcap_{s<0} \bigcup_{t>0} [\mathcal{V}([s + t, \infty)), \mathcal{M}]_{cl} \]
\[ \subseteq \bigcup_{t<0} [\mathcal{V}([t, \infty)), \mathcal{M}]_{cl} = \mathcal{M}^{(+)} , \]
so that \( \mathcal{M}^{(+)} \) is left-normalized. Clearly, \( \mathcal{M}^{(+)} \) is the unique, minimal, left-normalized, invariant subspace for \( \mathcal{A} \) containing \( \mathcal{M} \). A minor modification of this argument, which we omit, shows that \( \mathcal{M}^{(-)} \) is the unique, maximal, right-normalized invariant subspace for \( \mathcal{A} \) contained in \( \mathcal{M} \).

Suppose now that \( f \) is a unit vector in \( \mathcal{M}^{(+)} \ominus \mathcal{M}^{(-)} \). The inclusion (5.3) shows that for all \( s > 0 \), the linear manifold \( \mathcal{V}([s, \infty)) f \) is contained in \( \mathcal{M}^{(-)} \). Consequently, the vector state \( \rho \) determined by \( f \) annihilates the ultra-weak closure of \( \bigcup_{s>0} \mathcal{V}([s, \infty)) \). By Corollary 3.13, \( \rho \) is invariant, and the proof is complete.

Since the following corollaries are immediate, we shall omit the proofs.

**COROLLARY 5.6.** If \( \mathcal{M} \) is an invariant subspace for \( \mathcal{A} \), then \( (\mathcal{M}^{(+)})^{(+)} = \mathcal{M}^{(+)} \), \( (\mathcal{M}^{(+)})^{(-)} = \mathcal{M}^{(-)} \), and \( (\mathcal{M}^{(-)})^{(-)} = \mathcal{M}^{(-)} \). If \( \mathcal{H}^{(-)} = \mathcal{H} \), then \( (\mathcal{M}^{(-)})^{(+)} = \mathcal{M}^{(+)} \) also. If \( \mathcal{H}^{(-)} = \mathcal{H} \), and if \( \mathcal{M}^{(+)} \) does not reduce \( \mathcal{A} \), then \( \mathcal{M}^{(-)} = F((0, \infty)) E \mathcal{H} \oplus E_{1} \mathcal{H} \) and \( \mathcal{M}^{(+)} \ominus \mathcal{M}^{(-)} = F([0]) E \mathcal{H} \) where \( E, E_{1} \), and \( F \) are the objects associated with \( \mathcal{M}^{(+)} \) in Theorem 5.2.

The next corollary implies, for example, that every invariant subspace for \( H^{\alpha}(\mathbb{R}) \) is completely normalized.

**COROLLARY 5.7.** If \( \mathcal{V} \) is completely non-R-finite with respect to \( \{\alpha_{t}\}_{t \in \mathbb{R}} \) then every invariant subspace for \( \mathcal{A} \) is completely normalized.

We now apply our analysis to the construction of some reductive algebras.

**DEFINITION 5.8.** An algebra \( \mathcal{A} \) of operators on a Hilbert space \( \mathcal{H} \) is called **reductive** in case every subspace of \( \mathcal{H} \) invariant under \( \mathcal{A} \) is also invariant under \( \mathcal{A}^{*} \), i.e., in case every invariant subspace for \( \mathcal{A} \) reduces \( \mathcal{A} \).

Equivalently, an algebra is reductive in case it has precisely the same invariant subspaces as the von Neumann algebra it generates.

Recall that the **reductive algebra question**, a generalization of the invariant subspace question, asks whether there are any non-self-adjoint, reductive algebras which are _weakly_ closed. Our objective is to utilize Theorem 5.2 to exhibit examples of non-self-adjoint reductive algebras which are ultraweakly closed. Unfortunately (or fortunately), these algebras are not weakly closed, and, in fact, they are weakly dense in the von Neumann algebras they generate. Although these examples do not quite settle the reductive algebra question,
they do provide strong evidence for a negative answer. Moreover, they constitute examples showing that spectral subspaces need not be weakly closed even though the automorphism group is weakly continuous.

The examples fall into two classes; the ones in the first classe generate certain II\(_\infty\) factors while the examples in the second class generate finite von Neumann algebras of type II. It should be noted that the only von Neumann algebras in which our approach will produce examples of non-self-adjoint reductive algebras (at least when the automorphism group is a representation of \(\mathbb{R}\)) are those of type II. To see this, first note that if \(\mathcal{B}\) is type III, then every linear functional in \(\mathcal{B}\) is of the form \(\omega_{x,y}\) where \(\omega_{x,y}(B) = (Bx, y),\) \(B \in \mathcal{B}\) [11, Corollary 10, Chap. III Sect. 8, and Corollary 3, Chap. I, Sect. 6]. So if \(\mathcal{A}\) were any non-self-adjoint, ultraweakly closed subalgebra of \(\mathcal{B}\), which generates \(\mathcal{A}\) as a von Neumann algebra, then there would exist a non-zero \(\omega_{x,y}\) annihilating \(\mathcal{A}\). But then \([\mathcal{A}\mathcal{B}]^{\mathcal{B}}\) would be a subspace invariant under \(\mathcal{A}\), but not under \(\mathcal{B}\). Thus when \(\mathcal{A}\) is type III, no \(\mathcal{A}\) can be ultraweakly closed, reductive, and non-self-adjoint. On the other hand, the analysis presented in Section 4.1 shows that if \(\mathcal{B}\) is of type I, then it is possible to find a projection \(E \in \mathcal{B}'\) such that on \(E\mathcal{H}\) there is a strongly continuous unitary representation of \(\mathbb{R}\) \(U = \{U_t\}_{t \in \mathbb{R}}\) which consists of more than just the identity operator, provided \(\{\alpha_t\}_{t \in \mathbb{R}}\) consists of more than the identity automorphism, such that \((\pi_E, U)\) is a covariant representation of \((\mathcal{B}, \alpha)\). It follows from Theorem 2.13 that in this case, too, the algebra of analytic operators will not be reductive.

The following proposition is a corollary of Theorem 5.2 and Proposition 5.5 and is basic for the construction of our examples.

**Proposition 5.8.** Suppose that for each nonzero projection \(E \in \mathcal{B}'\), \(\{\alpha_t\}_{t \in \mathbb{R}}\) is not spatially implementable on \(E\mathcal{H}\), i.e., suppose that there does not exist a strongly continuous unitary representation \(U = \{U_t\}_{t \in \mathbb{R}}\) of \(\mathbb{R}\) on \(E\mathcal{H}\) such that \((\pi_E, U)\) is a covariant representation of \((\mathcal{B}, \alpha)\). Then \(\mathcal{A}\) is an ultraweakly closed, non-self-adjoint, reductive subalgebra of \(\mathcal{B}\).

**Proof.** We know that \(\mathcal{A}\) is ultraweakly closed and that \(\mathcal{A} \cap \mathcal{A}^* = \mathcal{B}^\mathcal{A}([0])\). Since the trivial representation is spatially implementable, the hypothesis implies that \(\{\alpha_t\}_{t \in \mathbb{R}}\) is nontrivial. Hence \(\mathcal{A}\) is larger than \(\mathcal{B}^\mathcal{A}([0])\) and therefore non-self-adjoint.

Suppose \(\mathcal{A}\) is not reductive and let \(\mathcal{M}\) be a subspace of \(\mathcal{H}\) which is invariant under \(\mathcal{A}\) but not under \(\mathcal{B}\). By Theorem 5.2, \(\mathcal{M}\) is not left-normalized. On the other hand, by Proposition 5.5 \(\mathcal{M}\) is contained in a unique minimal left-normalized invariant subspace for \(\mathcal{A}\), \(\mathcal{M}^{(+)}\), which by Theorem 5.2 and hypothesis, must reduce \(\mathcal{A}\). Consequently, there is a unit vector, say \(f\), in \(\mathcal{M}^{(+)} \cap \mathcal{M}\), and by Proposition 5.5, the state \(f\) determines is invariant. Let \(E\) be the projection onto \([\mathcal{B}f]^{\mathcal{B}}\). Then \(E \in \mathcal{B}'\), and if for vectors of the form \(g = Bf\) in \(E\mathcal{H}\) we define \(U_t g = \alpha_t(B)f, t \in \mathbb{R}\), then because the state determined by \(f\)
is invariant, \( \{ U_t \}_{t \in \mathbb{R}} \) extends uniquely to a strongly continuous unitary representation of \( \mathbb{R} \) on \( E_X \) which implements \( \{ \alpha_t \}_{t \in \mathbb{R}} \). Since this is contrary to hypothesis, \( \mathfrak{M} \) must be reductive.

**Notation 5.9.** For \( n = 1, 2, \ldots \), we shall write \( \mathcal{B}(n) \) for \( \mathcal{B} \otimes \mathbb{C}I_n \) and identify it, as is customary, with the set of \( n \times n \) operator matrices

\[
\begin{bmatrix}
    B & 0 \\
    B & \ddots \\
    0 & \ddots & B
\end{bmatrix}
\]

with \( B \in \mathcal{B} \). Similarly, we shall denote the automorphism group on \( \mathcal{B}(n) \) which \( \{ \alpha_t \}_{t \in \mathbb{R}} \) induces, by \( \{ n \cdot \alpha_t \}_{t \in \mathbb{R}} \), where by definition,

\[
n \cdot \alpha_t = \begin{bmatrix}
    B & 0 \\
    B & \ddots \\
    0 & \ddots & B
\end{bmatrix}
\]

Finally, we shall write \( \mathfrak{A}^{(n)} \) for the subalgebra \( \mathfrak{A} \otimes \mathbb{C}I_n \) of \( \mathcal{B}(n) \).

**Corollary 5.10.** If, for each positive integer \( n \), \( \mathcal{B}(n) \) and \( \{ n \cdot \alpha_t \}_{t \in \mathbb{R}} \) satisfy the hypothesis of Proposition 5.8, then \( \mathfrak{A} \) is strongly dense in \( \mathcal{B} \).

**Proof.** Proposition 5.8 implies that \( \mathfrak{A}^{(n)} \) has the same invariant subspaces as \( \mathcal{B}(n) \) and so the result follows from Theorem 7.1 of [45], a variant of the double commutant theorem.

The next theorem was shown to us by Masamichi Takesaki, who attributed the argument to Kadison [25].

**Theorem 5.11.** Let \( \mathcal{B} \) be a \( \Pi_\omega \) factor with a \( \Pi_1 \) commutant and suppose that \( \{ \alpha_t \}_{t \in \mathbb{R}} \) does not preserve a faithful, normal, semifinite trace on \( \mathcal{B} \). Then for each positive integer \( n \), \( \mathcal{B}^{(n)} \) and \( \{ n \cdot \alpha_t \}_{t \in \mathbb{R}} \) satisfy the hypothesis of Proposition 5.8.

**Proof.** Since \( \mathcal{B} \) is a factor, any two faithful, normal, semifinite trace on \( \mathcal{B} \) are proportional. Thus if one is not preserved by \( \{ \alpha_t \}_{t \in \mathbb{R}} \), none are. We fix one for the proof and denote it by \( \text{tr} \). Also, we denote by \( \text{tr}' \) the unique faithful normal finite trace on \( \mathcal{B}' \) such that \( \text{tr}'(1) = 1 \).

Observe that for each positive integer \( n \), \( \mathcal{B}^{(n)} \) and \( \{ n \cdot \alpha_t \}_{t \in \mathbb{R}} \) satisfy the same hypotheses as \( \mathcal{B} \) and \( \{ \alpha_t \}_{t \in \mathbb{R}} \). Thus it suffices to show that \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is not spatially implemented on any \( \mathcal{B} \) invariant subspace. On the other hand, if \( E \) is nonzero projection in \( \mathcal{B}' \), then \( \mathcal{B}_E = \pi_E(\mathcal{B}) \) is a \( \Pi_\infty \) factor and its commutant is the
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reduced von Neumann algebra $\mathcal{B}_E [11$, Propositions 1, 2, and Corollary, Chap. I, Sect. 2] which is clearly a finite factor. A faithful normal semifinite trace on $\mathcal{B}_E$ is defined by the formula $\text{tr} \circ \pi_E^{-1}$ (recall that because $\mathcal{B}$ is a factor, $\pi_E$ is an isomorphism) and this trace is not preserved by the induced action $\{\alpha_t^E\}_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{B}_E$ defined by the formula $\alpha_t^E = \pi_E \circ \alpha_t \circ \pi_E^{-1}$. Thus we see that for all nonzero projections $E$ in $\mathcal{B}_E$, $\mathcal{B}_E$ and $\{\alpha_t^E\}_{t \in \mathbb{R}}$ satisfy the same hypotheses as $\mathcal{B}$ and $\{\alpha_t\}_{t \in \mathbb{R}}$. Hence, it suffices to prove that $\{\alpha_t\}_{t \in \mathbb{R}}$ is not spatially implemented on $\mathcal{H}$.

To this end, suppose to the contrary that $\{U_t\}_{t \in \mathbb{R}}$ is a unitary representation of $\mathbb{R}$ on $\mathcal{H}$ such that $c(A) = U_tA^*U_t^*$ for all $A \in \mathcal{B}$, $t \in \mathbb{R}$. By the uniqueness of $\text{tr}$, there is a nonzero $\lambda$ in $\mathbb{R}$ such that $\text{tr}(U_tA^*U_t^*) = e^{\lambda t} \text{tr}(A)$ for all $A \in \mathcal{B}$, $t \in \mathbb{R}$. Let $\{\alpha_t\}_{t \in \mathbb{R}}$ be the automorphism group on $\mathcal{B}'$ determined by $\{U_t\}_{t \in \mathbb{R}}$ via conjugation, and note that by the uniqueness of $\text{tr}'$, $\text{tr}' \circ \alpha'_t = \text{tr}'$ for all $t$. Since $\mathcal{B}$ is properly infinite, we may apply [11, Corollary 10, Chap. III, Sect. 8] to assert that there is a vector $f$ in $\mathcal{H}$ such that $\text{tr}'(A) = (Af, f)$ for all $A \in \mathcal{B}'$.

The following corollary is an immediate consequence of Theorem 5.11, Proposition 5.8, and Corollary 5.10.

**COROLLARY 5.12.** Under the hypotheses of Theorem 5.11, $\mathcal{B}$ is a non-self-adjoint, ultraweakly closed, reductive algebra which is weakly dense in $\mathcal{B}$.

It may appear that factors and automorphism groups satisfying the hypotheses of Theorem 5.11 are hard to come by. However, Takesaki's duality theory for crossed products [57] indicates that they exist quite commonly. Rather than appeal to this theory for examples, we take a more pedestrian approach and present the following example which was shown to us by Takesaki.

**EXAMPLE 5.13.** Let $X = \mathbb{R} \times (0, \infty)$, let $d\mu = dx \times dy$, where both $dx$ and $dy$ are Lebesgue measure, and let $G = \{(a, b) = (\alpha, \beta) \mid a \in \mathbb{Q}^+, b \in \mathbb{Q}\}$. Define an action of $G$ on $X$ by the formula $(a, b)(x, y) = (ax + b, (1/a)y)$,
and note that $\mu$ is invariant and ergodic under this action. It is easy to see that the group $G$ and the measure space $(X, \mu)$ satisfy the hypotheses of Proposition 3 in [11, Chap. 1, Sect. 9] and so the von Neumann algebra $\mathfrak{B}_0$ which is the crossed product of $L_x(X, \mu)$ with $G$ is a $\Pi_\infty$ factor. Now $\mathbb{R}$ acts on $X$ via the formula $(x, y) + t = (x, e^t y)$, $t \in \mathbb{R}$, and for each Borel set $M$ in $X$

$$\mu(M + t) = e^t \mu(M), \quad t \in \mathbb{R}. \quad (5.4)$$

This action of $\mathbb{R}$ on $X$ commutes with the action of $G$ and so lifts in a canonical way to an ultraweakly continuous representation of $\mathbb{R}$ as a group $\{\alpha_t\}_{t \in \mathbb{R}}$ of $*$-automorphisms of $\mathfrak{B}_0$ which, by (5.4) and the construction in [11], does not preserve the trace $\pi$. The commutant of $\mathfrak{B}_0$ is not finite, however, so to complete the example, one needs merely to choose a finite projection $E$ in $\mathfrak{B}_0$ and to set $\mathfrak{B} = \pi_E(\mathfrak{B}_0)$ and $\alpha_t = \pi_E \circ \alpha_t \circ \pi_E^{-1}$, $t \in \mathbb{R}$.

We turn now to the problem of constructing reductive algebras in finite von Neumann algebras. Here, too, some of the arguments are due to Takesaki.

Let $X$ be a standard Borel space with a $\sigma$-finite measure $\mu$ and assume that $\mathbb{R}$ acts measurably on $X$ leaving $\mu$ quasi-invariant and ergodic. For definiteness, one may take $X$ to be either $\mathbb{R}$ or $T$ with Lebesgue measure and the action of $\mathbb{R}$ may be taken to be the usual one. Let $\mathfrak{M}$ be a $\Pi_1$-factor in standard form, i.e., assume there is a Hilbert space anti-isomorphism $J$ such that $\mathfrak{M} = JM'J$, let $\mathfrak{B}_0 = L^\infty(X) \otimes \mathfrak{M}$, identify $\mathfrak{B}_0$ with the algebra of all bounded measurable functions with values in $\mathfrak{M}$ (resp. $\mathfrak{B}'$), and let $\mathfrak{H}_0$ be the Hilbert space on which $\mathfrak{B}_0$ acts. Choose a measurable partition $\{X_n\}_{n=0}^\infty$ of $X$ such that $\mu(X_n) > 0$ for all $n$ and choose a projection $F$ in $\mathfrak{B}_0$ with the property that $\text{tr}'(F(x)) = 2^{-n}$ when $x \in X_n$, where $\text{tr}'$ is the trace on $\mathfrak{M}'$. Since $\mathfrak{M}'$ is a $\Pi_1$ factor, such a choice is clearly possible. Our basic von Neumann algebra will be $\mathfrak{B} = \pi_F(\mathfrak{B}_0)$ acting on $\mathfrak{H} = F \mathfrak{H}_0$. Note that since the central support of $F$ is $\mathfrak{I}$, $\pi_F$ is an isomorphism. We let $\{\alpha_t\}_{t \in \mathbb{R}}$ denote the automorphism group of $\mathfrak{B}_0$ defined by the formula $(\alpha_t(A))(x) = A(x + t)$, $A \in \mathfrak{B}_0$, and we set $\alpha_t = \pi_F \circ \alpha_t \circ \pi_F^{-1}$, $t \in \mathbb{R}$.

**Theorem 5.14.** For each positive integer $n$, $\mathfrak{B}^{(n)}$ and $\{n \cdot \alpha_t\}_{t \in \mathbb{R}}$ satisfy the hypothesis of Proposition 5.8.

**Proof.** Since $\mathfrak{M}$ is assumed to be in standard form, $\mathfrak{B}_0$ is also standard [11, Proposition 10, Chap. I, Sect. 5] and so the coupling operator for $\mathfrak{B}_0$ is one [11, Proposition 4, Chap. III, Sect. 6]. By [11, Proposition 2, Chap. III, Sect. 6], the coupling operator $c$ for $\mathfrak{B}$ is given by the formula $c(x) = 2^{-n}$, $x \in X_n$. By the same result, for each projection $E$ in $\mathfrak{B}'$, the coupling operator for $\mathfrak{B}_E$, $c_E$, is given by the formula $c_E = c \cdot \Phi'(E)$ where $\Phi'$ is the (unique) normalized center-valued trace on $\mathfrak{B}'$. Thus, for each projection $E$ in $\mathfrak{B}'$ with central support equal to $I$, $c_E$ is never bounded below by a positive constant,
and in particular, it is not possible to find a projection $E$ in $\mathcal{B}'$ with central support equal to $I$ such that $c_E$ is a positive constant. But the coupling operator for a finite von Neumann algebra is fixed by all spatial automorphisms of the algebra, and since $\{\alpha_t\}_{t \in \mathbb{R}}$ acts ergodically on the center of $\mathcal{B}$, the only time $\{\alpha_t\}_{t \in \mathbb{R}}$ could possibly be spatially implemented on some space $E \mathcal{M}$, $E \in \mathcal{B}'$, $E \neq 0$, is when $E$ has central support equal to $I$ and $c_E$ is constant. Since this never happens by construction, $\{\alpha_t\}_{t \in \mathbb{R}}$ is never spatially implemented on any $\mathcal{B}$ invariant subspace.

To see that $\mathcal{B}^{(n)}$ and $\{n \cdot \alpha_t\}_{t \in \mathbb{R}}$ satisfy the hypothesis of Proposition 5.8 for all $n \geq 2$, simply note that the coupling operator for $\mathcal{B}^{(n)}$ is $n \cdot c$ [51, Corollary 29, Chap. 2, Sect. 2] and so the preceding argument may be repeated to yield the desired result.

The following corollary is an immediate consequence of Proposition 5.8, Corollary 5.10, and Theorem 5.14.

**Corollary 5.15.** The algebra $\mathcal{A}$ of analytic operators in the von Neumann algebra $\mathcal{B}$ constructed above is an ultraweakly closed, non-self-adjoint, reductive subalgebra of $\mathcal{B}$ which is weakly dense in $\mathcal{B}$.

**Remarks 5.16.** (a) Some of the results of this section were announced in [32].

(b) The fact that there are no nontrivial reductive algebras in a type III von Neumann algebra was also noticed by Hoover [24]. The same argument applies when the algebra is finite and type II with an infinite commutant.

(c) By a result of Størmer [54, Theorem 1] the von Neumann algebra constructed in Theorem 5.15 will be $\mathbb{R}$-finite if and only if the measure $\mu$ is equivalent to a finite invariant measure. (We note, too, that by the same result, a von Neumann algebra with automorphism group satisfying the hypotheses of Theorem 5.11 is completely non-$\mathbb{R}$-finite.) Thus whether or not a von Neumann algebra is $\mathbb{R}$-finite seems to have no direct bearing on whether or not the algebra of analytic operators is reductive.

*Note added in proof.* Since this paper was submitted, two papers which overlap it have appeared. They are: (i) S. Kawamura and J. Tomiyama, On subdiagonal algebras associated with flows in operator algebras, *J. Math. Soc. Japan* 29 (1977), 73–90; and (ii) L. Zsido, Spectral and ergodic properties of the analytic generators, *J. Approximation Theory* 20 (1977), 77–138. Also, we note that in a recent preprint, “Non-Self-Adjoint Crossed Products,” by M. McAsey, K.-S. Sato, and the second author, an extensive analysis of the examples discussed in Section 4.3 is made.

**Acknowledgments**

We would like to thank L. Brown and C. Moore for correspondence and conversations which were helpful to us in the writing of this paper, especially Section 4.1. We would
like to thank, too, R. Coifman and G. Weiss, who spotted an error in an earlier draft, and we would like to thank G. Pisier for showing us that the error involved a fundamental obstruction and cannot be salvaged—at least not without substantial modification. Finally, we are particularly grateful to Masamichi Takesaki for all the help he gave us. He showed us the examples in Section 5.

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