Single elements of matrix incidence algebras

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Abstract

An element \( s \) of an algebra \( \mathcal{A} \) is called a single element of \( \mathcal{A} \) if \( asb = 0 \) and \( a, b \in \mathcal{A} \) imply that \( as = 0 \) or \( sb = 0 \). Let \( n \in \mathbb{Z}^+ \), let \( K \) be a field and let \( \preceq \) be a partial order on \{1, 2, \ldots, n\}. Let \( \mathcal{A}_n(\preceq) \) be the matrix incidence algebra consisting of those \( n \times n \) matrices \( A = (a_{i,j}) \) with entries in \( K \), satisfying \( a_{i,j} = 0 \) whenever \( i \not\preceq j \). An element \( S = (s_{i,j}) \) of \( \mathcal{A}_n(\preceq) \) is a single element if and only if (i) \( r_i = 0 \) and \( c_j = 0 \) imply \( s_{i,j} = 0 \), (ii) \( i \preceq j \) and \( i \preceq j \) for some \( i \Rightarrow r_{j_1} \) and \( r_{j_2} \) are linearly dependent, (iii) \( i_1 \preceq j \) and \( i_2 \preceq j \) for some \( j \Rightarrow c_{i_1} \) and \( c_{i_2} \) are linearly dependent. Here \( r_i \) and \( c_j \) denote the \( i \)th row and the \( j \)th column of \( S \), respectively. If \( |K| \geq 3 \), the maximum rank of a single element of \( \mathcal{A}_n(\preceq) \) is the largest positive integer \( m \) for which there exist sets \( X, Y \) of minimal, respectively, maximal, elements with \( |X| = |Y| = m \) satisfying \( x \preceq y \) for every \( x \in X, y \in Y \). © 2000 Elsevier Science Inc.

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1. Preliminaries

For any locally finite partially ordered set \( (S, \preceq) \) and field \( K \) the incidence algebra \( \mathcal{A}(S) \) of \( S \) over \( K \) is the set of all functions \( f: S \times S \rightarrow K \) with the property that \( f(x, y) = 0 \) whenever \( x \not\preceq y \). (Here, local finiteness means that every interval \([x, y] = \{u \in S: x \preceq u \preceq y\}\) is finite.) \( \mathcal{A}(S) \) becomes an associative \( K \)-algebra with

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the pointwise operations of addition and scalar multiplication and with the Dirichlet product:
\[(f * g)(x, y) = \sum_{x \leq u \leq y} f(x, u)g(u, y).
\]

The Kronecker delta function is the multiplicative identity of \(A(S)\). Rota [15] proposed the idea of such algebras as a basis for a unified study of combinatorial theory. Certain subalgebras of \(A(S)\) were considered in [4] (see also [5]), and generators were considered in [11]. Here we consider single elements of \(A(S)\).

An element \(s\) of an (abstract) algebra \(A\) is called a single element of \(A\) if \(asb = 0\) and \(a, b \in A\) imply that \(as = 0\) or \(sb = 0\). This definition was first given by Erdos [1], though the notion itself was used earlier in [13,14]. It has been found useful in the representation theory of normed algebras [2] and the (algebraic) nature of isomorphisms between operator algebras (for example, see [6,8,9,12]). The reason, basically, for this is twofold: firstly, every operator of rank 1 is a single element in any operator algebra containing it and, secondly, the property of being single is purely algebraic and so is preserved under monomorphisms. The notion of ‘single element’ may prove to be useful in other fields. It is hoped that the present note may encourage its wider usage. Here we consider single elements from a purely algebraic viewpoint, characterising them and describing their maximum rank, as elements of incidence algebras of the form \(A(S)\), where \(S\) is a finite set (our operator algebras are matrix algebras).

If \(n \in \mathbb{Z}^+\) and \(\preceq\) is a partial order on \([1, 2, \ldots, n]\), the corresponding incidence algebra can be identified in a natural way with the algebra of \(n \times n\) matrices (with entries in \(K\)) \(A = (a_{i,j})\) satisfying \(a_{i,j} = 0\) whenever \(i \not\preceq j\), with the usual matrix operations. (The Dirichlet product becomes matrix multiplication.) As in [11], we will call such matrix algebras matrix incidence algebras and, with a slight change of notation, use \(A_n(\preceq)\) to denote the matrix incidence algebra described immediately above. Notice that \(A_n(=)\) is the algebra of diagonal matrices, and \(A_n(\preceq)\) (where \(\preceq\) denotes the natural order) is the algebra of upper-triangular matrices. More generally, if \(\preceq\) is consistent with the natural order on \([1, 2, \ldots, n]\) in the sense that \(i \preceq j\) implies \(i \leq j\), then \(A_n(\preceq)\) is an algebra of upper-triangular matrices. (In the operator-theoretic context it is natural to restrict attention to such ‘consistent’ partial orders. This is not a severe restriction. Any incidence algebra arising from a finite partially ordered set is isomorphic to some matrix incidence algebra \(A_n(\preceq)\), where \(\preceq\) is consistent with the natural order. Indeed, if \((S, \ll\) is a finite partially ordered set and \(x_1, x_2, \ldots, x_n\) is an enumeration of \(S\) satisfying: \(x_i \ll x_j\) implies \(i \leq j\), then \(A(S)\) is isomorphic to \(A_n(\preceq)\), where \(\preceq\) is defined by \(i \preceq j\) if \(x_i \ll x_j\), by the map \(f \mapsto (f(x_i, x_j))\).

Throughout what follows, \(K\) will denote a fixed but arbitrary field in which \(0 \neq 1\), and all matrices will be assumed to have entries in \(K\), unless otherwise specified.

For any given matrix incidence algebra \(A_n(\preceq)\) we give a matricial characterisation (Corollary 1) of when an element of it is single. We then use this to show how
some single elements can be constructed (Lemma 1), and, if $K$ has at least three elements, to determine the maximum rank of a single element of $\mathcal{A}_m(\leq)$ (Theorem 2), in terms of the partial order. A characterisation of when every non-zero single element has rank 1 follows (Corollary 2). In Section 3, we discuss, amongst other things, some results concerning single elements that have their origins in operator theory.

2. Main results

Let $m, n \in \mathbb{Z}^+$ and let $\preceq_1$ and $\preceq_2$ be partial orders on $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$, respectively. Let $\mathcal{A}_m(\preceq_1)$ and $\mathcal{A}_n(\preceq_2)$ be the corresponding matrix incidence algebras (over $K$). In the following theorem $E_{n,q}$ will denote the matrix having $(p, q)$th entry equal to 1, with all other entries equal to 0. Also, sometimes $(A)_{i,j}$ is used to denote the $(i, j)$th entry of a matrix $A$, and $A^t$ denotes the transpose of $A$.

**Theorem 1.** Let $S = (s_{i,q})$ be an $m \times n$ matrix over the field $K$. Then $S$ has the property

$$A \in \mathcal{A}_m(\preceq_1), \quad B \in \mathcal{A}_n(\preceq_2) \quad \text{and} \quad ASB = 0$$

if and only if:

(i) $r_i \neq 0$ and $c_q \neq 0$ ($1 \leq i \leq m$, $1 \leq q \leq n$) $\Rightarrow$ $s_{i,q} \neq 0$;

(ii) $i \preceq_1 j_1$ and $i \preceq_1 j_2$ for some $1 \leq i \leq m$ $\Rightarrow$ $r_{j_1}$ and $r_{j_2}$ are linearly dependent;

(iii) $p_1 \preceq_2 q$ and $p_2 \preceq_2 q$ for some $1 \leq q \leq n$ $\Rightarrow$ $c_{p_1}$ and $c_{p_2}$ are linearly dependent.

Here $r_i$ and $c_q$ denote the $i$th row and the $q$th column of $S$, respectively.

**Proof.** Suppose first that $S$ has property (*).

Let $r_i \neq 0$ and $c_q \neq 0$. Now $E_{i,j} \in \mathcal{A}_m(\preceq_1)$ and $E_{q,q} \in \mathcal{A}_n(\preceq_2)$. Since $E_{i,j}S \neq 0$ and $SE_{q,q} \neq 0$, $E_{i,j}SE_{q,q} \neq 0$. Hence, $s_{i,q} \neq 0$. This proves (i).

Next, let $i \preceq_1 j_1$ and $i \preceq_1 j_2$. We may suppose that $r_{j_1}$ and $r_{j_2}$ are non-zero, and that $j_1 < j_2$. Note that, if $A \in \mathcal{A}_m(\preceq_1)$ and $Ac_k = 0$, with $1 \leq k \leq n$ and $c_k \neq 0$, then $AS = 0$. For then $ASE_{k,k} = (0, \ldots, 0, c_k, 0, \ldots, 0)$ and $ASE_{k,k} = (0, \ldots, 0, A c_k, 0, \ldots, 0) = 0$ so $AS = 0$. Let $c_k$ be any non-zero column of $S$. Then, by (i), $s_{j_1,k} \neq 0$ and $s_{j_2,k} \neq 0$. Let $A$ be the $m \times m$ matrix whose only non-zero row is the $i$th, that being (0, 0, 0, 0, 0, 0, 0, 0, 0), and whose non-zero entries (reading from left to right) occur in the $j_1$th and $j_2$th positions. Then $A \in \mathcal{A}_m(\preceq_1)$ and $Ac_k = 0$. Hence, $AS = 0$. In particular, the $i$th row of $AS$ is zero. This gives $s_{j_2,k}r_{j_1} = s_{j_1,k}r_{j_2}$. This proves (ii).

Let $p_1 \preceq_2 q$ and $p_2 \preceq_2 q$. We may suppose that $c_{p_1}$ and $c_{p_2}$ are non-zero, and that $p_1 < p_2$. Note that, if $B \in \mathcal{A}_n(\preceq_2)$ and $r_uB = 0$, with $1 \leq u \leq m$ and $r_u \neq 0$, then $SB = 0$. For then $E_{n,u}S = (0, \ldots, 0, r_u, 0, \ldots, 0)^t$ and $E_{n,u}SB = (0, \ldots, 0,$
(r_a B)^t, 0, \ldots, 0)^t = 0$, so $SB = 0$. Let $r_a$ be any non-zero row of $S$. Then, by (i), $s_{a, p_1} \neq 0$ and $s_{a, p_2} \neq 0$. Let $B$ be the $n \times n$ matrix whose only non-zero column is the $q$th, that being $(0, \ldots, 0, s_{a, p_2}, 0, \ldots, 0, -s_{a, p_1}, 0, \ldots, 0)^t$, where the non-zero entries (reading from top to bottom) occur in the $p_1$th and $p_2$th positions. Then $B \in \mathcal{A}_n(\leq 2)$ and $r_a B = 0$. Hence, $SB = 0$. In particular, the $q$th column of $SB$ is zero. This gives $s_{a, p_1} c_{p_1} = s_{a, p_2} c_{p_2}$. This proves (iii).

Conversely, suppose that conditions (i), (ii) and (iii) hold. For every $1 \leq i \leq m$ let $J_i = \{ j \in \{1, 2, \ldots, m\} : i \leq 1 \} \setminus \{ j \}$. If $J_i = \emptyset$, denote the (numerically) smallest element of $J_i$ by $\alpha_i$.

Claim 1. For every $A \in \mathcal{A}_m(\leq 1)$, there exist elements $A_i$, $i = 1, 2, \ldots, m$ of $K$ such that

\[ \text{ith row of } AS = A_i r_{j_i} \quad (1 \leq i \leq m), \tag{1} \]

where we take $A_i = 0$ if $J_i = \emptyset$.

Proof. Let $A = (a_{i, j}) \in \mathcal{A}_m(\leq 1)$. For $1 \leq i \leq m$ and $1 \leq p \leq n$,

\[
(A S)_{i, p} = \sum_{j=1}^{m} a_{i, j} s_{j, p} \\
= \sum_{j : i \leq 1} a_{i, j} s_{j, p} \quad \text{(since } a_{i, j} = 0 \text{ if } i \not\leq 1 \text{)} \\
= \sum_{j \in J_i} a_{i, j} s_{j, p} \quad \text{(since } s_{j, p} = 0 \text{ if } r_j = 0),
\]

where we take the empty sum to be zero. Clearly, if $J_i = \emptyset$ the $i$th row of $AS$ is zero. Suppose that $J_i \neq \emptyset$. By the definition of $J_i$, every element of $\{ r_j : j \in J_i \}$ is non-zero, and, by (ii), every pair of such elements is linearly dependent. Hence, there exist elements $\lambda_i, j (j \in J_i)$ of $K$ such that $r_j = \lambda_i, j r_{j_i}$ for every $j \in J_i$. Thus, $s_{j, p} = \lambda_i, j s_{j_i, p}$ and

\[
(A S)_{i, p} = \left( \sum_{j \in J_i} a_{i, j} \lambda_i, j \right) s_{j_i, p}
\]

for every $1 \leq p \leq n$. This proves the claim, taking $A_i = \sum_{j \in J_i} a_{i, j} \lambda_i, j$.

For every $1 \leq q \leq n$ let $P_q = \{ p \in \{1, 2, \ldots, n\} : p \leq q \text{ and } c_p \neq 0 \}$. If $P_q = \emptyset$ denote the (numerically) greatest element of $P_q$ by $p_q$. □

Claim 2. For every $B \in \mathcal{A}_n(\leq 2)$, there exist elements $\Gamma_q$, $q = 1, 2, \ldots, n$ of $K$ such that

\[ \text{qth column of } SB = \Gamma_q c_{p_q} \quad (1 \leq q \leq n), \tag{2} \]

where we take $\Gamma_q = 0$ if $P_q = \emptyset$. 

Proof. Let $B = (b_{p,q}) \in \mathcal{A}_n(\leq_2)$. For $1 \leq q \leq n$ and $1 \leq j \leq m,$

$$(SB)_{j,q} = \sum_{p=1}^{n} s_{j,p}b_{p,q}$$

$$(SB)_{j,q} = \sum_{(p: p \leq q)} s_{j,p}b_{p,q} \quad \text{(since } b_{p,q} = 0 \text{ if } p \leq 2 \text{ q)}$$

$$(SB)_{j,q} = \sum_{p \in P_q} s_{j,p}b_{p,q} \quad \text{(since } s_{j,p} = 0 \text{ if } c_p = 0),$$

where again the empty sum is taken to be zero. Clearly, the desired result follows if $P_q = \emptyset$. Suppose that $P_q \neq \emptyset$. By the definition of $P_q$, every element of $\{c_p: p \in P_q\}$ is non-zero, and, by (iii), every pair of such elements is linearly dependent. Hence, there exist elements $\gamma_{q,p} (p \in P_q)$ of $K$ such that $c_p = \gamma_{q,p}c_{pq}$ for every $p \in P_q$. Thus, $s_{j,p} = \gamma_{q,p}s_{j,pq}$ and

$$(SB)_{j,q} = \left( \sum_{p \in P_q} b_{p,q}\gamma_{q,p} \right) s_{j,pq}$$

for every $1 \leq j \leq m$. This proves the claim, taking $\Gamma_q = \sum_{p \in P_q} b_{p,q}\gamma_{q,p}$. \hfill \Box

Proof of Theorem 1 (continued). Now let $A \in \mathcal{A}_m(\leq_1)$ and let $B \in \mathcal{A}_n(\leq_2)$ with $AS \neq 0$ and $SB \neq 0$. We show that $ASB \neq 0$.

Since $AS \neq 0$, one of its rows, say the $i$th, is non-zero. Then, by (1), $A_i \neq 0$ and $r_{ji} \neq 0$. Since $SB \neq 0$, one of its columns, say the $q$th, is non-zero. Then, by (2), $\Gamma_q \neq 0$ and $c_{pq} \neq 0$. By (i), $s_{j,p} \neq 0$. Now

$$(ASB)_{i,q} = \sum_{u=1}^{n} (AS)_{i,u}b_{u,q}$$

$$(ASB)_{i,q} = A_i \sum_{u=1}^{n} s_{j,u}b_{u,q} \quad \text{(by (1))}$$

$$(ASB)_{i,q} = A_i (SB)_{j,q}$$

$$(ASB)_{i,q} = A_i \Gamma_q s_{j,pq} \quad \text{(by (2))},$$

so $ASB \neq 0$. This completes the proof. \hfill \Box

The symmetry of conditions (i), (ii) and (iii) in the statement of the preceding theorem is a reflection of the symmetry of condition ($*$) and the fact that, for every $n \in \mathbb{Z}^+$ and every partial order $\leq$ on $\{1, 2, \ldots, n\}$,

$$\mathcal{A}_n(\leq^d) = (\mathcal{A}_n(\leq))^t = \{A^t: A \in \mathcal{A}_n(\leq)\},$$
where \( \preceq^d \) is the partial order on \( \{ 1, 2, \ldots, n \} \) dual to \( \preceq \), that is, defined by \( i \preceq^d j \) if \( j \preceq i \).

Taking \( m = n \) and \( \preceq^1 = \preceq^2 \) in the theorem gives the following corollary.

**Corollary 1.** Let \( \mathcal{A}_n(\preceq) \) be a matrix incidence algebra over a field \( K \). The matrix \( S \in \mathcal{A}_n(\preceq) \) is a single element of \( \mathcal{A}_n(\preceq) \) if and only if:

(i) \( r_i \neq 0 \) and \( c_j \neq 0 \) for \( 1 \leq i \leq n \) \( \Rightarrow s_{i,j} \neq 0 \);

(ii) \( i \leq j_1 \) and \( i \leq j_2 \) for some \( 1 \leq i \leq n \) \( \Rightarrow r_{j_1} \) and \( r_{j_2} \) are linearly dependent;

(iii) \( i_1 \leq j \) and \( i_2 \leq j \) for some \( 1 \leq j \leq n \) \( \Rightarrow c_{i_1} \) and \( c_{i_2} \) are linearly dependent.

Here \( r_i \) and \( c_j \) denote the \( i \)th row and the \( j \)th column of \( S \), respectively.

Note that it follows almost immediately from this corollary that every rank 1 element of a matrix incidence algebra is a single element (this is just as easily proved directly from the definition). It also follows, by condition (i), that if \( S = (s_{i,j}) \) is a single element of \( \mathcal{A}_n(\preceq) \), then there exist subsets \( I \) and \( J \) of \( \{ 1, 2, \ldots, n \} \) such that \( s_{i,j} \neq 0 \) if and only if \( (i, j) \in I \times J \). Clearly then, in any matrix incidence algebra over the field \( \mathbb{Z}/2 \) (that is, the integers modulo 2), every non-zero single element has rank 1. Over other fields, singles of higher rank may exist as we shall soon see. Note also that, if \( S = (s_{i,j}) \in \mathcal{A}_n(\preceq) \), the existence of subsets \( I \) and \( J \) of \( \{ 1, 2, \ldots, n \} \) such that \( s_{i,j} \neq 0 \) if and only if \( (i, j) \in I \times J \) does not guarantee that \( S \) is a single element. For example, if \( a \in K \) and \( a \neq 0, 1 \) the matrix

\[
\begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & a \\
0 & 0 & 0
\end{bmatrix}
\]

is not a single element of \( \mathcal{A}_n(\preceq) \). (Every non-zero single element has rank 1 in this case, by Corollary 3.) However, the next result gives a partial converse and a way of constructing single elements.

**Lemma 1.** Let \( \mathcal{A}_n(\preceq) \) be a matrix incidence algebra over a field \( K \). Let \( X_0 \), respectively \( Y_0 \), be the set of minimal elements, respectively maximal elements, of the poset \( (\{ 1, 2, \ldots, n \}, \preceq) \). Let \( X \subseteq X_0 \) and \( Y \subseteq Y_0 \) be non-empty subsets satisfying \( x \preceq y \) for every \( x \in X \), \( y \in Y \). Every \( n \times n \) matrix \( S = (s_{i,j}) \) satisfying \( s_{i,j} \neq 0 \) if and only if \( (i, j) \in X \times Y \) is a single element of \( \mathcal{A}_n(\preceq) \).

**Proof.** Clearly \( S \in \mathcal{A}_n(\preceq) \) and condition (i) of the preceding corollary holds.

Let \( i \leq j_1 \) and \( i \leq j_2 \), for some \( 1 \leq i \leq n \), with \( j_1 \neq j_2 \). By minimality, at least one of \( j_1, j_2 \) does not belong to \( X \), so at least one of the rows \( r_{j_1}, r_{j_2} \) is zero. Hence, \( r_{j_1} \) and \( r_{j_2} \) are linearly dependent. This proves that condition (ii) holds.

The proof that condition (iii) holds is similar, and the proof is complete. \( \square \)

The following theorem is analogous to Theorem 3 of [10]. In its proof we use the easily verifiable fact that, if \( s \) is a single element of an (abstract) algebra \( \mathcal{A} \), then
as and sb are also single elements for every a, b ∈ A. We also use the fact that, for every field with at least three elements and every k ∈ \(\mathbb{Z}^+\), there exists a \(k \times k\) matrix over the field, of rank k and with all its entries non-zero. Indeed, if \(a \neq 0, 1\), the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & a & 1 & \cdots & 1 & 1 \\
1 & 1 & a & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & a \\
\end{bmatrix}
\]

is such a matrix.

**Theorem 2.** Let \(\mathcal{A}_n(\leq)\) be a matrix incidence algebra over a field \(K\) with \(|K| \geq 3\). Let \(X_0\), respectively \(Y_0\), be the set of minimal elements, respectively maximal elements, of the poset \(([1, 2, \ldots, n], \leq)\). Let \(m\) be the largest positive integer for which there exist subsets \(X \subseteq X_0\) and \(Y \subseteq Y_0\) with \(|X| = |Y| = m\) satisfying \(x \leq y\) for every \(x \in X\), \(y \in Y\). For every \(0 \leq k \leq m\), \(\mathcal{A}_n(\leq)\) contains a single element of rank \(k\). Moreover, every single element of \(\mathcal{A}_n(\leq)\) has rank less than or equal to \(m\).

**Proof.** The preceding lemma, and our remark immediately preceding this theorem, show that \(\mathcal{A}_n(\leq)\) contains a single element of rank \(k\) for every \(0 \leq k \leq m\).

Let \(S = (s_{i,j})\) be a non-zero single element of \(\mathcal{A}_n(\leq)\) with rank \(k\). Then, as remarked earlier, there exist subsets \(I\) and \(J\) of \(\{1, 2, \ldots, n\}\) such that \(s_{i,j} \neq 0\) if and only if \((i, j) \in I \times J\). Using the fact that \(ASB\) is a single element of \(\mathcal{A}_n(\leq)\), for all \(0, 1\)-diagonal matrices \(A\) and \(B\), we may suppose that \(|I| = |J| = k\). Since \(S \in \mathcal{A}_n(\leq)\), \(i \leq j\) for every \(i \in I\), \(j \in J\). By condition (ii) of Corollary 1, if \(i_1, i_2 \in I\) and \(i_1 \neq i_2\), the set of minimal elements less than or equal to \(i_1\) is disjoint from the set of minimal elements less than or equal to \(i_2\) (with respect to the partial order \(\leq\)). For each \(i \in I\), let \(x_i\) be a minimal element satisfying \(x_i \leq i\), and let \(X = \{x_i: i \in I\}\). Then \(|X| = k\).

Similarly, using condition (iii) of the corollary, there is a set of maximal elements \(Y = \{y_j: j \in J\}\) such that \(|Y| = k\) and \(j \leq y_j\) for every \(j \in J\). Then \(x \leq y\) for every \(x \in X\), \(y \in Y\). For, \(x = x_i\) and \(y = y_j\) for some \(i \in I\), \(j \in J\), so \(x = x_i \leq i \leq j \leq y_j = y\) gives \(x \leq y\). It follows that \(k \leq m\) and the proof is complete.}

The next result is an algebraic version of the answer, given in [10], to a question raised by R. L. Moore. It is slightly easier to state than that given in [10].

**Corollary 2.** Let \(\mathcal{A}_n(\leq)\) be a matrix incidence algebra over a field \(K\) with \(|K| \geq 3\). Every non-zero single element of \(\mathcal{A}_n(\leq)\) has rank 1 if and only if the following condition fails to hold:

There exist distinct minimal elements \(x_1, x_2\) and distinct maximal elements \(y_1, y_2\) such that \(x_i \leq y_j\) for every \(1 \leq i, j \leq 2\). (**)
Proof. By Theorem 2, \( \mathcal{A}_n(\leq) \) contains a single element of rank greater than 1 if and only if it contains a single element of rank 2. This is, in turn, equivalent to condition (**) in Theorem 2. 

A second corollary follows immediately.

**Corollary 3.** Let \( \mathcal{A}_n(\leq) \) be a matrix incidence algebra over a field \( K \). If the poset \( (\{1, 2, \ldots, n\}, \leq) \) has a unique minimal element or a unique maximal element, then every non-zero single element of \( \mathcal{A}_n(\leq) \) has rank 1.

### 3. Concluding comments

1. Corollary 2 and an earlier remark above show that every non-zero single element of \( \mathcal{A}_n(=) \), that is, the algebra of all diagonal \( n \times n \) matrices over any field \( K \), has rank 1. On the other hand, Corollary 3 shows that the same result holds for non-zero single elements of \( \mathcal{A}_n(\leq) \), that is, the algebra of all \( n \times n \) upper-triangular matrices over \( K \). (The analogous operator-theoretic versions of these facts were first proved in [6] and [14], respectively.)

2. The simplest example of an incidence algebra containing a single element of rank 2 is the algebra, over any field \( K \) with at least three elements, arising from the poset shown in Fig. 1(a). This algebra consists of all matrices of the form

\[
\begin{bmatrix}
* & 0 & * & * \\
0 & * & * & * \\
0 & ** & 0 & 0 \\
0 & 0 & 0 & *
\end{bmatrix}
\]

where ‘*’ denotes an arbitrary element of \( K \). By Lemma 1,

\[
\begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

is a single element of this algebra, whenever \( a \neq 0, 1 \). (This example was first given, for the complex field, in [3].)

3. The simplest example of an incidence algebra containing a single element of rank \( m \) is the algebra, over any field with at least three elements, arising from the poset shown in Fig. 1(b). This algebra consists of all \( 2m \times 2m \) matrices of the form

\[
\begin{bmatrix}
A & C \\
0 & B
\end{bmatrix}
\]

where \( A, B \) are arbitrary \( m \times m \) diagonal matrices and \( C \) is an arbitrary \( m \times m \) matrix. By Lemma 1,
is a single element of the algebra if every entry of $C$ is non-zero. (This example was first given, for the complex field, in [7].)

4. Neither of the three conditions in the statement of Corollary 1 is redundant. Once again, consider incidence algebras over fields with at least three elements.

(a) Condition (i) is not redundant: if $S$ is any $n \times n$ (diagonal) matrix of rank greater than 1, then $S$ is not a single element of $\mathcal{A}_n(=)$ yet conditions (ii) and (iii) are (automatically) satisfied.

(b) Neither condition (ii) nor condition (iii) is redundant: in the incidence algebra arising from the poset shown in Fig. 1(c), the matrix

$$
\begin{bmatrix}
0 & C \\
0 & 0
\end{bmatrix}
$$

where $a \neq 0, 1$, satisfies conditions (i) and (iii) but not (ii). On the other hand, in the incidence algebra arising from the poset shown in Fig. 1(d), the same matrix satisfies conditions (i) and (ii), but not (iii). (Note that every non-zero single element in each of these algebras has rank 1, by Corollary 3.)
References