

# *Representations of Cuntz-Pimsner Algebras*

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ABSTRACT. Let  $X$  be a Hilbert bimodule over a  $C^*$ -algebra  $A$ . We analyse the structure of the associated Cuntz-Pimsner algebra  $\mathcal{O}_X$  and related algebras using representation-theoretic methods. In particular, we study the ideals  $\mathcal{I}(I)$  in  $\mathcal{O}_X$  induced by appropriately invariant ideals  $I$  in  $A$ , and identify the quotients  $\mathcal{O}_X/\mathcal{I}(I)$  as relative Cuntz-Pimsner algebras of Muhly and Solel. We also prove a gauge-invariant uniqueness theorem for  $\mathcal{O}_X$ , and investigate the relationship between  $\mathcal{O}_X$  and an alternative model proposed by Doplicher, Pinzari and Zuccante.

Let  $A$  be a  $C^*$ -algebra, and let  $X$  be a Hilbert bimodule over  $A$ , in the sense that  $X$  is a right Hilbert  $A$ -module with a left action of  $A$  by adjointable operators. In [27], Pimsner constructed  $C^*$ -algebras  $\mathcal{O}_X$  in such a way that, for particular choices of  $A$  and  $X$ , one recovers the Cuntz-Krieger algebras or crossed products by  $\mathbb{Z}$  or  $\mathbb{N}$ . Since then, other important classes of  $C^*$ -algebras have been shown to fit Pimsner's model (see, for example, [12, 16, 30]). Thus the *Cuntz-Pimsner algebras*  $\mathcal{O}_X$  have recently attracted a good deal of attention.

The algebras  $\mathcal{O}_X$  were originally constructed in a very concrete way: Pimsner introduced first a Toeplitz algebra  $\mathcal{T}_X$  acting on an analogue of Fock space, and took for  $\mathcal{O}_X$  a particular quotient of  $\mathcal{T}_X$ . Nevertheless, one of his main results identifies a universal property of  $\mathcal{O}_X$ ; in our language, he shows that Cuntz-Pimsner covariant representations  $(\psi, \pi)$  of  $X$  give representations  $\psi \times \pi$  of  $\mathcal{O}_X$  [27, Theorem 3.12]. Here we study these covariant representations. We use representation-theoretic methods to analyse ideals and quotients of Cuntz-Pimsner algebras, and give criteria under which a given representation  $\psi \times \pi$  is faithful.

Our interest in Cuntz-Pimsner algebras derives partly from their connection with the graph algebras of [19, 4, 28], and our analysis is motivated by what we know to be true for graph algebras. Thus we seek not just a class of  *$X$ -invariant ideals*  $I$  in  $A$  which give rise to ideals  $\mathcal{I}(I)$  in  $\mathcal{O}_X$ , but also to identify Cuntz-Pimsner algebras which are, respectively, Morita equivalent to  $\mathcal{I}(I)$  and isomorphic to  $\mathcal{O}_X/\mathcal{I}(I)$ . This we can do when  $A$  acts on the left by compact operators, but

in general the quotient is one of the relative Cuntz-Pimsner algebras introduced in [25] rather than a Cuntz-Pimsner algebra. The appropriate setting for our analysis, therefore, is that of relative Cuntz-Pimsner algebras, and we work in this generality from the start.

Much of the current literature on Cuntz-Pimsner algebras concerns bimodules which satisfy additional hypotheses, and of course this is often appropriate for the particular examples or applications authors have in mind. For the bimodules associated to graph algebras, however, these additional hypotheses always seem to impose substantial restrictions on the underlying graph. Thus, for example,  $A$  acts on the left by compact operators precisely when the graph is row-finite, and  $A$  has an identity only when the graph has finitely many vertices; the effect of other standard hypotheses is analysed in [12, Section 5]. So we have tried to avoid making any additional assumptions on the bimodules we consider. Thus our main theorem about faithful representations, for example, is an analogue of the gauge-invariant uniqueness theorem for graph algebras, which requires no structural hypotheses on the graph (see [4, Theorem 2.1] and [28, Theorem 2.7]).

We begin in Section 1 with a review of the relative Cuntz-Pimsner algebras and their representation theory. We take the universal approach of [12], so that the Cuntz-Pimsner algebra is by definition the  $C^*$ -algebra generated by a universal Cuntz-Pimsner covariant representation of  $X$ , and one studies  $\mathcal{O}_X$  by analysing its representations. We describe the main sets of examples of interest to us, namely the bimodules associated to graphs and endomorphisms. In Section 2, we discuss  $X$ -invariant ideals in  $A$ ; this notion was introduced by Kajiwara, Pinzari and Watatani in [16]. To each such ideal is associated a submodule  $XI$  which is a Hilbert bimodule over  $I$ , and whose quotient  $X/XI$  is a bimodule over  $A/I$ .

Our main theorem (Theorem 3.1) says that the ideal  $\mathcal{I}(I)$  in  $\mathcal{O}_X$  generated by an  $X$ -invariant ideal is Morita equivalent to  $\mathcal{O}_{XI}$ , and identifies the quotient as a relative Cuntz-Pimsner algebra for the bimodule  $X/XI$ . This result extends Theorem 4.3 of [16], by identifying the Morita equivalence class of  $\mathcal{I}(I)$ , and by lifting structural hypotheses from the algebra and bimodule (in [16],  $A$  is unital and  $X$  is full, finitely generated projective, and satisfies an analogue of Cuntz's Condition (II)). The main ingredient in our proof is a construction, something like dilation, which allows us to extend a covariant representation of  $XI$  to one of  $X$ .

Our analogue of the gauge-invariant uniqueness theorem is Theorem 4.1; it extends part of [9, Theorem 3.3] as well as the various gauge-invariant uniqueness theorems for graph algebras. In Section 5, we give some applications. In particular, we use Theorem 4.1 to identify the  $C^*$ -envelope of the tensor algebra of  $X$ , thus settling a problem left open in [25, Section 6]. In the final section, we consider the alternative approach to  $\mathcal{O}_X$  taken in [9]. We show that the algebra  $\mathcal{DR}_X$  in [9], which is modelled on Doplicher-Roberts algebras rather than Cuntz algebras, is in general larger than the Cuntz-Pimsner algebra, and identify the representations of  $\mathcal{O}_X$  which extend to  $\mathcal{DR}_X$  (Theorem 6.6).

**Conventions.** If  $X$  is a right Hilbert  $A$ -module, we denote by  $\mathcal{L}(X)$  the  $C^*$ -algebra of adjointable operators on  $X$ . For  $x, y \in X$ , we define  $\Theta_{x,y}^X(z) := x \cdot \langle y, z \rangle_A$ ; we drop the superscript  $X$  if ambiguity seems unlikely. Then  $\Theta_{x,y} \in \mathcal{L}(X)$  with  $\Theta_{x,y}^* = \Theta_{y,x}$ , and  $\mathcal{K}(X) := \overline{\text{span}}\{\Theta_{x,y} : x, y \in X\}$  is an ideal in  $\mathcal{L}(X)$ . In general, if  $M$  is a subobject of  $N$ , we write  $q^M$  for the quotient map of  $N$  onto  $N/M$ .

### 1. REPRESENTATIONS OF HILBERT BIMODULES

Let  $A$  be a  $C^*$ -algebra, let  $X$  be a right Hilbert  $A$ -module, and let  $\varphi : A \rightarrow \mathcal{L}(X)$  be a homomorphism. Then  $a \cdot x := \varphi(a)x$  defines a left action of  $A$  on  $X$ , and we call  $X$  a *Hilbert bimodule* over  $A$ . A *Toeplitz representation*  $(\psi, \pi)$  of  $X$  in a  $C^*$ -algebra  $B$  consists of a linear map  $\psi : X \rightarrow B$  and a homomorphism  $\pi : A \rightarrow B$  such that

$$\psi(x \cdot a) = \psi(x)\pi(a), \quad \psi(x)^* \psi(y) = \pi(\langle x, y \rangle_A), \quad \text{and} \quad \psi(a \cdot x) = \pi(a)\psi(x)$$

for  $x, y \in X$  and  $a \in A$ . (It is important in our applications that we do not require  $\pi$  to be nondegenerate: see the comments in [12, page 178] and Example 3.13 below. In allowing  $\pi$  to be degenerate, we are departing from the conventions in [25].) Given such a representation, there is a homomorphism  $\pi^{(1)} : \mathcal{K}(X) \rightarrow B$  which satisfies

$$(1.1) \quad \pi^{(1)}(\Theta_{x,y}^X) = \psi(x)\psi(y)^* \quad \text{for all } x, y \in X,$$

and we then have

$$\pi^{(1)}(T)\psi(x) = \psi(Tx) \quad \text{for all } T \in \mathcal{K}(X) \text{ and } x \in X.$$

(See [27, page 202], [16, Lemma 2.2], and [12, Remark 1.7] for details.) If  $\rho : B \rightarrow C$  is a homomorphism of  $C^*$ -algebras, then  $(\rho \circ \psi, \rho \circ \pi)$  is a Toeplitz representation of  $X$ , and since

$$(\rho \circ \pi)^{(1)}(\Theta_{x,y}) = \rho \circ \psi(x)\rho \circ \psi(y)^* = \rho \circ \pi^{(1)}(\Theta_{x,y}) \quad \text{for all } x, y \in X,$$

by linearity and continuity we have

$$(1.2) \quad (\rho \circ \pi)^{(1)} = \rho \circ \pi^{(1)}.$$

**Definition 1.1.** If  $X$  is a Hilbert bimodule over  $A$ , we define

$$J(X) := \varphi^{-1}(\mathcal{K}(X)),$$

which is a closed two-sided ideal in  $A$ . Let  $K$  be an ideal in  $J(X)$ . We say that a Toeplitz representation  $(\psi, \pi)$  of  $X$  is *coisometric on  $K$*  if

$$(1.3) \quad \pi^{(1)}(\varphi(a)) = \pi(a) \quad \text{for all } a \in K.$$

When  $(\psi, \pi)$  is coisometric on all of  $J(X)$ , we say it is *Cuntz-Pimsner covariant*.

**Remark 1.2.** Our use of the term “coisometric” follows [25], and is motivated in part by the simplest possible example, in which  $X = A = \mathbb{C}$ : if  $(\psi, \pi)$  is a Toeplitz representation, then  $\psi(1)$  is an isometry, and  $\psi(1)$  is a coisometry if and only if  $(\psi, \pi)$  is Cuntz-Pimsner covariant. See Example 1.6 for further discussion.

**Proposition 1.3.** *Let  $X$  be a Hilbert bimodule over  $A$ , and let  $K$  be an ideal in  $J(X)$ . Then there is a  $C^*$ -algebra  $\mathcal{O}(K, X)$  and a Toeplitz representation  $(k_X, k_A) : X \rightarrow \mathcal{O}(K, X)$  which is coisometric on  $K$  and satisfies:*

- (a) *for every Toeplitz representation  $(\psi, \pi)$  of  $X$  which is coisometric on  $K$ , there is a homomorphism  $\psi \times_K \pi$  of  $\mathcal{O}(K, X)$  such that  $(\psi \times_K \pi) \circ k_X = \psi$  and  $(\psi \times_K \pi) \circ k_A = \pi$ ; and*
- (b)  *$\mathcal{O}(K, X)$  is generated as a  $C^*$ -algebra by  $k_X(X) \cup k_A(A)$ .*

*The triple  $(\mathcal{O}(K, X), k_X, k_A)$  is unique: if  $(B, k'_X, k'_A)$  has similar properties, there is an isomorphism  $\theta : \mathcal{O}(K, X) \rightarrow B$  such that  $\theta \circ k_X = k'_X$  and  $\theta \circ k_A = k'_A$ . There is a strongly continuous gauge action  $\gamma : \mathbb{T} \rightarrow \text{Aut } \mathcal{O}(K, X)$  which satisfies  $\gamma_z(k_A(a)) = k_A(a)$  and  $\gamma_z(k_X(x)) = zk_X(x)$  for  $a \in A, x \in X$ .*

**Remark 1.4.** The algebra  $\mathcal{O}(\{0\}, X)$  is the Toeplitz algebra  $\mathcal{T}_X$ , and  $\mathcal{O}(J(X), X)$  is the Cuntz-Pimsner algebra  $\mathcal{O}_X$ . The algebra  $\mathcal{O}(K, X)$  is called the relative Cuntz-Pimsner algebra determined by  $K$ , and was introduced by Muhly and Solel in [25]. In [27] and [25], the Toeplitz algebra  $\mathcal{T}_X$  was studied in its Fock representation, and  $\mathcal{O}_X$  and  $\mathcal{O}(K, X)$  were quotients by ideals generated by certain adjointable operators on the Fock module  $\bigoplus_{n=0}^{\infty} X^{\otimes n}$ . Our approach here, as in [12], is somewhat different: we define these algebras abstractly by their universal properties, and one of our goals is to give conditions which determine whether or not a given representation is faithful. This is accomplished for the Toeplitz algebra in [12, Theorems 2.1 and 3.1], and for  $\mathcal{O}_X$  in our Theorem 4.1.

*Proof of Proposition 1.3.* This is proved for the Toeplitz algebra in [12, Proposition 1.3]; we write  $(i_X, i_A) : X \rightarrow \mathcal{T}_X$  for the universal Toeplitz representation of  $X$ . Let  $\mathcal{I}$  be the ideal in  $\mathcal{T}_X$  generated by  $\{i_A(k) - i_A^{(1)}(\varphi(k)) : k \in K\}$ , and define  $\mathcal{O}(K, X) := \mathcal{T}_X/\mathcal{I}$ . Let  $q$  be the quotient map of  $\mathcal{T}_X$  onto  $\mathcal{O}(K, X)$ , and define  $k_X := q \circ i_X$  and  $k_A := q \circ i_A$ . Using (1.2), we see that  $(k_X, k_A)$  is a Toeplitz representation such that

$$k_A(k) - k_A^{(1)}(\varphi(k)) = q(i_A(k) - i_A^{(1)}(\varphi(k))) = 0 \quad \text{for all } k \in K;$$

that is,  $(k_X, k_A)$  is coisometric on  $K$ . To see that it is universal, suppose  $(\psi, \pi)$  is another Toeplitz representation of  $X$  which is coisometric on  $K$ . Then  $(\psi, \pi)$  induces a representation  $\psi \times \pi$  of  $\mathcal{T}_X$  such that  $(\psi \times \pi) \circ i_X = \psi$  and  $(\psi \times \pi) \circ i_A = \pi$ . By (1.2) we have

$$\psi \times \pi(i_A(k) - i_A^{(1)}(\varphi(k))) = \pi(k) - \pi^{(1)}(\varphi(k)) = 0 \quad \text{for all } k \in K,$$

so  $\psi \times \pi$  annihilates the ideal  $\mathcal{I}$  and hence descends to a homomorphism  $\psi \times_K \pi$  of  $\mathcal{O}(K, X)$  with the required properties. The assertions regarding uniqueness and the gauge action are established as in [12, Proposition 1.3].  $\square$

**Example 1.5** (Graph algebras). Let  $E = (E^0, E^1, r, s)$  be a directed graph with vertex set  $E^0$ , edge set  $E^1$ , and range and source maps  $r, s: E^1 \rightarrow E^0$ . A *Toeplitz-Cuntz-Krieger  $E$ -family* in a  $C^*$ -algebra  $B$  consists of partial isometries  $\{S_e: e \in E^1\}$  with commuting range projections and mutually orthogonal projections  $\{P_v: v \in E^0\}$  satisfying  $S_e^* S_e = P_{r(e)}$  and  $S_e S_e^* \leq P_{s(e)}$ . If  $\{S_e, P_v\}$  also satisfies

$$(1.4) \quad P_v = \sum_{\{e: s(e)=v\}} S_e S_e^*$$

for every vertex  $v$  with  $0 < \#\{e: s(e) = v\} < \infty$ , we call  $\{S_e, P_v\}$  a *Cuntz-Krieger  $E$ -family* [18, 19, 11]. The *graph algebra*  $C^*(E)$  is generated by a universal Cuntz-Krieger  $E$ -family  $\{s_e, p_v\}$ .

Recall from [12, Example 1.2] that the Cuntz-Krieger bimodule  $X(E)$  is the set of functions  $x: E^1 \rightarrow \mathbb{C}$  such that

$$v \in E^0 \mapsto \sum_{\{e \in E^1: r(e)=v\}} |x(e)|^2$$

takes finite values and belongs to  $A := c_0(E^0)$ , with Hilbert-bimodule structure given by

$$\begin{aligned} (x \cdot a)(e) &:= x(e)a(r(e)) \quad \text{for } e \in E^1, \\ \langle x, y \rangle_A(v) &:= \sum_{\{e \in E^1: r(e)=v\}} \overline{x(e)}y(e) \quad \text{for } v \in E^0, \text{ and} \\ (a \cdot x)(e) &:= a(s(e))x(e) \quad \text{for } e \in E^1. \end{aligned}$$

If  $(i_X, i_A)$  is the universal Toeplitz representation of  $X(E)$  in the Toeplitz algebra  $\mathcal{T}_{X(E)}$ , then  $\{i_X(\delta_e), i_A(\delta_v)\}$  is a universal Toeplitz-Cuntz-Krieger  $E$ -family which generates  $\mathcal{T}_{X(E)}$  [12, Theorem 4.1]. Then

$$J(X(E)) = \overline{\text{span}}\{\delta_v: |s^{-1}(v)| < \infty\}$$

[12, Proposition 4.4], and hence when  $E$  has no sinks, the Cuntz-Pimsner algebra  $\mathcal{O}_{X(E)}$  is canonically isomorphic to  $C^*(E)$  [11, Proposition 12].

The relative Cuntz-Pimsner algebras of  $X(E)$  interpolate between  $\mathcal{T}_{X(E)}$  and  $\mathcal{O}_{X(E)}$ . Let  $K \triangleleft J(X(E))$ , let  $(\mathcal{O}(K, X(E)), k_X, k_A)$  be universal for Toeplitz representations of  $X(E)$  which are coisometric on  $K$ , let  $s_e := k_X(\delta_e)$ , and let  $p_v := k_A(\delta_v)$ . Then  $K = K_F = \overline{\text{span}}\{\delta_v: v \in F\}$  for some set  $F$  of vertices which emit finitely many edges, and  $(\mathcal{O}(K, X(E)), s_e, p_v)$  is universal for Toeplitz-Cuntz-Krieger  $E$ -families which satisfy (1.4) for every  $v \in F$ .

**Example 1.6** (Crossed products by endomorphisms). Let  $\alpha$  be an endomorphism of a  $C^*$ -algebra  $A$ . We assume for simplicity that  $A$  has an identity 1, though it suffices that  $\alpha$  extends to an endomorphism  $\bar{\alpha}$  of  $M(A)$ : just replace  $\alpha(1)$  by  $\bar{\alpha}(1_{M(A)})$  throughout. We consider the Hilbert bimodule over  $A$  with underlying space  $X := \alpha(1)A$  and

$$a \cdot x := \alpha(a)x, \quad x \cdot a := xa, \quad \text{and} \quad \langle x, y \rangle_A = x^*y.$$

Suppose  $(\psi, \pi)$  is a Toeplitz representation of  $X$  on  $\mathcal{H}$ . We notice first that for each  $x$ , we have  $\psi(x) = \psi(\alpha(1)x1) = \psi(1 \cdot x \cdot 1) = \pi(1)\psi(x)\pi(1)$ , so both  $\psi$  and  $\pi$  vanish on the complement of  $\pi(1)H$ , and it suffices to consider pairs  $(\psi, \pi)$  in which  $\pi$  is nondegenerate. Then  $V := \psi(\alpha(1))^*$  satisfies

$$(1.5) \quad \begin{aligned} \pi(\alpha(a)) &= \pi(\langle \alpha(1), \alpha(a) \rangle_A) = \psi(\alpha(1))^* \psi(\alpha(a)) \\ &= V\psi(a \cdot \alpha(1)) = V\pi(a)V^*. \end{aligned}$$

Taking  $a = 1$  in (1.5) shows that  $V^n(V^*)^n = \pi(\alpha^n(1))$  is a projection, so  $V^n$  is a partial isometry for all  $n$ ; in other words,  $V$  is a power partial isometry. Another computation shows that

$$V^* \pi(\alpha(a)) = \psi(\alpha(1)\alpha(a)) = \psi(\alpha(a)\alpha(1)) = \pi(a)\psi(\alpha(1)) = \pi(a)V^*;$$

this implies, first, by taking adjoints, that  $\pi(\alpha(a))V = V\pi(a)$  for all  $a \in A$ , and, second, that  $V^*V$  commutes with  $\pi(a)$ . Conversely, it is not hard to verify that if  $\pi: A \rightarrow B(\mathcal{H})$  is nondegenerate and  $V$  is a power partial isometry such that  $V^*V$  commutes with each  $\pi(a)$ , then setting  $\psi(x) := V^*\pi(x)$  gives a Toeplitz representation  $(\psi, \pi)$ ; see [23] for further discussion of such pairs  $(\pi, V)$ .

We next observe that the operator  $\varphi(a): x \mapsto a \cdot x$  is just  $\Theta_{\alpha(a), \alpha(1)}$ . Thus  $J(X) = A$ , and the Toeplitz representation  $(\psi, \pi)$  corresponding to  $(\pi, V)$  is Cuntz-Pimsner covariant if and only if

$$(1.6) \quad \pi(a) = \pi^{(1)}(\Theta_{\alpha(a), \alpha(1)}) = \psi(\alpha(a))\psi(\alpha(1))^* = V^*\pi(\alpha(a))V$$

for all  $a \in A$ ;

in view of (1.5), this is equivalent to  $\pi(a) = V^*V\pi(a)V^*V = V^*V\pi(a)$ , and hence to  $V^*V = 1$ . Thus  $(\psi, \pi)$  is Cuntz-Pimsner covariant if and only if  $V = \psi(\alpha(1))^*$  is an isometry. (We wrote  $V$  for  $\psi(\alpha(1))^*$  rather than  $\psi(\alpha(1))$  to make (1.5) look like the usual covariance relation for semigroup crossed products. However, we could equally say that  $(\psi, \pi)$  is Cuntz-Pimsner covariant if and only if  $\psi(\alpha(1))$  is a coisometry.)

In conclusion: the Cuntz-Pimsner algebra  $\mathcal{O}_X$  is the semigroup crossed product  $A \rtimes_{\alpha} \mathbb{N}$ , as in, for example, [2] or [21] (see also [26, Lemma 12]). The Toeplitz algebra  $\mathcal{T}_X$  is one of the crossed products  $A \rtimes \mathbb{N}$  of [23], in which  $\mathbb{N}$  acts by partial isometries rather than isometries. As we shall see in the next example, there can be lots of relative Cuntz-Pimsner algebras in between.

**Example 1.7.** We consider the forward shift endomorphism  $\tau$  on the  $C^*$ -algebra  $c$  of convergent sequences, and the bimodule  $X := \tau(1)c$  of the preceding example. It is shown in the proof of [12, Proposition 5.3] that  $(\psi, \pi) \mapsto \psi(\tau(1))^*$  sets up a bijection between the Toeplitz representations of  $X$  and the power partial isometries  $V$  on Hilbert space: for this  $X$ , the operator  $V$  determines the representation  $\pi$  because  $c$  is spanned by the functions  $\tau^n(1)$  and  $\pi(\tau^n(1)) = V^n(V^*)^n$ . As in the previous example,  $(\psi, \pi)$  is Cuntz-Pimsner covariant if and only if  $V^*V = 1$ . Thus the Cuntz-Pimsner algebra  $\mathcal{O}_X$  is universal for isometries, and hence is isomorphic to the usual Toeplitz algebra.

Now let  $K$  be the ideal  $c_0$  in  $J(X) = c$ . The Toeplitz representation  $(\psi_V, \pi_V)$  determined by  $V$  is coisometric on  $K$  when  $\pi_V(a) = V^*\pi_V(\tau(a))V$  for all  $a \in K$ , or equivalently for all  $a$  of the form  $\tau^n(1) - \tau^{n+1}(1)$ . Since  $\pi_V(\tau^n(1)) = V^n(V^*)^n$ , and since the range and source projections of the  $V^n$  form a commuting family, it follows that  $(\psi_V, \pi_V)$  is coisometric on  $K$  if and only if

$$(1.7) \quad V^*V(V^n(V^*)^n - V^{n+1}(V^*)^{n+1}) = V^n(V^*)^n - V^{n+1}(V^*)^{n+1}$$

for all  $n \geq 0$ . The structure theorem for power partial isometries in [13] says that  $V$  is a direct sum of a unitary, a multiple of the unilateral shift  $S$ , a multiple of  $S^*$ , and multiples of the truncated shifts  $J_m$  on  $\mathbb{C}^m$ . Unitaries, isometries and coisometries all satisfy (1.7). However, for the truncated shift  $J_m$ , we have  $J_m^m = 0$ ,  $J_m^*J_m$  is the projection on the first  $m - 1$  variables, and  $J_m^n(J_m^*)^n$  is the projection on the last  $m - n$  variables, so (1.7) is satisfied for  $n \neq m - 1$  but not for  $n = m - 1$ . Thus the relative Cuntz-Pimsner algebra  $\mathcal{O}(c_0, X)$  is universal for power partial isometries which are direct sums of an isometry and a coisometry (we can absorb a unitary factor into either the isometry or the coisometry).

More generally, suppose  $R$  is a subset of  $\mathbb{N}$  and

$$K_R := \{x \in c_0 : x_n = 0 \text{ when } n \notin R\} = \overline{\text{span}}\{1_n - 1_{n+1} : n \in R\}.$$

Then  $(\psi_V, \pi_V)$  is coisometric on  $K_R$  if and only if (1.7) holds for all  $n \in R$ . Since  $J_m$  satisfies (1.7) for  $n \neq m - 1$ , the relative Cuntz-Pimsner algebra  $\mathcal{O}(K_R, X)$  is universal for power partial isometries which have no summands equivalent to  $J_m$  for  $m - 1 \in R$ .

**Example 1.8.** We now consider the truncated shift endomorphism  $\tau_m$  of the  $C^*$ -algebra  $\mathbb{C}^m$  of sequences of length  $m$ . As in the previous example, Toeplitz representations  $(\psi, \pi)$  of the bimodule  $\tau_m(1)\mathbb{C}^m$  are determined by a power partial isometry  $V = \psi(\tau_m(1))^*$ , which because  $\tau_m^m = 0$  has to satisfy  $V^m = 0$ . A non-trivial example is provided by taking for  $\pi$  the representation of  $\mathbb{C}^m$  by multiplication operators on  $\mathbb{C}^m$ , and for  $V$  the truncated shift  $J_m$ . The representation  $(\psi, \pi)$  is Cuntz-Pimsner covariant precisely when  $\pi(a) = V^*\pi(a)V$  for all  $a$ ; however, this is only possible if  $\pi(a) = (V^*)^m\pi(\tau_m^m(a))V^m = 0$  for all  $a \in A$ . Thus for this bimodule,  $\mathcal{O}_X = 0$ .

We will need the following spatial characterization of coisometric representations.

**Lemma 1.9.** *Let  $(\psi, \pi)$  be a Toeplitz representation of  $X$  on a Hilbert space  $\mathcal{H}$ , and let  $K$  be an ideal in  $J(X)$ . Then  $(\psi, \pi)$  is coisometric on  $K$  if and only if*

$$\overline{\pi(K)\mathcal{H}} \subseteq \overline{\psi(X)\mathcal{H}} := \overline{\text{span}\{\psi(x)h : x \in X, h \in \mathcal{H}\}}.$$

*Proof.* Since  $\pi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$  for all  $x, y \in X$ , the essential subspace of  $\pi^{(1)}$  is contained in  $\overline{\psi(X)\mathcal{H}}$ . Hence if  $(\psi, \pi)$  is coisometric on  $K$ , then

$$\overline{\pi(K)\mathcal{H}} = \overline{\pi^{(1)}(\varphi(K))\mathcal{H}} \subseteq \overline{\pi^{(1)}(\mathcal{K}(X))\mathcal{H}} \subseteq \overline{\psi(X)\mathcal{H}}.$$

On the other hand, for all  $a \in K$  and  $x \in X$  we have

$$\pi^{(1)}(\varphi(a))\psi(x) = \psi(\varphi(a)x) = \pi(a)\psi(x),$$

so if  $\overline{\pi(K)\mathcal{H}} \subseteq \overline{\psi(X)\mathcal{H}}$ , then  $(\psi, \pi)$  is coisometric on  $K$ . □

## 2. INVARIANT IDEALS

Let  $I$  be an ideal in  $A$ . The closed submodule

$$X_I := \{x \in X : \langle x, y \rangle_A \in I \text{ for all } y \in X\}$$

is a right Hilbert  $I$ -module. We claim that  $X_I = XI := \{x \cdot i : x \in X, i \in I\}$ . Indeed, since  $\langle x \cdot i, y \rangle_A = i^* \langle x, y \rangle_A \in I$ , we have  $XI \subseteq X_I$ , and by the Hewitt-Cohen Factorization Theorem each  $x \in X_I$  can be written as  $x = y \cdot i$  for some  $y \in X_I$  and  $i \in I$ , so we also have  $X_I \subseteq XI$ .

Let  $q^I : A \rightarrow A/I$  and  $q^{XI} : X \rightarrow X/XI$  be the quotient maps.

**Lemma 2.1.**  *$X/XI$  is a right Hilbert  $A/I$ -module with*

$$\begin{aligned} q^{XI}(x) \cdot q^I(a) &:= q^{XI}(x \cdot a) \quad \text{for } x \in X \text{ and } a \in A, \text{ and} \\ \langle q^{XI}(x), q^{XI}(y) \rangle_{A/I} &:= q^I(\langle x, y \rangle_A) \quad \text{for } x, y \in X. \end{aligned}$$

*Proof.* When  $X$  is full, this is standard [29, Proposition 3.25]. When  $X$  is not full, it is still routine to check that the action and inner product are well-defined, so we only need to show that  $X/XI$  is complete in the  $A/I$ -norm. Let  $A_0 := \overline{\text{span}\{\langle x, y \rangle_A : x, y \in X\}}$ , and let  $I_0$  be the ideal  $I \cap A_0$ . Since

$$\begin{aligned} (2.1) \quad XI_0 &= \{x \in X : \langle x, y \rangle_A \in I_0 \text{ for all } y \in X\} \\ &= \{x \in X : \langle x, y \rangle_A \in I \text{ for all } y \in X\} = XI, \end{aligned}$$



we can view  $X/XI$  as either an  $A/I$ -module or an  $A_0/I_0$ -module. But  $X$  is full as a Hilbert  $A_0$ -module, so  $X/XI_0$  is complete as an inner-product  $A_0/I_0$ -module; it therefore suffices to show that the  $A/I$ -norm on  $X/XI$  agrees with the  $A_0/I_0$ -norm. The map  $a \in A_0 \mapsto q^I(a) \in A/I$  has kernel  $I_0$ , and hence induces an isometric homomorphism  $A_0/I_0 \rightarrow A/I$ ; thus

$$\|q^I(a)\| = \|q^{I_0}(a)\| \quad \text{for } a \in A_0.$$

For each  $x \in X$  we thus have

$$\begin{aligned} \|q^{XI}(x)\|_{A/I}^2 &= \|\langle q^{XI}(x), q^{XI}(x) \rangle_{A/I}\| = \|q^I(\langle x, x \rangle_A)\| \\ &= \|q^{I_0}(\langle x, x \rangle_{A_0})\| = \|\langle q^{XI}(x), q^{XI}(x) \rangle_{A_0/I_0}\| \\ &= \|q^{XI}(x)\|_{A_0/I_0}^2, \end{aligned}$$

so the two norms on  $X/XI$  coincide. □

For any ideal  $I$  in  $A$ ,  $\varphi_I(i) := \varphi(i)|_{XI}$  defines a homomorphism  $\varphi_I : I \rightarrow \mathcal{L}(XI)$  which gives  $XI$  the structure of a Hilbert bimodule over  $I$ . To define a left action on  $X/XI$ , more structure is required.

**Definition 2.2.** An ideal  $I$  in  $A$  is *X-invariant* if  $\varphi(I)X \subseteq XI$ .

**Lemma 2.3** ([16, Proposition 4.2]). *If  $I$  is an X-invariant ideal in  $A$ , then there is a homomorphism  $\varphi_{A/I} : A/I \rightarrow \mathcal{L}(X/XI)$  satisfying*

$$\varphi_{A/I}(q^I(a))(q^{XI}(x)) = q^{XI}(\varphi(a)x) \quad \text{for } a \in A \text{ and } x \in X,$$

so that  $X/XI$  is a Hilbert bimodule over  $A/I$ .

*Proof.* For fixed  $a \in A$  we have  $\varphi(a)XI \subseteq XI$ , and hence  $q^{XI}(x) \mapsto q^{XI}(\varphi(a)x)$  is well-defined on  $X/XI$ . When  $I$  is  $X$ -invariant, this map depends only on the equivalence class of  $a$  in  $A/I$ , so  $\varphi_{A/I}$  is well-defined. It is routine to check that each  $\varphi_{A/I}(q^I(a))$  is adjointable with adjoint  $\varphi_{A/I}(q^I(a^*))$ , and that  $\varphi_{A/I}$  is a homomorphism. □

**Example 2.4.** Let  $X = X(E)$  be the Cuntz-Krieger bimodule of a directed graph  $E$ , as in Example 1.5. Each ideal  $I$  in  $A = c_0(E^0)$  has the form  $I_V = \overline{\text{span}}\{\delta_v : v \in V\}$  for some subset  $V$  of  $E^0$ , and then

$$XI_V = \overline{\text{span}}\{\delta_e \cdot \delta_v : e \in E^1, v \in V\} = \overline{\text{span}}\{\delta_e : r(e) \in V\},$$

and similarly

$$\varphi(I_V)X = \overline{\text{span}}\{\delta_e : s(e) \in V\}.$$

Thus  $I_V$  is  $X$ -invariant if and only if  $V$  is hereditary in the sense that  $s(e) \in V \Rightarrow r(e) \in V$ . When  $V \subseteq E^0$  is hereditary,  $E \setminus V := (E^0 \setminus V, r^{-1}(E^0 \setminus V))$  is a subgraph

of  $E$ , and it is easy to see that  $X/XI_V$  is canonically isomorphic to  $X(E \setminus V)$ . As it stands, the submodule  $XI_V$  differs slightly from the bimodule of a subgraph of  $E$ :  $XI_V$  contains the functions  $\delta_e$  supported on edges  $e$  with  $r(e) \in V$  but  $s(e) \notin V$ .

**Remark 2.5.** We have been careful not to assume that  $\varphi$  is injective, or that  $X$  is essential as a left  $A$ -module (i.e.,  $X = \overline{\text{span}}\{\varphi(a)x : a \in A, x \in X\}$ ), and the previous example shows why. If  $r(e) \in V$  and  $s(e) \notin V$ , then  $\delta_e \in XI$  is orthogonal to  $\varphi_I(i)x$  for every  $i \in I$  and  $x \in XI$ , and hence  $XI$  is not essential as a left  $I$ -module.

The following lemma is the key to understanding the relations among coisometric Toeplitz representations of the bimodules  $X$ ,  $XI$  and  $X/XI$ .

**Lemma 2.6.** *Suppose  $X$  is a right Hilbert  $A$ -module and  $I \triangleleft A$ .*

(1) *There is an isometric embedding  $\iota^{\mathcal{K}} : \mathcal{K}(XI) \rightarrow \mathcal{K}(X)$  such that*

$$\iota^{\mathcal{K}}(\Theta_{x,y}^{XI}) = \Theta_{x,y}^X \quad \text{for } x, y \in XI.$$

*Moreover, for  $T \in \mathcal{K}(XI)$ ,  $\iota^{\mathcal{K}}(T)$  is the unique extension of  $T$  to an operator in  $\mathcal{L}(X)$  whose range is contained in  $XI$ .*

(2) *There is a surjective homomorphism  $q_{\mathcal{K}} : \mathcal{K}(X) \rightarrow \mathcal{K}(X/XI)$  such that*

$$q_{\mathcal{K}}(R)(q^{XI}(x)) = q^{XI}(Rx) \quad \text{for } R \in \mathcal{K}(X) \text{ and } x \in X,$$

*and then*

$$(2.2) \quad q_{\mathcal{K}}(\Theta_{x,y}^X) = \Theta_{q^{XI}(x), q^{XI}(y)}^{X/XI} \quad \text{for all } x, y \in X.$$

(3)  $0 \rightarrow \mathcal{K}(XI) \xrightarrow{\iota^{\mathcal{K}}} \mathcal{K}(X) \xrightarrow{q_{\mathcal{K}}} \mathcal{K}(X/XI) \rightarrow 0$  *is exact.*

*Proof.*

(1) We have

$$\left\| \sum \Theta_{x_i, y_i}^X \right\| = \sup \left\{ \left\| \sum \Theta_{x_i, y_i}^X(x) \right\| : x \in X, \|x\| \leq 1 \right\},$$

and since  $\sum \Theta_{x_i, y_i}^X(x) = \sum x_i \cdot \langle y_i, x \rangle_A \in X \cdot I \subseteq XI$ , it follows that

$$\begin{aligned} & \left\| \sum \Theta_{x_i, y_i}^X \right\| \\ &= \sup \left\{ \left\| \left\langle \sum \Theta_{x_i, y_i}^X(x), y \right\rangle_A \right\| : x \in X, \|x\| \leq 1, y \in XI, \|y\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \left\langle x, \sum \Theta_{y_i, x_i}^X(y) \right\rangle_A \right\| : x \in X, \|x\| \leq 1, y \in XI, \|y\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \sum \Theta_{y_i, x_i}^X(y) \right\| : y \in XI, \|y\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \sum \Theta_{y_i, x_i}^{XI}(y) \right\| : y \in XI, \|y\| \leq 1 \right\} \\ &= \left\| \sum \Theta_{y_i, x_i}^{XI} \right\|. \end{aligned}$$

We deduce that  $\sum \Theta_{x_i, y_i}^{XI} \mapsto \sum \Theta_{x_i, y_i}^X$  is well-defined and extends to an isometric linear map  $\iota^{\mathcal{K}}$  on  $\mathcal{K}(XI)$ , and it is obvious that  $\iota^{\mathcal{K}}$  is a homomorphism. When  $T = \sum \Theta_{x_i, y_i}^{XI}$  as above, observe that  $\iota^{\mathcal{K}}(T)$  extends  $T$  and maps  $X$  into  $XI$ ; by continuity these properties hold for every  $T \in \mathcal{K}(XI)$ . Suppose  $S \in \mathcal{L}(X)$  also extends  $T$  and maps  $X$  into  $XI$ . Let  $(e_\lambda)$  be an approximate identity for  $I$ . For any  $x \in X$  we have  $Sx \in XI$ , so  $Sx = \lim(Sx) \cdot e_\lambda = \lim S(x \cdot e_\lambda) = \lim T(x \cdot e_\lambda)$ ; similarly,  $\iota^{\mathcal{K}}(T)x = \lim T(x \cdot e_\lambda)$ . Thus  $S = \iota^{\mathcal{K}}(T)$ .

- (2) Let  $A_0 := \overline{\text{span}}\{\langle x, y \rangle_A : x, y \in A\}$  and  $I_0 := I \cap A_0$ , and consider the imprimitivity bimodule  ${}_{\mathcal{K}(X)}X_{A_0}$ . Since  $XI_0 = XI$  (see Equation 2.1), the image  $\iota^{\mathcal{K}}(\mathcal{K}(XI))$  is the ideal in  $\mathcal{K}(X)$  which corresponds to  $I_0$  under the Rieffel correspondence, and we deduce that  $X/XI_0$  is a  $\mathcal{K}(X)/\iota^{\mathcal{K}}(\mathcal{K}(XI))$ - $A_0/I_0$  imprimitivity bimodule. Since  $\mathcal{K}((X/XI_0)_{A_0/I_0}) = \mathcal{K}((X/XI)_{A/I})$ , the desired map  $q_{\mathcal{K}}$  is simply the composition of the quotient map

$$\mathcal{K}(X) \rightarrow \mathcal{K}(X)/\iota^{\mathcal{K}}(\mathcal{K}(XI))$$

with the left action of  $\mathcal{K}(X)/\iota^{\mathcal{K}}(\mathcal{K}(XI))$  as compact operators on  $X/XI$ . Verifying (2.2) is routine: for every  $z \in X$ ,

$$\begin{aligned} q_{\mathcal{K}}(\Theta_{x,y}^X)(q^{XI}(z)) &= q^{XI}(\Theta_{x,y}^X(z)) = q^{XI}(x \cdot \langle y, z \rangle_A) \\ &= q^{XI}(x) \cdot q^I(\langle y, z \rangle_A) \\ &= q^{XI}(x) \cdot \langle q^{XI}(y), q^{XI}(z) \rangle_{A/I} \\ &= \Theta_{q^{XI}(x), q^{XI}(y)}^{X/XI}(q^{XI}(z)). \end{aligned}$$

- (3) Since  $X/XI$  is full as a left Hilbert  $\mathcal{K}(X)/\iota^{\mathcal{K}}(\mathcal{K}(XI))$ -module, the left action of  $\mathcal{K}(X)/\iota^{\mathcal{K}}(\mathcal{K}(XI))$  is injective, and hence  $\ker q_{\mathcal{K}} = \iota^{\mathcal{K}}(\mathcal{K}(XI))$ .  $\square$

**Lemma 2.7.** *Let  $I$  be an  $X$ -invariant ideal in  $A$ . Then*

$$(2.3) \quad \varphi(a) = \iota^{\mathcal{K}}(\varphi_I(a)) \quad \text{for } a \in J(XI),$$

and  $J(XI) = J(X) \cap I$ . Also,

$$(2.4) \quad \varphi_{A/I} \circ q^I(a) = q_{\mathcal{K}} \circ \varphi(a) \quad \text{for } a \in J(X),$$

and  $q^I(J(X)) \subseteq J(X/XI)$ . If  $\varphi(A) \subseteq \mathcal{K}(X)$ , then  $q^I(J(X)) = J(X/XI)$ .

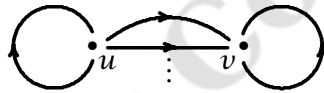
*Proof.* Suppose  $a \in J(XI)$ . Since  $a$  belongs to the  $X$ -invariant ideal  $I$ ,  $\varphi(a)$  maps  $X$  into  $XI$ ; since  $\varphi(a)$  extends the compact operator  $\varphi_I(a)$ , (2.3) follows from the uniqueness assertion of Lemma 2.6(1). By (2.3), we have  $J(XI) \subseteq$

$J(X) \cap I$ . For the reverse inclusion, suppose  $a \in J(X) \cap I$ . Then  $\varphi(a)$  belongs to  $\mathcal{K}(X)$ , the domain of  $q_{\mathcal{K}}$ ; moreover, since  $a$  belongs to the  $X$ -invariant ideal  $I$ , we have  $\varphi(a) \in \ker q_{\mathcal{K}}$ . By Lemma 2.6(3), this means that  $\varphi(a) = \iota^{\mathcal{K}}(T)$  for some  $T \in \mathcal{K}(XI)$ , and since  $\varphi(a)$  extends  $\varphi_I(a)$ , we have  $T = \varphi_I(a)$ . Thus  $a \in J(XI)$ .

Equation (2.4) follows immediately from the definition of  $q_{\mathcal{K}}$ , and shows that  $q^I(J(X)) \subseteq J(X/XI)$ ; the reverse inclusion is trivial if  $\varphi(A) \subseteq \mathcal{K}(X)$ .  $\square$

**Example 2.8.** As in Example 2.4, let  $X = X(E)$  be the Cuntz-Krieger bimodule of a directed graph  $E$ , let  $V$  be a hereditary subset of  $E^0$ , and identify  $X/XI_V$  with  $X(E \setminus V)$ . If  $u \in E^0 \setminus V$ , then  $\delta_u \in J(X/XI_V)$  if and only if  $u$  emits at most finitely many edges in  $(E^0 \setminus V, r^{-1}(E^0 \setminus V))$  [12, Proposition 4.4]. Since  $u$  may emit infinitely many edges in the original graph  $E$ ,  $\delta_u$  may not belong to the ideal  $J(X)$ ; hence the inclusion  $q^I(J(X)) \subseteq J(X/XI)$  of Lemma 2.7 can be proper when  $\varphi(A) \not\subseteq \mathcal{K}(X)$ .

For a specific example, consider the graph



with infinitely many edges from  $u$  to  $v$ . Then  $I := \mathbb{C}\delta_v$  is  $X$ -invariant, and since  $J(X) = I$ , the ideal  $q^I(J(X))$  vanishes. But the subgraph  $(E^0 \setminus V, r^{-1}(E^0 \setminus V))$  consists of just the one vertex  $u$  with a loop, and hence  $J(X/XI) = \mathbb{C}\delta_u$  is nontrivial. Observe that this example exhibits no pathology other than that  $\varphi(A) \not\subseteq \mathcal{K}(X)$ :  $X$  is full and essential as a left  $A$ -module and the left action is injective.

**Lemma 2.9.** *Let  $I$  be an  $X$ -invariant ideal in  $A$ , let  $K$  be an ideal in  $J(X)$ , and let  $B$  be a  $C^*$ -algebra.*

(1) *If  $(\psi, \pi)$  is a Toeplitz representation of  $X$  in  $B$ , then  $(\psi|_{XI}, \pi|_I)$  is a Toeplitz representation of  $XI$  such that*

$$(2.5) \quad (\pi|_I)^{(1)} = \pi^{(1)} \circ \iota^{\mathcal{K}}.$$

*If  $(\psi, \pi)$  is coisometric on  $K$ , then  $(\psi|_{XI}, \pi|_I)$  is coisometric on  $K \cap I$ .*

(2) *Suppose  $\psi_0: X/XI \rightarrow B$  is linear and  $\pi_0: A/I \rightarrow B$  is a homomorphism. Then  $(\psi_0, \pi_0)$  is a Toeplitz representation of  $X/XI$  if and only if  $(\psi_0 \circ q^{XI}, \pi_0 \circ q^I)$  is a Toeplitz representation of  $X$ , in which case*

$$(2.6) \quad (\pi_0 \circ q^I)^{(1)} = \pi_0^{(1)} \circ q_{\mathcal{K}}.$$

*Moreover,  $(\psi_0 \circ q^{XI}, \pi_0 \circ q^I)$  is coisometric on  $K$  if and only if  $(\psi_0, \pi_0)$  is coisometric on  $q^I(K)$ .*

*Proof.*

- (1) It is obvious that  $(\psi|_{XI}, \pi|_I)$  is a Toeplitz representation of  $XI$ . If  $x, y \in XI$ , then

$$\begin{aligned} (\pi|_I)^{(1)}(\Theta_{x,y}^{XI}) &= \psi|_{XI}(x)\psi|_{XI}(y)^* = \psi(x)\psi(y)^* \\ &= \pi^{(1)}(\Theta_{x,y}^X) = \pi^{(1)} \circ \iota^{\mathcal{K}}(\Theta_{x,y}^{XI}), \end{aligned}$$

and (2.5) follows by linearity and continuity. If  $(\psi, \pi)$  is coisometric on  $K$ , then for any  $a \in K \cap I$  we have from (2.5) and (2.3) that

$$(\pi|_I)^{(1)}(\varphi_I(a)) = \pi^{(1)} \circ \iota^{\mathcal{K}}(\varphi_I(a)) = \pi^{(1)}(\varphi(a)) = \pi(a) = \pi|_I(a),$$

so  $(\psi|_{XI}, \pi|_I)$  is coisometric on  $K \cap I$ .

- (2) It is routine to check that  $(\psi_0, \pi_0)$  is a Toeplitz representation of  $X/XI$  if and only if  $(\psi_0 \circ q^{XI}, \pi_0 \circ q^I)$  is a Toeplitz representation of  $X$ . For any  $x, y \in X$  we have

$$\begin{aligned} (2.7) \quad (\pi_0 \circ q^I)^{(1)}(\Theta_{x,y}^X) &= (\psi_0 \circ q^{XI}(x))(\psi_0 \circ q^{XI}(y))^* \\ &= \pi_0^{(1)}(\Theta_{q^{XI}(x), q^{XI}(y)}^{X/XI}) = \pi_0^{(1)} \circ q_{\mathcal{K}}(\Theta_{x,y}^X), \end{aligned}$$

and (2.6) follows by linearity and continuity. This and (2.4) give

$$(2.8) \quad (\pi_0 \circ q^I)^{(1)}(\varphi(k)) = \pi_0^{(1)} \circ q_{\mathcal{K}}(\varphi(k)) = \pi_0^{(1)}(\varphi_{A/I}(q^I(k))) \quad \text{for } k \in K.$$

If  $(\psi_0, \pi_0)$  is coisometric on  $q^I(K)$ , then the right-hand side equals  $\pi_0 \circ q^I(k)$ , and hence  $(\psi_0 \circ q^{XI}, \pi_0 \circ q^I)$  is coisometric on  $K$ . If  $(\psi_0 \circ q^{XI}, \pi_0 \circ q^I)$  is coisometric on  $K$ , then the left-hand side equals  $\pi_0(q^I(k))$ , and therefore  $(\psi_0, \pi_0)$  is coisometric on  $q^I(K)$ .  $\square$

### 3. A STRUCTURE THEOREM

We can now state our main theorem.

**Theorem 3.1.** *Suppose  $X$  is a Hilbert bimodule over  $A$ ,  $K$  is an ideal in  $J(X)$ , and  $I$  is an  $X$ -invariant ideal in  $A$ . Then the ideal  $\mathfrak{I}(I)$  in  $\mathcal{O}(K, X)$  generated by  $k_A(I)$  is Morita equivalent to  $\mathcal{O}(K \cap I, XI)$ , and the quotient  $\mathcal{O}(K, X)/\mathfrak{I}(I)$  is canonically isomorphic to  $\mathcal{O}(q^I(K), X/XI)$ .*

In particular, taking  $K = \{0\}$  and  $K = J(X)$  gives us results about the Toeplitz algebra and the Cuntz-Pimsner algebra.

**Corollary 3.2.** *If  $I$  is an  $X$ -invariant ideal in  $A$ , then the ideal  $\mathfrak{I}(I)$  of  $\mathcal{T}_X$  generated by  $k_X(I)$  is Morita equivalent to  $\mathcal{T}_{XI}$ , and the quotient  $\mathcal{T}_X/\mathfrak{I}(I)$  is canonically isomorphic to  $\mathcal{T}_{X/XI}$ .*

**Corollary 3.3.** *If  $I$  is an  $X$ -invariant ideal in  $A$ , then the ideal  $\mathcal{I}(I)$  of  $\mathcal{O}_X$  generated by  $k_X(I)$  is Morita equivalent to  $\mathcal{O}_{XI}$ , and the quotient  $\mathcal{O}_X/\mathcal{I}(I)$  is canonically isomorphic to the relative Cuntz-Pimsner algebra  $\mathcal{O}(q^1(J(X)), X/XI)$ . If  $\varphi(A) \subseteq \mathcal{K}(X)$ , then  $\mathcal{O}_X/\mathcal{I}(I) \cong \mathcal{O}_{X/XI}$ .*

*Proof.* The last statement follows from Lemma 2.7.  $\square$

In the proof of Theorem 3.1, the main problem is to show that  $\mathcal{O}(K \cap I, XI)$  embeds in  $\mathcal{O}(K, X)$ . To prove this, we show how to extend a representation of  $\mathcal{O}(K \cap I, XI)$  to a representation of  $\mathcal{O}(K, X)$ .

**Proposition 3.4.** *Let  $X$  be a Hilbert bimodule over  $A$ , let  $I$  be an  $X$ -invariant ideal in  $A$ , and let  $K$  be an ideal in  $J(X)$ . If  $(\psi, \pi)$  is a Toeplitz representation of  $XI$  on  $\mathcal{H}$  which is coisometric on  $(K \cap I)$ , then there are a Hilbert space  $\mathcal{M}$ , an isometry  $U : \mathcal{H} \rightarrow \mathcal{M}$ , and a Toeplitz representation  $(\bar{\psi}, \bar{\pi})$  of  $X$  on  $\mathcal{M}$  which is coisometric on  $K$  and satisfies*

$$(3.1) \quad \begin{aligned} \bar{\psi}|_{XI} &= \text{Ad } U \circ \psi, \\ \bar{\pi}|_I &= \text{Ad } U \circ \pi. \end{aligned}$$

Of particular interest are the Cuntz-Pimsner covariant representations.

**Corollary 3.5.** *If  $I$  is an  $X$ -invariant ideal in  $A$ , then every Cuntz-Pimsner covariant representation of  $XI$  can be extended to a Cuntz-Pimsner covariant representation of  $X$ .*

*Proof.* Immediate from Lemma 2.7 since  $J(XI) = J(X) \cap I$ .  $\square$

Before proving Proposition 3.4, we establish some useful notation and identify a collection of monomials which span  $\mathcal{O}(K, X)$ . Recall that for each  $n \geq 1$ , the  $n$ -fold tensor product  $X^{\otimes n} := X \otimes_A \cdots \otimes_A X$  is also a Hilbert bimodule over  $A$ ; see [25, Section 2.2] for details.

**Lemma 3.6.**

- (1) *Suppose  $(\psi, \pi)$  is a Toeplitz representation of  $X$  in a  $C^*$ -algebra  $B$ . For each  $n \geq 1$ , there is a linear map  $\psi^{\otimes n} : X^{\otimes n} \rightarrow B$  which satisfies*

$$\psi^{\otimes n}(x_1 \otimes_A \cdots \otimes_A x_n) = \psi(x_1) \cdots \psi(x_n) \quad \text{for all } x_1, \dots, x_n \in X,$$

*and  $(\psi^{\otimes n}, \pi)$  is a Toeplitz representation of  $X^{\otimes n}$ . By convention, we take  $X^{\otimes 0} := A$  and  $\psi^{\otimes 0} = \pi$ .*

- (2) *Let  $K$  be an ideal in  $J(X)$ . If  $(k_X, k_A)$  is universal for Toeplitz representations of  $X$  which are coisometric on  $K$ , then*

$$\mathcal{O}(K, X) = \overline{\text{span}}\{k_X^{\otimes r}(x)k_A(a)k_X^{\otimes s}(y)^* : r, s \geq 0, x \in X^{\otimes r}, y \in X^{\otimes s}, a \in A\}.$$

*Proof.* (1) is a special case of [12, Proposition 1.8(1)], and (2) is proved exactly as in [12, Lemma 2.4].  $\square$

**Remark 3.7.** When  $m \geq 1$ , we have natural isomorphisms  $X^{\otimes m} \otimes_A X^{\otimes n} \cong X^{\otimes(m+n)}$  which carry  $\psi^{\otimes m} \otimes_A \psi^{\otimes n}$  into  $\psi^{\otimes(m+n)}$ . When  $m = 0$  we have to be a little careful because  $A \otimes_A X$  is isomorphic to the essential subspace  $A \cdot X$  rather than  $X$ .

*Proof of Proposition 3.4.* We will show that  $\mathcal{T}_{XI}$  embeds naturally as a hereditary subalgebra of  $\mathcal{T}_X$ , and obtain  $\bar{\psi} \times \bar{\pi}$  by inducing  $\psi \times \pi$  from  $\mathcal{T}_{XI}$  to  $\mathcal{T}_X$ . Let  $(\mathcal{T}_X, i_X, i_A)$  be universal for Toeplitz representations of  $X$ . We claim that

$$\iota := i_X|_{XI} \times i_A|_I : \mathcal{T}_{XI} \rightarrow \mathcal{T}_X$$

is injective. For this, let  $(\omega, \rho)$  be a Toeplitz representation of  $X$  on a Hilbert space  $\mathcal{H}$  such that  $\rho$  acts faithfully on  $(\omega(X)\mathcal{H})^\perp$ ; for the existence of such a representation, see [12, Corollary 2.2]. Then  $\rho|_I$  acts faithfully on the larger space  $(\omega(XI)\mathcal{H})^\perp$ , so by [12, Theorem 2.1],  $\omega|_{XI} \times \rho|_I : \mathcal{T}_{XI} \rightarrow B(\mathcal{H})$  is faithful. But  $\omega|_{XI} \times \rho|_I = (\omega \times \rho) \circ \iota$ , so  $\iota$  is injective.

Let  $B := \iota(\mathcal{T}_{XI})$ , and let

$$L := \overline{\text{span}}\{i_X^{\otimes r}(x)i_A(i)i_X^{\otimes s}(y)^* : r, s \geq 0, x \in X^{\otimes r}, y \in (XI)^{\otimes s}, i \in I\}.$$

We claim that  $L$  is a closed left ideal in  $\mathcal{T}_X$  such that  $L^*L = B$  and  $BL \subseteq B$ . Since  $\mathcal{T}_X$  is generated by  $i_X(X) \cup i_A(A)$ , to see that  $L$  is a left ideal it suffices to check that the generating monomials of  $L$  are invariant under left multiplication by  $i_X(X)$ ,  $i_A(A)$ , and  $i_X(X)^*$ . Most cases are routine since  $(i_X, i_A)$  is a Toeplitz representation and  $I$  is an ideal; the only subtle case is when a monomial of the form  $i_A(i)i_X^{\otimes s}(y)^*$  is multiplied by  $i_X(z)^*$  for some  $z \in X$ :

$$i_X(z)^*i_A(i)i_X^{\otimes s}(y)^* = i_X(\varphi(i^*)z)^*i_X^{\otimes s}(y)^* = i_X^{\otimes(s+1)}(y \otimes \varphi(i^*)z)^*$$

which belongs to  $i_X^{\otimes(s+1)}((XI)^{\otimes(s+1)})^*$  because  $I$  is  $X$ -invariant, and hence to  $L$  because  $i_X(XI)^* = i_A(I)i_X(XI)^*$ . Thus  $L$  is a left ideal.

Next we prove that  $L^*L = B$ . By Lemma 3.6(2),  $B$  is spanned by monomials of the form  $i_X^{\otimes r}(x)i_A(i)i_X^{\otimes s}(y)^*$  with  $r, s \geq 0$ ,  $x \in (XI)^{\otimes r}$ ,  $y \in (XI)^{\otimes s}$  and  $i \in I$ , from which it is obvious that  $B \subseteq L$ ; hence  $B = B^*B \subseteq L^*L$ . For the reverse inclusion, it suffices to check that  $\ell_1^*\ell_2 \in B$  whenever  $\ell_1$  and  $\ell_2$  are generating monomials of  $L$ , which reduces to checking that

$$i_A(j)i_X^{\otimes t}(w)^*i_X^{\otimes r}(x)i_A(i) \in B$$

whenever  $r, t \geq 0$ ,  $x \in X^{\otimes r}$ ,  $w \in X^{\otimes t}$ , and  $i, j \in I$ . This is trivial if  $r = t = 0$ , and the relation  $i_X(x)^*i_X(y) = i_A(\langle x, y \rangle_A)$  allows us to assume that either  $r = 0$  or  $t = 0$ . Without loss of generality  $t = 0$ , and then

$$i_A(j)i_X^{\otimes t}(w)^*i_X^{\otimes r}(x)i_A(i) = i_A(j)i_X^{\otimes r}(x \cdot i).$$

An elementary argument using the  $X$ -invariance of  $I$  and the equality  $I = I^2$  allows us to deduce that  $i_A(j)i_X^{\otimes r}(x \cdot i) \subseteq i_X^{\otimes r}((XI)^{\otimes r}) \subseteq B$ , completing the proof that  $L^*L = B$ .

Since a typical generating monomial  $i_X^{\otimes r}(x)i_A(i)i_X^{\otimes s}(y)^* \in L$  belongs to  $B$  if  $r = 0$ , to prove that  $BL \subseteq B$  it suffices to show that  $Bi_X(X) \subseteq B$ . Since  $I$  is  $X$ -invariant we have

$$\begin{aligned} i_A(I)i_X(X) &= i_X(\varphi(I)X) \subseteq i_X(XI) \subseteq B; \\ i_X(XI)i_X(X) &= i_X(XI)i_A(I)i_X(X) \\ &\subseteq i_X(XI)i_X(\varphi(I)X) \subseteq i_X(XI)^2 \subseteq B; \\ i_X(XI)^*i_X(X) &= i_A(\langle XI, X \rangle_A) \subseteq i_A(I) \subseteq B. \end{aligned}$$

Since  $B$  is generated as an algebra by  $i_A(I) \cup i_X(XI) \cup i_X(XI)^*$ , we deduce that  $BL \subseteq B$ .

Since  $L$  is a closed left ideal in  $\mathcal{T}_X$  such that  $L^*L = B$ ,  $L$  is a right-Hilbert  $\mathcal{T}_X$ - $B$  bimodule. Inducing the representation  $\sigma := (\psi \times \pi) \circ \iota^{-1}: B \rightarrow B(\mathcal{H})$  gives a representation  $\text{Ind } \sigma$  of  $\mathcal{T}_X$  on  $\mathcal{M} := L \otimes_B \mathcal{H}$ , and we let  $(\bar{\psi}, \bar{\pi})$  be the Toeplitz representation of  $X$  defined by

$$\bar{\psi} := (\text{Ind } \sigma) \circ i_X \quad \text{and} \quad \bar{\pi} := (\text{Ind } \sigma) \circ i_A.$$

There is no harm in assuming that  $(\psi, \pi)$  (and hence  $\sigma$ ) is nondegenerate, in which case there is a unique isometry  $U: \mathcal{H} \rightarrow \mathcal{M}$  which satisfies

$$U\sigma(b)h = b \otimes_B h \quad \text{for } b \in B \text{ and } h \in \mathcal{H}.$$

Since  $\bar{\psi}|_{XI} \times \bar{\pi}|_I = ((\text{Ind } \sigma)|_B) \circ \iota$ , the essential subspace of  $\bar{\psi}|_{XI} \times \bar{\pi}|_I$  is

$$\overline{(\text{Ind } \sigma)(B)(L \otimes_B \mathcal{H})} = \overline{BL \otimes_B \mathcal{H}} \subseteq B \otimes_B \mathcal{H} = U\mathcal{H}.$$

Hence to verify (3.1) it suffices to prove that

$$(3.2) \quad \bar{\psi}(x)U = U\psi(x) \text{ for } x \in XI, \quad \text{and} \quad \bar{\pi}(i)U = U\pi(i) \text{ for } i \in I.$$

Suppose  $x \in XI$ . Then for  $b \in B$  and  $h \in \mathcal{H}$  we have

$$\begin{aligned} \bar{\psi}(x)U\sigma(b)h &= \text{Ind } \sigma(i_X(x))(b \otimes_B h) = i_X(x)b \otimes_B h \\ &= \iota(i_{XI}(x)) \otimes_B \sigma(b)h \\ &= U\psi \times \pi(i_{XI}(x))\sigma(b)h = U\psi(x)\sigma(b)h, \end{aligned}$$

and since  $\sigma$  is nondegenerate this gives the first equation in (3.2); a similar calculation gives the second.



It remains to show that  $(\bar{\psi}, \bar{\pi})$  is coisometric on  $K$ , for which we use Lemma 1.9; that is, we will show that

$$\overline{\bar{\pi}(K)\mathcal{M}} \subseteq \overline{\bar{\psi}(X)\mathcal{M}} := \overline{\text{span}\{\bar{\psi}(x)m : x \in X, m \in \mathcal{M}\}}.$$

For this, it suffices to show that

$$(3.3) \quad \bar{\pi}(k)(\ell \otimes_B h) \in \overline{\bar{\psi}(X)\mathcal{M}} \quad \text{for all } k \in K, \ell \in L, \text{ and } h \in \mathcal{H}.$$

We consider two cases.

CASE 1:  $\ell = i_X(x)\ell'$  with  $x \in X$  and  $\ell' \in L$ . Then

$$\begin{aligned} \bar{\pi}(k)(\ell \otimes_B h) &= i_A(k)i_X(x)\ell' \otimes_B h = i_X(\varphi(k)x)\ell' \otimes_B h \\ &= \bar{\psi}(\varphi(k)x)(\ell' \otimes_B h) \subseteq \overline{\bar{\psi}(X)\mathcal{M}}. \end{aligned}$$

CASE 2:  $\ell = i_A(i)b$  with  $i \in I$  and  $b \in B$ . Since  $ki \in KI = K \cap I$ , we can express  $ki = cd$  with  $c, d \in K \cap I$ , and then

$$\begin{aligned} \bar{\pi}(k)(\ell \otimes_B h) &= i_A(k)i_A(i)b \otimes_B h \\ &= i_A(c)i_A(d) \otimes_B \sigma(b)h \\ &= i_A(c)\iota(i_I(d)) \otimes_B \sigma(b)h \\ &= i_A(c) \otimes_B \psi \times \pi(i_I(d))\sigma(b)h \\ &= i_A(c) \otimes_B \pi(d)\sigma(b)h \\ &\in i_A(I) \otimes_B \overline{\bar{\psi}(XI)\mathcal{H}} \quad (\text{by Lemma 1.9}) \\ &= i_A(I) \otimes_B \overline{\psi \times \pi(i_{XI}(XI))\mathcal{H}} \\ &\subseteq \overline{i_A(I)i_X(XI) \otimes_B \mathcal{H}} \\ &\subseteq \overline{i_X(XI)i_A(I) \otimes_B \mathcal{H}} \\ &\subseteq \overline{i_X(X)L \otimes_B \mathcal{H}} \\ &= \overline{\bar{\psi}(X)\mathcal{M}}. \end{aligned}$$

Since the vectors  $\ell$  considered in these two cases span a dense subspace of  $L$ , this gives (3.3), and hence  $(\bar{\psi}, \bar{\pi})$  is coisometric on  $K$ .  $\square$

**Corollary 3.8.** *If  $I$  is an  $X$ -invariant ideal in  $A$  and  $K$  is an ideal in  $J(X)$ , then  $\mathcal{O}(K \cap I, XI)$  embeds canonically in  $\mathcal{O}(K, X)$ .*

*Proof.* Let  $(\mathcal{O}(K, X), k_X, k_A)$  be universal for Toeplitz representations of  $X$  which are coisometric on  $K$ . By Lemma 2.9,  $(k_X|_{XI}, k_A|_I)$  is a Toeplitz representation of  $XI$  which is coisometric on  $K \cap I$ , and hence induces a homomorphism

$$k_X|_{XI} \times_{K \cap I} k_A|_I : \mathcal{O}(K \cap I, XI) \rightarrow \mathcal{O}(K, X).$$

Let  $(\psi, \pi)$  be a  $(K \cap I)$ -coisometric Toeplitz representation of  $XI$  on  $\mathcal{H}$  such that  $\psi \times_{K \cap I} \pi : \mathcal{O}(K \cap I, XI) \rightarrow B(\mathcal{H})$  is faithful, and let  $(\bar{\psi}, \bar{\pi})$  be its extension to a Toeplitz representation of  $X$  which is coisometric on  $K$ , as in Proposition 3.4. Since

$$(\bar{\psi} \times_K \bar{\pi}) \circ (k_X|_{XI} \times_{K \cap I} k_A|_I) = (\text{Ad } U) \circ (\psi \times_{K \cap I} \pi)$$

is faithful,  $k_X|_{XI} \times_{K \cap I} k_A|_I$  is injective.  $\square$

**Corollary 3.9.** *If  $I$  is an  $X$ -invariant ideal in  $A$ , then  $\mathcal{O}_{XI}$  embeds canonically in  $\mathcal{O}_X$ .*

*Proof.* By Lemma 2.7,  $\mathcal{O}(J(X) \cap I, XI) = \mathcal{O}(J(XI), XI) = \mathcal{O}_{XI}$ .  $\square$

*Proof of Theorem 3.1.* Let  $(\mathcal{O}(K, X), k_X, k_A)$  be the relative Cuntz-Pimsner algebra determined by  $K$ , and let

$$L := \overline{\text{span}}\{k_X^{\otimes r}(x)k_A(i)k_X^{\otimes s}(y)^* : r, s \geq 0, x \in X^{\otimes r}, y \in XI^{\otimes s}, i \in I\}.$$

From the proof of Proposition 3.4, we deduce that  $L$  is a closed left ideal in  $\mathcal{O}(K, X)$  such that  $L^*L$  is the canonical embedded image of  $\mathcal{O}(K \cap I, XI)$  in  $\mathcal{O}(K, X)$  (see Corollary 3.8). Moreover,  $LL^*$  is a two-sided ideal which contains  $k_A(I)$ , and since  $L \subseteq \mathcal{I}(I)$ , this implies that  $LL^* = \mathcal{I}(I)$ . Thus  $L$  implements a Morita equivalence between  $\mathcal{I}(I)$  and  $\mathcal{O}(K \cap I, XI)$ .

Let  $\psi := k_{X/XI} \circ q^{XI}$  and  $\pi := k_{A/I} \circ q^I$ . By Lemma 2.9,  $(\psi, \pi)$  is a Toeplitz representation of  $X$  which is coisometric on  $K$ . The induced homomorphism  $\psi \times_K \pi : \mathcal{O}(K, X) \rightarrow \mathcal{O}(q^I(K), X/XI)$  annihilates the ideal  $\mathcal{I}(I)$  since

$$\psi \times_K \pi(k_A(I)) = \pi(I) = k_{A/I} \circ q^I(I) = \{0\},$$

and hence induces a map  $(\psi \times_K \pi)_* : \mathcal{O}(K, X)/\mathcal{I}(I) \rightarrow \mathcal{O}(q^I(K), X/XI)$ . To see that  $(\psi \times_K \pi)_*$  is an isomorphism, we construct its inverse.

Since  $k_X(XI) = k_X(X)k_A(I) \subseteq \mathcal{I}(I)$ , there is a linear map  $\psi_0 : X/XI \rightarrow \mathcal{O}(K, X)/\mathcal{I}(I)$  such that  $\psi_0 \circ q^{XI} = q \circ k_X$ . Similarly, since  $I \subseteq \ker(q \circ k_A)$ , there is a homomorphism  $\pi_0 : A/I \rightarrow \mathcal{O}(K, X)/\mathcal{I}(I)$  such that  $\pi_0 \circ q^I = q \circ k_A$ . Certainly  $(q \circ k_X, q \circ k_A)$  is a Toeplitz representation of  $X$ , and it is coisometric on  $K$  because (1.2) gives

$$(q \circ k_A)^{(1)}(\varphi(k)) = q \circ k_A^{(1)}(\varphi(k)) = q \circ k_A(k) \quad \text{for all } k \in K.$$

By Lemma 2.9(2), we deduce that  $(\psi_0, \pi_0)$  is a Toeplitz representation of  $X/XI$  which is coisometric on  $q^I(K)$ . The induced homomorphism

$$\psi_0 \times_{q^I(K)} \pi_0 : \mathcal{O}(q^I(K), X/XI) \rightarrow \mathcal{O}(K, X)/\mathcal{I}(I)$$

is surjective since it maps the generating set  $k_{X/XI}(X/XI) \cup k_{A/I}(A/I)$  onto the generating set  $q(k_X(X)) \cup q(k_A(A))$ . By checking on generators, it is routine to

check that  $(\psi \times_K \pi)_* \circ (\psi_0 \times_{q^l(K)} \pi_0)$  is the identity on  $\mathcal{O}(q^l(K), X/XI)$ , so that  $(\psi \times_K \pi)_*$  and  $\psi_0 \times_{q^l(K)} \pi_0$  are isomorphisms. For example,

$$\begin{aligned} (\psi \times_K \pi)_* \circ (\psi_0 \times_{q^l(K)} \pi_0) \circ k_{X/XI} \circ q^{XI} &= (\psi \times_K \pi)_* \circ \psi_0 \circ q^{XI} \\ &= (\psi \times_K \pi)_* \circ q \circ k_X \\ &= (\psi \times_K \pi) \circ k_X \\ &= \psi \\ &= k_{X/XI} \circ q^{XI}, \end{aligned}$$

so  $(\psi \times_K \pi)_* \circ (\psi_0 \times_{q^l(K)} \pi_0)$  is the identity on the range of  $k_{X/XI}$ . We can see in similar fashion that it is the identity on the range of  $k_{A/I}$ , and hence on all of  $\mathcal{O}(q^l(K), X/XI)$ .  $\square$

**Example 3.10.** Suppose that  $X = X(E)$  is the bimodule of a graph  $E$ ,  $V$  is a hereditary subset of  $E^0$ , and  $I_V$  is the  $X$ -invariant ideal discussed in Example 2.4. Let  $X(V)$  be the bimodule of the graph  $V := (V, s^{-1}(V))$ . We claim that there is a natural isomorphism of  $\mathcal{T}_{X(V)}$  onto a full corner in  $\mathcal{T}_{XI_V}$ . To justify this claim, consider the universal Toeplitz representation  $(i_{XI_V}, i_{I_V})$  in  $\mathcal{T}_{XI_V}$ . Then the elements  $\{i_{XI_V}(\delta_e) : s(e) \in V\}$  and  $\{i_{I_V}(\delta_v) : v \in V\}$  form a Cuntz-Krieger  $V$ -family in  $\mathcal{T}_{XI_V}$ , and hence there is a homomorphism  $\mu$  of  $\mathcal{T}_{X(V)}$  into  $\mathcal{T}_{XI_V}$ . To see that  $\mu$  is injective, we show that every representation of  $\mathcal{T}_{X(V)}$  factors through  $\mu$ . So let  $\{S_e, P_v\}$  be a Toeplitz-Cuntz-Krieger  $V$ -family on  $\mathcal{H}$  with corresponding representation  $\pi_{S,P} : \mathcal{T}_{X(V)} \rightarrow B(\mathcal{H})$ . For each  $e \in E^1$  with  $r(e) \in V$  but  $s(e) \notin V$ , we choose a unitary isomorphism  $S_e$  of  $P_{r(e)}\mathcal{H}$  onto a new Hilbert space  $\mathcal{H}_e$ . As in [12, Example 1.2], there is a Toeplitz representation  $(\psi, \pi)$  of  $XI_V$  on  $\mathcal{H} \oplus (\bigoplus_{e \in r^{-1}(V) \setminus s^{-1}(V)} \mathcal{H}_e)$  such that  $\psi(\delta_e) = S_e$  for all  $e \in r^{-1}(V)$  and  $\pi(\delta_v) = P_v$  for all  $v \in V$ , and then  $\pi_{S,P}$  is the restriction of  $(\psi \times \pi) \circ \mu$  to the invariant subspace  $\mathcal{H}$ . Thus  $\mu$  is injective. The image of  $\mu$  is the corner associated to the projection  $p := \sum_{v \in V} i_{I_V}(\delta_v)$  in  $M(\mathcal{T}_{XI_V})$ , which is full because all the generators  $b$  for  $\mathcal{T}_{XI_V}$  satisfy  $b = bp$  and hence lie in the ideal generated by  $p$ . We have now proved the claim.

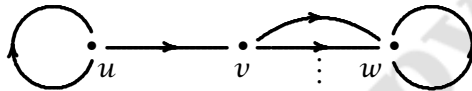
As we saw in Example 1.5, the ideals  $K$  in  $J(X)$  have the form  $K_F$  for some set  $F$  of vertices which emit finitely many edges, and  $K \cap I_V = K_{F \cap V}$ . Since passing to the relative Cuntz-Pimsner algebra  $\mathcal{O}(K_{F \cap V}, X(V))$  involves imposing relations at the vertices in  $F \cap V$ , the embedding of  $\mathcal{T}_{X(V)}$  in  $\mathcal{T}_{XI_V}$  induces an isomorphism of  $\mathcal{O}(K_{F \cap V}, X(V))$  onto a full corner in  $\mathcal{O}(K_{F \cap V}, XI_V)$ . Thus Theorem 3.1 implies that  $\mathcal{O}(K_F, X(E))$  contains an ideal Morita equivalent to  $\mathcal{O}(K_{F \cap V}, X(V))$  with quotient isomorphic to  $\mathcal{O}(K_{F \setminus V}, X(E \setminus V))$ . This generalises results for the graph algebras of row-finite graphs in [4, Theorem 4.1, parts (2) and (3)].

**Remark 3.11.** When  $E$  is a row-finite graph there is a bijection between the saturated hereditary subsets of  $E^0$  and the gauge invariant ideals in  $C^*(E)$  [4, Theorem 4.1, part (1)]. Since the ideals  $\mathcal{I}(I)$  in  $\mathcal{O}_X$  associated to  $X$ -invariant

ideals are certainly gauge-invariant, it is natural to ask for a concept of  $X$ -saturation under which one can prove an analogue of [4, Theorem 4.1, part (1)]. One such concept is used in [16] in the context of finitely generated Hilbert modules: they ask that

$$(3.4) \quad a \in A \text{ and } \varphi(a)X \subset XI \Rightarrow a \in I.$$

If  $I_V$  is the ideal in  $X(E)$  corresponding to a hereditary subset  $V$  in a row-finite graph, then  $I_V$  has this property precisely when  $V$  is saturated in the sense of [4]. For non-row-finite graphs this concept is inappropriate: for example, consider the graph



with infinitely many edges from  $v$  to  $w$ . Here  $I_w$  is  $X(E)$ -invariant and does not satisfy (3.4), but generates a gauge-invariant ideal which is strictly smaller than the ideal  $I_{v,w}$  which does satisfy (3.4). This example and the analyses of ideals in graph algebras in [10] and [5] suggest that this question could be quite subtle.

**Example 3.12.** Let  $X = \alpha(1)A$  be the bimodule associated to an endomorphism  $\alpha$  of  $A$ , as in Example 1.6. If  $I$  is an ideal in  $A$ , then  $\alpha(I)X = \alpha(I)A$ ,  $XI = \alpha(1)I$ , and hence  $I$  is  $X$ -invariant precisely when  $\alpha(I) \subset I$ . To get from an invariant ideal to an ideal in the crossed product, we need to know that the ideal is *extendibly invariant*: the endomorphism  $\alpha|_I$  extends to an endomorphism  $\tilde{\alpha}$  of  $M(I)$  in such a way that  $\tilde{\alpha}(1_{M(I)})$  coincides with the canonical image of  $\alpha(1) \in A$  in  $M(I)$  (see [1, 22]). If so,  $XI$  is the bimodule  $\tilde{\alpha}(1_{M(I)})I$  associated to  $\alpha|_I$ , and  $\mathcal{O}_{XI} \cong I \times_{\alpha} \mathbb{N}$ . The quotient map  $q^I$  carries  $X/XI$  onto the module  $\alpha^{A/I}(1)(A/I)$  associated to the induced endomorphism  $\alpha^{A/I}$  of  $A/I$ , and  $\mathcal{O}_X/\mathcal{O}_{XI} \cong (A/I) \times_{\alpha^{A/I}} \mathbb{N}$ , as predicted by the results in [1] and [22]. When the invariant ideal is not extendibly invariant, quite different things can happen, as the next example shows.

**Example 3.13.** For an extreme example of a non-extendibly invariant ideal, we consider  $A := c_0(\mathbb{Z} \cup \{\infty\})$ , the forward shift  $\tau$  on  $A$ , and the ideal  $I \cong c$  of sequences whose negative terms are all zero. The bimodule associated to  $\alpha$  is  $X = A_A$ , with  $a \cdot x = \alpha(a)x$  as usual, and  $XI$  is the Hilbert bimodule  $I_I$  in which the left action is given by  $\tau|_I$ . This is not quite the usual bimodule for the one-sided shift—indeed, if  $(\psi, \pi)$  is a Toeplitz representation and  $\pi$  is nondegenerate, then  $\pi$  has to vanish on  $c_0$ . (The element  $e_0 := 1_I - \tau(1_I)$  satisfies  $a \cdot e_0 = 0$  for all  $a \in I$ , so  $\pi(1_I)\psi(e_0) = \psi(\tau(1_I)e_0) = 0$ . If  $\pi$  is nondegenerate, then  $\psi(e_0) = 0$ ,  $\pi(e_0) = \pi(\langle e_0, e_0 \rangle_I) = 0$ , and  $\pi(\tau^n(e_0)) = V^n\pi(e_0)(V^*)^n = 0$  for all  $n \geq 0$ . In general,  $\psi(e_0)$  is a partial isometry with initial projection  $\pi(1_I - \tau(1_I))$  and range projection orthogonal to  $\pi(1_I)$ .) However, the Cuntz-Pimsner algebra  $\mathcal{O}_{XI}$

contains  $c \times_{\tau} \mathbb{N}$  as the full corner determined by  $k_{XI}(1_I)$ . Since  $c \times_{\tau} \mathbb{N}$  is a full corner in  $A \times_{\tau} \mathbb{Z}$ , it follows that the ideal  $\mathcal{I}(I)$  generated by  $\mathcal{O}_{XI}$  is all of  $\mathcal{O}_X$ .

Theorem 3.1 therefore implies that the Cuntz-Pimsner algebra  $\mathcal{O}_{X/XI}$  is trivial. To confirm this, note that  $X/XI$  is  $(A/I)_{A/I}$  with left multiplication given by the induced endomorphism  $\tau^{A/I}$  of  $A/I$ . Suppose  $(\psi, \pi)$  is a Cuntz-Pimsner covariant representation of  $X/XI$ , and  $V$  is the isometry  $\psi(\tau^{A/I}(1_{A/I}))^*$ . For  $b \in \tau^{-n}(I)$ ,  $(\tau^{A/I})^n(q^l(b)) = 0$ , and the covariance relation (1.5) implies that  $\pi(q^l(b)) = (V^*)^n \pi((\tau^{A/I})^n(q^l(b)))V^n = 0$ ; since  $\bigcup_n \tau^{-n}(I)$  is dense in  $A$ , it follows that  $\pi = 0$ . Thus  $X/XI$  has no nonzero covariant representations, and  $\mathcal{O}_{X/XI} = 0$ .

This example is not an anomalous one: the same analysis applies whenever  $(A, \alpha)$  is the minimal dilation of  $(I, \alpha|_I)$ , as in [20]. The moral is that extendible invariance is important.

We next identify the kernel of the quotient map  $\mathcal{O}(q^l(K), X/XI)$  onto  $\mathcal{O}_{X/XI}$ , which is isomorphic to the subquotient  $\ker\{(\mathcal{O}(K, X) \rightarrow \mathcal{O}_{X/XI})\}/\mathcal{I}(I)$  of Theorem 3.1.

**Proposition 3.14.** *Suppose  $X$  is full and  $L \triangleleft K \triangleleft J(X)$ . The kernel of the quotient map  $\Theta : \mathcal{O}(L, X) \rightarrow \mathcal{O}(K, X)$  is Morita equivalent to  $K/L$ .*

**Remark 3.15.** If  $X$  is not full, then  $\ker(\mathcal{O}(L, X) \rightarrow \mathcal{O}(K, X))$  is Morita equivalent to  $(K \cap A_0)/(L \cap A_0)$ , where  $A_0 = \overline{\text{span}}\{\langle x, y \rangle_A : x, y \in X\}$ .

*Proof of Proposition 3.14.* Let  $(T, \varphi_{\infty})$  be the Fock representation of  $X$ . Thus,  $(T, \varphi_{\infty})$  is the Toeplitz representation of  $X$  as adjointable operators on the Fock module  $F(X) := \bigoplus_{n=0}^{\infty} X^{\otimes n}$  in which  $T(x)$  tensors on the left by  $x$ , and  $\varphi_{\infty}(a)$  is the diagonal left action induced by  $\varphi(a)$ ; see [27, Section 1], [25, Section 2.2], and [12, Example 1.4 and Remark 1.5] for more details. Let  $\tau_K : \mathcal{T}_X \rightarrow \mathcal{O}(K, X)$  be the quotient map. We claim that

$$(3.5) \quad T \times \varphi_{\infty}(\ker \tau_K) = \iota(\mathcal{K}(F(X)K)),$$

where  $\iota$  is the canonical embedding of  $\mathcal{K}(F(X)K)$  in  $\mathcal{K}(F(X))$ ; see Lemma 2.6. Pimsner proves this in a bit less generality in [27, Theorem 3.13], and we essentially follow his proof. Let  $Q_0$  be the projection of  $F(X)$  onto  $A = X^{\otimes 0}$ . Since  $\ker \tau_K$  is the ideal generated by  $\{i_A(k) - i_A^{(1)}(\varphi(k)) : k \in K\}$ , and since

$$T \times \varphi_{\infty}(i_A(k) - i_A^{(1)}(\varphi(k))) = \varphi_{\infty}(k) - \varphi_{\infty}^{(1)}(\varphi(k)) = \varphi_{\infty}(k)Q_0$$

for every  $k \in K$ ,  $T \times \varphi_{\infty}(\ker \tau_K)$  is generated as an ideal by  $\{\varphi_{\infty}(k)Q_0 : k \in K\}$ . But  $T \times \varphi_{\infty}(\mathcal{T}_X)$  is spanned by monomials of the form  $T^{\otimes r}(x)\varphi_{\infty}(a)T^{\otimes s}(y)^*$ , and since  $Q_0T(x') = 0$  for  $x' \in X$ , we deduce that  $T \times \varphi_{\infty}(\ker \tau_K)$  is

$$\overline{\text{span}}\{T^{\otimes r}(x)\varphi_{\infty}(k_1)Q_0\varphi_{\infty}(k_2^*)T^{\otimes s}(y)^* : r, s \geq 0, x \in X^{\otimes r}, y \in X^{\otimes s}, a, b \in K\}.$$

Since  $T^{\otimes r}(x)\varphi_\infty(k_1)Q_0\varphi_\infty(k_2^*)T^{\otimes s}(y)^* = \Theta_{x \cdot k_1, y \cdot k_2}^{F(X)}$ , this gives (3.5). We have  $\Theta \circ \tau_L = \tau_K$ , and since these maps are all surjections we have  $\ker \Theta \cong \ker \tau_K / \ker \tau_L$ . Since  $T \times \varphi_\infty$  is injective by [27, Theorem 3.4] or [12, Corollary 2.2], we deduce from (3.5) that

$$(3.6) \quad \ker \Theta \cong \frac{\ker \tau_K}{\ker \tau_L} \cong \frac{\iota(\mathcal{K}(F(X)K))}{\iota(\mathcal{K}(F(X)L))} \cong \frac{\mathcal{K}(F(X)K)}{\iota_0(\mathcal{K}(F(X)L))},$$

where  $\iota_0$  is the canonical embedding of  $\mathcal{K}(F(X)L)$  in  $\mathcal{K}(F(X)K)$ . Since  $X$  is full, so is  $F(X)$ , and hence  $F(X)K$  is full as a Hilbert  $K$ -module. The ideal in  $\mathcal{K}(F(X)K)$  which corresponds to  $L$  under the Rieffel correspondence is  $\iota_0(\mathcal{K}(F(X)L))$ , and hence  $F(X)K/F(X)L$  is a  $\mathcal{K}(F(X)K)/\iota_0(\mathcal{K}(F(X)L))$ - $K/L$  imprimitivity bimodule. The result now follows immediately from (3.6).  $\square$

#### 4. A GAUGE-INVARIANT UNIQUENESS THEOREM

Recall that we denote by  $\gamma$  the gauge action of  $\mathbb{T}$  on  $\mathcal{O}_X$ .

**Theorem 4.1.** *Suppose that  $X$  is a Hilbert bimodule over  $A$ . Suppose that  $(\psi, \pi)$  is a Cuntz-Pimsner covariant representation of  $X$  in a  $C^*$ -algebra  $B$  such that  $\pi$  is faithful, and that there is a strongly continuous action  $\beta : \mathbb{T} \rightarrow \text{Aut}(\psi \times \pi(\mathcal{O}_X))$  such that  $\beta_z \circ (\psi \times \pi) = (\psi \times \pi) \circ \gamma_z$  for  $z \in \mathbb{T}$ . Then the homomorphism  $\psi \times \pi$  of  $\mathcal{O}_X$  into  $B$  is injective.*

This result was motivated by the gauge-invariant uniqueness theorems for graph algebras in [15] and [4]. The important point here is that the theorem requires no structural hypothesis on the bimodule, as opposed to the uniqueness theorems of Cuntz-Krieger type which have hypotheses on the graph or matrix but no hypothesis on the gauge action. (Though notice that the existence of such a covariant representation forces  $\varphi : A \rightarrow \mathcal{L}(X)$  to be injective.) This theorem also generalises Theorem 3.3 of [9] (where we think the authors may have intended to impose a left-annihilator condition—see Remark 6.12). We can also deduce from this result that Pimsner’s concretely defined algebra is isomorphic to ours.

Our basic strategy is a familiar one: we show that the representation  $\psi \times \pi$  is faithful on the fixed-point algebra  $\mathcal{O}_X^\gamma$  for the gauge action, and then extend this to  $\mathcal{O}_X$  by averaging over the action  $\beta$ . Carrying out the first step requires a careful analysis of the fixed-point algebra  $\mathcal{O}_X^\gamma$ . Throughout we write  $(i_X, i_A)$  for the universal Cuntz-Pimsner covariant representation of  $X$  in  $\mathcal{O}_X$ .

In the following Lemma we reorganize some results from [27, Section 3].

**Lemma 4.2.** *Suppose  $X$  and  $Y$  are right Hilbert  $A$ -modules and  $\varphi : A \rightarrow \mathcal{L}(X)$  is injective. Then:*

- (1)  $\theta : S \mapsto S \otimes_A 1$  is an isometric homomorphism of  $\mathcal{L}(Y)$  into  $\mathcal{L}(Y \otimes_A X)$ ;
- (2) if  $S \otimes_A 1 \in \mathcal{K}(Y \otimes_A X)$ , then  $S$  is compact, and  $\text{ran } S \subseteq YJ(X)$ .

*Proof.*

- (1) Let  $\pi$  be a faithful representation of  $A$  on Hilbert space. Then  $X$ -Ind  $\pi$  is faithful because  $\varphi$  is, and the representation  $(Y \otimes_A X)$ -Ind  $\pi$  of  $\mathcal{L}(Y)$ , which is unitarily equivalent to  $(Y$ -Ind( $X$ -Ind  $\pi$ ))  $\circ \theta$ , is also faithful. Thus  $\theta$  is injective, hence isometric.
- (2) Let  $(u_\lambda)$  be an approximate identity for  $\mathcal{K}(Y)$ . Since  $\mathcal{K}(Y)$  acts nondegenerately on  $Y$ ,  $u_\lambda$  converges to 1 in the strong operator topology on  $\mathcal{L}(Y)$ . Calculations like

$$(u_\lambda \otimes_A 1) \Theta_{\mathcal{Y}_1 \otimes_A X_1, \mathcal{Y}_2 \otimes_A X_2} = \Theta_{u_\lambda \mathcal{Y}_1 \otimes_A X_1, \mathcal{Y}_2 \otimes_A X_2} \rightarrow \Theta_{\mathcal{Y}_1 \otimes_A X_1, \mathcal{Y}_2 \otimes_A X_2}$$

show that  $(u_\lambda \otimes_A 1)$  converges strictly to  $1 \in \mathcal{L}(Y \otimes_A X) = M(\mathcal{K}(Y \otimes_A X))$ . Thus if  $S \otimes_A 1$  is compact, then by part (1) we have

$$0 = \lim \|S \otimes_A 1 - (u_\lambda \otimes_A 1)(S \otimes_A 1)\| = \lim \|S - u_\lambda S\|,$$

so  $S$  is compact. For each  $\xi \in Y$ , define  $T_\xi : X \rightarrow Y \otimes_A X$  by  $T_\xi(x) := \xi \otimes x$ . Calculations like

$$T_\xi^* \Theta_{\mathcal{Y}_1 \otimes_A X_1, \mathcal{Y}_2 \otimes_A X_2} T_\eta = \Theta_{\langle \langle \xi, \mathcal{Y}_1 \rangle_A \rangle_{X_1}, \langle \langle \eta, \mathcal{Y}_2 \rangle_A \rangle_{X_2}} \in \mathcal{K}(X)$$

show that  $T_\xi^* K T_\eta \in \mathcal{K}(X)$  for every  $K \in \mathcal{K}(Y \otimes_A X)$  and  $\xi, \eta \in Y$ . Thus if  $S \otimes_A 1$  is compact, then  $T_\xi^*(S \otimes_A 1)T_\eta = \varphi(\langle \langle \xi, S\eta \rangle_A \rangle)$  is compact, and  $\langle \xi, S\eta \rangle_A \in J(X)$ . We deduce that  $\text{ran } S \subseteq YJ(X)$ .  $\square$

We will need the following notation in some of our spatial arguments.

**Notation 4.3.** Suppose  $(\psi, \pi)$  is a Toeplitz representation of  $X$  on a Hilbert space  $\mathcal{H}$ . By Lemma 3.6(1),  $(\psi^{\otimes n}, \pi)$  is a Toeplitz representation of  $X^{\otimes n}$ , so there is a representation  $\pi^{(n)} : \mathcal{K}(X^{\otimes n}) \rightarrow B(\mathcal{H})$  such that

$$\pi^{(n)}(\Theta_{x, y}) = \psi^{\otimes n}(x)\psi^{\otimes n}(y)^* \quad \text{for } x, y \in X^{\otimes n}.$$

Let  $\overline{\pi^{(n)}}$  be the extension of  $\pi^{(n)}$  to  $\mathcal{L}(X^{\otimes n}) = M(\mathcal{K}(X^{\otimes n}))$  with the same essential subspace as  $\pi^{(n)}$ ;  $\overline{\pi^{(n)}}$  is precisely the representation  $\rho^{\psi^{\otimes n}, \pi}$  of [12, Proposition 1.6]. Denote by  $P_n$  the projection onto the essential subspace of  $\pi^{(n)}$ . Since  $\mathcal{K}(X^{\otimes n})$  acts nondegenerately on  $X^{\otimes n}$ , we have

$$P_n \mathcal{H} = \overline{\text{span}}\{\psi^{\otimes n}(x)h : x \in X^{\otimes n}, h \in \mathcal{H}\}.$$

Write  $1^k$  for the identity operator on  $X^{\otimes k}$ , and for  $S \in \mathcal{L}(X^{\otimes n})$  write  $S \otimes_A 1^k$  for the adjointable operator on  $X^{\otimes(n+k)}$  which satisfies

$$S \otimes_A 1^k(x \otimes_A y) = Sx \otimes_A y \quad \text{for } x \in X^{\otimes n} \text{ and } y \in X^{\otimes k}.$$

The following Lemma collects some results from [12].

**Lemma 4.4.** *Let  $(\psi, \pi)$  be a Toeplitz representation of  $X$  on a Hilbert space  $\mathcal{H}$ , and suppose  $n \geq 1$ ,  $S \in \mathcal{L}(X^{\otimes n})$ ,  $x \in X^{\otimes n}$ , and  $k \geq 0$ . Then*

- (1)  $\overline{\pi^{(n)}(S)}\psi^{\otimes n}(x) = \psi^{\otimes n}(Sx)$ ;
- (2)  $\overline{\pi^{(n)}(S)}P_{n+k} = \overline{\pi^{(n+k)}(S \otimes_A 1^k)} = P_{n+k}\overline{\pi^{(n)}(S)}$ ;
- (3)  $\psi^{\otimes n}(x)P_k = P_{n+k}\psi^{\otimes n}(x)$ .

*Proof.* See [12, Proposition 1.6(1), Proposition 1.8(2), Lemma 2.5(2)].  $\square$

**Lemma 4.5** ([27, Lemma 3.10]). *Suppose  $(\psi, \pi)$  is a Cuntz-Pimsner covariant representation of  $X$  in a  $C^*$ -algebra  $B$ . Let  $n \geq 1$ . If  $S \in \mathcal{L}(X^{\otimes n})$  and  $S \otimes_A 1 \in \mathcal{K}(X^{\otimes(n+1)})$ , then  $S \in \mathcal{K}(X^{\otimes n})$  and  $\pi^{(n+1)}(S \otimes_A 1) = \pi^{(n)}(S)$ .*

*Proof.* By Lemma 4.2,  $S$  is compact and  $S(X^{\otimes n}) \subseteq X^{\otimes n}J(X)$ . Represent  $B$  faithfully on a Hilbert space  $\mathcal{H}$ , and adopt Notation 4.3. Since  $(\psi, \pi)$  is Cuntz-Pimsner covariant, Lemma 1.9 gives  $\pi(J(X))(1 - P_1) = 0$ , and hence

$$\begin{aligned} \pi^{(n)}(S)(I - P_{n+1})\psi^{\otimes n}(X^{\otimes n}) \\ &= \pi^{(n)}(S)\psi^{\otimes n}(X^{\otimes n})(I - P_1) = \psi^{\otimes n}(S(X^{\otimes n}))(I - P_1) \\ &\subseteq \psi^{\otimes n}(X^{\otimes n}J(X))(I - P_1) = \psi^{\otimes n}(X^{\otimes n})\pi(J(X))(I - P_1) = \{0\}. \end{aligned}$$

Thus  $0 = \pi^{(n)}(S)(I - P_{n+1})P_n = \pi^{(n)}(S) - \pi^{(n)}(S)P_{n+1}$ , and Lemma 4.4(2) gives  $\pi^{(n)}(S) = \pi^{(n+1)}(S \otimes_A 1)$ , as required.  $\square$

We now aim to identify the core  $\mathcal{O}_X^y$ . When  $n \geq 1$ , write  $A \otimes_A 1^n$  for  $\varphi(A) \otimes_A 1^{n-1} \subseteq \mathcal{L}(X^{\otimes n})$ , and define

$$C_n := A \otimes_A 1^n + \mathcal{K}(X) \otimes_A 1^{n-1} + \cdots + \mathcal{K}(X^{\otimes n}) \quad \text{for all } n \geq 0;$$

then  $C_n$  is a  $C^*$ -subalgebra of  $\mathcal{L}(X^{\otimes n})$ .

**Proposition 4.6** (cf. [27, Proposition 3.11]). *Suppose  $(\psi, \pi)$  is a Cuntz-Pimsner covariant representation of  $X$  in a  $C^*$ -algebra  $B$ . Then there is a homomorphism  $\kappa_n = \kappa_n^{\psi, \pi} : C_n \rightarrow B$  such that*

$$\kappa_n(k_0 \otimes_A 1^n + k_1 \otimes_A 1^{n-1} + \cdots + k_n) = \pi(k_0) + \pi^{(1)}(k_1) + \cdots + \pi^{(n)}(k_n).$$

*If  $\pi$  is faithful, so is  $\kappa_n$ .*

*Proof.* First suppose  $n \geq 1$  and  $c := k_0 \otimes_A 1^n + k_1 \otimes_A 1^{n-1} + \cdots + k_n$  is compact; we claim that

$$(4.1) \quad \pi^{(n)}(c) = \pi(k_0) + \pi^{(1)}(k_1) + \cdots + \pi^{(n)}(k_n).$$



When  $n = 1$ ,  $c = \varphi(k_0) + k_1 \in \mathcal{K}(X)$  implies that  $k_0 \in J(X)$ , and hence  $\pi^{(1)}(c) = \pi(k_0) + \pi^{(1)}(k_1)$  as required. Assume inductively that (4.1) holds for  $n - 1$  for some  $n \geq 2$ . Since  $(k_0 \otimes_A 1^{n-1} + \cdots + k_{n-1}) \otimes_A 1 = c - k_n$  is compact, by Lemma 4.5  $k_0 \otimes_A 1^{n-1} + \cdots + k_{n-1}$  is compact and

$$\pi^{(n)}((k_0 \otimes_A 1^{n-1} + \cdots + k_{n-1}) \otimes_A 1) = \pi^{(n-1)}(k_0 \otimes_A 1^{n-1} + \cdots + k_{n-1}).$$

By induction we thus have

$$\pi^{(n)}(c - k_n) = \pi(k_0) + \pi^{(1)}(k_1) + \cdots + \pi^{(n-1)}(k_{n-1}),$$

and the claim follows. Applying the claim to  $c = 0$  gives

$$\pi(k_0) + \pi^{(1)}(k_1) + \cdots + \pi^{(n)}(k_n) = \pi^{(n)}(c) = 0,$$

from which it is clear that  $\kappa_n$  is a well-defined  $*$ -linear map.

It remains to show that  $\kappa_n$  is multiplicative. For this we represent  $B$  faithfully on  $\mathcal{H}$  and resume Notation 4.3. Using Lemma 4.4 we can see that

$$(4.2) \quad \kappa_{n-1}(c)P_n = \overline{\pi^{(n)}}(c \otimes_A 1) = P_n \kappa_{n-1}(c) \quad \text{for } n \geq 1 \text{ and } c \in C_{n-1}.$$

Thus if  $c \in C_{n-1}$  and  $k \in \mathcal{K}(X^{\otimes n})$ , then

$$(4.3) \quad \begin{aligned} \kappa_n(k)\kappa_{n-1}(c) &= \pi^{(n)}(k)P_n \kappa_{n-1}(c) = \pi^{(n)}(k)\overline{\pi^{(n)}}(c \otimes_A 1) \\ &= \pi^{(n)}(k(c \otimes_A 1)) = \kappa_n(k(c \otimes_A 1)), \end{aligned}$$

and taking adjoints gives

$$(4.4) \quad \kappa_n((c \otimes_A 1)k) = \kappa_{n-1}(c)\kappa_n(k).$$

Also, observe that

$$(4.5) \quad \kappa_n(c \otimes_A 1) = \kappa_{n-1}(c) \quad \text{for } c \in C_{n-1}.$$

To prove that  $\kappa_n$  is multiplicative we induct on  $n$ . For  $n = 0$ , recall that  $C_0 = A$  and  $\kappa_0 = \pi$ . Assume inductively that  $\kappa_{n-1}$  is multiplicative for some  $n \geq 1$ . Let  $c, c' \in C_{n-1}$  and  $k, k' \in \mathcal{K}(X^{\otimes n})$ , so that  $c \otimes_A 1 + k$  and  $c' \otimes_A 1 + k'$  are typical elements of  $C_n$ . Using (4.3)-(4.5), we have

$$\begin{aligned} &\kappa_n((c \otimes_A 1 + k)(c' \otimes_A 1 + k')) \\ &= \kappa_n(cc' \otimes_A 1 + k(c' \otimes_A 1) + (c \otimes_A 1)k' + kk') \\ &= \kappa_{n-1}(c)\kappa_{n-1}(c') + \kappa_n(k)\kappa_{n-1}(c') + \kappa_{n-1}(c)\kappa_n(k') + \kappa_n(k)\kappa_n(k') \\ &= (\kappa_{n-1}(c) + \kappa_n(k))(\kappa_{n-1}(c') + \kappa_n(k')) \\ &= \kappa_n(c \otimes_A 1 + k)\kappa_n(c' \otimes_A 1 + k'). \end{aligned}$$

Hence  $\kappa_n$  is multiplicative.

Finally, suppose  $\pi$  is faithful. Then  $\overline{\pi^{(n)}}$  is also faithful; see [12, Proposition 1.6(2)]. If  $\kappa_n(c) = 0$ , then by (4.2) we have  $\overline{\pi^{(n)}}(c) = P_n \kappa_n(c) = 0$ , and hence  $c = 0$ . Thus  $\kappa_n$  is also faithful.  $\square$

**Definition 4.7.** Let  $C$  denote the inductive limit  $\varinjlim C_n$  under the isometric homomorphisms  $c \in C_n \mapsto c \otimes_A 1 \in C_{n+1}$ .

**Corollary 4.8.** Suppose  $(\psi, \pi)$  is a Cuntz-Pimsner covariant representation of  $X$  in a  $C^*$ -algebra  $B$ . The homomorphisms  $\kappa_n^{\psi, \pi} : C_n \rightarrow B$  induce a homomorphism  $\kappa^{\psi, \pi} : C \rightarrow B$ , and  $\kappa^{\psi, \pi}$  is faithful if  $\pi$  is.

*Proof.* Immediate since  $\kappa_n^{\psi, \pi}(c \otimes_A 1) = \kappa_{n-1}^{\psi, \pi}(c)$  for every  $c \in C_{n-1}$ .  $\square$

For the following corollary we need to know that the universal map  $i_A : A \rightarrow \mathcal{O}_X$  is injective when  $\varphi$  is injective. This is proved in [25, Proposition 2.21], and we give another proof in Corollary 6.2.

**Corollary 4.9.** There is a homomorphism  $\kappa : C \rightarrow \mathcal{O}_X$  which satisfies

$$\kappa(k_0 \otimes 1^n + k_1 \otimes 1^{n-1} + \cdots + k_n) = i_A(k_0) + i_A^{(1)}(k_1) + \cdots + i_A^{(n)}(k_n).$$

Moreover,  $\kappa$  is injective and maps onto the fixed-point algebra  $\mathcal{O}_X^y$ .

*Proof.* Apply Corollary 4.8 to the universal representation  $(i_X, i_A)$  of  $X$  in  $\mathcal{O}_X$  and take  $\kappa := \kappa^{i_X, i_A}$ . Since  $i_A$  is injective so is  $\kappa$ . Define  $e : \mathcal{O}_X \rightarrow \mathcal{O}_X^y$  by  $e(d) := \int_{\mathbb{T}} \gamma_z(d) dz$ . Since  $\mathcal{O}_X$  is spanned by monomials of the form  $d := i_X^{\otimes r}(x) i_A(a) i_X^{\otimes s}(y)^*$ , and since  $e(d) = d$  if  $r = s$  and  $e(d) = 0$  if  $r \neq s$ , we deduce from the continuity of  $e$  that  $\mathcal{O}_X^y$  is spanned by the monomials in which  $r = s$ . But  $i_X^{\otimes r}(x) i_A(a) i_X^{\otimes r}(y)^* = \kappa(\mathcal{O}_{X \cdot a, y})$ , so  $\kappa$  maps onto  $\mathcal{O}_X^y$ .  $\square$

**Proof of Theorem 4.1.** The homomorphism  $\kappa^{\psi, \pi} : C \rightarrow B$  is faithful by Corollary 4.8, and since  $\kappa^{\psi, \pi} = (\psi \times \pi) \circ \kappa^{i_X, i_A}$ , this shows that  $\psi \times \pi$  is faithful on  $\mathcal{O}_X^y$ . Define  $E : B \rightarrow B^{\beta}$  by  $E(b) := \int_{\mathbb{T}} \beta_z(b) dz$ . Since  $\psi \times \pi$  intertwines  $\gamma$  and  $\beta$ , and since  $e : \mathcal{O}_X \rightarrow \mathcal{O}_X^y$  is faithful on positive elements, we deduce that  $\psi \times \pi$  is injective from the following chain:

$$(4.6) \quad \begin{aligned} \psi \times \pi(d) = 0 &\Rightarrow E(\psi \times \pi(d^* d)) = 0 \Rightarrow \psi \times \pi(e(d^* d)) = 0 \\ &\Rightarrow e(d^* d) = 0 \Rightarrow d^* d = 0 \Rightarrow d = 0. \end{aligned} \quad \square$$

**Corollary 4.10.** If  $\varphi : A \rightarrow \mathcal{L}(X)$  is injective, then every nonzero gauge-invariant ideal in  $\mathcal{O}_X$  has nonzero intersection with  $i_A(A)$ .

5. APPLICATIONS OF GAUGE-INVARIANT UNIQUENESS

We first apply Theorem 4.1 to the graph algebra  $C^*(E)$  discussed in Example 1.5. The resulting gauge-invariant uniqueness theorem is stronger than the one in [4]; it can also be deduced from the version in [4] using the approximation techniques of [28] (see [28, Theorem 2.7]).

**Corollary 5.1.** *Let  $E$  be a directed graph with no sinks. Suppose  $\{S_e, P_v\}$  is a Cuntz-Krieger  $E$ -family in a  $C^*$ -algebra  $B$  such that each  $P_v \neq 0$ , and suppose there is a strongly continuous action  $\beta: \mathbb{T} \rightarrow \text{Aut } B$  such that  $\beta_z(S_e) = zS_e$  for all  $e \in E^1$  and  $\beta_z(P_v) = P_v$  for all  $v \in E^0$ . Then the homomorphism  $\pi_{S,P}: C^*(E) \rightarrow B$  is injective.*

For an application of Theorem 4.1 to the  $C^*$ -algebras of continuous graphs, see [7].

We can also apply Theorem 4.1 to the bimodule  $\alpha(1)A$  associated to an endomorphism of  $A$ , where it yields the following variant of [6, Proposition 2.1].

**Corollary 5.2.** *Suppose  $\alpha$  is an injective endomorphism of a  $C^*$ -algebra  $A$ , and  $(\pi, V)$  is a covariant homomorphism of the semigroup dynamical system  $(A, \mathbb{N}, \alpha)$  into a  $C^*$ -algebra  $B$ . If  $\pi$  is injective and there is an action  $\beta: \mathbb{T} \rightarrow \text{Aut } B$  such that  $\beta_z(\pi(a)) = \pi(a)$  and  $\beta_z(V) = zV$ , then  $\pi \times V$  is an injective homomorphism of  $A \times_{\alpha} \mathbb{N}$  into  $B$ .*

As our most important new application of Theorem 4.1, we settle a problem left open in [25]. We recall some terminology and background. Let  $\mathcal{A}$  be a non-self-adjoint subalgebra of a  $C^*$ -algebra  $\mathcal{B}$ . We shall assume that  $\mathcal{A}$  is either unital or contains a contractive approximate identity. We shall also assume that  $\mathcal{A}$  generates  $\mathcal{B}$  as a  $C^*$ -algebra and that  $\mathcal{A}$  acts essentially on  $\mathcal{B}$ , meaning that  $\mathcal{A}\mathcal{B} := \overline{\text{span}}\{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$  and  $\mathcal{B}\mathcal{A} := \overline{\text{span}}\{ba : a \in \mathcal{A}, b \in \mathcal{B}\}$  are dense in  $\mathcal{B}$ . We shall simply say that  $\mathcal{A}$  is an *essential* subalgebra of  $\mathcal{B}$  to describe this situation. Of course, if  $\mathcal{A}$  is an essential subalgebra of  $\mathcal{B}$  and if  $\mathcal{A}$  is unital, then so is  $\mathcal{B}$  and the unit of  $\mathcal{A}$  is the unit of  $\mathcal{B}$ . If  $\mathcal{A}$  is only approximately unital, then an approximate unit for  $\mathcal{A}$  serves as a (not-necessarily-self-adjoint) contractive approximate unit for  $\mathcal{B}$ . A (2-sided) ideal  $J$  of  $\mathcal{B}$  is called a *boundary ideal* for  $\mathcal{A}$  in case the quotient map  $q: \mathcal{B} \rightarrow \mathcal{B}/J$  is completely isometric when restricted to  $\mathcal{A}$ . There is a boundary ideal  $J_0$  that contains all other boundary ideals; it is called the *Shilov boundary ideal*, and the quotient  $\mathcal{B}/J_0$  is called the  *$C^*$ -envelope of  $\mathcal{A}$* . This terminology is due to Arveson [3], who proved the existence of the Shilov boundary ideal and the  $C^*$ -envelope in special cases. The complete result was proved by Hamana in [14]. (Actually, both Arveson and Hamana worked only in the unital setting. The details for non-unital algebras do not differ substantially from the those for unital algebras, but they are scattered throughout the literature. See [24] for references.) The  $C^*$ -envelope of  $\mathcal{A}$  is unique in the following sense: Suppose  $j: \mathcal{A} \rightarrow \mathcal{B}_1$  is a completely isometric homomorphism of  $\mathcal{A}$  onto an essential subalgebra of a  $C^*$ -algebra  $\mathcal{B}_1$  and suppose that the Shilov boundary

ideal in  $\mathcal{B}_1$  for  $j(\mathcal{A})$  vanishes. Then there is a  $C^*$ -isomorphism  $\pi: \mathcal{B}/\mathcal{J}_0 \rightarrow \mathcal{B}_1$  such that  $\pi \circ q|_{\mathcal{A}} = j$ . In particular, we see that all completely isometric automorphisms of  $\mathcal{A}$  are restrictions to  $\mathcal{A}$  of  $*$ -automorphisms of  $\mathcal{B}/\mathcal{J}_0$ .

In the setting of this paper, suppose we are given a  $C^*$ -bimodule  $X$  over a  $C^*$ -algebra  $A$ , let  $(j_X, j_A)$  be the universal Toeplitz representation of  $X$  in  $\mathcal{T}_X$  and let  $\mathcal{T}_{+X}$  denote the closed subalgebra of  $\mathcal{T}_X$  generated by  $j_A(A)$  and  $j_X(X)$ . Then  $\mathcal{T}_{+X}$  is called the *tensor algebra* of  $X$ . It is clear that  $\mathcal{T}_{+X}$  is an essential subalgebra of  $\mathcal{T}_X$ : it generates  $\mathcal{T}_X$  as a  $C^*$ -algebra. Also, if  $A$  is unital, then the image under  $j_A$  of  $1_A$  is the common unit of  $\mathcal{T}_{+X}$  and  $\mathcal{T}_X$ ; similarly,  $j_A$  carries an approximate unit for  $A$  to an approximate unit for both  $\mathcal{T}_{+X}$  and  $\mathcal{T}_X$ . In Theorem 6.4 of [25], the authors consider a module  $X$  which is both faithful, in the sense that  $\varphi$  is injective, and strict, in the sense that the essential submodule  $\varphi(A)X$  is a summand of  $X$ ; they prove that  $\mathcal{I}(J(X))$  is a boundary ideal in  $\mathcal{T}_X$  for  $\mathcal{T}_{+X}$ . Consequently, the quotient map  $q: \mathcal{T}_X \rightarrow \mathcal{T}_X/\mathcal{I}(J(X)) = \mathcal{O}_X$  restricts to a completely isometric homomorphism on  $\mathcal{T}_{+X}$  and so the  $C^*$ -envelope of  $\mathcal{T}_{+X}$  is a quotient of  $\mathcal{O}_X$ .

**Theorem 5.3.** *If  $X$  is a faithful, strict Hilbert bimodule over  $A$ , then  $\mathcal{I}(J(X))$  is the Shilov boundary ideal for  $\mathcal{T}_{+X}$  in  $\mathcal{T}_X$  and  $\mathcal{O}_X$  is the  $C^*$ -envelope of  $\mathcal{T}_{+X}$ .*

*Proof.* We may identify  $\mathcal{T}_{+X}$  with its image under the quotient map  $q: \mathcal{T}_X \rightarrow \mathcal{T}_X/\mathcal{I}(J) = \mathcal{O}_X$ , and then we may think of  $\mathcal{T}_{+X}$  as the closed algebra  $\mathcal{O}_X$  generated by  $i_X(X)$  and  $i_A(A)$ , where  $(i_X, i_A) = (q \circ j_X, q \circ j_A)$  is the universal Cuntz-Pimsner covariant Toeplitz representation of  $X$ . Observe that the gauge automorphism group of  $\mathcal{O}_X$  fixes  $\mathcal{T}_{+X}$ . Suppose  $I_0$  is the Shilov boundary ideal in  $\mathcal{O}_X$  for  $\mathcal{T}_{+X}$ . We want to show that  $I_0 = 0$ . Since each gauge automorphism  $\gamma_z$  maps  $\mathcal{T}_{+X}$  onto  $\mathcal{T}_{+X}$ ,  $\gamma_z(I_0)$  is a boundary ideal in  $\mathcal{O}_X$  for  $\mathcal{T}_{+X}$ . However,  $I_0$  contains all boundary ideals. Therefore  $\gamma_z(I_0) \subseteq I_0$ ,  $z \in \mathbb{T}$ , and so  $I_0$  is gauge invariant. By Corollary 4.10 and the fact that  $i_A$  is injective (Corollary 6.2), we conclude that  $I_0 \cap i_A(A)$  is non-zero, unless  $I_0 = 0$ . Since the quotient map from  $\mathcal{O}_X$  onto  $\mathcal{O}_X/I_0$  is completely isometric when restricted to  $\mathcal{T}_{+X}$ , it is faithful when restricted to  $i_A(A)$ . Thus  $I_0 \cap i_A(A)$  vanishes and so, then, does  $I_0$ .  $\square$

**Remark 5.4.** One can weaken the hypothesis in Theorem 5.3 that  $X$  is faithful to conclude that the  $C^*$ -envelope of  $\mathcal{T}_{+X}$  is a relative Cuntz-Pimsner algebra  $\mathcal{O}(K, X)$ . (Indeed, that is why the authors of [25] introduced the concept of relative Cuntz-Pimsner algebras.) However, when this is done, it appears that a stronger hypothesis than “strict” is necessary. Whether this is simply an artifact of the proofs in [25], we do not know. The analysis of ideals in relative Cuntz-Pimsner algebras that we have given may help to clarify the situation.

## 6. THE DOPLICHER-ROBERTS ALGEBRA OF A HILBERT BIMODULE

When the homomorphism  $\varphi: A \rightarrow \mathcal{L}(X)$  given by the left action on  $X$  is injective, so are the maps  $T \mapsto T \otimes_{\mathcal{A}} 1$  of  $\mathcal{L}(X^{\otimes n}, X^{\otimes(n+k)})$  into

$\mathcal{L}(X^{\otimes(n+1)}, X^{\otimes(n+k+1)})$ . For  $k \in \mathbb{Z}$ , let  $\mathcal{DR}_X^{(k)}$  be the Banach-space direct limit

$$\mathcal{DR}_X^{(k)} := \varinjlim \mathcal{L}(X^{\otimes n}, X^{\otimes(n+k)})$$

under the embeddings  $T \mapsto T \otimes_A 1$ . For  $m, n \geq 1$ , we write  $\iota^{m,n}$  for the canonical embedding of  $\mathcal{L}(X^{\otimes m}, X^{\otimes n})$  in  $\mathcal{DR}_X^{(n-m)}$ . The algebraic direct sum  $\bigoplus_{k \in \mathbb{Z}} \mathcal{DR}_X^{(k)}$  has a natural structure as  $\mathbb{Z}$ -graded  $*$ -algebra (see [8, page 180]); the *Doplicher-Roberts algebra*  $\mathcal{DR}_X$  of  $X$  is the  $C^*$ -algebra obtained by completing  $\bigoplus_{k \in \mathbb{Z}} \mathcal{DR}_X^{(k)}$  in the unique  $C^*$ -norm for which the automorphic action of  $\mathbb{T}$  defined by the grading is isometric (see [8, Theorem 4.2]). We identify  $\mathcal{DR}_X^{(k)}$  with its canonical image in  $\mathcal{DR}_X$ .

Our study of  $\mathcal{DR}_X$  was motivated by [9] (where  $\mathcal{DR}_X$  is denoted  $\mathcal{O}_{X_A}$ ), and our aim is to generalize [9, Theorem 4.1]. We begin as in [9] by demonstrating that  $\mathcal{O}_X$  embeds in  $\mathcal{DR}_X$ . Define  $L: X \rightarrow \mathcal{L}(X, X^{\otimes 2})$  by  $L_X(y) := x \otimes_A y$ , and set

$$j_X := \iota^{1,2} \circ L: X \rightarrow \mathcal{DR}_X \quad \text{and} \quad j_A := \iota^{1,1} \circ \varphi: A \rightarrow \mathcal{DR}_X.$$

**Lemma 6.1.**  $(j_X, j_A)$  is a Cuntz-Pimsner covariant representation of  $X$  in  $\mathcal{DR}_X$ .

*Proof.* For  $x, y \in X$  and  $a \in A$ , we have

$$\begin{aligned} j_X(x \cdot a) &= \iota^{1,2}(L_{x \cdot a}) = \iota^{1,2}(L_x \varphi(a)) \\ &= \iota^{1,2}(L_x) \iota^{1,1}(\varphi(a)) = j_X(x) j_A(a), \\ j_X(x)^* j_X(y) &= \iota^{1,2}(L_x)^* \iota^{1,2}(L_y) = \iota^{2,1}(L_x^*) \iota^{1,2}(L_y) \\ &= \iota^{1,1}(L_x^* L_y) = \iota^{1,1}(\varphi(\langle x, y \rangle_A)) = j_A(\langle x, y \rangle_A), \end{aligned}$$

and

$$\begin{aligned} j_X(\varphi(a)x) &= \iota^{1,2}(L_{\varphi(a)x}) = \iota^{1,2}((\varphi(a) \otimes_A 1)L_x) \\ &= \iota^{2,2}(\varphi(a) \otimes_A 1) \iota^{1,2}(L_x) = \iota^{1,1}(\varphi(a)) \iota^{1,2}(L_x) = j_A(a) j_X(x). \end{aligned}$$

Thus  $(j_X, j_A)$  is a Toeplitz representation of  $X$  in  $\mathcal{DR}_X$ . To see that  $(j_X, j_A)$  is Cuntz-Pimsner covariant, we first establish

$$(6.1) \quad j_A^{(1)} = \iota^{1,1}|_{\mathcal{K}(X)}.$$

For any  $x, y \in X$ ,

$$\begin{aligned} j_A^{(1)}(\Theta_{x,y}) &= j_X(x) j_X(y)^* = \iota^{1,2}(L_x) \iota^{1,2}(L_y)^* \\ &= \iota^{2,2}(L_x L_y^*) = \iota^{2,2}(\Theta_{x,y} \otimes_A 1) = \iota^{1,1}(\Theta_{x,y}), \end{aligned}$$

and (6.1) follows by linearity and continuity. Thus for  $a \in J(X)$ , we have  $j_A^{(1)}(\varphi(a)) = \iota^{1,1}(\varphi(a)) = j_A(a)$ , and  $(j_X, j_A)$  is Cuntz-Pimsner covariant.  $\square$

**Corollary 6.2** ([25, Proposition 2.21]). *If  $\varphi$  is injective, then the universal map  $i_A: A \rightarrow \mathcal{O}_X$  is injective.*

*Proof.* Since  $j_A = (j_X \times j_A) \circ i_A$  is injective, so is  $i_A$ .  $\square$

**Proposition 6.3** ([9, Proposition 3.2]).  *$j_X \times j_A: \mathcal{O}_X \rightarrow \mathcal{DR}_X$  is injective.*

*Proof.*  $j_A$  is injective, and  $j_X \times j_A$  intertwines the gauge actions on  $\mathcal{O}_X$  and  $\mathcal{DR}_X$ , so the result follows from Theorem 4.1.  $\square$

**Definition 6.4.** A Toeplitz representation  $(\psi, \pi)$  of  $X$  on  $\mathcal{H}$  is called *fully coisometric* if  $\psi(X)\mathcal{H}$  is total in  $\mathcal{H}$ .

**Remark 6.5.** By Lemma 1.9, fully coisometric Toeplitz representations are Cuntz-Pimsner covariant. For  $n \geq 1$ , let  $P_n$  be the orthogonal projection of  $\mathcal{H}$  onto  $\overline{\text{span}} \psi^{\otimes n}(X^{\otimes n})\mathcal{H}$ . Then  $(\psi, \pi)$  is fully coisometric if and only if  $P_1 = 1$ , in which case it is easy to see that  $P_n = 1$  for every  $n \geq 1$ .

Since the left action of  $\mathcal{K}(X^{\otimes n})$  on  $X^{\otimes n}$  is nondegenerate, the essential subspace of the representation  $\pi^{(n)}: \mathcal{K}(X^{\otimes n}) \rightarrow B(\mathcal{H})$  is precisely  $P_n\mathcal{H}$ . Thus  $(\psi, \pi)$  is fully coisometric if and only if  $\pi^{(1)}$  is nondegenerate, in which case  $\pi^{(n)}$  is nondegenerate for every  $n \geq 1$ .

We denote by  $\gamma$  the gauge actions of  $\mathbb{T}$  on both  $\mathcal{O}_X$  and  $\mathcal{DR}_X$ ; this should cause no problems since the embedding  $j_X \times j_A$  is equivariant.

**Theorem 6.6.** *Suppose  $(\psi, \pi)$  is a fully coisometric Toeplitz representation of  $X$  on a Hilbert space  $\mathcal{H}$ .*

(1) *There is a unique representation  $\overline{\psi \times \pi}$  of  $\mathcal{DR}_X$  on  $\mathcal{H}$  such that*

$$(6.2) \quad (\overline{\psi \times \pi}) \circ (j_X \times j_A) = \psi \times \pi.$$

(2) *A representation  $\rho$  of  $\mathcal{DR}_X$  has the form  $\overline{\psi \times \pi}$  for some fully coisometric Toeplitz representation  $(\psi, \pi)$  of  $X$  if and only if  $\rho \circ \iota^{1,1}|_{\mathcal{K}(X)}$  is nondegenerate.*

(3) *If  $\pi$  is faithful and*

$$(6.3) \quad \left\| \psi \times \pi \left( \int_{\mathbb{T}} \gamma_z(b) dz \right) \right\| \leq \|\psi \times \pi(b)\| \quad \text{for } b \in \mathcal{O}_X,$$

*then  $\overline{\psi \times \pi}$  is faithful.*

To prove this theorem, we need some preliminary results.

**Lemma 6.7.** *Let  $\{A^{(k)}: k \in \mathbb{Z}\}$  be a  $\mathbb{Z}$ -graded  $C^*$ -algebra. Endow  $\bigoplus_{k \in \mathbb{Z}} A^{(k)}$  with the unique  $C^*$ -norm for which the automorphic action of  $\mathbb{T}$  defined by the grading is isometric, as in [8, Theorem 4.2]. Then every  $*$ -homomorphism of  $\bigoplus_{k \in \mathbb{Z}} A^{(k)}$  into a  $C^*$ -algebra is contractive.*

*Proof.* Suppose  $\sigma$  is a  $*$ -homomorphism of  $\bigoplus A^{(k)}$  into a  $C^*$ -algebra. If  $a = \bigoplus a_k \in \bigoplus A^{(k)}$ , then

$$\|\sigma(\gamma_z(a))\| = \left\| \bigoplus z^k \sigma(a_k) \right\| \leq \sum \|\sigma(a_k)\|$$

for each  $z \in \mathbb{T}$ , so

$$\|a\|'_y := \sup (\{\|a\|_y\} \cup \{\|\sigma(\gamma_z(a))\| : z \in \mathbb{T}\})$$

defines a  $C^*$ -norm  $\|\cdot\|'_y$  on  $\bigoplus A^{(k)}$ . Since  $\|\gamma_z(a)\|'_y = \|a\|'_y$  for  $a \in \bigoplus A^{(k)}$  and  $z \in \mathbb{T}$ , it follows from [8, Theorem 4.2] that  $\|\cdot\|'_y = \|\cdot\|_y$ . Thus

$$\|\sigma(a)\| = \|\sigma(\gamma_0(a))\| \leq \|a\|'_y = \|a\|_y$$

for every  $a \in \bigoplus A^{(k)}$ . □

**Lemma 6.8.** *Let  $X, Y$  and  $Z$  be Hilbert  $A$ -modules, and suppose  $\pi$  is a representation of  $A$  on a Hilbert space  $\mathcal{H}$ . Then there is a contractive linear map  $T \mapsto T \otimes_A 1$  of  $\mathcal{L}(X, Y)$  into  $\mathcal{B}(X \otimes_A \mathcal{H}, Y \otimes_A \mathcal{H})$  such that*

$$T \otimes_A 1(x \otimes_A h) := Tx \otimes_A h.$$

We then have  $(T \otimes_A 1)^* = T^* \otimes_A 1$ , and if  $S \in \mathcal{L}(Y, Z)$ , then

$$(6.4) \quad (S \otimes_A 1)(T \otimes_A 1) = ST \otimes_A 1.$$

**Remark 6.9.** Since  $X \otimes_A \mathcal{H}$  is canonically isomorphic to  $X \otimes_A \pi(A)\mathcal{H}$ , we do not need to assume that  $\pi$  is nondegenerate.

*Proof of Lemma 6.8.* The proof of [29, Proposition 2.66] carries over provided we use Remark 2.23 instead of Corollary 2.22 to see that  $T^*T \leq \|T\|^2$  in  $\mathcal{L}(X)$ . If  $x \in X, y \in Y$  and  $h, k \in \mathcal{H}$ , then

$$\begin{aligned} & (T \otimes_A 1(x \otimes_A h) \mid y \otimes_A k) \\ &= (Tx \otimes_A h \mid y \otimes_A k) = (h \mid \pi(\langle Tx, y \rangle_A)k) \\ &= (x \otimes_A h \mid T^*y \otimes_A k) = (x \otimes_A h \mid T^* \otimes_A 1(y \otimes_A k)), \end{aligned}$$

so  $(T \otimes_A 1)^* = T^* \otimes_A 1$ , and (6.4) follows from

$$\begin{aligned} (S \otimes_A 1)(T \otimes_A 1)(x \otimes_A h) &= (S \otimes_A 1)(Tx \otimes_A h) \\ &= STx \otimes_A h = ST \otimes_A 1(x \otimes_A h). \end{aligned}$$
□

**Definition 6.10.** Let  $C$  be the  $C^*$ -category with object set  $\mathbb{N}$  and morphisms  $\text{Hom}(m, n) := \mathcal{L}(X^{\otimes m}, X^{\otimes n})$ . (See [8, Section 1] for the definition of a  $C^*$ -category.) A  $*$ -representation of  $C$  on a Hilbert space  $\mathcal{H}$  is a collection of linear maps  $\rho^{m,n} : \text{Hom}(m, n) \rightarrow B(\mathcal{H})$  such that

$$(6.5) \quad \rho^{n,p}(S)\rho^{m,n}(T) = \rho^{m,p}(ST) \quad \text{and} \quad \rho^{m,n}(T)^* = \rho^{n,m}(T^*).$$

**Proposition 6.11.** Let  $(\psi, \pi)$  be a Toeplitz representation of  $X$  on a Hilbert space  $\mathcal{H}$ . With the convention that  $\psi^{\otimes 0} := \pi$ , there is a unique  $*$ -representation  $[\psi, \pi]$  of  $C$  on  $\mathcal{H}$  such that, for  $T \in \mathcal{L}(X^{\otimes m}, X^{\otimes n})$ ,

$$\begin{aligned} [\psi, \pi]^{m,n}(T)\psi^{\otimes m}(x)h &= \psi^{\otimes n}(Tx)h & \text{for } x \in X^{\otimes m}, h \in \mathcal{H}, \\ [\psi, \pi]^{m,n}(T)k &= 0 & \text{for } k \perp \{\psi^{\otimes m}(x)h : x \in X^{\otimes m}, h \in \mathcal{H}\}. \end{aligned}$$

*Proof.* For  $n \in \mathbb{N}$ , let  $U^{(n)} : X^{\otimes n} \otimes_A \mathcal{H} \rightarrow \mathcal{H}$  be the isometry given by

$$U^{(n)}(x \otimes_A h) := \psi^{\otimes n}(x)h \quad \text{for } x \in X^{\otimes n}, h \in \mathcal{H}.$$

We obtain a linear map  $[\psi, \pi]^{m,n}$  with the stated properties by defining

$$[\psi, \pi]^{m,n}(T) := U^{(n)}(T \otimes_A 1)(U^{(m)})^* \quad \text{for } T \in \mathcal{L}(X^{\otimes m}, X^{\otimes n}),$$

and (6.5) follows immediately from Lemma 6.8.  $\square$

*Proof of Theorem 6.6.*

(1) First we claim that the  $*$ -representation of Proposition 6.11 satisfies

$$(6.6) \quad [\psi, \pi]^{m+1, n+1}(T \otimes_A 1) = [\psi, \pi]^{m,n}(T) \quad \text{for } T \in \mathcal{L}(X^{\otimes m}, X^{\otimes n}).$$

Since  $(\psi, \pi)$  is fully coisometric, it suffices to check this equation on vectors of the form  $\psi^{\otimes(m+1)}(x \otimes_A y)h$ , where  $x \in X^{\otimes m}$ ,  $y \in X$ , and  $h \in \mathcal{H}$ :

$$\begin{aligned} & [\psi, \pi]^{m+1, n+1}(T \otimes_A 1)\psi^{\otimes(m+1)}(x \otimes_A y)h \\ &= \psi^{\otimes(n+1)}(Tx \otimes_A y)h \\ &= \psi^{\otimes n}(Tx)\psi(y)h \\ &= [\psi, \pi]^{m,n}(T)\psi^{\otimes m}(x)\psi(y)h \\ &= [\psi, \pi]^{m,n}(T)\psi^{\otimes(m+1)}(x \otimes_A y)h, \end{aligned}$$

justifying the claim.

Since  $[\psi, \pi]$  is a  $*$ -representation of a  $C^*$ -category, the linear maps  $[\psi, \pi]^{m,n}$  are contractive. Thus by (6.6) there is a contractive linear map  $[\psi, \pi]^{(k)} : \mathcal{DR}_X^{(k)} \rightarrow B(\mathcal{H})$  such that

$$[\psi, \pi]^{(k)} \circ \iota^{n, n+k} = [\psi, \pi]^{n, n+k}.$$



We claim that  $\sigma := \bigoplus_{k \in \mathbb{Z}} [\psi, \pi]^{(k)}$  is a  $*$ -representation of  $\bigoplus_{k \in \mathbb{Z}} \mathcal{DR}_X^{(k)}$  on  $\mathcal{H}$ . For  $T \in \mathcal{L}(X^{\otimes m}, X^{\otimes(m+k)})$ , we have

$$\begin{aligned} [\psi, \pi]^{(k)}(\iota^{m,m+k}(T))^* &= [\psi, \pi]^{m,m+k}(T)^* = [\psi, \pi]^{m+k,m}(T^*) \\ &= [\psi, \pi]^{(-k)}(\iota^{m+k,m}(T^*)), \end{aligned}$$

so if  $t_j \rightarrow t \in \mathcal{DR}_X^{(k)}$  and  $t_j \in \iota^{m,m+k}(\mathcal{L}(X^{\otimes m}, X^{\otimes(m+k)}))$ , then by norm continuity of  $[\psi, \pi]^{(k)}$ ,  $[\psi, \pi]^{(-k)}$  and the involution, we have

$$\begin{aligned} \sigma(t)^* &= [\psi, \pi]^{(k)}(t)^* = \lim [\psi, \pi]^{(k)}(t_j)^* \\ &= \lim [\psi, \pi]^{(-k)}(t_j^*) = \sigma(t^*). \end{aligned}$$

Similarly, for  $T \in \mathcal{L}(X^{\otimes m}, X^{\otimes n})$  and  $S \in \mathcal{L}(X^{\otimes n}, X^{\otimes p})$ , we have

$$\begin{aligned} [\psi, \pi]^{(p-m)}(\iota^{n,p}(S)\iota^{m,n}(T)) \\ &= [\psi, \pi]^{(p-m)}(\iota^{m,p}(ST)) \\ &= [\psi, \pi]^{m,p}(ST) = [\psi, \pi]^{n,p}(S)[\psi, \pi]^{m,n}(T) \\ &= [\psi, \pi]^{(p-n)}(\iota^{n,p}(S))[\psi, \pi]^{(n-m)}(\iota^{m,n}(T)), \end{aligned}$$

and by norm continuity we conclude that  $\sigma(st) = \sigma(s)\sigma(t)$  whenever  $s \in \mathcal{DR}_X^{(p-n)}$  and  $t \in \mathcal{DR}_X^{(n-m)}$ . By Lemma 6.7,  $\sigma$  is contractive, and hence extends to a representation  $\overline{\psi \times \pi}$  of  $\mathcal{DR}_X$  on  $\mathcal{H}$ .

We now verify (6.2). Let  $(i_X, i_A)$  be the universal representation in  $\mathcal{O}_X$ . Since  $(\psi, \pi)$  is fully coisometric, the calculation

$$[\psi, \pi]^{1,2}(L_X)\psi(y)h = \psi^{\otimes 2}(x \otimes_A y)h = \psi(x)\psi(y)h$$

shows that  $[\psi, \pi]^{1,2} \circ L = \psi$ . But

$$\overline{\psi \times \pi} \circ (j_X \times j_A) \circ i_X = \overline{\psi \times \pi} \circ j_X = \sigma \circ \iota^{1,2} \circ L = [\psi, \pi]^{1,2} \circ L,$$

so  $\overline{\psi \times \pi} \circ (j_X \times j_A) \circ i_X = \psi$ . Similarly, the calculation

$$[\psi, \pi]^{1,1}(\varphi(a))\psi(y)h = \psi(\varphi(a)y)h = \pi(a)\psi(y)h$$

shows that  $[\psi, \pi]^{1,1} \circ \varphi = \pi$ , and since

$$\overline{\psi \times \pi} \circ (j_X \times j_A) \circ i_A = \overline{\psi \times \pi} \circ j_A = \sigma \circ \iota^{1,1} \circ \varphi = [\psi, \pi]^{1,1} \circ \varphi,$$

we also have  $\overline{\psi \times \pi} \circ (j_X \times j_A) \circ i_A = \pi$  and (6.2) follows.

For the uniqueness assertion, we require the following generalization of (6.1), which can be easily checked on elementary tensors:

$$j_X^{\otimes n}(x)j_X^{\otimes m}(y)^* = \iota^{m,n}(\Theta_{x,y}) \quad \text{for all } x \in X^{\otimes n} \text{ and } y \in X^{\otimes m}.$$

It follows that

$$(6.7) \quad j_A^{(m)} = \iota^{m,m} |_{\mathcal{K}(X^{\otimes m})} \quad \text{for all } m \geq 1,$$

and that

$$(6.8) \quad \iota^{m,n}(\mathcal{K}(X^{\otimes m}, X^{\otimes n})) \subseteq j_X \times j_A(\mathcal{O}_X).$$

Suppose  $\rho$  is a representation of  $\mathcal{DR}_X$  such that  $\rho \circ (j_X \times j_A) = \psi \times \pi$ ; that is, such that  $\rho \circ j_X = \psi$  and  $\rho \circ j_A = \pi$ . We will show that for  $T \in \mathcal{L}(X^{\otimes m}, X^{\otimes n})$ , the operator  $\rho(\iota^{m,n}(T))$  is determined by the restriction of  $\rho$  to  $j_X \times j_A(\mathcal{O}_X)$ . Since the operators  $\iota^{m,n}(T)$  span a dense subspace of  $\mathcal{DR}_X$ , it will then follow that  $\rho = \overline{\psi \times \pi}$ .

Since  $(\psi, \pi)$  is fully coisometric, the representation  $\pi^{(m)}$  of  $\mathcal{K}(X^{\otimes m})$  is nondegenerate; hence  $\rho(\iota^{m,n}(T))$  is determined by its action on vectors of the form  $\pi^{(m)}(K)h$  for  $K \in \mathcal{K}(X^{\otimes m})$ . Applying  $\rho$  to (6.7) gives  $\pi^{(m)}(K) = \rho(\iota^{m,m}(K))$ , and hence

$$(6.9) \quad \rho(\iota^{m,n}(T))\pi^{(m)}(K) = \rho(\iota^{m,n}(T)\iota^{m,m}(K)) = \rho(\iota^{m,n}(TK)).$$

Since  $TK \in \mathcal{K}(X^{\otimes m}, X^{\otimes n})$ , (6.9) and (6.8) show that  $\rho(\iota^{m,n}(T))$  is indeed determined by the restriction of  $\rho$  to  $j_X \times j_A(\mathcal{O}_X)$ , as claimed.

(2) Suppose  $\rho$  is a representation of  $\mathcal{DR}_X$  on  $\mathcal{H}$ . Then  $\psi := \rho \circ j_X$  and  $\pi := \rho \circ j_A$  form a Cuntz-Pimsner covariant representation of  $X$  such that  $\psi \times \pi = \rho \circ (j_X \times j_A)$ . Since  $\pi^{(1)}\rho \circ \iota^{1,1}|_{\mathcal{K}(X)}$ ,  $(\psi, \pi)$  is fully coisometric if and only if  $\rho \circ \iota^{1,1}|_{\mathcal{K}(X)}$  is nondegenerate, in which case  $\rho = \overline{\psi \times \pi}$  by uniqueness of  $\overline{\psi \times \pi}$ .

(3) If  $\pi$  is faithful, then each  $[\psi, \pi]^{n,n} : \mathcal{L}(X^{\otimes n}) \rightarrow B(\mathcal{H})$  is faithful [12, Proposition 1.6], and hence so is  $[\psi, \pi]^{(0)}$ . To complete the proof it thus suffices to show that

$$(6.10) \quad \|\overline{\psi \times \pi}(E(b))\| \leq \|\overline{\psi \times \pi}(b)\| \quad \text{for } b \in \mathcal{DR}_X,$$

where  $E$  is the expectation on  $\mathcal{DR}_X$  obtained by averaging over  $\mathcal{Y}$ ; given this, a chain like (4.6) shows that  $\overline{\psi \times \pi}$  is faithful.

It suffices to prove (6.10) for  $b \in \text{span}\{\iota^{m,n}(\mathcal{L}(X^{\otimes m}, X^{\otimes n})) : m, n \geq 0\}$ . Given such a  $b$ , there exists  $r \geq 1$  such that  $b$  is the finite sum  $\sum b_n$  of elements  $b_n = \iota^{r,r+n}(B_n)$  for some  $B_n \in \mathcal{L}(X^{\otimes r}, X^{\otimes(r+n)})$ . Suppose  $K \in \mathcal{K}(X^{\otimes r})$ , and let  $k := \iota^{r,r}(K)$ . Then if  $e$  is the expectation onto  $\mathcal{O}_X$ ,

$$b_n k = \iota^{r,r+n}(B_n K) \in \iota^{r,r+n}(\mathcal{K}(X^{\otimes r}, X^{\otimes(r+n)})),$$

so by (6.8) and the injectivity of  $j_X \times j_A$  (Proposition 6.3), there is a unique  $c_n \in \mathcal{O}_X$  such that  $b_n k = j_X \times j_A(c_n)$ . Let  $c := \sum c_n$ . Then

$$b_0 k = E(bk) = E(j_X \times j_A(c)) = j_X \times j_A(e(c)),$$

so

$$\begin{aligned} \|\overline{\psi \times \pi}(b_0 k)\| &= \|\overline{\psi \times \pi}(j_X \times j_A(e(c)))\| = \|\psi \times \pi(e(c))\| \\ &\leq \|\psi \times \pi(c)\| = \|\overline{\psi \times \pi}(j_X \times j_A(c))\| \\ &= \|\overline{\psi \times \pi}(bk)\|. \end{aligned}$$

Now suppose  $\xi \in \mathcal{H}$  and  $\varepsilon > 0$ . Since  $(\psi, \pi)$  is fully coisometric,  $\pi^{(r)} = \overline{\psi \times \pi} \circ \iota^{r,r}|_{\mathcal{K}(X^{\otimes r})}$  is nondegenerate, so by the Hewitt-Cohen Factorization Theorem  $\xi = \overline{\psi \times \pi}(k)\eta$  for some  $k = \iota^{r,r}(K)$  and  $\eta \in \mathcal{H}$  satisfying

$$\|\overline{\psi \times \pi}(k)\|\|\eta\| \leq (1 + \varepsilon)\|\xi\|.$$

Then

$$\begin{aligned} \|\overline{\psi \times \pi}(E(b))\xi\| &= \|\overline{\psi \times \pi}(b_0)\overline{\psi \times \pi}(k)\eta\| \\ &\leq \|\overline{\psi \times \pi}(b_0 k)\|\|\eta\|\|\overline{\psi \times \pi}(bk)\|\|\eta\| \\ &\leq \|\overline{\psi \times \pi}(b)\|\|\overline{\psi \times \pi}(k)\|\|\eta\| \\ &\leq \|\overline{\psi \times \pi}(b)\|(1 + \varepsilon)\|\xi\|, \end{aligned}$$

so  $\|\overline{\psi \times \pi}(E(b))\| \leq (1 + \varepsilon)\|\overline{\psi \times \pi}(b)\|$ . Since  $\varepsilon$  was arbitrary, this gives (6.10), completing the proof.  $\square$

**Remark 6.12.** Our theorem is a generalisation of [9, Theorem 4.1]: we do not assume that  $\pi$  is faithful, nor do we require any sort of basis for  $X$ , and their left-annihilator condition says that  $(\psi, \pi)$  is fully coisometric. To see this last point, let  $(\psi, \pi)$  be a Toeplitz representation of  $X$  on  $\mathcal{H}$  which is nondegenerate in the sense that the  $C^*$ -algebra  $C^*(\psi, \pi)$  generated by  $\psi(X) \cup \pi(A)$  is nondegenerate. (If  $\varphi: A \rightarrow \mathcal{L}(X)$  is nondegenerate, then this is equivalent to requiring that  $\pi$  be nondegenerate.) Let  $\mathcal{D}$  be the  $C^*$ -algebra generated by  $\{[\psi, \pi]^{m,n}(\mathcal{L}(X^{\otimes m}, X^{\otimes n})) : m, n \geq 0\}$ . We claim that the left annihilator of  $\psi(X)$  in  $\mathcal{D}$  is zero if and only if  $(\psi, \pi)$  is fully coisometric.

First suppose that  $T \in \mathcal{D}$  is in the left annihilator of  $\psi(X)$ . Then  $T\psi(x)h=0$  for every  $x \in X$  and  $h \in \mathcal{H}$ , and when  $(\psi, \pi)$  is fully coisometric this forces  $T = 0$ .

Conversely, suppose the left annihilator of  $\psi(X)$  in  $\mathcal{D}$  is zero and  $T \in \mathcal{D}$ . Then

$$T[\psi, \pi]^{1,1}(1)\psi(x) = T\psi(x), \quad x \in X,$$

so that  $T[\psi, \pi]^{1,1}(1) = T$ . Hence  $[\psi, \pi]^{1,1}(1)$  is an identity for  $\mathcal{D}$ . Since  $C^*(\psi, \pi)$  is nondegenerate so is  $\mathcal{D}$ , and thus  $[\psi, \pi]^{1,1}(1)$  is the identity on  $\mathcal{H}$ . But  $[\psi, \pi]^{1,1}(1) = UU^*$ , where  $U: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$  is the isometry given by  $U(x \otimes_A h) := \psi(x)h$ , so  $P_1 = UU^* = [\psi, \pi]^{1,1}(1)$  is the identity. Thus  $(\psi, \pi)$  is fully coisometric.

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