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The Slice Map Problem and Approximation Properties

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We show that a σ-weakly closed subspace $\mathscr{S}$ of the bounded operators $B(\mathscr{H})$ on an infinite dimensional Hilbert space $\mathscr{H}$ has Property S_σ (the Fubini product $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \bar{\otimes} \mathscr{T}$ for all σ-weakly closed subspaces $\mathscr{T} \subset B(\mathscr{H})$) if and only if $\mathscr{S}$ satisfies a certain approximation property, which we call the σ-weak approximation property. The σ-weak approximation property is implied by (but does not imply) the (weak-$*$) completely bounded approximation property. Moreover, if $\mathscr{S}$ has the σ-weak approximation property, then the predual $\mathscr{S}_*$ has the approximation property for Banach spaces. We also prove analagous results for Property S (and variations of Property S) for C^*-algebras. As an application of our characterization of subspaces with Property S_σ, we show that the reflexive algebra tensor product formula

$$\text{alg}\, \mathscr{L}_1 \bar{\otimes} \text{alg}\, \mathscr{L}_2 = \text{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$$

is not always valid. In fact, we show that for each of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$, there is a separably acting factor $\mathscr{M} = \text{alg}\, \mathscr{L}_1$ of that type and a reflexive algebra $\mathscr{A} = \text{alg}\, \mathscr{L}_2$ such that $\text{alg}\, \mathscr{L}_1 \bar{\otimes} \text{alg}\, \mathscr{L}_2$ is strictly contained in $\text{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$.

0. INTRODUCTION

One of the fundamental results in the theory of tensor products of von Neumann algebras is Tomita's commutation theorem: if $\mathscr{M}_1$ and $\mathscr{M}_2$ are von Neumann algebras, then

$$(\mathscr{M}_1)' \bar{\otimes} (\mathscr{M}_2)' = (\mathscr{M}_1 \bar{\otimes} \mathscr{M}_2)'. \tag{0.1}$$

Gilfeather, Hopenwasser, and Larson observed in [22] that if we let $\mathscr{L}_i$ denote the projection lattice of $\mathscr{M}_i$ $(i = 1, 2)$, then (0.1) can be rewritten as

$$\text{alg}\, \mathscr{L}_1 \bar{\otimes} \text{alg}\, \mathscr{L}_2 = \text{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2), \tag{0.2}$$

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where $\mathscr{L}_1 \otimes \mathscr{L}_2$ denotes the subspace lattice generated by $\{e_1 \otimes e_2 \mid e_i \in \mathscr{L}_i\}$. Equation (0.2) makes sense for arbitrary pairs of reflexive algebras, and the following reflexive algebra tensor product problem was raised in [22]: For which pairs of reflexive algebras alg $\mathscr{L}_1$ and alg $\mathscr{L}_2$ is (0.2) valid? Slice maps have proved to be very useful in studying this problem [29, 30, 34–36].

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, and φ is a σ-weakly continuous linear functional on $B(\mathscr{H}_1)$ (so $\varphi \in B(\mathscr{H}_1)_*$), the right slice map R_φ associated with φ is the (unique) σ-weakly continuous linear map from $B(\mathscr{H}_1) \bar{\otimes} B(\mathscr{H}_2)$ to $B(\mathscr{H}_2)$ such that

$$R_\varphi(a \otimes b) = \langle a, \varphi \rangle b \qquad (a \in B(\mathscr{H}_1),\ b \in B(\mathscr{H}_2)).$$

The left slice maps $L_\psi \colon B(\mathscr{H}_1) \bar{\otimes} B(\mathscr{H}_2) \to B(\mathscr{H}_1)$ ($\psi \in B(\mathscr{H}_2)_*$) are similarly defined. If $\mathscr{S} \subset B(\mathscr{H}_1)$ and $\mathscr{T} \subset B(\mathscr{H}_2)$ are σ-weakly closed subspaces, the Fubini product $F(\mathscr{S}, \mathscr{T})$ of $\mathscr{S}$ and $\mathscr{T}$ is the set of all operators x in $B(\mathscr{H}_1) \bar{\otimes} B(\mathscr{H}_2)$ all of whose right slices $R_\varphi(x)$ are in $\mathscr{T}$, and all of whose left slices $L_\psi(x)$ are in $\mathscr{S}$. It is immediate from its definition that $F(\mathscr{S}, \mathscr{T})$ contains the algebraic tensor product $\mathscr{S} \otimes \mathscr{T}$ of $\mathscr{S}$ and $\mathscr{T}$, and it is easy to show that $F(\mathscr{S}, \mathscr{T})$ is σ-weakly closed. Hence $F(\mathscr{S}, \mathscr{T})$ always contains the σ-weak closure $\mathscr{S} \bar{\otimes} \mathscr{T}$ of $\mathscr{S} \otimes \mathscr{T}$. The slice map problem is to find all pairs of subspaces $\mathscr{S}$ and $\mathscr{T}$ for which

$$\mathscr{S} \bar{\otimes} \mathscr{T} = F(\mathscr{S}, \mathscr{T}). \tag{0.3}$$

The slice map problem is of interest because a number of questions concerning tensor products of σ-weakly closed subspaces are special cases of the slice map problem. Tomiyama proved in [52] that Tomita's theorem is equivalent to the fact that (0.3) is valid whenever $\mathscr{S}$ and $\mathscr{T}$ are von Neumann algebras. Moreover, it was shown by the author in [34] that

$$F(\operatorname{alg} \mathscr{L}_1, \operatorname{alg} \mathscr{L}_2) = \operatorname{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2) \tag{0.4}$$

for all pairs of reflexive algebras alg $\mathscr{L}_1$ and alg $\mathscr{L}_2$. Hence the tensor product problem for reflexive algebras is a special case of the slice map problem. Other special cases of the slice map problem are discussed in [29, 34–36].

A σ-weakly closed subspace $\mathscr{S} \subset B(\mathscr{H}_1)$ is said to have Property S_σ [34] if (0.3) is valid for all σ-weakly closed subspaces $\mathscr{T} \subset B(\mathscr{H}_2)$ (where $\mathscr{H}_2$ can be any Hilbert space). It follows from (0.4) that if alg $\mathscr{L}_1$ has Property S_σ, then (0.2) is valid for alg $\mathscr{L}_1$ and all reflexive algebras alg $\mathscr{L}_2$. The author showed in [36] that the converse is true. (See also Remark 1.1 below.) A number of classes of reflexive algebras have been shown to have Property S_σ [29, 34–36]. However, it has remained an open question whether every reflexive algebra has Property S_σ, and hence whether (0.2) is always valid.

One of the main results of this paper is that not only are there reflexive algebras without Property S_σ, but there are von Neumann algebras without Property S_σ. It was shown in [34] that all type I von Neumann algebras have Property S_σ. In contrast to this we show that for each of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$, there is a separably acting factor of that type without Property S_σ. It follows that the reflexive algebra tensor product formula (0.2) can fail even when one of the reflexive algebras is a von Neumann algebra.

Section 1 contains a discussion of the previously known results concerning the tensor product problem for reflexive algebras, as well as some basic definitions and notation.

The main result in Section 2 is that a subspace $\mathscr{S}$ has Property S_σ if and only if it has a certain approximation property, which we call the σ-weak approximation property. The completely bounded approximation property (CBAP) implies the σ-weak approximation property (Theorem 2.10), but there are von Neumann algebras which have the σ-weak approximation property (and so have Property S_σ), but do not have the CBAP (Example 2.11).

In Section 3 we prove the result concerning von Neumann algebras without Property S_σ mentioned above. We first show that if $\mathscr{S}$ is a σ-weakly closed subspace of $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, and $\mathscr{S}$ has Property S_σ, then $\mathscr{S}_*$ (the Banach space of σ-weakly continuous linear functionals on $\mathscr{S}$) has the approximation property for Banach spaces. Since $B(\mathscr{H})$ does not have the approximation property if $\mathscr{H}$ is infinite dimensional [48], $B(\mathscr{H})^*$ also does not have the approximation property, and so $B(\mathscr{H})^{**}$ is a von Neumann algebra without Property S_σ. Of course $B(\mathscr{H})^{**}$ is not a factor, and is not separably acting. In order to prove the existence of separably acting factors without Property S_σ, we use various stability properties of the class of von Neumann algebras with Property S_σ to prove that if for any of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$ all separably acting factors of that type have Property S_σ, then every von Neumann algebra has Property S_σ, which contradicts our result about $B(\mathscr{H})^{**}$. A stability result which we prove in Section 3 which is of interest in its own right is that if almost all the factors in the central decomposition of a separably acting von Neumann algebra $\mathscr{M}$ have Property S_σ, then $\mathscr{M}$ has Property S_σ. The proof of this result makes use of the direct integral theory for strongly closed algebras developed by Azoff, Fong, and Gilfeather in [2].

In Section 4 we consider the question of which singly generated unital algebras have Property S_σ. The author showed in [36] that if $t \in B(\mathscr{H})$ is a subnormal operator or an operator of class $\mathbf{A}(\mathscr{H})$ [3], then the σ-weakly closed unital algebra $\mathscr{A}(t)$ generated by t has Property S_σ. Using a technique of Wogen [60], we show that the existence of a subspace of

$B(\mathscr{H})$ without Property S_σ implies that there is an operator t for which $\mathscr{A}(t)$ does not have Property S_σ. We also show, in a positive direction, that if t is a injective weighted shift, then $\mathscr{A}(t)$ has the CCAP, and so has Property S_σ.

In Section 5 we consider the slice map problem for C^*-algebras. We define subspace versions of the slice map conjecture and of Property S (which were defined for C^*-subalgebras in [57]), and prove that a C^*-algebra A has Property S for subspaces of the compact operators K (i.e., (A, K, T) verifies the slice map conjecture whenever T is a norm closed subspace of K) if and only if A has the approximation property for operator spaces defined by Effros and Ruan in [21]. This result is of interest because Effros and Ruan show in [21] that their approximation property is the natural analogue for operator spaces of Grothendieck's approximation property for Banach spaces. In particular, just as there are a number of properties of Banach spaces that are equivalent to the approximation property, there are a number of analogous properties of operator spaces that are equivalent to Effros and Ruan's approximation property. Moreover, as noted in [21], the σ-weak approximation property (which we called the complete pointwise approximation property in [37]) is just the normal version of the approximation property for operator spaces.

The main results in this paper were announced in [37].

1. Preliminaries and Notation

Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace, and let $\mathscr{K}$ be a Hilbert space. Then for any $\varphi \in \mathscr{S}_*$ and any $x \in \mathscr{S} \bar{\otimes} B(\mathscr{K})$, the map $\psi \to \langle x, \varphi \otimes \psi \rangle$ is a continuous linear functional on $B(\mathscr{K})_*$, and so defines an element $R_\varphi(x)$ of $B(\mathscr{K})$. It is easily checked that R_φ is a σ-weakly continuous linear map from $\mathscr{S} \bar{\otimes} B(\mathscr{K})$ to $B(\mathscr{K})$, and that

$$R_\varphi(s \otimes b) = \langle s, \varphi \rangle b \qquad (s \in \mathscr{S}, b \in B(\mathscr{K})). \tag{1.1}$$

If $\mathscr{S} \neq B(\mathscr{H})$ and $\varphi \in \mathscr{S}_*$, then φ does not have a unique extension to a σ-weakly continuous linear functional on $B(\mathscr{H})$. However, if ρ is any element of $B(\mathscr{H})_*$ that extends φ, then it follows from (1.1) that the right slice map R_ρ (from $B(\mathscr{H}) \bar{\otimes} B(\mathscr{K})$ to $B(\mathscr{K})$) agrees with R_φ on $\mathscr{S} \bar{\otimes} B(\mathscr{K})$. Moreover, since $B(\mathscr{K})$ has Property S_σ, $F(\mathscr{S}, \mathscr{T}) \subset \mathscr{S} \bar{\otimes} B(\mathscr{K})$ whenever $\mathscr{T}$ is a σ-weakly closed subspace of $B(\mathscr{K})$ [34, Remark 1.5]. Hence for any σ-weakly closed subspace $\mathscr{T} \subset B(\mathscr{K})$ we have that

$$F(\mathscr{S}, \mathscr{T}) = \{x \in \mathscr{S} \bar{\otimes} B(\mathscr{K}) \mid R_\varphi(x) \in \mathscr{T} \text{ for all } \varphi \in \mathscr{S}_*\}.$$

If $\mathcal{S} \subset B(\mathcal{H})$ is a σ-weakly closed subspace, and $\mathcal{N} \subset B(\mathcal{H})$ is a von Neumann algebra, we say that $\mathcal{S}$ has *Property* S_σ *for* $\mathcal{N}$ if $F(\mathcal{S}, \mathcal{T}) = \mathcal{S} \bar{\otimes} \mathcal{T}$ for all σ-weakly closed subspaces of $\mathcal{N}$. (An argument similar to that in the preceding paragraph shows that this definition does not depend on what Hilbert space $\mathcal{N}$ is realized on. See also Remark 1.2 in [34].) This concept is of interest because we will show in Section 3 that $\mathcal{S}_*$ has the approximation property if and only if $\mathcal{S}$ has Property S_σ for $l^\infty(\mathbf{N})$. Of course $\mathcal{S}$ has Property S_σ in the usual sense if and only if it has Property S_σ for all von Neumann algebras. There are no examples known of subspaces which have Property S_σ for $l^\infty(\mathbf{N})$ but do not have Property S_σ, although it seems likely that such examples exist. However, we will show in Section 2 that if $\mathcal{K}$ is a separable infinite dimensional Hilbert space, and if $\mathcal{S}$ has Property S_σ for $\mathcal{N} = B(\mathcal{K})$, then $\mathcal{S}$ has Property S_σ.

Let $\mathcal{H}$ be a Hilbert space. A collection $\mathcal{L}$ of (orthogonal) projections on $\mathcal{H}$ is said to be a *subspace lattice* if it is strongly closed, contains 0 and the identity operator 1, and is closed under the usual lattice operations for projections. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is said to be a *commutative subspace lattice* (or CSL). If $\mathcal{L}$ is a subspace lattice, $\operatorname{alg} \mathcal{L}$ denotes the set of operators in $B(\mathcal{H})$ that leave the ranges of all the projections in $\mathcal{L}$ invariant. It is easily checked that $\operatorname{alg} \mathcal{L}$ is a σ-weakly closed unital subalgebra of $B(\mathcal{H})$, and that

$$\operatorname{alg} \mathcal{L} = \{a \in B(\mathcal{H}) \mid ae = eae \text{ for all } e \in \mathcal{L}\}.$$

If $\mathcal{L}$ is a CSL, $\operatorname{alg} \mathcal{L}$ is said to be a *CSL algebra*.

A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be *reflexive* if $\mathcal{A} = \operatorname{alg} \operatorname{lat} \mathcal{A}$, where $\operatorname{lat} \mathcal{A}$ denotes the subspace lattice consisting of the projections left invariant by all the operators in $\mathcal{A}$. The reflexive algebras are precisely the algebras of the form $\operatorname{alg} \mathcal{L}$ for some subspace lattice $\mathcal{L}$. Every von Neumann algebra is reflexive, and every self-adjoint reflexive algebra is a von Neumann algebra.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are reflexive algebras, we say that the *reflexive algebra tensor product formula* (the RTPF) is valid for $\mathcal{A}_1$ and $\mathcal{A}_2$ if $F(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{A}_1 \bar{\otimes} \mathcal{A}_2$. It follows from Eq. (0.4) that the RTPF is valid for $\mathcal{A}_1$ and $\mathcal{A}_2$ if and only if $\operatorname{alg} \mathcal{L}_1 \bar{\otimes} \operatorname{alg} \mathcal{L}_2 = \operatorname{alg}(\mathcal{L}_1 \otimes \mathcal{L}_2)$ whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices such that $\mathcal{A}_1 = \operatorname{alg} \mathcal{L}_1$ and $\mathcal{A}_2 = \operatorname{alg} \mathcal{L}_2$. As noted in the Introduction, Tomita's theorem is equivalent to the statement that the RTPF is valid for every pair of von Neumann algebras.

If $\mathcal{A}$ is a reflexive algebra with Property S_σ, and if $\mathcal{B}$ is any reflexive algebra, then $F(\mathcal{A}, \mathcal{B}) = \mathcal{A} \bar{\otimes} \mathcal{B}$, and so the RTPF is valid for $\mathcal{A}$ and $\mathcal{B}$. It was shown in [35] that $\operatorname{alg} \mathcal{L}$ has Property S_σ whenever $\mathcal{L}$ is a completely distributive CSL. This generalized results in [22, 30, 33]. It was shown in [29] that if $\mathcal{M}$ is a von Neumann algebra such that $\mathcal{N}' \cap \mathcal{M}$ has

Property S_σ whenever $\mathcal{N}$ is an abelian von Neumann subalgebra of $\mathcal{M}$ (in which case $\mathcal{M}$ is said to have Property *RC*), and if $\mathcal{L} \subset \mathcal{M}$ is a finite width CSL (i.e., if $\mathcal{L}$ is generated by a finite number of commuting chains of projections in $\mathcal{M}$), then the reflexive algebra (alg $\mathcal{L}$) $\cap \mathcal{M}$ has Property S_σ. It was also shown in [29] that the class of von Neumann algebras with Property *RC* is closed under taking direct sums, and includes all injective von Neumann algebras and all finite von Neumann algebras with Property S_σ. In particular, $B(\mathcal{H})$ has Property *RC*, and so alg $\mathcal{L}$ has Property S_σ whenever $\mathcal{L}$ is a finite width CSL. It is also known that certain classes of reflexive algebras that are singly generated (as unital σ-weakly closed algebras) have Property S_σ. These results are discussed in Section 4 below.

Remark 1.1. Let $\mathcal{A}$ be a reflexive algebra without Property S_σ, and let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. We will show in Section 2 that there is a σ-weakly closed subspace $\mathcal{T} \subset B(\mathcal{H})$ such that $F(\mathcal{A}, \mathcal{T}) \neq \mathcal{A} \bar{\otimes} \mathcal{T}$. The subspace $\mathcal{T}$ need not be a reflexive algebra. However, it was shown in the proof of Theorem 2.2 in [36] that one can associate an abelian reflexive subalgebra $\mathcal{B} \subset B(\mathcal{H})$ to $\mathcal{T}$ which has the property that if $F(\mathcal{S}, \mathcal{T}) \neq \mathcal{S} \bar{\otimes} \mathcal{T}$ (where $\mathcal{S}$ is a σ-weakly closed subspace of $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$), then $F(\mathcal{S}, \mathcal{B}) \neq \mathcal{S} \bar{\otimes} \mathcal{B}$. In particular, the RTPF is not valid for $\mathcal{A}$ and $\mathcal{B}$. Hence to find a pair of reflexive algebras for which the RTPF is not valid, it suffices to find a reflexive algebra without Property S_σ, which we will do in Section 3. In the proof of Theorem 3.10 below we will need to use the fact that $\mathcal{B}$ has the additional property that if $F(\mathcal{S}, \mathcal{T}) = \mathcal{S} \bar{\otimes} \mathcal{T}$, then $F(\mathcal{S}, \mathcal{B}) = \mathcal{S} \bar{\otimes} \mathcal{B}$. For the convenience of the reader, we will review the construction of $\mathcal{B}$ and give a proof of this fact.

Let $\mathcal{T} \subset B(\mathcal{H})$ be a σ-weakly closed subspace, let $\mathcal{H}_1$ denote the direct sum of two copies of $\mathcal{H}$, and let $\mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}$. Viewing the elements of $B(\mathcal{H}_1)$ as 2×2 matrices with entries in $B(\mathcal{H})$ in the usual way, we first define an algebra $\mathcal{B}_0 \subset B(\mathcal{H}_1)$ by

$$\mathcal{B}_0 = \{ [b_{ij}] \mid b_{11} = b_{22} \in \mathbf{C}1,\ b_{21} = 0, \text{ and } b_{12} \in \mathcal{T} \}.$$

Let $\mathcal{B}_1 = \mathcal{B}_0 \bar{\otimes} \mathbf{C}1$. Then $\mathcal{B}_1$ is a an abelian reflexive subalgebra of $B(\mathcal{H}_2)$ (see the proof of Theorem 2.2. in [36]). Let $\mathcal{K}$ be a Hilbert space, and let $\mathcal{S} \subset B(\mathcal{K})$ be a σ-weakly closed subspace. Then it is easy to see that each element of $\mathcal{S} \bar{\otimes} B(\mathcal{H}_1) \bar{\otimes} \mathbf{C}1$ can be written uniquely in the form $x \otimes 1$, where $x = [x_{ij}]$ is a 2×2 matrix with entries in $\mathcal{S} \bar{\otimes} B(\mathcal{H})$, and that

$$\mathcal{S} \bar{\otimes} \mathcal{B}_1 = \{ x \otimes 1 \mid x_{11} = x_{22} \in \mathcal{S} \bar{\otimes} \mathbf{C}1,\ x_{21} = 0, \text{ and } x_{12} \in \mathcal{S} \bar{\otimes} \mathcal{T} \}. \quad (1.2)$$

For $\varphi \in B(\mathcal{K})_*$, let R_φ (respectively r_φ) denote the associated right slice map from $\mathcal{S} \bar{\otimes} B(\mathcal{H}_1) \bar{\otimes} \mathbf{C}1$ to $B(\mathcal{H}_1) \bar{\otimes} \mathbf{C}1$ (respectively from $\mathcal{S} \bar{\otimes} B(\mathcal{H})$

to $B(\mathscr{H})$). A straightforward calculation shows that if x_{ij} is in the algebraic tensor product $\mathscr{S} \otimes B(\mathscr{H})$ for $1 \leqslant i, j \leqslant 2$, and if $x = [x_{ij}]$, then

$$R_\varphi(x \otimes 1) = [r_\varphi(x_{ij})] \otimes 1. \tag{1.3}$$

Since R_φ and r_φ are σ-weakly continuous linear maps, it is immediate that (1.3) is valid for any operator $x = [x_{ij}]$ in $\mathscr{S} \bar{\otimes} B(\mathscr{H}_1)$. Hence, by the definition of $\mathscr{B}_1$, $R_\varphi(x \otimes 1) \in \mathscr{B}_1$ if and only if $r_\varphi(x_{11}) = r_\varphi(x_{22}) \in \mathbf{C}1$, $r_\varphi(x_{21}) = 0$, and $r_\varphi(x_{12}) \in \mathscr{T}$. Since this is true for every φ in $B(\mathscr{K})_*$, and since $F(\mathscr{S}, \mathbf{C}1) = \mathscr{S} \bar{\otimes} \mathbf{C}1$ and $F(\mathscr{S}, \{0\}) = \{0\}$, we have that

$$F(\mathscr{S}, \mathscr{B}_1) = \{x \otimes 1 \mid x_{11} = x_{22} \in \mathscr{S} \bar{\otimes} \mathbf{C}1,\ x_{21} = 0, \text{ and } x_{12} \in F(\mathscr{S}, \mathscr{T})\}. \tag{1.4}$$

A comparison of (1.2) and (1.4) shows that $F(\mathscr{S}, \mathscr{B}_1) = \mathscr{S} \bar{\otimes} \mathscr{B}_1$ if and only if $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \bar{\otimes} \mathscr{T}$. Now let u be a unitary operator from $\mathscr{H}$ onto $\mathscr{H}_2$, and let $\mathscr{B} = u^* \mathscr{B}_1 u$. Then it is easily checked that $\mathscr{B}$ is an abelian reflexive subalgebra of $B(\mathscr{H})$ and that $F(\mathscr{S}, \mathscr{B}) = \mathscr{S} \bar{\otimes} \mathscr{B}$ if and only if $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \bar{\otimes} \mathscr{T}$, as claimed.

If we assume further that $\mathscr{S}$ is a reflexive algebra, then $\mathscr{S} \bar{\otimes} \mathscr{B}_1$ is a reflexive algebra (see Remark 2.4 of [36]), and hence $\mathscr{S} \bar{\otimes} \mathscr{B}$ is also a reflexive algebra, since it is unitarily equivalent to $\mathscr{S} \bar{\otimes} \mathscr{B}_1$.

Remark 1.2. An important open question concerning the reflexive algebra tensor product problem is whether the RTPF is valid for every pair of CSL algebras. Of course if every CSL algebra has Property S_σ, then this is true. However, the RTPF is valid for every pair of von Neumann algebras, even though there are von Neumann algebras without Property S_σ. Hence it is possible that there are CSL algebras without Property S_σ (so that (0.2) can fail even when one of $\mathscr{L}_1$ or $\mathscr{L}_2$ is a CSL), but that the RTPF is valid for every pair of CSL algebras (so (0.2) always holds when both $\mathscr{L}_1$ and $\mathscr{L}_2$ are CSLs).

2. An Approximation Property Equivalent to Property S_σ

For a norm closed subspace S of $B(\mathscr{H})$, we let $\mathbf{M}_n(S)$ denote the space of $n \times n$ matrices with entries in S, with the norm inherited from $\mathbf{M}_n(B(\mathscr{H}))$. If $S \subset B(\mathscr{H}_1)$ and $T \subset B(\mathscr{H}_2)$ are norm closed subspaces, and Φ is a bounded (linear) map from S to T, then for each positive integer n we let Φ_n denote the map from $\mathbf{M}_n(S)$ to $\mathbf{M}_n(T)$ defined by $\Phi_n([s_{ij}]) = [\Phi(s_{ij})]$. The map Φ is said to be *completely positive* if each Φ_n is positive, *completely contractive* if each Φ_n is a contraction, and *completely bounded* if $\sup\{\|\Phi_n\| : n \in \mathbf{N}\} < \infty$. If Φ is completely bounded then its completely bounded norm is defined by $\|\Phi\|_{cb} = \sup\{\|\Phi_n\| : n \in \mathbf{N}\}$. We denote the space of all completely bounded maps from S to T by $CB(S, T)$, and write $CB(S)$ for $CB(S, S)$. A net $\{\Phi_\alpha\}$ in $CB(S, T)$ is said to be *bounded* if $\sup \|\Phi_\alpha\|_{cb} < \infty$.

Now suppose that $\mathscr{S} \subset B(\mathscr{H}_1)$ and $\mathscr{T} \subset B(\mathscr{H}_2)$ are σ-weakly closed subspaces. Then we denote the space of all σ-weakly continuous maps in $CB(\mathscr{S}, \mathscr{T})$ by $CB_\sigma(\mathscr{S}, \mathscr{T})$, and the space of all σ-weakly continuous maps in $CB(\mathscr{S})$ by $CB_\sigma(\mathscr{S})$. If $\mathscr{N}$ is a von Neumann algebra and $\Phi \in CB_\sigma(\mathscr{S}, \mathscr{T})$, it follows from a straightforward modification of the proof of Lemma 1.5 in [13] that there is a (unique) σ-weakly continuous map $\tilde{\Phi}$ from $\mathscr{S} \bar{\otimes} \mathscr{N}$ to $\mathscr{T} \bar{\otimes} \mathscr{N}$ such that

$$\tilde{\Phi}(s \otimes b) = \Phi(s) \otimes b \qquad (s \in \mathscr{S},\ b \in \mathscr{N}).$$

Moreover, $\|\tilde{\Phi}\| \leqslant \|\Phi\|_{cb}$, with equality if $\mathscr{N}$ is infinite dimensional. (Of course $\tilde{\Phi}$ depends on $\mathscr{N}$ as well as Φ, but it will always be clear from context what the domain of $\tilde{\Phi}$ is.)

If $\psi \in \mathscr{N}_*$, $s \in \mathscr{S}$, and $b \in \mathscr{N}$, then

$$L_\psi(\tilde{\Phi}(s \otimes b)) = \langle b, \psi \rangle\, \Phi(s) = \Phi(\langle b, \psi \rangle s) = \Phi(L_\psi(s \otimes b)). \tag{2.1}$$

Since L_ψ, Φ, and $\tilde{\Phi}$ are σ-weakly continuous linear maps, it follows immediately from (2.1) that

$$L_\psi(\tilde{\Phi}(x)) = \Phi(L_\psi(x)) \qquad (x \in \mathscr{S} \bar{\otimes} \mathscr{N},\ \psi \in \mathscr{N}_*,\ \Phi \in CB_\sigma(\mathscr{S}, \mathscr{T})). \tag{2.2}$$

Let $F_\sigma(\mathscr{S})$ denote the collection of all σ-weakly continuous finite rank maps from $\mathscr{S}$ to $\mathscr{S}$. Then $\Phi \in F_\sigma(\mathscr{S})$ if and only if for some $n \in \mathbf{N}$ there are $s_1, ..., s_n$ in $\mathscr{S}$ and $\varphi_1, ..., \varphi_n$ in $\mathscr{S}_*$ such that

$$\Phi(s) = \sum_{i=1}^{i=n} \langle s, \varphi_i \rangle\, s_i$$

for all s in $\mathscr{S}$. It follows immediately from this and Corollary 3.4 in [19] that $F_\sigma(\mathscr{S}) \subset CB_\sigma(\mathscr{S})$. If $x \in \mathscr{S} \bar{\otimes} \mathscr{N}$, we let $F_\sigma(x)$ denote the σ-weak closure of the linear space $\{\tilde{\Phi}(x) \mid \Phi \in F_\sigma(\mathscr{S})\}$.

A subspace $\mathscr{S}$ is said to have the (σ-*weak*) *completely bounded approximation property* (CBAP) if there is a bounded net $\{\Phi_\alpha\}$ in $F_\sigma(\mathscr{S})$ such that

$$\Phi_\alpha(s) \to s \qquad \sigma\text{-weakly for all } s \in \mathscr{S}. \tag{2.3}$$

If the Φ_α's can be chosen to be complete contractions, then $\mathscr{S}$ is said to have the (σ-*weak*) *completely contractive approximation property* (CCAP), and if the Φ_α's can be chosen to be completely positive and completely contractive, then $\mathscr{S}$ is said to have the (σ-*weak*) *completely positive approximation property* (CPAP).

It is immediate that if $\mathscr{S}$ has the CPAP then it has the CCAP, and if it has the CCAP then it has the CBAP. Moreover, if $\mathscr{S}$ is a von Neumann algebra, then the existence of a net $\{\Phi_\alpha\}$ of completely positive maps in $F_\sigma(\mathscr{S})$ satisfying (2.3) implies that $\mathscr{S}$ is semidiscrete [18], i.e., that the Φ_α's can be chosen so that we also have $\Phi_\alpha(1)=1$ [55, p. 105]. Since any unital completely positive map is completely contractive [41, Proposition 3.5], semidiscreteness in turn implies the CPAP. De Cannière and Haagerup have shown there are von Neumann algebras with the CCAP which do not have the CPAP [13, 24], and Cowling and Haagerup have shown that there are von Neumann algebras with the CBAP which do not have the CCAP [12]. For example, the group von Neumann algebra $\mathscr{M}(\mathbf{F}_2)$ of the free group on two generators has the CCAP but not the CPAP, and if Γ is a lattice in $\mathrm{Sp}(1, n)$ (with $n \geqslant 2$) then $\mathscr{M}(\Gamma)$ has the CBAP but not the CCAP.

Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace, and let $\mathscr{K}$ be an Hilbert space. We say that $\mathscr{S}$ has the *(σ-weak) complete pointwise approximation property* (CPWAP) for $\mathscr{K}$ if there is a net $\{\Phi_\alpha\}$ in $F_\sigma(\mathscr{S})$ such that

$$\tilde{\Phi}_\alpha(x) \to x \qquad \sigma\text{-weakly for every } x \in \mathscr{S} \mathbin{\bar{\otimes}} B(\mathscr{K}).$$

It is clear that if $\mathscr{S}$ has the CPWAP for a Hilbert space $\mathscr{K}$, then it has the CPWAP for every norm closed subspace of $\mathscr{K}$. It is also clear that if $\mathscr{K}_1$ and $\mathscr{K}_2$ are Hilbert spaces that are unitarily equivalent, then $\mathscr{S}$ has the CPWAP for $\mathscr{K}_1$ if and only if it has the CPWAP for $\mathscr{K}_2$. We will show below that if $\mathscr{S}$ has the CPWAP for a separable infinite dimensional Hilbert space, then it has the CPWAP for all Hilbert spaces. The main result of this section is that $\mathscr{S}$ has Property S_σ if and only if it has the CPWAP. The next two propositions give some useful equivalent conditions for the CPWAP.

PROPOSITION 2.1. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace and let $\mathscr{K}$ be an infinite dimensional Hilbert space. The following are equivalent.*

(a) *$\mathscr{S}$ has the CPWAP for $\mathscr{K}$.*

(b) *$x \in F_\sigma(x)$ for every $x \in \mathscr{S} \mathbin{\bar{\otimes}} B(\mathscr{K})$.*

(c) *For every $x \in \mathscr{S} \mathbin{\bar{\otimes}} B(\mathscr{K})$, there is a net $\{\Phi_\alpha\}$ in $F_\sigma(\mathscr{S})$ such that $\tilde{\Phi}_\alpha(x) \to x$ σ-weakly.*

Proof. It is immediate that (a) $\Rightarrow$ (b) and that (b) $\Rightarrow$ (c).

Suppose that (c) holds, and let $x_1, \dots, x_n$ be any elements of $\mathscr{S} \mathbin{\bar{\otimes}} B(\mathscr{K})$. Let $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n$. Then x is an element of $\mathscr{S} \mathbin{\bar{\otimes}} B(\mathscr{K}^{(n)})$, where $\mathscr{K}^{(n)}$ denotes the direct sum of n copies of $\mathscr{K}$. Since $\mathscr{K}^{(n)}$ is unitarily equivalent to $\mathscr{K}$, condition (c) is also valid when $\mathscr{K}$ is replaced by $\mathscr{K}^{(n)}$, and so there is a net $\{\Phi_\alpha\}$ in $F_\sigma(\mathscr{S})$ such that $\tilde{\Phi}_\alpha(x)$ converges to x

σ-weakly. It follows easily from this that $\tilde{\Phi}_\alpha(x_i) \to x_i$ σ-weakly for all i. Hence if F is any finite subset of $\mathscr{S} \bar{\otimes} B(\mathscr{K})$ and $\mathscr{U}$ is any σ-weak neighborhood of 0 in $\mathscr{S} \bar{\otimes} B(\mathscr{K})$, then there is a map $\Phi_{(F, \mathscr{U})} \in F_\sigma(\mathscr{S})$ such that

$$\tilde{\Phi}_{(F, \mathscr{U})}(x) \in \mathscr{U} + x \qquad \text{for all} \quad x \in F. \tag{2.4}$$

Let $\mathscr{I}$ denote the set of all pairs $(F, \mathscr{U})$. Define a partial order on $\mathscr{I}$ by $(F_1, \mathscr{U}_1) \leqslant (F_2, \mathscr{U}_2)$ if $F_1 \subset F_2$ and $\mathscr{U}_2 \subset \mathscr{U}_1$. Then $\mathscr{I}$ is a directed set, and it follows immediately from (2.4) that $\tilde{\Phi}_{(F, \mathscr{U})}(x) \to x$ σ-weakly for every $x \in \mathscr{S} \bar{\otimes} B(\mathscr{K})$. Hence (a) holds. ∎

If $\mathscr{S} \subset B(\mathscr{H})$ is a σ-weakly closed subspace, we denote the set of all norm continuous finite rank maps from $\mathscr{S}_*$ to itself by $F(\mathscr{S}_*)$. If $\Phi \in F_\sigma(\mathscr{S})$, then the restriction Φ_* of Φ^* to $\mathscr{S}_*$ is in $F(\mathscr{S}_*)$, and if $\Psi \in F(\mathscr{S}_*)$, then $\Psi^* \in F_\sigma(\mathscr{S})$. If $\mathscr{N}$ is a von Neumann algebra, if $\Psi \in F(\mathscr{S}_*)$, and if we set $\Phi = \Psi^*$, then

$$(\tilde{\Phi})_* (\varphi \otimes \psi) = \Psi(\varphi) \otimes \psi \qquad (\varphi \in \mathscr{S}_*, \psi \in \mathscr{N}_*).$$

Hence if we set $\tilde{\Psi} = (\tilde{\Phi})_*$, then $\tilde{\Psi}$ is an extension of $\Psi \otimes id$ to $(\mathscr{S} \bar{\otimes} \mathscr{N})_*$, and

$$\{\tilde{\Psi} \mid \Psi \in F(\mathscr{S}_*)\} = \{(\tilde{\Phi})_* \mid \Phi \in F_\sigma(\mathscr{S})\}. \tag{2.5}$$

If $\rho \in (\mathscr{S} \bar{\otimes} \mathscr{N})_*$, we let $F(\rho)$ denote the norm closure (= the weak closure) of the subspace $\{\tilde{\Psi}(\rho) \mid \Psi \in F(\mathscr{S}_*)\}$ of $(\mathscr{S} \bar{\otimes} \mathscr{N})_*$.

PROPOSITION 2.2. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace, and let $\mathscr{K}$ be an infinite dimensional Hilbert space. The following are equivalent:*

(a) *$\mathscr{S}$ has the CPWAP for $\mathscr{K}$.*

(b) *$\rho \in F(\rho)$ for every $\rho \in (\mathscr{S} \bar{\otimes} B(\mathscr{K}))_*$.*

(c) *For every $\rho \in (\mathscr{S} \bar{\otimes} B(\mathscr{K}))_*$ there is a net $\{\Psi_\alpha\}$ in $F(\mathscr{S}_*)$ such that $\tilde{\Psi}_\alpha(\rho) \to \rho$ in norm.*

(d) *There is a net $\{\Psi_\alpha\}$ in $F(\mathscr{S}_*)$ such that $\tilde{\Psi}_\alpha(\rho) \to \rho$ in norm for every $\rho \in (\mathscr{S} \bar{\otimes} B(\mathscr{K}))_*$.*

Proof. (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (b) are immediate, and the proof of (c) $\Rightarrow$ (d) is similar to the proof of (c) $\Rightarrow$ (a) in Proposition 2.1.

Next observe that it follows from (2.5) that if $x \in \mathscr{S} \bar{\otimes} B(\mathscr{K})$ and $\rho \in (\mathscr{S} \bar{\otimes} B(\mathscr{K}))_*$, then x annihilates $F(\rho)$ if and only if ρ annihilates $F_\sigma(x)$. Using the duality between $\mathscr{S} \bar{\otimes} B(\mathscr{K})$ and $(\mathscr{S} \bar{\otimes} B(\mathscr{K}))_*$, it follows easily from this and Proposition 2.1 that (a) and (b) are equivalent. ∎

PROPOSITION 2.3. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace. If $\mathscr{S}$ has the CPWAP for some infinite dimensional Hilbert space, then it has the CPWAP for all infinite dimensional Hilbert spaces.*

Proof. Let $\mathscr{K}$ be an infinite dimensional Hilbert space, and let $\rho \in (\mathscr{S} \bar{\otimes} B(\mathscr{K}))_*$. Then since ρ is a countable sum of vector functionals, and each vector in $\mathscr{H} \otimes \mathscr{K}$ is a linear combination of a countable number of basis vectors, there is a projection e in $B(\mathscr{K})$ with at most countably infinite dimensional range such that

$$\langle (1 \otimes e) x (1 \otimes e), \rho \rangle = \langle x, \rho \rangle \qquad \text{for all} \quad x \in \mathscr{S} \bar{\otimes} B(\mathscr{K}). \tag{2.6}$$

Let $\mathscr{K}_0$ denote the range of e, and let ρ_0 denote the restriction of ρ to $\mathscr{S} \bar{\otimes} B(\mathscr{K}_0)$. Since $\mathscr{S}$ has the CPWAP for some infinite dimensional Hilbert space $\mathscr{K}_1$, and since $\mathscr{K}_0$ is unitarily equivalent to a norm closed subspace of $\mathscr{K}_1$, $\mathscr{S}$ has the CPWAP for $\mathscr{K}_0$. Hence, by Proposition 2.2, there is a net $\{\Psi_\alpha\}$ in $F(\mathscr{S}_*)$ such that $\tilde{\Psi}_\alpha(\rho_0) \to \rho_0$ in norm. It follows from (2.6) that

$$\langle (1 \otimes e) x (1 \otimes e), \tilde{\Psi}_\alpha(\rho_0) \rangle = \langle x, \tilde{\Psi}_\alpha(\rho) \rangle \qquad \text{for all} \quad x \in \mathscr{S} \bar{\otimes} B(\mathscr{K})$$

(where we view $(1 \otimes e) x (1 \otimes e)$ as an element of $\mathscr{S} \bar{\otimes} B(\mathscr{K}_0)$). Hence $\tilde{\Psi}_\alpha(\rho) \to \rho$ weakly, and so $\rho \in F(\rho)$. Thus $\mathscr{S}$ has the CPWAP for $\mathscr{K}$ by Proposition 2.2. ∎

If $\mathscr{S} \subset B(\mathscr{H})$ is a σ-weakly closed subspace, and $\mathscr{H}_0$ is a separable infinite dimensional Hilbert space, then $\mathscr{S} \bar{\otimes} B(\mathscr{H}_0)$ can be viewed as a space of $\infty \times \infty$ matrices with entries in $\mathscr{S}$. Following [20], we let $\mathbf{M}_\infty(\mathscr{S})$ denote the linear space of matrices $[s_{ij}]_{i,j \in \mathbf{N}}$ with entries in $\mathscr{S}$ which are bounded in the sense that

$$\|s\| = \sup\{ \| [s_{ij}]_{i,j \leq n} \| : n \in \mathbf{N} \} \tag{2.7}$$

is finite. We can identify $\mathbf{M}_\infty(\mathscr{S})$ in the obvious way with the space of bounded operators $s = [s_{ij}]$ on $\mathscr{H}^{(\infty)}$ with entries in $\mathscr{S}$. The norm given by (2.7) is then the operator norm.

If $\mathscr{S} \subset B(\mathscr{H}_1)$ and $\mathscr{T} \subset \mathscr{B}(\mathscr{H}_2)$ are σ-weakly closed subspaces and $\Phi \in CB_\sigma(\mathscr{S}, \mathscr{T})$, we can define a map Φ_∞ from $\mathbf{M}_\infty(\mathscr{S})$ to $\mathbf{M}_\infty(\mathscr{T})$ by $\Phi_\infty([s_{ij}]) = [\Phi(s_{ij})]$. Then $\|\Phi_\infty\| = \|\Phi\|_{cb}$. Moreover, under the natural isomorphisms, $\mathbf{M}_\infty(\mathscr{S})$ is isomorphic (and σ-weakly homeomorphic) to $\mathscr{S} \bar{\otimes} B(\mathscr{H}_0)$, $\mathbf{M}_\infty(\mathscr{T})$ is isomorphic to $\mathscr{T} \bar{\otimes} B(\mathscr{H}_0)$, and Φ_∞ is sent to $\tilde{\Phi}$. In analogy with [21], we say that a net $\{\Phi_\alpha\}$ in $CB_\sigma(\mathscr{S}, \mathscr{T})$ converges to $\Phi \in CB_\sigma(\mathscr{S}, \mathscr{T})$ in the *stable point σ-weak topology* if and only if $[(\Phi_\alpha)_\infty](s)$ converges σ-weakly to $\Phi_\infty(s)$ for every s in $\mathbf{M}_\infty(\mathscr{S})$. We say that $\mathscr{S}$ has the *σ-weak approximation property* if the identity map from $\mathscr{S}$

to $\mathscr{S}$ is the limit of finite rank σ-weakly continuous completely bounded maps in the stable point σ-weak topology.

Remark 2.4. It follows immediately from the above discussion and Proposition 2.3 that $\mathscr{S}$ has the σ-weak approximation property if and only if has the CPWAP for some infinite dimensional Hilbert space if and only if it has the CPWAP for all infinite dimensional Hilbert spaces.

The next proposition plays a crucial role in the proof that the σ-weak approximation property is equivalent to Property S_σ.

PROPOSITION 2.5. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace, and let $\mathscr{N}$ be a von Neumann algebra. Let $x \in \mathscr{S} \bar{\otimes} \mathscr{N}$, and let $\mathscr{T}$ denote the σ-weak closure of the linear space $\{R_\varphi(x) \mid \varphi \in \mathscr{S}_*\}$. Then $F_\sigma(x) = \mathscr{S} \bar{\otimes} \mathscr{T}$.*

Proof. Let $\Phi \in F_\sigma(\mathscr{S})$. Then, as noted above, for some $n \in \mathbf{N}$ there are $s_1, ..., s_n$ in $\mathscr{S}$ and $\varphi_1, ..., \varphi_n$ in $\mathscr{S}_*$ such that

$$\Phi(s) = \sum_{i=1}^{i=n} \langle s, \varphi_i \rangle s_i$$

for all s in $\mathscr{S}$. Hence if $b \in \mathscr{N}$, then

$$\tilde{\Phi}(s \otimes b) = \sum_{i=1}^{i=n} s_i \otimes \langle s, \varphi_i \rangle b = \sum_{i=1}^{i=n} s_i \otimes R_{\varphi_i}(s \otimes b). \tag{2.8}$$

Since $\tilde{\Phi}(\cdot)$ and $\sum_{i=1}^{i=n} s_i \otimes R_{\varphi_i}(\cdot)$ are both σ-weakly continuous linear maps, it follows from (2.8) that they are equal on all of $\mathscr{S} \bar{\otimes} \mathscr{N}$. In particular, $\tilde{\Phi}(x) = \sum_{i=1}^{i=n} s_i \otimes R_{\varphi_i}(x)$, and so $\tilde{\Phi}(x) \in \mathscr{S} \otimes \mathscr{T}$. Since $\mathscr{S} \bar{\otimes} \mathscr{T}$ is σ-weakly closed, $F_\sigma(x) \subset \mathscr{S} \bar{\otimes} \mathscr{T}$.

Now let $s_0 \in \mathscr{S}$ and $\varphi \in \mathscr{S}_*$. For $s \in \mathscr{S}$, let $\Phi(s) = \langle s, \varphi \rangle s_0$. Then $\Phi \in F_\sigma(\mathscr{S})$, and so $s_0 \otimes R_\varphi(x) = \tilde{\Phi}(x) \in F_\sigma(x)$. Since $F_\sigma(x)$ is a σ-weakly closed subspace of $\mathscr{S} \bar{\otimes} \mathscr{N}$, $\mathscr{S} \bar{\otimes} \mathscr{T} \subset F_\sigma(x)$. Hence $\mathscr{S} \bar{\otimes} \mathscr{T} = F_\sigma(x)$. ∎

THEOREM 2.6. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace. Then the following are equivalent*:

(a) *$\mathscr{S}$ has Property S_σ.*

(b) *$\mathscr{S}$ has the σ-weak approximation property.*

(c) *$\mathscr{S}$ has the CPWAP for some infinite dimensional Hilbert space*

(d) *$\mathscr{S}$ has the CPWAP for every Hilbert space.*

Proof. By Remark 2.4, (b), (c), and (d) are equivalent, so it suffices to show that (a) and (d) are equivalent.

(a)$\Rightarrow$(d). Let $\mathscr{K}$ be a Hilbert space, let $x \in \mathscr{S} \bar{\otimes} B(\mathscr{K})$, and let $\mathscr{T}$ denote the σ-weakly closed linear span of $\{R_\varphi(x) \mid \varphi \in \mathscr{S}_*\}$. Since $\mathscr{S}$ has Property S_σ, $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \bar{\otimes} \mathscr{T}$. But $x \in F(\mathscr{S}, \mathscr{T})$ by the definition of $\mathscr{T}$, and $\mathscr{S} \bar{\otimes} \mathscr{T} = F_\sigma(x)$ by Proposition 2.5, so $x \in F_\sigma(x)$. Hence $\mathscr{S}$ has the CPWAP for $\mathscr{K}$ by Proposition 2.1.

(d)$\Rightarrow$(a). Let $\mathscr{K}$ be a Hilbert space, let $\mathscr{T}$ be a σ-weakly closed subspace of $B(\mathscr{K})$, and let $x \in F(\mathscr{S}, \mathscr{T})$. Then $\{R_\varphi(x) \mid \varphi \in \mathscr{S}_*\} \subset \mathscr{T}$, so $F_\sigma(x) \subset \mathscr{S} \bar{\otimes} \mathscr{T}$ by Proposition 2.5. Since $\mathscr{S}$ has the CPWAP for $\mathscr{K}$, $x \in F_\sigma(x)$. Hence $F(\mathscr{S}, \mathscr{T}) \subset \mathscr{S} \bar{\otimes} \mathscr{T}$. The reverse inclusion is always valid, and so $F(\mathscr{S}, \mathscr{T}) = \mathscr{S} \bar{\otimes} \mathscr{T}$. Hence $\mathscr{S}$ has Property S_σ. ∎

Remark 2.7. Suppose $\mathscr{S} \subset B(\mathscr{H})$ is a σ-weakly closed subspace without Property S_σ, and let $\mathscr{K}$ be an infinite dimensional Hilbert space. By Theorem 2.6, $\mathscr{S}$ does not have the CPWAP for $\mathscr{K}$. Hence it follows from Proposition 2.1 that there is an x in $\mathscr{S} \bar{\otimes} B(\mathscr{K})$ such that x is not in $F_\sigma(x)$. Let $\mathscr{T}$ denote the σ-weakly closed linear span of $\{R_\varphi(x) \mid \varphi \in \mathscr{S}_*\}$. Then $x \in F(\mathscr{S}, \mathscr{T})$ by definition, while x is not in $\mathscr{S} \bar{\otimes} \mathscr{T}$, since $F_\sigma(x) = \mathscr{S} \bar{\otimes} \mathscr{T}$ by Proposition 2.5. Hence if $\mathscr{S}$ does not have Property S_σ, then for any infinite dimensional Hilbert space $\mathscr{K}$ there is a σ-weakly closed subspace $\mathscr{T} \subset B(\mathscr{K})$ such that $\mathscr{S} \bar{\otimes} \mathscr{T} \neq F(\mathscr{S}, \mathscr{T})$.

The proof of the next result can be obtained by making obvious modifications to the proofs of Proposition 2.2 and Theorem 2.6, and is left to the reader,

THEOREM 2.8. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace, and let $\mathscr{N}$ be a von Neumann algebra. The following are equivalent*:

(a) *$\mathscr{S}$ has Property S_σ for $\mathscr{N}$.*

(b) *$x \in F_\sigma(x)$ for every $x \in \mathscr{S} \bar{\otimes} \mathscr{N}$.*

(c) *$\rho \in F(\rho)$ for every $\rho \in (\mathscr{S} \bar{\otimes} \mathscr{N})_*$.*

We will make use of the next proposition a number of times in applications of Theorems 2.6 and 2.8.

PROPOSITION 2.9. *Let $\mathscr{S} \subset B(\mathscr{H}_1)$ and $\mathscr{T} \subset B(\mathscr{H}_2)$ be σ-weakly closed subspaces, and suppose that $\{\Phi_\alpha\}$ is a bounded net in $CB_\sigma(\mathscr{S}, \mathscr{T})$ that converges pointwise σ-weakly to an element Φ of $CB_\sigma(\mathscr{S}, \mathscr{T})$. Then for any von Neumann algebra $\mathscr{N}$ we have that $\tilde{\Phi}_\alpha(x) \to \tilde{\Phi}(x)$ σ-weakly for every $x \in \mathscr{S} \bar{\otimes} \mathscr{N}$.*

Proof. Let $\mathscr{N}$ be a von Neumann algebra, let $x \in \mathscr{S} \bar{\otimes} \mathscr{N}$, and let $\psi \in \mathscr{N}_*$. Then $L_\psi(x) \in \mathscr{S}$, so $\Phi_\alpha(L_\psi(x)) \to \Phi(L_\psi(x))$ σ-weakly. Combining this fact with (2.2) and the definition of L_ψ, we get that

$$\langle \tilde{\Phi}_\alpha(x), \varphi \otimes \psi \rangle \to \langle \tilde{\Phi}(x), \varphi \otimes \psi \rangle \qquad (\varphi \in \mathscr{S}_*, \psi \in \mathscr{N}_*). \tag{2.9}$$

Since $\|\tilde{\Phi}_\alpha\| \leqslant \|\Phi_\alpha\|_{cb}$ for every α, the net $\{\tilde{\Phi}_\alpha(x)\}$ is bounded in norm, and so (2.9) implies that $\langle \tilde{\Phi}_\alpha(x), \rho \rangle \to \langle \tilde{\Phi}(x), \rho \rangle$ for every ρ in the norm closure of the algebraic tensor product $\mathscr{S}_* \otimes \mathscr{N}_*$. But $\mathscr{S}_* \otimes \mathscr{N}_*$ is norm dense in $(\mathscr{S} \bar{\otimes} \mathscr{N})_*$, so $\tilde{\Phi}_\alpha(x) \to \tilde{\Phi}(x)$ σ-weakly. ∎

The next result is an immediate consequence of Proposition 2.9 (and Theorem 2.6).

THEOREM 2.10. *Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace. If $\mathscr{S}$ has the CBAP, then $\mathscr{S}$ has the σ-weak approximation property, and so has Property S_σ.*

EXAMPLE 2.11. Let Γ denote the semidirect product of $\mathbf{Z}^2$ with $SL(2, \mathbf{Z})$ under the natural action of $SL(2, \mathbf{Z})$ on $\mathbf{Z}^2$. Haagerup has shown [25] that the group von Neumann algebra $\mathscr{M}(\Gamma)$ does not have the CBAP. He has also shown that there is a bounded net $\{\Phi_\alpha\}$ in $CB_\sigma(\mathscr{M}(\Gamma))$ such that $\Phi_\alpha(a) \to a$ σ-weakly for every a in $\mathscr{M}(\Gamma)$, and such that each Φ_α is the limit in the pointwise σ-weak topology of a bounded net in $F_\sigma(\mathscr{M}(\Gamma))$. (More generally, such a net exists whenever Γ is the semidirect product of two discrete groups whose group von Neumann algebras have the CBAP [26].) Hence if $\mathscr{H}$ is any Hilbert space and $x \in \mathscr{M}(\Gamma) \bar{\otimes} B(\mathscr{H})$, then by Proposition 2.9, x is in the σ-weak closure of $\{\tilde{\Phi}_\alpha(x)\}$. It also follows from Proposition 2.9 that $\tilde{\Phi}_\alpha(x) \in F_\sigma(x)$ for all α. Thus $x \in F_\sigma(x)$ for all $x \in \mathscr{M}(\Gamma) \bar{\otimes} B(\mathscr{H})$, and so $\mathscr{M}(\Gamma)$ has the σ-weak approximation property by Proposition 2.1 and Remark 2.4. Hence the σ-weak approximation property does not imply the CBAP.

Remark 2.12. We will show in the next section that there are separably acting factors without Property S_σ, and hence without the σ-weak approximation property. However, the proof that such factors exist is very indirect, and it would be of great interest to find concrete examples of separably acting von Neumann algebras without the σ-weak approximation property. A good place to look for such examples is among the group von Neumann algebras of discrete groups. In [26] it is shown that if Γ is a discrete group, then $\mathscr{M}(\Gamma)$ has the σ-weak approximation property if and only if for every locally compact group H, there is a net $\{\varphi_\alpha\}$ of functions in the Fourier algebra $A(\Gamma)$ of Γ with finite support such that $(\varphi_\alpha \times 1_H)\zeta \to \zeta$ in the $A(\Gamma \times H)$ norm for every $\zeta \in A(\Gamma \times H)$. (This should be compared with Haagerup's characterization in [25] of those discrete groups (called weakly amenable groups in [12]) whose group von Neumann algebras have the CBAP.) It is also shown in [26] that the class of discrete groups whose group von Neumann algebras have the σ-weak approximation property is closed under taking semidirect products. (This is not true of the smaller class of weakly amenable discrete groups, as

Example 2.11 shows.) We do not know of any examples of discrete groups whose group von Neumann algebras do not have the σ-weak approximation property, but a likely candidate is $SL(3, \mathbf{Z})$.

PROPOSITION 2.13. *Let $\mathscr{S}$ be a σ-weakly closed subspace of $B(\mathscr{H})$, and let $\mathscr{N}$ be a von Neumann algebra. Suppose that $\{\mathscr{S}_\alpha \subset B(\mathscr{H}_\alpha)\}$ is a net of σ-weakly closed subspaces with Property S_σ for $\mathscr{N}$, and that there are nets $\{\Phi_\alpha\}$ in $CB_\sigma(\mathscr{S}, \mathscr{S}_\alpha)$ and $\{\Psi_\alpha\}$ in $CB_\sigma(\mathscr{S}_\alpha, \mathscr{S})$ such that $(\tilde{\Psi}_\alpha \circ \tilde{\Phi}_\alpha)(x) \to x$ σ-weakly for every x in $\mathscr{S} \bar{\otimes} \mathscr{N}$. Then $\mathscr{S}$ has Property S_σ for $\mathscr{N}$.*

Proof. Let $x \in \mathscr{S} \bar{\otimes} \mathscr{N}$. Then by assumption,

$$(\tilde{\Psi}_\alpha \circ \tilde{\Phi}_\alpha)(x) \to x \qquad \sigma\text{-weakly}. \tag{2.10}$$

For each α, $\tilde{\Phi}_\alpha(x) \in \mathscr{S}_\alpha \bar{\otimes} \mathscr{N}$, and $\mathscr{S}_\alpha$ has Property S_σ for $\mathscr{N}$, so by Theorem 2.8, $\tilde{\Phi}_\alpha(x)$ is in the σ-weak closure of $\{(\tilde{\Phi} \circ \tilde{\Phi}_\alpha)(x) \mid \Phi \in F_\sigma(\mathscr{S}_\alpha)\}$. Since $\tilde{\Psi}_\alpha$ is σ-weakly continuous, $(\tilde{\Psi}_\alpha \circ \tilde{\Phi}_\alpha)(x)$ is in the σ-weak closure of $\{(\tilde{\Psi}_\alpha \circ \tilde{\Phi} \circ \tilde{\Phi}_\alpha)(x) \mid \Phi \in F_\sigma(\mathscr{S}_\alpha)\}$. Moreover, $\Psi_\alpha \circ \Phi \circ \Phi_\alpha$ is in $F_\sigma(\mathscr{S})$ whenever $\Phi \in F_\sigma(\mathscr{S}_\alpha)$. Hence $(\tilde{\Psi}_\alpha \circ \tilde{\Phi}_\alpha)(x) \in F_\sigma(x)$ for all α. But $F_\sigma(x)$ is σ-weakly closed, so it follows from (2.10) that $x \in F_\sigma(x)$. Hence $\mathscr{S}$ has Property S_σ for $\mathscr{N}$ by Theorem 2.8. ∎

COROLLARY 2.14. *Let $\mathscr{S}$ be a σ-weakly closed subspace of $B(\mathscr{H})$, and let $\mathscr{N}$ be a von Neumann algebra. Suppose that $\{\mathscr{S}_\alpha \subset B(\mathscr{H}_\alpha)\}$ is a net of σ-weakly closed subspaces with Property S_σ for $\mathscr{N}$, and that there are nets $\{\Phi_\alpha\}$ in $CB_\sigma(\mathscr{S}, \mathscr{S}_\alpha)$ and $\{\Psi_\alpha\}$ in $CB_\sigma(\mathscr{S}_\alpha, \mathscr{S})$ such that the net $\{\Psi_\alpha \circ \Phi_\alpha\}$ is bounded and converges pointwise σ-weakly to the identity map of $\mathscr{S}$. Then $\mathscr{S}$ has Property S_σ for $\mathscr{N}$.*

Proof. This follows immediately from Propositions 2.9 and 2.13. ∎

We will make use of the next result in Section 3.

PROPOSITION 2.15. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras, and suppose $\mathscr{M}$ is σ-finite. If every von Neumann subalgebra of $\mathscr{M}$ with separable predual has Property S_σ for $\mathscr{N}$, then $\mathscr{M}$ has Property S_σ for $\mathscr{N}$.*

Proof. Since $\mathscr{M}$ is σ-finite, it has a faithful normal state ω [50, Proposition II.3.19]. Moreover, one can find an increasing family $\{\mathscr{M}_\alpha\}$ of von Neumann subalgebras of $\mathscr{M}$ such that each $\mathscr{M}_\alpha$ has separable predual and is invariant under the modular automorphism group σ^ω of ω, and such that the union of the $\mathscr{M}_\alpha$'s is σ-weakly dense in $\mathscr{M}$ [27, proof of Proposition 9.7]. Since $\mathscr{M}_\alpha$ is invariant under σ^ω, there is a normal conditional expectation E_α from $\mathscr{M}$ onto $\mathscr{M}_\alpha$ [47, Theorem 10.1], and since the union of the $\mathscr{M}_\alpha$'s is σ-weakly dense in $\mathscr{M}$, $E_\alpha(a) \to a$ σ-weakly for every $a \in \mathscr{M}$

[11, Lemma 2]. The E_α's are completely positive and of norm one [47, Propositions 9.2 and 9.3], and so are complete contractions [41, Proposition 3.5]. Finally, each $\mathscr{M}_\alpha$ has separable predual, and so has Property S_σ for $\mathscr{N}$. Hence it follows from Corollary 2.14 (setting $\mathscr{S} = \mathscr{M}$, $\mathscr{S}_\alpha = \mathscr{M}_\alpha$, $\Phi_\alpha = E_\alpha$, and letting Ψ_α denote the inclusion map from $\mathscr{M}_\alpha$ to $\mathscr{M}$) that $\mathscr{M}$ has Property S_σ for $\mathscr{N}$. ∎

The next result and its corollary generalize results in [35, Sect. 2].

PROPOSITION 2.16. *Let $\mathscr{S}$ be a σ-weakly closed subspace of $B(\mathscr{H})$. Suppose there is a net $\{r_\alpha\}$ of finite rank operators in $B(\mathscr{H})$ such that $r_\alpha \to 1$ σ-weakly and such that $r_\alpha \mathscr{S} \subset \mathscr{S}$ and $\mathscr{S} r_\alpha \subset \mathscr{S}$ for all α. Then $\mathscr{S}$ has Property S_σ.*

Proof. Let $\mathscr{K}$ be a separable infinite dimensional Hilbert space, and let r_β be a fixed element of the net $\{r_\alpha\}$. For each α, let $\mathscr{S}_\alpha = r_\alpha \mathscr{S} r_\beta$, let $\Phi_\alpha(sr_\beta) = r_\alpha s r_\beta$, and let Ψ_α denote the inclusion map from $\mathscr{S}_\alpha$ to $\mathscr{S} r_\beta$. Since r_α and r_β are finite rank, $\mathscr{S}_\alpha$ is finite dimensional, and so has Property S_σ [34, Proposition 1.7]. Moreover, $\tilde{\Phi}_\alpha(x) = (r_\alpha \otimes 1)x$ for all $x \in (\mathscr{S} r_\beta) \bar{\otimes} B(\mathscr{K})$ (where we also let 1 denote the identity operator on $\mathscr{K}$), and so $(\tilde{\Psi}_\alpha \circ \tilde{\Phi}_\alpha)(x) \to x$ σ-weakly for every $x \in (\mathscr{S} r_\beta) \bar{\otimes} B(\mathscr{K})$. Hence $\mathscr{S} r_\beta$ has Property S_σ by Proposition 2.13. Another application of Proposition 2.13 (setting $\mathscr{S}_\alpha = \mathscr{S} r_\alpha$, $\Phi_\alpha(s) = s r_\alpha$ $(s \in \mathscr{S})$, and letting Ψ_α denote the inclusion map from $\mathscr{S}_\alpha$ to $\mathscr{S}$) shows that $\mathscr{S}$ has Property S_σ. ∎

COROLLARY 2.17. *Let $\mathscr{L}$ be a completely distributive commutative subspace lattice on a separable Hilbert space $\mathscr{H}$, and let $\mathscr{A} = \operatorname{alg} \mathscr{L}$. Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed $\mathscr{A}$-bimodule. Then $\mathscr{S}$ has Property S_σ.*

Proof. By a result of Laurie and Longstaff [38], the set of finite rank operators in $\mathscr{A}$ is σ-weakly dense in $\mathscr{A}$. Hence, since $1 \in \mathscr{A}$, there is a net $\{r_\alpha\}$ of finite rank operators in $\mathscr{A}$ such that $r_\alpha \to 1$ σ-weakly. Since $\mathscr{S}$ is an $\mathscr{A}$-bimodule, $\mathscr{A}\mathscr{S} \subset \mathscr{S}$ and $\mathscr{S}\mathscr{A} \subset \mathscr{S}$. Hence the net $\{r_\alpha\}$ satisfies the hypotheses of Proposition 2.16, and so $\mathscr{S}$ has Property S_σ. ∎

3. SUBSPACES WITHOUT PROPERTY S_σ

The main result of this section is that for each of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$, there is a separably acting factor of that type without Property S_σ. The proof of this result requires a number of steps, but the key ingredient is the observation, due to Uffe Haagerup, that if a subspace $\mathscr{S}$ has the σ-weak approximation property, then its predual $\mathscr{S}_*$ has the (Banach space) approximation property.

Recall that a Banach space X has the approximation property (AP) if for every compact subset K of X and every $\varepsilon > 0$ there is a finite rank norm continuous linear map T from X to itself such that $\|Tx - x\| < \varepsilon$ for all $x \in K$. The approximation property was introduced by Grothendieck in [23], where a number of equivalent conditions for the AP are given.

For a Banach space X, we let $l^p(\mathbf{N}, X)$ denote the space of X-valued functions $x(\cdot)$ on $\mathbf{N}$ such that the functions $\{\|x(\cdot)\|\}$ are $l^p(\mathbf{N})$, $1 \leqslant p \leqslant \infty$. With the obvious norm, $l^p(\mathbf{N}, X)$ is a Banach space. If $\mathscr{H}$ is a Hilbert space, then $l^\infty(\mathbf{N}, B(\mathscr{H}))$ is a von Neumann algebra, and

$$(l^\infty(\mathbf{N}, B(\mathscr{H})))_* = l^1(\mathbf{N}, B(\mathscr{H})_*), \tag{3.1}$$

where the duality is given by

$$\langle x, \varphi \rangle = \sum \langle x(n), \varphi(n) \rangle \qquad (x \in l^\infty(\mathbf{N}, B(\mathscr{H})), \varphi \in l^1(\mathbf{N}, B(\mathscr{H})_*)). \tag{3.2}$$

Moreover, there is a *-isomorphism Λ from $B(\mathscr{H}) \bar{\otimes} l^\infty(\mathbf{N})$ onto $l^\infty(\mathbf{N}, B(\mathscr{H}))$ such that

$$\Lambda(x \otimes \lambda(\cdot)) = \lambda(\cdot)x \qquad (x \in B(\mathscr{H}), \lambda(\cdot) \in l^\infty(\mathbf{N})). \tag{3.3}$$

(Proofs of these facts can be found in Section IV.7 of [50].) In what follows we will use Λ to identify $B(\mathscr{H}) \bar{\otimes} l^\infty(\mathbf{N})$ and $l^\infty(\mathbf{N}, B(\mathscr{H}))$.

Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace. Then it follows easily from (3.1) and (3.2) that

$$\mathscr{S} \bar{\otimes} l^\infty(\mathbf{N}) = l^\infty(\mathbf{N}, \mathscr{S}). \tag{3.4}$$

Since for each $\varphi \in \mathscr{S}_*$ we can choose a $\rho \in B(\mathscr{H})_*$ such that $\rho = \varphi$ on $\mathscr{S}$ and such that $\|\rho\| < 2\|\varphi\|$ (see, e.g., Lemma 2.4 in [28]), it also follows from (3.1) and (3.2) (and (3.4)) that

$$(\mathscr{S} \bar{\otimes} l^\infty(\mathbf{N}))_* = l^1(\mathbf{N}, \mathscr{S}_*). \tag{3.5}$$

As noted above, the next result is essentially due to Uffe Haagerup, as are Example 3.2 and Theorem 3.3. We are grateful to Professor Haagerup for allowing us to include these results in this paper.

THEOREM 3.1. *Let $\mathscr{S}$ be a σ-weakly closed subspace of $B(\mathscr{H})$. Then $\mathscr{S}$ has Property S_σ for $l^\infty(\mathbf{N})$ if and only if $\mathscr{S}_*$ has the approximation property.*

Proof. Let $L = L(\mathscr{S}_*, \mathscr{S}_*)$ denote the space of all bounded linear operators from $\mathscr{S}_*$ to itself, and put on L the topology τ of uniform convergence on compact sets in $\mathscr{S}_*$. Then $\mathscr{S}_*$ has the AP if and only if the identity map *id* from $\mathscr{S}_*$ to itself is in the τ-closure of $F(\mathscr{S}_*)$. Grothendieck proved in [23] that the dual of (L, τ) can be identified with the projective tensor product $\mathscr{S}_* \hat{\otimes} \mathscr{S}$. The identification is described as follows. If $\zeta \in \mathscr{S}_* \hat{\otimes} \mathscr{S}$, then there is a $\varphi = \varphi(\cdot)$ in $l^1(\mathbf{N}, \mathscr{S}_*)$ and an $s = s(\cdot)$ in $l^\infty(\mathbf{N}, \mathscr{S})$ such that

$$\zeta = \sum \varphi(n) \otimes s(n) \tag{3.6}$$

[45, Theorem III.6.4]. Using the representation (3.6) of ζ, we can define a linear functional ζ_* on L by

$$\langle \Psi, \zeta_* \rangle = \sum \langle s(n), \Psi(\varphi(n)) \rangle. \tag{3.7}$$

Grothendieck's theorem is that ζ_* is a τ-continuous linear functional and that every τ-continuous linear functional is of the form ζ_* for some $\zeta \in \mathscr{S}_* \hat{\otimes} \mathscr{S}$.

Next observe that if $\Psi \in F(\mathscr{S}_*)$, then it follows from the definition of $\tilde{\Psi}$, from (3.5), and from the way we are identifying $\mathscr{S} \bar{\otimes} l^\infty(\mathbf{N})$ and $l^\infty(\mathbf{N}, \mathscr{S})$, that

$$[\tilde{\Psi}(\varphi)](n) = \Psi(\varphi(n)) \qquad (\varphi \in l^1(\mathbf{N}, \mathscr{S}_*)).$$

Hence if ζ is given by (3.6), then

$$\langle \Psi, \zeta_* \rangle = \langle s, \tilde{\Psi}(\varphi) \rangle.$$

Thus $\mathscr{S}_*$ has the AP if and only if whenever $\varphi \in l^1(\mathbf{N}, \mathscr{S}_*)$ and $s \in l^\infty(\mathbf{N}, \mathscr{S})$ and

$$\langle s, \tilde{\Psi}(\varphi) \rangle = 0 \qquad \text{for all} \quad \Psi \in F(\mathscr{S}_*), \tag{3.8}$$

then

$$\sum \langle s(n), (id)(\varphi(n)) \rangle = \sum \langle s(n), \varphi(n) \rangle = \langle s, \varphi \rangle = 0. \tag{3.9}$$

Moreover, it follows from the duality between $l^1(\mathbf{N}, \mathscr{S}_*)$ and $l^\infty(\mathbf{N}, \mathscr{S})$ that (3.8) implies (3.9) for all pairs $\varphi \in l^1(\mathbf{N}, \mathscr{S}_*)$ and $s \in l^\infty(\mathbf{N}, \mathscr{S})$ if and only if $\varphi \in F(\varphi)$ for all $\varphi \in l^1(\mathbf{N}, \mathscr{S}_*) = (\mathscr{S} \bar{\otimes} l^\infty(\mathbf{N}))_*$. Hence it follows immediately from Theorem 2.8 that $\mathscr{S}_*$ has the AP if and only if $\mathscr{S}$ has Property S_σ for $l^\infty(\mathbf{N})$. ▮

EXAMPLE 3.2. Let X be any Banach space without the AP. Let K denote the closed unit ball of X, and define a map Φ from X^* to $l^\infty(K)$ by

$$[\Phi(x^*)](x) = \langle x, x^* \rangle \qquad (x^* \in X^*, x \in K).$$

Then Φ is obviously an isometric linear map, and it's easy to show that the restriction of Φ to the unit ball of X^* is weak*-continuous. It follows from this and standard facts about the dual of Banach spaces (see, e.g., Section V.5.5 of [16] or Section 3 of [20]) that Φ is a weak*-homeomorphism of X^* onto a weak* (= σ-weakly) closed subspace of $l^\infty(K)$. Let $\mathscr{S} = \Phi(X^*)$. Then $\mathscr{S}$ is a σ-weakly closed subspace of $B(l^2(K))$, and the restriction of Φ^* to $\mathscr{S}_*$ is an isometric isomorphism from $\mathscr{S}_*$ onto X. Hence $\mathscr{S}_*$ doesn't have the AP, and so $\mathscr{S}$ does not have Property S_σ for $l^\infty(\mathbf{N})$. If X is separable, then we can replace K in the above construction by a countable dense subset of K, in which case $\mathscr{S} \subset l^\infty(\mathbf{N})$. Since there are separable Banach spaces without the AP (in fact separable C^*-algebras without the AP [48]), there are subspaces of $l^\infty(\mathbf{N})$ without Property S_σ for $l^\infty(\mathbf{N})$. Hence there exist subspaces $\mathscr{S}$ and $\mathscr{T}$ of $l^\infty(\mathbf{N})$ such that $\mathscr{S} \bar{\otimes} \mathscr{T} \neq F(\mathscr{S}, \mathscr{T})$.

In [48], Szankowski proved the remarkable result that $B(\mathscr{H})$ does not have the AP if $\mathscr{H}$ is infinite dimensional. Using this result, Christensen and Sinclair proved in [9] that if $\mathscr{M}$ is an injective von Neumann algebra with separable predual and $\mathscr{M}$ is not finite type I of bounded degree, then $\mathscr{M}$ does not have the AP. Their result has been generalized by Robertson and Wassermann, who prove in [43] that a von Neumann algebra has the AP if and only if it is the finite direct sum of finite type I von Neumann algebras. It is a standard fact that if X^* has the AP, then so does X [39, Theorem 1.e.7]. Combining these facts with Theorem 3.1 and Theorem 2.6 we obtain our next result.

THEOREM 3.3. *If $\mathscr{M}$ is a von Neumann algebra which is not the finite direct sum of finite type I von Neumann algebras, then $\mathscr{M}^{**}$ does not have Property S_σ for $l^\infty(\mathbf{N})$. Hence $\mathscr{M}^{**}$ does not have Property S_σ and so does not have the σ-weak approximation property.*

None of the von Neumann algebras $\mathscr{M}^{**}$ of Theorem 3.3 are factors, and since none of them have separable predual, none of them have a faithful representation on a separable Hilbert space. As noted above, we will show that there are separably acting factors without Property S_σ. The proof will involve a number of steps, and makes use of the stability properties of the class of von Neumann algebras with Property S_σ. Most of these stability properties are also valid for the class of von Neumann algebras with Property S_σ for $\mathscr{N}$, where $\mathscr{N}$ is some fixed (infinite dimensional) von Neumann algebra. The case of most interest, of course, is when $\mathscr{N} = l^\infty(\mathbf{N})$. (We will in fact show that there are separably acting factors without Property S_σ for $l^\infty(\mathbf{N})$.) One of the stability properties that we will make use of several times below concerns tensor products. We state it here for convenient reference.

PROPOSITION 3.4. *Let $\mathscr{S}_1 \subset B(\mathscr{H}_1)$ and $\mathscr{S}_2 \subset B(\mathscr{H}_2)$ be nonzero σ-weakly closed subspaces, and let $\mathscr{N}$ be an infinite dimensional von Neumann algebra. If $\mathscr{S}_1 \bar{\otimes} \mathscr{S}_2$ has Property S_σ for $\mathscr{N}$, then $\mathscr{S}_1$ and $\mathscr{S}_2$ both have Property S_σ for $\mathscr{N}$. If $\mathscr{S}_1$ has Property S_σ and $\mathscr{S}_2$ has Property S_σ for $\mathscr{N}$, then $\mathscr{S}_1 \bar{\otimes} \mathscr{S}_2$ has Property S_σ for $\mathscr{N}$.*

Proof. The first statement of the proposition follows from the proof of the "only if" direction of Proposition 1.15 in [34]. So assume that $\mathscr{S}_1$ has Property S_σ and $\mathscr{S}_2$ has Property S_σ for $\mathscr{N}$. Let $\mathscr{T}$ be a σ-weakly closed subspace of $\mathscr{N}$, and let $x \in F(\mathscr{S}_1 \bar{\otimes} \mathscr{S}_2, \mathscr{T})$. It suffices to show that $x \in \mathscr{S}_1 \bar{\otimes} \mathscr{S}_2 \bar{\otimes} \mathscr{T}$. It follows from the proof of Lemma 14 in [57] (replacing C^*-algebras by σ-weakly closed subspaces, duals by preduals, and spatial C^*-tensor products by von Neumann algebra tensor products) that $R_\varphi(x) \in F(\mathscr{S}_2, \mathscr{T})$ for all $\varphi \in B(\mathscr{H}_1)_*$ (where R_φ is the right slice map from $\mathscr{S}_1 \bar{\otimes} \mathscr{S}_2 \bar{\otimes} \mathscr{N}$ to $\mathscr{S}_2 \bar{\otimes} \mathscr{N}$). Since $\mathscr{S}_2$ has Property S_σ for $\mathscr{N}$, $R_\varphi(x) \in \mathscr{S}_2 \bar{\otimes} \mathscr{T}$ for all $\varphi \in B(\mathscr{H}_1)_*$, so $x \in F(\mathscr{S}_1, \mathscr{S}_2 \bar{\otimes} \mathscr{T})$. Since $\mathscr{S}_1$ has Property S_σ, $x \in \mathscr{S}_1 \bar{\otimes} \mathscr{S}_2 \bar{\otimes} \mathscr{T}$, as required. ▌

The next result is the first step in our proof that there are separably acting factors without Property S_σ for $l^\infty(\mathbf{N})$.

LEMMA 3.5. *Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. Then there is a von Neumann algebra $\mathscr{M} \subset B(\mathscr{H})$ which does not have Property S_σ for $l^\infty(\mathbf{N})$.*

Proof. By Theorem 3.3, there is a von Neumann algebra without Property S_σ for $l^\infty(\mathbf{N})$. Moreover, every von Neumann algebra is the direct sum of von Neumann algebras of the form $\mathscr{N} \bar{\otimes} B(\mathscr{K})$, where $\mathscr{N}$ is σ-finite (combine Lemma 7 of [15, Part III, Chap. 1] with Proposition 5(ii) of [15, Part I, Chap. 2]), and the direct sum of von Neumann algebras with Property S_σ for $l^\infty(\mathbf{N})$ has Property S_σ for $l^\infty(\mathbf{N})$, as the proof of Proposition 1.12 in [34] shows. Hence there is a σ-finite von Neumann algebra $\mathscr{N}$ and a Hilbert space $\mathscr{K}$ such that $\mathscr{N} \bar{\otimes} B(\mathscr{K})$ does not have Property S_σ for $l^\infty(\mathbf{N})$. By Proposition 3.4, $\mathscr{N}$ does not have Property S_σ for $l^\infty(\mathbf{N})$. Since $\mathscr{N}$ is σ-finite, it follows from Proposition 2.15 that there is a von Neumann subalgebra $\mathscr{M}$ of $\mathscr{N}$ with separable predual which does not have Property S_σ for $l^\infty(\mathbf{N})$. Finally, since $\mathscr{M}$ has separable predual, $\mathscr{M}$ has a faithful normal representation on $\mathscr{H}$. ▌

Remark 3.6. Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space, and suppose $\mathscr{M} \subset B(\mathscr{H})$ is a von Neumann algebra without Property S_σ. Let $S_\sigma(\mathscr{M})$ denote the collection of all σ-weakly closed subspaces $\mathscr{T}$ of $B(\mathscr{H})$ for which $F(\mathscr{M}, \mathscr{T}) = \mathscr{M} \bar{\otimes} \mathscr{T}$. Then $S_\sigma(\mathscr{M})$ obviously contains all subspaces with Property S_σ. However, it also contains subspaces without Property S_σ, since it contains all von Neumann subalgebras of $B(\mathscr{H})$ by

Tomita's Theorem. In particular, $\mathcal{M}$ itself is in $S_\sigma(\mathcal{M})$. Let $V(\mathcal{H})$ (resp. $R(\mathcal{H})$; resp. $S(\mathcal{H})$) denote the collection all von Neumann algebras (resp. reflexive algebras; resp. σ-weakly closed operator systems) acting on $\mathcal{H}$. (Recall that a subspace of $B(\mathcal{H})$ is said to be an operator system if it is self-adjoint and contains the identity operator [8].) Then $R(\mathcal{H}) \cap S(\mathcal{H}) = V(\mathcal{H})$ is contained in $S_\sigma(\mathcal{M})$. By Remark 1.1, $S_\sigma(\mathcal{M})$ does not contain $R(\mathcal{H})$. It also does not contain $S(\mathcal{H})$. To see this, let $\mathcal{T}$ be any σ-weakly closed subspace of $B(\mathcal{H})$ that is not in $S_\sigma(\mathcal{M})$, and (using the notation of Remark 1.1) let $\mathcal{S}_0$ denote the subspace of $B(\mathcal{H}_1)$ consisting of all 2×2 operator matrices $a=[a_{ij}]$ with $a_{11}=a_{22}\in \mathbf{C}1$, $a_{12}\in\mathcal{T}$, and $a_{21}\in\mathcal{T}^*$ (where $\mathcal{T}^*=\{t^* \mid t\in\mathcal{T}\}$). Then it is obvious that $\mathcal{S}_0\in S(\mathcal{H}_1)$. Moreover, it is easy to modify the arguments in Remark 1.1 to show that if we write the elements of $\mathcal{M} \bar{\otimes} B(\mathcal{H}_1)$ as 2×2 operator matrices $x=[x_{ij}]$ with $x_{ij}\in\mathcal{M} \bar{\otimes} B(\mathcal{H})$, then

$$F(\mathcal{M}, \mathcal{S}_0) = \{x \mid x_{11}=x_{22}\in\mathcal{M} \bar{\otimes} \mathbf{C}I,\ x_{12}\in F(\mathcal{M},\mathcal{T}) \text{ and } x_{21}\in F(\mathcal{M},\mathcal{T}^*)\}. \tag{3.10}$$

Since $F(\mathcal{M},\mathcal{T}) \neq \mathcal{M} \bar{\otimes} \mathcal{T}$, it follows from (3.10) that $F(\mathcal{M},\mathcal{S}_0) \neq \mathcal{M} \bar{\otimes} \mathcal{S}_0$. Now let u be any unitary operator from $\mathcal{H}$ onto $\mathcal{H}_1$, and let $\mathcal{S}=u^*\mathcal{S}_0 u$. Then it is easily checked that $\mathcal{S}\in S(\mathcal{H})$, but $\mathcal{S}$ is not in $S_\sigma(\mathcal{M})$.

Finally, we note that $S_\sigma(\mathcal{M})$ is closed under taking adjoints. To see this, first observe that if $x\in B(\mathcal{H}) \bar{\otimes} B(\mathcal{H})$ and if φ and ψ are in $B(\mathcal{H})_*$, then

$$\begin{aligned}\langle R_\varphi(x^*), \psi\rangle &= \langle x^*, \varphi\otimes\psi\rangle = \overline{\langle x, \varphi^*\otimes\psi^*\rangle}\\ &= \overline{\langle R_{\varphi^*}(x), \psi^*\rangle} = \langle (R_{\varphi^*}(x))^*, \psi\rangle\end{aligned}$$

so $R_\varphi(x^*) = (R_{\varphi^*}(x))^*$. The same formula is valid for left slice maps, so

$$F(\mathcal{S}^*, \mathcal{T}^*) = (F(\mathcal{S},\mathcal{T}))^* \tag{3.11}$$

for all pairs of σ-weakly closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $B(\mathcal{H})$. Applying (3.11) with $\mathcal{S}=\mathcal{M}=\mathcal{M}^*$, we conclude that $\mathcal{T}\in S_\sigma(\mathcal{M})$ if and only if $\mathcal{T}^*\in S_\sigma(\mathcal{M})$, as claimed.

By Lemma 3.5, there is a separably acting von Neumann algebra without Property S_σ for $l^\infty(\mathbf{N})$. It is clear that to show there is a factor without Property S_σ for $l^\infty(\mathbf{N})$, one needs to make use of direct integral theory. In fact, it suffices to show that if $\lambda \to \mathcal{M}(\lambda)$ is the central decomposition of a separably acting von Neumann algebra $\mathcal{M}$, and if the factors $\mathcal{M}(\lambda)$ all have Property S_σ for $l^\infty(\mathbf{N})$, then $\mathcal{M}$ also has Property S_σ for $l^\infty(\mathbf{N})$. If all the $\mathcal{M}(\lambda)$'s have Property S_σ for $l^\infty(\mathbf{N})$, and if $\mathcal{T}$ is a σ-weakly closed subspace of $l^\infty(\mathbf{N})$, then $F(\mathcal{M}(\lambda),\mathcal{T}) = \mathcal{M}(\lambda) \bar{\otimes} \mathcal{T}$ for all λ. If $\mathcal{T}$ is a von Neumann algebra, then one can show, using the usual direct integral theory, that the direct integral decomposition of the von Neumann algebra $F(\mathcal{M}, \mathcal{T})$ with respect to the center of $\mathcal{M} \bar{\otimes} B(l^2(\mathbf{N}))$ is $\lambda \to F(\mathcal{M}(\lambda), \mathcal{T}) = \mathcal{M}(\lambda) \bar{\otimes} \mathcal{T}$, and it follows from this that $F(\mathcal{M}, \mathcal{T}) = \mathcal{M} \bar{\otimes} \mathcal{T}$. The problem with this argument, of course, is that if $F(\mathcal{M}, \mathcal{T}) \neq \mathcal{M} \bar{\otimes} \mathcal{T}$, then $\mathcal{T}$ can't be a von Neumann algebra. The way around this difficulty is to make use of Remark 1.1, which implies that if all the $\mathcal{M}(\lambda)$'s have Property S_σ for $l^\infty(\mathbf{N})$, but $\mathcal{M}$ does not have Property S_σ for $l^\infty(\mathbf{N})$, then there is a separably acting reflexive algebra $\mathcal{B}$ such that $F(\mathcal{M}(\lambda), \mathcal{B}) = \mathcal{M}(\lambda) \bar{\otimes} \mathcal{B}$ for all λ, but $F(\mathcal{M}, \mathcal{B}) \neq \mathcal{M} \bar{\otimes} \mathcal{B}$. Since $\mathcal{B}$ is a strongly closed algebra, we can make use of the direct integral theory for strongly closed algebras developed by Azoff, Fong, and Gilfeather in [2]. Using results from [2], we will show below that since $F(\mathcal{M}(\lambda), \mathcal{B}) = \mathcal{M}(\lambda) \bar{\otimes} \mathcal{B}$ for all λ, we must have that $F(\mathcal{M}, \mathcal{B}) = \mathcal{M} \bar{\otimes} \mathcal{B}$, and this contradiction shows that $\mathcal{M}$ has Property S_σ for $l^\infty(\mathbf{N})$.

We will assume the reader is familiar with [2], as well as the usual direct integral theory for von Neumann algebras. All of the facts about direct integral theory for von Neumann algebras that we use without giving a specific reference can be found in [15, Part II].

Recall that if $\mathcal{M}$ is a von Neumann algebra acting on a separable infinite dimensional Hilbert space $\mathcal{H}$, then there is a compact metrizable space Λ, a (complete) σ-finite regular Borel measure μ on Λ, a measurable field $\lambda \to \mathcal{H}(\lambda)$ of Hilbert spaces, and a measurable field $\lambda \to \mathcal{M}(\lambda)$ of factors (where $\mathcal{M}(\lambda)$ acts on $\mathcal{H}(\lambda)$) such that $\mathcal{H}$ is the direct integral of the $\mathcal{H}(\lambda)$'s, and $\mathcal{M}$ is the direct integral of the $\mathcal{M}(\lambda)$'s. We will refer to this as the central decomposition of $\mathcal{M}$. Let $\mathcal{K}$ be a separable Hilbert space, and let $\lambda \to \mathcal{K}(\lambda) = \mathcal{K}$ be the corresponding constant field over Λ. Then $\lambda \to \mathcal{H}(\lambda) \otimes \mathcal{K}$ is a measurable field of Hilbert spaces, and we can identify

$$\left(\int^{\oplus} \mathcal{H}(\lambda)\, d\mu(\lambda)\right) \otimes \mathcal{K} \qquad \text{and} \qquad \int^{\oplus} (\mathcal{H}(\lambda) \otimes \mathcal{K})\, d\mu(\lambda).$$

Moreover, with this identification we have that

$$\mathcal{M} \bar{\otimes} B(\mathcal{K}) = \left(\int^{\oplus} \mathcal{M}(\lambda)\, d\mu(\lambda)\right) \bar{\otimes} B(\mathcal{K}) = \int^{\oplus} (\mathcal{M}(\lambda) \bar{\otimes} B(\mathcal{K}))\, d\mu(\lambda). \tag{3.12}$$

It follows from (3.12) that for every operator $x \in \mathcal{M} \bar{\otimes} B(\mathcal{K})$ there is an essentially bounded measurable field of operators $\lambda \to x(\lambda)$ such that $x(\lambda) \in \mathcal{M}(\lambda) \bar{\otimes} B(\mathcal{K})$ almost everywhere and such that

$$x = \int^{\oplus} x(\lambda)\, d\mu(\lambda).$$

Moreover, if

$$a = \int^{\oplus} a(\lambda)\, d\mu(\lambda)$$

is in $\mathcal{M}$, if $b \in B(\mathcal{K})$, and if $x = a \otimes b$, then

$$x = \int^{\oplus} (a(\lambda) \otimes b)\, d\mu(\lambda). \tag{3.13}$$

By Proposition 8.34 in [50, Chap. IV], for any $\varphi \in \mathcal{M}_*$ there is a unique integrable field $\lambda \to \varphi(\lambda) \in \mathcal{M}(\lambda)_*$ of normal functionals such that

$$\left\langle \int^{\oplus} a(\lambda)\, d\mu(\lambda),\, \varphi \right\rangle = \int \langle a(\lambda), \varphi(\lambda) \rangle\, d\mu(\lambda) \tag{3.14}$$

for all $a = \int^{\oplus} a(\lambda)\, d\mu(\lambda)$ in $\mathcal{M}$, and, conversely, if $\lambda \to \varphi(\lambda)$ is an integrable field of normal functionals, then (3.14) defines an normal linear functional φ on $\mathcal{M}$.

The proof of the next lemma is a straightforward exercise in direct integral theory, and is left to the reader.

LEMMA 3.7. *Let $\varphi \in \mathcal{M}_*$, and let $\lambda \to \varphi(\lambda) \in \mathcal{M}(\lambda)_*$ be the integrable field of normal functionals such that* (3.14) *holds for all $a \in \mathcal{M}$. Let $\psi \in B(\mathcal{K})$. Then $\lambda \to \varphi(\lambda) \otimes \psi \in (\mathcal{M}(\lambda) \bar{\otimes} B(\mathcal{K}))_*$ is an integrable field of normal functionals, and*

$$\left\langle \int^{\oplus} x(\lambda)\, d\mu(\lambda),\, \varphi \otimes \psi \right\rangle = \int \langle x(\lambda), \varphi(\lambda) \otimes \psi \rangle\, d\mu(\lambda) \tag{3.15}$$

for all $x = \int^{\oplus} x(\lambda)\, d\mu(\lambda)$ in $\mathcal{M} \bar{\otimes} B(\mathcal{K})$.

LEMMA 3.8. *Let $\mathcal{T} \subset B(\mathcal{K})$ be a σ-weakly closed subspace. Then $x = \int^{\oplus} x(\lambda)\, d\mu(\lambda)$ is in $F(\mathcal{M}, \mathcal{T})$ if and only if $x(\lambda) \in F(\mathcal{M}(\lambda), \mathcal{T})$ for almost all λ.*

Proof. First suppose that $x(\lambda) \in F(\mathcal{M}(\lambda), \mathcal{T})$ for almost all λ, and that $\psi \in \mathcal{T}_\perp$ (where $\mathcal{T}_\perp$ denotes the annihilator of $\mathcal{T}$ in $B(\mathcal{K})_*$). Let $\varphi \in \mathcal{M}_*$, and let $\lambda \to \varphi(\lambda) \in \mathcal{M}(\lambda)_*$ be the associated field of normal functionals. Then since $R_{\varphi(\lambda)}(x(\lambda)) \in \mathcal{T}$ for almost all λ,

$$\langle x(\lambda), \varphi(\lambda) \otimes \psi \rangle = \langle R_{\varphi(\lambda)}(x(\lambda)), \psi \rangle = 0$$

for almost all λ. It follows immediately from this and (3.15) that

$$\langle R_\varphi(x), \psi \rangle = \langle x, \varphi \otimes \psi \rangle = 0 \qquad (\psi \in \mathscr{T}_\perp).$$

Hence $R_\varphi(x) \in \mathscr{T}$ for all $\varphi \in \mathscr{M}_*$, and so $x \in F(\mathscr{M}, \mathscr{T})$.

Next suppose that $x \in F(\mathscr{M}, \mathscr{T})$. By Theorem 8.13 in [50, Chap. IV], for each $\lambda \in \Lambda$ there exists an isometry $u(\lambda)$ of $\mathscr{H}(\lambda)$ into $\mathscr{H}$ such that for each vector

$$\zeta = \int^{\oplus} \zeta(\lambda)\, d\mu(\lambda) \qquad \text{in} \quad \mathscr{H} = \int^{\oplus} \mathscr{H}(\lambda)\, d\mu(\lambda),$$

the functions $\lambda \to (u(\lambda)\, \zeta(\lambda) \mid \eta)$ are measurable for all $\eta \in \mathscr{H}$. It follows easily from this that for any $\eta \in \mathscr{H}$, the mapping $\lambda \to u(\lambda)^* \eta \in \mathscr{H}(\lambda)$ is a measurable vector field (although not square integrable in general).

Let $\zeta, \eta \in \mathscr{H}$, and let $\omega_{\zeta,\eta}$ denote, as usual, the element of $B(\mathscr{H})_*$ defined by $\omega_{\zeta,\eta}(a) = (a\zeta \mid \eta)$. For each $\lambda \in \Lambda$, let $\rho(\lambda)$ denote the restriction of $\omega_{u^*(\lambda)\zeta,\, u^*(\lambda)\eta}$ to $\mathscr{M}(\lambda)$. Then $\rho(\lambda) \in \mathscr{M}(\lambda)_*$, and since $\lambda \to u(\lambda)^* \zeta$ and $\lambda \to u(\lambda)^* \eta$ are measurable vector fields,

$$\lambda \to \langle a(\lambda), \rho(\lambda) \rangle = (a(\lambda)\, u(\lambda)^* \zeta \mid u(\lambda)^* \eta)$$

is a measurable function for every measurable operator field $\lambda \to a(\lambda) \in \mathscr{M}(\lambda)$. Hence $\lambda \to \rho(\lambda)$ is a measurable field of normal functionals. The field $\rho(\cdot)$ need not be integrable, but it is obvious that $\|\rho(\cdot)\|$ is bounded by $\|\zeta\|\, \|\eta\|$. Hence if E is any subset of Λ with finite measure, and χ_E denotes the characteristic function of E, then $\lambda \to \chi_E(\lambda)\, \rho(\lambda)$ is an integrable field of normal functionals and so defines an element of $\mathscr{M}_*$, which we will denote by ρ_E.

Now let $\psi \in \mathscr{T}_\perp$. Since $x \in F(\mathscr{M}, \mathscr{T})$,

$$\langle x, \rho_E \otimes \psi \rangle = \langle R_{\rho_E}(x), \psi \rangle = 0$$

for every $E \subset \Lambda$ with finite measure. Hence by (3.15) we have that

$$0 = \int \langle x(\lambda), \chi_E(\lambda)\, \rho(\lambda) \otimes \psi \rangle\, d\mu(\lambda) = \int_E \langle x(\lambda), \rho(\lambda) \otimes \psi \rangle\, d\mu(\lambda)$$

for every $E \subset \Lambda$ with finite measure, and so $\langle x(\lambda), \rho(\lambda) \otimes \psi \rangle = 0$ for almost all λ.

Since $\mathscr{K}$ is separable, we can choose a countable dense subset $\psi_1, \psi_2, \ldots$ of $\mathscr{T}_\perp$. For $i = 1, 2, \ldots$, let

$$N_i = \{\lambda \in \Lambda \mid \langle x(\lambda), \rho(\lambda) \otimes \psi_i \rangle \neq 0\},$$

and let $N_{\zeta,\eta}$ denote the union of the N_i's. Then $N_{\zeta,\eta}$ is a set of measure zero, and $\langle R_{\rho(\lambda)}(x(\lambda)), \psi_i \rangle = \langle x(\lambda), \rho(\lambda) \otimes \psi_i \rangle = 0$ for all λ not in $N_{\zeta,\eta}$ and for all $i \in \mathbf{N}$. Hence if $\lambda \notin N_{\zeta,\eta}$, then $\langle R_{\rho(\lambda)}(x(\lambda)), \psi \rangle = 0$ for all $\psi \in \mathscr{T}_\perp$, and so $R_{\rho(\lambda)}(x(\lambda)) \in \mathscr{T}$.

Now let $\zeta_1, \zeta_2, \ldots$ be an orthonormal basis for $\mathscr{H}$, and for each $\lambda \in \Lambda$, let $\rho_{i,j}(\lambda)$ denote the restriction of $\omega_{u^*(\lambda)\zeta_i, u^*(\lambda)\zeta_j}$ to $\mathscr{M}(\lambda)$. Let N denote the union of the sets N_{ζ_i,ζ_j}. Then N is a set of measure zero, and $R_{\rho_{i,j}(\lambda)}(x(\lambda)) \in \mathscr{T}$ for all λ not in N. Suppose that $\lambda \notin N$. Since $u(\lambda)$ is an isometry, $\mathscr{H}(\lambda)$ is the norm closed linear span of $\{u(\lambda)^* \zeta_i \mid i \in \mathbf{N}\}$. It follows easily from this that $\mathscr{M}(\lambda)_*$ is the norm closed linear span of the $\rho_{i,j}(\lambda)$'s. Hence $R_\varphi(x(\lambda)) \in \mathscr{T}$ for all $\varphi \in \mathscr{M}(\lambda)_*$, since $R_{\rho_{i,j}(\lambda)}(x(\lambda)) \in \mathscr{T}$ for all i and j, and since $\varphi \to R_\varphi(x(\lambda))$ is a bounded linear map from $\mathscr{M}(\lambda)_*$ to $B(\mathscr{K})$. Thus $x(\lambda) \in F(\mathscr{M}(\lambda), \mathscr{T})$ for all $\lambda \notin N$, and so $x(\lambda) \in F(\mathscr{M}(\lambda), \mathscr{T})$ for almost all λ. ∎

PROPOSITION 3.9. *Let $\mathscr{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$, and let $\mathscr{A}$ be a σ-weakly closed unital subalgebra of $B(\mathscr{H})$. Suppose that the components $\mathscr{M}(\lambda)$ of $\mathscr{M}$ in its central decomposition satisfy $F(\mathscr{M}(\lambda), \mathscr{A}) = \mathscr{M}(\lambda) \bar{\otimes} \mathscr{A}$ for almost all λ, that $\mathscr{M} \bar{\otimes} \mathscr{A}$ is strongly closed, and that $\mathscr{M}(\lambda) \bar{\otimes} \mathscr{A}$ is strongly closed for almost all λ. Then $F(\mathscr{M}, \mathscr{A}) = \mathscr{M} \bar{\otimes} \mathscr{A}$.*

Proof. Since the $\mathscr{M}(\lambda)$'s are the components of $\mathscr{M}$ in its central decomposition, we are in the situation described above, with $\mathscr{K} = \mathscr{H}$. Set $\mathscr{B} = \mathscr{M} \bar{\otimes} \mathscr{A}$. Then $\mathscr{B}$ is a strongly closed algebra by assumption, so the direct integral theory developed in [2] applies to $\mathscr{B}$. Since $\mathscr{B} \subset \mathscr{M} \bar{\otimes} B(\mathscr{H})$, all the operators in $\mathscr{B}$ are decomposable.

Let $\{s_i\}$ be a countable σ-weakly dense complex-rational unital *-subalgebra of $\mathscr{M}$, let $\{t_j\}$ be a countable σ-weakly dense subset of $\mathscr{A}$, and let $\mathscr{B}_0$ denote the complex-rational linear span of $\{s_i \otimes t_j\}$. Then $\mathscr{B}_0$ is σ-weakly dense in $\mathscr{B}$, and hence is strongly dense in $\mathscr{B}$, since $\mathscr{B}$ is strongly closed. Let $\{b_k\}$ be an enumeration of the elements of $\mathscr{B}_0$. Then it follows from Propositions 3 and 8 in [15, Part II, Chap. 2] that, for almost all λ, $\{s_i(\lambda)\}$ is a complex-rational unital *-algebra and $\{b_k(\lambda)\}$ is the complex-rational linear span of $\{s_i(\lambda) \otimes t_j\}$. Moreover, by Theorem 1 and Proposition 1 in [15, Part II, Chap. 3], the set $\{s_i(\lambda)\}$ generates $\mathscr{M}(\lambda)$ as a von Neumann algebra for almost all λ. Hence $\{b_k(\lambda)\}$ is a σ-weakly dense complex-rational linear subspace of $\mathscr{M}(\lambda) \bar{\otimes} \mathscr{A}$ for almost all λ. Finally, since $\mathscr{M}(\lambda) \bar{\otimes} \mathscr{A}$ is strongly closed for almost all λ, $\mathscr{M}(\lambda) \bar{\otimes} \mathscr{A}$ is the strongly closed algebra generated by $\{b_k(\lambda)\}$ for almost all λ.

Let $Z(\mathscr{M})$ denote the center of $\mathscr{M}$, let $\mathscr{D} = Z(\mathscr{M}) \bar{\otimes} \mathbf{C}1$, and let

$$\mathscr{B} \sim \int^{\oplus} \mathscr{B}(\lambda)\, d\mu(\lambda)$$

be the decomposition of $\mathscr{B}$ with respect to $\mathscr{D}$. (See Section 3 of [2].) Since $\{b_k\}$ is a generating set for $\mathscr{B}$ (as a strongly closed algebra), $\mathscr{B}(\lambda)$ is, by definition, the strongly closed algebra generated by $\{b_k(\lambda)\}$. Hence

$$\mathscr{B}(\lambda)=\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{A}\qquad\text{for almost all }\lambda. \tag{3.16}$$

Since $\mathscr{D}\subset\mathscr{B}$, it follows from (3.16) and Proposition 3.3 in [2] that

$$x=\int^{\oplus} x(\lambda)\,d\mu(\lambda)\in\mathscr{B}\Leftrightarrow x(\lambda)\in\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{A}\qquad\text{for almost all }\lambda. \tag{3.17}$$

Now suppose that $x=\int^{\oplus} x(\lambda)\,d\mu(\lambda)$ is in $F(\mathscr{M},\mathscr{A})$. Then $x(\lambda)\in F(\mathscr{M}(\lambda),\mathscr{A})$ for almost all λ by Lemma 3.8. Moreover, by assumption, $F(\mathscr{M}(\lambda),\mathscr{A})=\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{A}$ for almost all λ. Hence $x(\lambda)\in\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{A}$ for almost all λ, and so x is in $\mathscr{B}=\mathscr{M}\,\bar{\otimes}\,\mathscr{A}$ by (3.17). Thus $F(\mathscr{M},\mathscr{A})\subset\mathscr{M}\,\bar{\otimes}\,\mathscr{A}$. But the reverse inclusion is always valid, so $F(\mathscr{M},\mathscr{A})=\mathscr{M}\,\bar{\otimes}\,\mathscr{A}$. ∎

THEOREM 3.10. *Let $\mathscr{M}$ and $\mathscr{N}$ be separably acting von Neumann algebras, and suppose that the components $\mathscr{M}(\lambda)$ of $\mathscr{M}$ in its central decomposition have Property S_σ for $\mathscr{N}$ for almost all λ. Then $\mathscr{M}$ has Property S_σ for $\mathscr{N}$.*

Proof. Suppose that $\mathscr{M}$ does not have Property S_σ for $\mathscr{N}$. Let $\mathscr{T}$ be a σ-weakly closed subspace of $\mathscr{N}$ such that $F(\mathscr{M},\mathscr{T})\neq\mathscr{M}\,\bar{\otimes}\,\mathscr{T}$. Let $\mathscr{B}$ be the reflexive algebra associated with $\mathscr{T}$ as in Remark 1.1. Then $\mathscr{M}\,\bar{\otimes}\,\mathscr{B}$ and all of the algebras $\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{B}$ are reflexive, and so strongly closed. Moreover, if $\mathscr{M}(\lambda)$ has Property S_σ for $\mathscr{N}$, then $F(\mathscr{M}(\lambda),\mathscr{T})=\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{T}$, so $F(\mathscr{M}(\lambda),\mathscr{B})=\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{B}$ by Remark 1.1. Hence $F(\mathscr{M}(\lambda),\mathscr{B})=\mathscr{M}(\lambda)\,\bar{\otimes}\,\mathscr{B}$ for almost all λ, and so $F(\mathscr{M},\mathscr{B})=\mathscr{M}\,\bar{\otimes}\,\mathscr{B}$ by Proposition 3.9. But this implies that $F(\mathscr{M},\mathscr{T})=\mathscr{M}\,\bar{\otimes}\,\mathscr{T}$ by Remark 1.1, so we have a contradiction. Hence $\mathscr{M}$ has Property S_σ for $\mathscr{N}$. ∎

Remark 3.11. Suppose that $\mathscr{M}$ is a separably acting von Neumann algebra, and suppose that the components $\mathscr{M}(\lambda)$ of $\mathscr{M}$ in its central decomposition have Property S_σ for almost all λ. Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. Then almost all of the $\mathscr{M}(\lambda)$'s have Property S_σ for $B(\mathscr{H})$, and hence $\mathscr{M}$ has Property S_σ for $B(\mathscr{H})$. Thus, by Theorem 2.6, $\mathscr{M}$ has Property S_σ. Moreover, since the σ-weak approximation property is equivalent to Property S_σ, it follows that if almost all of the $\mathscr{M}(\lambda)$'s have the σ-weak approximation property, then so does $\mathscr{M}$. Finally, we note that if follows from Theorems 3.1 and 3.10 that if $\mathscr{M}(\lambda)_*$ has the AP for almost all λ, then $\mathscr{M}_*$ has the AP.

It follows immediately from Lemma 3.5 and Theorem 3.10 that there is a separably acting factor without Property S_σ for $l^\infty(\mathbf{N})$. In order to show that there are separably acting factors of each of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$ without Property S_σ for $l^\infty(\mathbf{N})$, we need to make use of the fact that if a von Neumann algebra $\mathcal{M}$ does not have Property S_σ for $l^\infty(\mathbf{N})$, then its crossed product by its modular automorphism group also does not have Property S_σ for $l^\infty(\mathbf{N})$. This is a special case of the following result in [26]: if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, and if the crossed product $\mathcal{M} \otimes_\alpha G$ of $\mathcal{M}$ by an action α of a locally compact group G has Property S_σ for $\mathcal{N}$, then $\mathcal{M}$ has Property S_σ for $\mathcal{N}$.

THEOREM 3.12. *For each of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$, there is a separably acting factor $\mathcal{M}$ of that type such that $\mathcal{M}$ does not have Property S_σ for $l^\infty(\mathbf{N})$ (and so $\mathcal{M}_*$ does not have the AP).*

Proof. We will show that for each of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$, if there is a separably acting factor of that type without Property S_σ for $l^\infty(\mathbf{N})$, then there are separably acting factors of all of the types II_1, II_∞, and III_λ $(0 \leqslant \lambda \leqslant 1)$ without Property S_σ for $l^\infty(\mathbf{N})$. Since there is a separably acting factor without Property S_σ for $l^\infty(\mathbf{N})$, and since it can't be type I, this will complete the proof. In the rest of the proof $\mathcal{H}$ will always denote a separable infinite dimensional Hilbert space. We will split the proof into cases.

Case 1. Suppose that there is a factor $\mathcal{M} \subset B(\mathcal{H})$ of type II_1 without Property S_σ for $l^\infty(\mathbf{N})$. Then $\mathcal{M} \bar{\otimes} B(\mathcal{H})$ is a factor of type II_∞ [32, Theorem 6.7.10], and does not have Property S_σ for $l^\infty(\mathbf{N})$ by Proposition 3.4. Moreover, if $\mathcal{N} \subset B(\mathcal{H})$ is any factor of type III_λ $(0 \leqslant \lambda \leqslant 1)$, then $\mathcal{M} \bar{\otimes} \mathcal{N}$ is also a factor of type III_λ [10, Corollaire 3.2.8.], and another application of Proposition 3.4 shows that $\mathcal{M} \bar{\otimes} \mathcal{N}$ does not have Property S_σ for $l^\infty(\mathbf{N})$.

Case 2. Suppose that there is a factor $\mathcal{M} \subset B(\mathcal{H})$ of type II_∞ without Property S_σ for $l^\infty(\mathbf{N})$. Then by Theorem 6.7.10 in [32], there is a factor $\mathcal{N}$ of type II_1 such that $\mathcal{M}$ is isomorphic to $\mathcal{N} \bar{\otimes} B(\mathcal{H})$. Since $B(\mathcal{H})$ has Property S_σ, it follows from Proposition 3.4 that $\mathcal{N}$ does not have Property S_σ for $l^\infty(\mathbf{N})$. Hence there is a separably acting factor of type II_1 without Property S_σ for $l^\infty(\mathbf{N})$, and so by Case 1 there are separably acting factors of type III_λ without Property S_σ for all λ $(0 \leqslant \lambda \leqslant 1)$.

Case 3. Suppose that $0 \leqslant \lambda \leqslant 1$, and that there is a factor $\mathcal{M} \subset B(\mathcal{H})$ of type III_λ without Property S_σ for $l^\infty(\mathbf{N})$. Let σ be the modular automorphism group of $\mathcal{M}$ corresponding to some faithful normal state of $\mathcal{M}$, and let $\mathcal{N} = \mathcal{M} \otimes_\sigma \mathbf{R}$ be the associated crossed product. By the result mentioned above, $\mathcal{N}$ does not have Property S_σ for $l^\infty(\mathbf{N})$. Moreover, $\mathcal{N}$ is separably acting and is of type II_∞ [32, Theorem 13.3.7]. Let $\lambda \to \mathcal{N}(\lambda)$ be the central decomposition of $\mathcal{N}$. Then $\mathcal{N}(\lambda)$ is a type II_∞ factor for almost all λ [32, Corollary 14.2.3], and by Theorem 3.10 there is a set $E \subset \Lambda$ with positive measure such that $\mathcal{N}(\lambda)$ does not have Property S_σ for $l^\infty(\mathbf{N})$ for any $\lambda \in E$. Hence there is a separably acting type II_∞ factor without Property S_σ for $l^\infty(\mathbf{N})$, and so, by Case 2, there are also separably acting factors of type II_1 and type III_λ (for all λ, $0 \leqslant \lambda \leqslant 1$) without Property S_σ for $l^\infty(\mathbf{N})$. ∎

The next result is an immediate consequence of Theorem 3.12 and Remark 1.1 (or apply Corollary 2.3 in [36]).

THEOREM 3.13. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then for each of the types II_1, II_∞, and III_λ* $(0 \leqslant \lambda \leqslant 1)$, *there is a factor $\mathcal{M} \subset B(\mathcal{H})$ of that type and a reflexive algebra $\mathcal{B} \subset B(\mathcal{H})$ such that if* $\operatorname{alg} \mathcal{L}_1 = \mathcal{M}$ *and* $\operatorname{alg} \mathcal{L}_2 = \mathcal{B}$, *then*

$$\operatorname{alg} \mathcal{L}_1 \mathbin{\bar{\otimes}} \operatorname{alg} \mathcal{L}_2 \neq \operatorname{alg}(\mathcal{L}_1 \otimes \mathcal{L}_2).$$

Remark 3.14. As noted in Remark 1.1, the reflexive algebras $\mathcal{B}$ in Theorem 3.13 can be chosen so that $\operatorname{alg} \mathcal{L}_1 \mathbin{\bar{\otimes}} \operatorname{alg} \mathcal{L}_2 = \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{B}$ is a reflexive algebra. Thus, although $F(\mathcal{A}, \mathcal{B})$ is always a reflexive algebra whenever $\mathcal{A} = \operatorname{alg} \mathcal{L}_1$ and $\mathcal{B} = \operatorname{alg} \mathcal{L}_2$ are reflexive algebras (since it equals $\operatorname{alg}(\mathcal{L}_1 \otimes \mathcal{L}_2)$), it need not be the smallest reflexive algebra containing $\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{B}$, even when one (but not both) of $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras. It remains an open question (first raised by Radjavi and Rosenthal in 1969 in [42]) whether the tensor product of reflexive algebras is always reflexive.

Every abelian von Neumann algebra is type I, so every abelian von Neumann algebra has Property S_σ. However, there are abelian reflexive algebras without Property S_σ. In fact, we have the following result.

PROPOSITION 3.15. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then there are abelian reflexive subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $B(\mathcal{H})$ such that $\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{B} \neq F(\mathcal{A}, \mathcal{B})$.*

Proof. Since there is a σ-weakly closed subspace of $B(\mathcal{H})$ without Property S_σ, there is an abelian reflexive algebra $\mathcal{A} \subset B(\mathcal{H})$ without Property S_σ [36, Theorem 2.1]. By Remark 2.7, there is a σ-weakly closed subspace $\mathcal{T} \subset B(\mathcal{H})$ such that $\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{T} \neq F(\mathcal{A}, \mathcal{T})$. It follows from this and Remark 1.1 that there is an abelian reflexive algebra $\mathcal{B} \subset B(\mathcal{H})$ such that $\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{B} \neq F(\mathcal{A}, \mathcal{B})$. ∎

Remark 3.16. *If* $\mathscr{A} \subset B(\mathscr{H})$ and $\mathscr{B} \subset B(\mathscr{K})$ are σ-weakly closed abelian algebras, then $\mathscr{A} \bar{\otimes} \mathscr{B}$ is also an abelian algebra, since multiplication is separately continuous in the σ-weak topology. However, it can be shown that if $\mathscr{H}$ and $\mathscr{K}$ are both infinite dimensional, then multiplication is not separately continuous in the locally convex topology on $B(\mathscr{H}) \bar{\otimes} B(\mathscr{K})$ generated by the seminorms of the form $\rho(x) = |(\varphi \otimes \psi)(x)|$ ($\varphi \in B(\mathscr{H})_*$, $\psi \in B(\mathscr{K})_*$). Since $F(\mathscr{A}, \mathscr{B})$ is the closure of $\mathscr{A} \otimes \mathscr{B}$ in this topology, we can't imitate the argument for $\mathscr{A} \bar{\otimes} \mathscr{B}$ to show that $F(\mathscr{A}, \mathscr{B})$ is an abelian algebra.

It is not known whether $F(\mathscr{A}, \mathscr{B})$ is always abelian. However, as was pointed out to the author by Professor Tadasi Huruya, $F(\mathscr{A}, \mathscr{B})$ is always an algebra, since

$$F(\mathscr{A}, \mathscr{B}) = F(\mathscr{A}, B(\mathscr{K})) \cap F(B(\mathscr{H}), \mathscr{B}) = (\mathscr{A} \bar{\otimes} B(\mathscr{K})) \cap (B(\mathscr{H}) \bar{\otimes} \mathscr{B})$$

and $\mathscr{A} \bar{\otimes} B(\mathscr{K})$ and $B(\mathscr{H}) \bar{\otimes} \mathscr{B}$ are algebras. ∎

4. Singly Generated Algebras and Property S_σ

In this section only, we will denote elements of $B(\mathscr{H})$ by upper-case italic letters. For $T \in B(\mathscr{H})$, let $\mathscr{A}(T)$ denote the σ-weakly closed unital subalgebra of $B(\mathscr{H})$ generated by T. It was shown in [36] that for many "nice" operators T, $\mathscr{A}(T)$ has Property S_σ. This is the case if T is a subnormal operator or if T is in the class $\mathbf{A}(\mathscr{H})$. When T is subnormal, $\mathscr{A}(T)$ is reflexive [40], and $\mathscr{A}(T)$ is also reflexive for many of the operators in $\mathbf{A}(\mathscr{H})$ (see [4]). Hence in these cases, $\operatorname{alg} \mathscr{L}_1 \bar{\otimes} \operatorname{alg} \mathscr{L}_2 = \operatorname{alg}(\mathscr{L}_1 \otimes \mathscr{L}_2)$ when $\mathscr{L}_1$ is the lattice of projections left invariant by T (in which case $\mathscr{A}(T) = \operatorname{alg} \mathscr{L}_1$), and when $\mathscr{L}_2$ is any subspace lattice. We will show below that if T is an injective weighted shift, then $\mathscr{A}(T)$ has the CCAP, and so has Property S_σ. First, however, we show that there is an operator T acting on a separable Hilbert space for which $\mathscr{A}(T)$ does not have Property S_σ. The proof makes use of a construction of Wogen that allows one to obtain "bad" $\mathscr{A}(T)$'s from "bad" subspaces [60].

Proposition 4.1. *Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. Then there is a $T \in B(\mathscr{H})$ such that $\mathscr{A}(T)$ does not have Property S_σ.*

Proof. Let $\mathscr{S} \subset B(\mathscr{H})$ be a σ-weakly closed subspace which does not have Property S_σ. Let $\mathscr{K}$ be the direct sum of a countable number of copies of $\mathscr{H}$. Let P_1 be the orthogonal projection of $\mathscr{K} \oplus \mathscr{H}$ onto the first summand of $\mathscr{H}$ in $\mathscr{K}$, and let P_∞ be the orthogonal projection of $\mathscr{K} \oplus \mathscr{H}$

onto $0 \oplus \mathscr{H}$. Each operator A in $B(\mathscr{K} \oplus \mathscr{H})$ admits a matrix representation $A = [A_{ij}]_{1 \leqslant i, j \leqslant \infty}$ where $A_{ij} \in B(\mathscr{H})$. Let

$$[\mathscr{S}]_{1,\infty} = \{A \in B(\mathscr{K} \oplus \mathscr{H}) \mid A_{1,\infty} \in \mathscr{S} \text{ and } A_{ij} = 0 \text{ if } (i, j) \neq (1, \infty)\}.$$

In Example 2 in [60], Wogen shows how to construct an operator $S \in B(\mathscr{K} \oplus \mathscr{H})$ such that $P_1 \mathscr{A}(S) P_\infty = [\mathscr{S}]_{1,\infty}$. Suppose that $\mathscr{A}(S)$ has Property S_σ. Then $P_1 \mathscr{A}(S) P_\infty$ has Property S_σ by Proposition 1.10 in [34], and so $\mathscr{S}$ has Property S_σ by Proposition 4.2 in [36]. But this contradicts the choice of $\mathscr{S}$, so $\mathscr{A}(S)$ does not have Property S_σ. Since $\mathscr{H}$ is infinite dimensional, there is a unitary operator U from $\mathscr{H}$ onto $\mathscr{K} \oplus \mathscr{H}$. Let $T = U^*SU$. Then $\mathscr{A}(T) = U^* \mathscr{A}(S) U$, so another application of Proposition 4.2 in [36] shows that $\mathscr{A}(T)$ does not have Property S_σ. ∎

Let $\mathscr{H}$ be a Hilbert space with orthonormal basis $\{e_n\}$. An operator $T \in B(\mathscr{H})$ is a *weighted shift* if there are complex numbers $\{w_n\}$ such that $Te_n = w_n e_{n+1}$. T is called a *unilateral weighted shift* if the index n runs over the nonnegative integers, and T is called a *bilateral weighted shift* if n runs over all the integers. We refer the reader to Shields excellent survey article [46] for a detailed treatment of shifts. All of the results quoted without proof in what follows are from this article.

THEOREM 4.2. *If T is an injective unilateral weighted shift, then $\mathscr{A}(T)$ has the CCAP, and so has Property S_σ.*

Proof. Let $\{w_n\}$ be the weight sequence of T. Define a sequence $\{\beta(n)\}$ by $\beta(0) = 1$ and $\beta(n) = w_0 \cdots w_{n-1}$ $(n \geqslant 0)$. Let $H^2(\beta)$ denote the Hilbert space whose elements are the functions $f = \{\hat{f}(n)\}$ such that

$$\sum |\hat{f}(n)|^2 [\beta(n)]^2 < \infty, \tag{4.1}$$

and whose inner product is given by

$$(f \mid g) = \sum \hat{f}(n) \, \hat{g}(n) [\beta(n)]^2. \tag{4.2}$$

Let M_z denote the operator defined by

$$\begin{aligned} (M_z f)\hat{}\,(n) &= \hat{f}(n-1) \qquad (n \geqslant 1), \\ &= 0 \qquad\qquad\quad (n = 0). \end{aligned}$$

Then T is unitarily equivalent to M_z [46, Proposition 7]. Hence $\mathscr{A}(T)$ is unitarily equivalent to $\mathscr{A}(M_z)$, so it suffices to show that $\mathscr{A}(M_z)$ has the

CCAP. The elements of $H^2(\beta)$ can be considered as formal power series in z, i.e., we can write

$$f(z) = \sum \hat{f}(n)\, z^n. \tag{4.3}$$

Then we can multiply f by another formal power series by using the convolution product (see [46, Sect. 4]). Let $H^\infty(\beta)$ denote the set of formal power series φ such that $\varphi H^2(\beta) \subset H^2(\beta)$. We will identify φ with the operator of multiplication by φ on $H^2(\beta)$. Then $H^\infty(\beta)$ is the commutant of M_z in $B(H^2(\beta))$ [46, Theorem 3]. It follows immediately from Theorem 12 in [46] that $H^\infty(\beta) \subset \mathscr{A}(M_z)$. Moreover, since T is injective, the commutant of M_z is a maximal abelian sublagebra of $B(H^2(\beta))$ [46, Corollary 1, p. 63]. Hence $\mathscr{A}(M_z) \subset H^\infty(\beta)$, so $\mathscr{A}(M_z) = H^\infty(\beta)$. Thus it suffices to show that $H^\infty(\beta)$ has the CCAP.

For $n \geqslant 0$ and for $\varphi \in H^\infty(\beta)$, let $\sigma_n(\varphi)$ denote the nth Cesàro mean of the partial sums of φ (see [46, p. 90]). By definition, σ_n is a finite rank linear map from $H^\infty(\beta)$ to itself. Furthermore, since $\varphi \to \hat{\varphi}(n)$ is σ-weakly continuous for each n, σ_n is also σ-weakly continuous for each n. Let $\mathbf{T}$ be the circle group $\{w \in \mathbf{C} \mid |w| = 1\}$, and for $w \in \mathbf{T}$ and $\varphi \in H^2(\beta)$ let $\varphi_w(z) = \varphi(wz)$. If we let $U_w(\varphi) = \varphi_w$, then U_w is a unitary operator, and for $\varphi \in H^\infty(\beta)$ we have

$$U_w \varphi (U_w)^* = \varphi_w. \tag{4.4}$$

For $n \geqslant 0$ let K_n denote the Fejer kernel, and let ds denote normalized Lebesgue measure on $\mathbf{T}$. It is shown in the proof of Theorem 12 in [46] that

$$\sigma_n(\varphi) = \int \varphi_w K_n(\bar{w})\, ds. \tag{4.5}$$

It follows easily from (4.4) and (4.5) that σ_n is a complete contraction. Finally, if $\varphi \in H^\infty(\beta)$, then $\sigma_n(\varphi) \to \varphi$ strongly [46, Theorem 12] and so σ-weakly. Hence $\mathscr{A}(M_z)$ has the CCAP. ∎

THEOREM 4.3. *Let T be an injective bilateral shift. Then $\mathscr{A}(T)$ has the CCAP, and so has Property S_σ. If T is invertible, then $\mathscr{A}(T, T^{-1})$ (the σ-weakly closed unital algebra generated by T and T^{-1}) also has the CCAP, and so has Property S_σ.*

Proof. The proof is similar to that of Theorem 4.2. Let $\{w_n\}$ be the weight sequence of T. Define a sequence $\{\beta(n)\}$ by $\beta(n) = w_0 \cdots w_{n-1}$ $(n > 0)$, $\beta(0) = 1$, and $\beta(-n) = (w_{-1} \cdots w_{-n})^{-1}$ $(n > 0)$. Let $L^2(\beta)$ denote the Hilbert space whose elements are the functions $\{\hat{f}(n)\}$ (where n runs over all integers) such that (4.1) holds, and whose inner product is given by (4.2). Let M_z denote the operator on $L^2(\beta)$ defined by

$$(M_z f)\hat{}(n) = \hat{f}(n-1) \qquad (\text{all } n).$$

Then T is unitarily equivalent to M_z [46, Proposition 7], and so to show $\mathscr{A}(T)$ has the CCAP it suffices to show that $\mathscr{A}(M_z)$ has the CCAP, and to show that $\mathscr{A}(T, T^{-1})$ has the CCAP it suffices to show that $\mathscr{A}(M_z, M_z^{-1})$ has the CCAP. We let $L^\infty(\beta)$ denote the set of formal Laurent series φ such that $\varphi L^2(\beta) \subset L^2(\beta)$ (where the elements of $L^2(\beta)$ are viewed as formal Laurent series using (4.3), and multiplication of formal Laurent series is given by the convolution product). We will identify φ with the operator of multiplication by φ on $L^2(\beta)$. Then $L^\infty(\beta)$ is the commutant of M_z in $L^2(\beta)$ [46, Theorem 3], and is a maximal abelian subalgebra of $B(L^2(\beta))$ [46, Corollary 1, p. 63]. The argument in the last paragraph of the proof of Theorem 4.2 (with $H^2(\beta)$ replaced by $L^2(\beta)$ and $H^\infty(\beta)$ replaced by $L^\infty(\beta)$) shows that $L^\infty(\beta)$ has the CCAP.

If T is not invertible and $\varphi \in L^\infty(\beta)$, then $\hat{\varphi}(n) = 0$ for $n < 0$ [46, Theorem 10′], and hence $\sigma_n(\varphi)$ is in $\mathscr{A}(M_z)$ for all $n \geqslant 0$. Since $\sigma_n(\varphi) \to \varphi$ σ-weakly, $\varphi \in \mathscr{A}(M_z)$. Combining this with the fact that $L^\infty(\beta)$ is maximal abelian, we conclude that $\mathscr{A}(M_z) = L^\infty(\beta)$, and so $\mathscr{A}(M_z)$ has the CCAP.

Finally, suppose that T is invertible. Since $L^\infty(\beta)$ is maximal abelian, $\mathscr{A}(M_z, M_z^{-1}) \subset L^\infty(\beta)$. Moreover, if $\varphi \in L^\infty(\beta)$ then $\sigma_n(\varphi)$ is in $\mathscr{A}(M_z, M_z^{-1})$ for all $n \geqslant 0$, and so $\varphi \in \mathscr{A}(M_z, M_z^{-1})$. Hence $\mathscr{A}(M_z, M_z^{-1}) = L^\infty(\beta)$, and so $\mathscr{A}(M_z, M_z^{-1})$ has the CCAP. Since the σ_n's are σ-weakly continuous, $\sigma_n(\mathscr{A}(M_z)) \subset \mathscr{A}(M_z)$ for all $n \geqslant 0$. The restriction of each σ_n to $\mathscr{A}(M_z)$ is a finite rank σ-weakly continuous contraction from $\mathscr{A}(M_z)$ to itself, and $\sigma_n(\varphi) \to \varphi$ σ-weakly for all $\varphi \in \mathscr{A}(M_z)$. Hence $\mathscr{A}(M_z)$ also has the CCAP. ∎

5. The Slice Map Problem for C^* Algebras

Many of the results in Section 2 and some of the results in Section 3 have analogues for C^*-algebras. We will discuss these and related results in this section. In this section only, if A and B are C^*-algebras, $A \otimes B$ will denote the spatial (or minimal) C^*-tensor product of A and B rather than the algebraic tensor product. Note that if $A \subset B(\mathscr{H})$ and $B \subset B(\mathscr{K})$, then $a \otimes b \in B(\mathscr{H} \otimes \mathscr{K})$ for all $a \in A$ and $b \in B$, and $A \otimes B$ is just the norm closed linear span of $\{a \otimes b \mid a \in A \text{ and } b \in B\}$ in $B(\mathscr{H} \otimes \mathscr{K})$.

If A and B are C^*-algebras, and $\varphi \in A^*$, there is a unique bounded linear map R_φ from $A \otimes B$ to B such that

$$R_\varphi(a \otimes b) = \langle a, \varphi \rangle b \qquad (a \in A, b \in B),$$

and if $\psi \in B^*$, there is a unique bounded linear map L_ψ from $A \otimes B$ to A such that

$$L_\psi(a \otimes b) = \langle b, \psi \rangle a \qquad (a \in A, b \in B)$$

[51]. As in the case of von Neumann algebras, the maps R_φ are called right slice maps and the maps L_ψ are called left slice maps. Slice maps have proved to be very useful in studying tensor products of C^*-algebras (see, e.g., [1, 31, 51–54, 57–59]).

If $S \subset A$ and $T \subset B$ are norm closed (linear) subspaces, we will denote by $S \otimes T$ the norm closed linear span of $\{s \otimes t \mid s \in S \text{ and } t \in T\}$ in $A \otimes B$. Note that if $A \subset B(\mathscr{H})$ and $B \subset B(\mathscr{K})$, then $S \otimes T$ is also the norm closed linear span of $\{s \otimes t \mid s \in S \text{ and } t \in T\}$ in $B(\mathscr{H} \otimes \mathscr{K})$. The *Fubini product of S and T with respect to* $A \otimes B$ is defined to be $\{x \in A \otimes B \mid R_\varphi(x) \in T, L_\psi(x) \in S \text{ for all } \varphi \in A^*, \psi \in B^*\}$, and is denoted by $F(S, T, A \otimes B)$. (See [53], where S and T are assumed to be C^*-subalgebras of A and B.) Note that if follows easily from the definitions that we always have that $S \otimes T \subset F(S, T, A \otimes B)$.

Remark 5.1. If $\mathscr{S} \subset B(\mathscr{H})$ and $\mathscr{T} \subset B(\mathscr{K})$ are σ-weakly closed subspaces, and if $\mathscr{M}$ and $\mathscr{N}$ are any von Neumann algebras containing $\mathscr{S}$ and $\mathscr{T}$, respectively, then with the obvious notation,

$$F(\mathscr{S}, \mathscr{T}, \mathscr{M} \mathbin{\bar{\otimes}} \mathscr{N}) = F(\mathscr{S}, \mathscr{T}, B(\mathscr{H}) \mathbin{\bar{\otimes}} B(\mathscr{K})). \tag{5.1}$$

Equation (5.1) is an easy consequence of Tomita's theorem (see Remark 1.2 in [34]). However, the analogue of Tomita's theorem does not hold for C^*-algebras! In particular, Wassermann showed in [59] that if $\mathscr{H}$ is a separable infinite dimensional Hilbert space, and if $K = K(\mathscr{H})$ denotes the algebra of compact operators on $\mathscr{H}$, then

$$B(\mathscr{H}) \otimes K \neq F(B(\mathscr{H}), K, B(\mathscr{H}) \otimes B(\mathscr{H})).$$

On the other hand, we obviously have that

$$B(\mathscr{H}) \otimes K = F(B(\mathscr{H}), K, B(\mathscr{H}) \otimes K).$$

Hence the analogue of Eq. (5.1) does not even hold for C^*-subalgebras of C^*-algebras. (However, two C^*-algebras always have a "largest" Fubini product. See [31].)

Let A and B be C^*-algebras, and let T be a norm closed subspace of B. Following Wassermann [57], we say that the triple (A, B, T) *verifies the slice map conjecture* if $F(A, T, A \otimes B) = A \otimes T$. We say that A has *Property S for subspaces of B* if (A, B, T) verifies the slice map conjecture whenever T is a norm closed subspace of B, and that A has *Property S for B* if

(A, B, D) verifies the slice map conjecture whenever D is a C^*-subalgebra of B. If A has Property S (resp. Property S for subspaces) for B for all C^*-algebras B, we say that A has *Property S* (resp. *Property S for subspaces*). Wassermann defined Property S in [57], and gave the first examples of C^*-algebras without Property S in [58], where he showed that the only von Neumann algebras with Property S are the finite sums of finite type I's.

For a norm closed subspace $S \subset A$, we let $\mathbf{M}_n(S)$ denote the space of $n \times n$ matrices with entries in S, with the norm inherited from $\mathbf{M}_n(A)$. If π is any faithful representation of A on a Hilbert space $\mathscr{H}$, then $\mathbf{M}_n(A)$ is isomorphic to $\mathbf{M}_n(\pi(A))$, so identifying S with $\pi(S)$, we can consider S as a subspace of $B(\mathscr{H})$. Then, as in Section 2, we let $CB(S)$ denote the space of all completely bounded maps from S to itself. If $\Phi \in CB(S)$, and if B is a C^*-algebra, then it follows from a straightforward modification of the proof of Lemma 1.5 in [13] that there is a (unique) bounded linear map Φ_B from $S \otimes B$ to itself such that

$$\Phi_B(s \otimes b) = \Phi(s) \otimes b \qquad (s \in S, b \in B).$$

Moreover, $\|\Phi_B\| \leqslant \|\Phi\|_{cb}$.

For a norm closed subspace S contained in a C^*-algebra A, we let $F(S)$ denote the collection of all bounded finite rank maps from S to S. Then $\Phi \in F(S)$ if and only if for some $n \in \mathbf{N}$ there are $s_1, ..., s_n$ in S and $\varphi_1, ..., \varphi_n$ in A^* such that

$$\Phi(s) = \sum_{i=1}^{i=n} \langle s, \varphi_i \rangle s_i$$

for all s in S. It follows immediately from this and Corollary 3.4 in [19] that $F(S) \subset CB(S)$. If B is a C^*-algebra and $x \in S \otimes B$, we let $F_B(x)$ denote the norm closure of the linear space $\{\Phi_B(x) \mid \Phi \in F(S)\}$.

In our next result, we will make use of the notion of a stable C^*-algebra. Recall that a C^*-algebra B is said to be stable [6] if B is *-isomorphic to $B \otimes K$, where K denotes the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. Since K is *-isomorphic to $K \otimes \mathbf{M}_n(\mathbf{C})$ for any integer n, it follows from Proposition 4.22 in [50, Chap. IV] and the associativity of the minimal tensor product that if B is stable, then B is *-isomorphic to $B \otimes \mathbf{M}_n(\mathbf{C})$ for any integer n.

THEOREM 5.2. *Let S be a norm closed subspace of a C^*-algebra A, and let B be a stable C^*-algebra. The following are equivalent:*

(a) *$x \in F_B(x)$ for every $x \in S \otimes B$.*

(b) *For every $x \in S \otimes B$, there is a sequence $\{\Phi_n\}$ in $F(S)$ such that $(\Phi_n)_B(x) \to x$ in norm.*

(c) *There is a net $\{\Phi_\alpha\}$ in $F(S)$ such that $(\Phi_\alpha)_B(x) \to x$ in norm for every $x \in S \otimes B$.*

Proof. It is immediate that (a) $\Rightarrow$ (b) and that (c) $\Rightarrow$ (a).

Suppose that (b) holds, and let $x_1, ..., x_n$ be any elements of $S \otimes B$. Let $C = B \otimes \mathbf{M}_n(\mathbf{C})$. Since B is stable, there is a *-isomorphism π from B onto C. By Proposition 4.22 in [50, Chap. IV], there is a *-isomorphism $\tilde{\pi}$ from $A \otimes B$ onto $A \otimes C$ such that

$$\tilde{\pi}(a \otimes b) = a \otimes \pi(b) \qquad (a \in A, b \in B). \tag{5.2}$$

It follows immediately from (5.2) that $\tilde{\pi}$ maps $S \otimes B$ onto $S \otimes C$, and that

$$\tilde{\pi} \circ \Phi_B = \Phi_C \circ \tilde{\pi} \qquad (\Phi \in CB(S)).$$

Let $y = \sum_{i=1}^{i=n} x_i \otimes e_{ii}$, where $\{e_{ij}\}_{1 \leq i,j \leq n}$ are the matrix units for $\mathbf{M}_n(\mathbf{C})$, and where we are identifying $A \otimes (B \otimes \mathbf{M}_n(\mathbf{C}))$ and $(A \otimes B) \otimes \mathbf{M}_n(\mathbf{C})$ in the usual way using the associativity of the minimal tensor product. Since $y = \tilde{\pi}(x)$ for some $x \in S \otimes B$, there is a sequence $\{\Phi_m\}$ in $F(S)$ such that $(\Phi_m)_C(y) \to y$ in norm, from which it follows easily that $(\Phi_m)_B(x_i) \to x_i$ in norm for each i. The proof of (b) $\Rightarrow$ (c) can now be completed by an argument similar to that in the last part of the proof of (c) $\Rightarrow$ (a) in Proposition 2.1. ∎

We do not know if the requirement that B be stable in Theorem 5.2 is necessary. However, Theorem 5.2 as stated in sufficient for our applications (Theorems 5.4 and 5.5).

The proofs of the next two results are similar to the proofs of Proposition 2.5 and Theorem 2.6, and are left to the reader.

PROPOSITION 5.3. *Let A and B be C^*-algebras. Let $x \in A \otimes B$, and let T denote the norm closure of the linear space $\{R_\varphi(x) \mid \varphi \in A^*\}$. Then $F_B(x) = A \otimes T$.*

THEOREM 5.4. *Let A and B be C^*-algebras. Consider the following conditions:*

(a) *A has Property S for subspaces of B.*

(b) *$x \in F_B(x)$ for every $x \in A \otimes B$.*

(c) *There is a net $\{\Phi_\alpha\}$ in $F(A)$ such that $(\Phi_\alpha)_B(x) \to x$ in norm for every $x \in A \otimes B$.*

(d) *A has Property S for B.*

(e) *x is in the C^*-algebra generated by $F_B(x)$ for every $x \in A \otimes B$.*

Then (a) *and* (b) *are equivalent,* (d) *and* (e) *are equivalent, and* (c) $\Rightarrow$ (a) $\Rightarrow$ (d). *If we also assume that B is stable, then* (c) *is equivalent to* (a) *and* (b).

As noted in the Introduction, Effros and Ruan have defined an approximation property for the category of operator spaces that is the natural analogue of Grothendieck's approximation property for the category of Banach spaces. As in Section 3, we will write AP for Grothendieck's approximation property. We will refer to Effros and Ruan's approximation property as the "operator space AP." A C^*-algebra A has the operator space AP if and only if there is a net $\{\Phi_\alpha\}$ in $F(A)$ such that $(\Phi_\alpha)_K(x) \to x$ in norm for every $x \in A \otimes K$. Since K is a stable C^*-algebra, our next result is an immediate consequence of Theorem 5.4.

THEOREM 5.5. *Let A be a C^*-algebra. Then A has the operator space AP if and only if A has Property S for subspaces of K.*

We do not know whether Property S implies Property S for subspaces, or whether Property S for subspaces of K implies Property S for subspaces. (However, it is shown in [26] that if A is locally reflexive [17], or if A is the reduced C^*-algebra of a discrete group, then A has Property S for subspaces if it has Property S for subspaces of K.) The next result (which is essentially due to Archbold and Batty), is of interest in connection with these two problems.

THEOREM 5.6. *Every C^*-algebra has Property S for K.*

Proof. Let A be a C^*-algebra. Since K is nuclear, it follows from Theorems 3.1 and 3.4 in [1] that (A, B, K) verifies the slice map conjecture whenever B is a nuclear C^*-subalgebra of K. However, every C^*-subalgebra of K is liminal (CCR) [14, Proposition 4.2.4], and hence type I [14, Theorem 5.5.2], and so is nuclear [49, Theorem 3]. Hence A has Property S for K. ∎

If B is any nuclear C^*-algebra all of whose C^*-subalgebras are nuclear, then it follows from the proof of Theorem 5.6 that every C^*-algebra has Property S for B. However, there are nuclear C^*-algebras with non-nuclear C^*-subalgebras [7]. It is an open question whether every C^*-algebra has Property S for every nuclear C^*-algebra.

Remark 5.7. Let $\mathcal{M}$ be a von Neumann algebra which is not the finite sum of finite type I von Neumann algebras. Then, as noted above, $\mathcal{M}$ does not have Property S. In view of Theorem 5.6, it is not obvious that this implies that $\mathcal{M}$ does not have Property S for subspaces of K. However, as noted in Section 3, $\mathcal{M}$ also does not have the AP. It follows from this and Theorem 5.8 below that $\mathcal{M}$ does not have Property S for subspaces of $c_0(\mathbf{N})$ (where, as usual, $c_0(\mathbf{N})$ denotes the C^*-subalgebra of $l^\infty(\mathbf{N})$ consisting of those sequences $\{x_n\}$ satisfying $\|x_n\| \to 0$ as $n \to \infty$). If we realize K as the compact operators on $l^2(\mathbf{N})$, and if we let the elements of $c_0(\mathbf{N})$ act on $l^2(\mathbf{N})$ in the usual way as multiplication operators, then $c_0(\mathbf{N})$ is a norm closed subspace of K. Hence $\mathcal{M}$ does not have Property S for subspaces of K.

The next result is the analogue for C^*-algebras of Theorem 3.1. It should also be compared to Tomiyama's result that (with the obvious definitions) a Banach space X has Property S for subspaces of Y for every Banach space Y (where the tensor product is the injective tensor product) if and only if X has the AP [54, Theorem 5.1].

THEOREM 5.8. *Let A be a C^*-algebra. Then A has the AP if and only if A has Property S for subspaces of $c_0(\mathbf{N})$.*

Proof. Since $c_0(\mathbf{N})$ is an abelian C^*-algebra, the minimal C^*-norm coincides with the injective norm on the algebraic tensor product of A and $c_0(\mathbf{N})$ [50, Theorem IV.4.14]. Hence, if T is any norm closed subspace of $c_0(\mathbf{N})$, then $A \otimes T = A \check{\otimes} T$, where $A \check{\otimes} T$ denotes the injective tensor product of A and T. Following Waelbroeck [56], we denote by $A\varphi T$ the space of linear mappings of A^* into T whose restrictions to the unit ball of A^* are weak-* continuous. With the obvious norm, $A\varphi T$ is a Banach space. For $a \in A$ and $b \in c_0(\mathbf{N})$, define an element $\tau(a \otimes b)$ in $A\varphi c_0(\mathbf{N})$ by $\tau(a \otimes b)(\varphi) = \langle a, \varphi \rangle b$ $(\varphi \in A^*)$. It is shown in [56] that τ extends to an isometric isomorphism of $A \check{\otimes} c_0(\mathbf{N}) = A \otimes c_0(\mathbf{N})$ onto $A\varphi c_0(\mathbf{N})$, and that A has the AP if and only $\tau(A \check{\otimes} T) = A\varphi T$ for all norm closed subspaces T of $c_0(\mathbf{N})$. On the other hand, it is easy to see that

$$\tau(x)(\varphi) = R_\varphi(x) \qquad (x \in A \otimes c_0(\mathbf{N}),\ \varphi \in A^*),$$

and so $\tau(x)$ is in $A\varphi T$ (i.e., $\tau(x)$ maps A^* into T) if and only if $x \in F(A, T, A \otimes c_0(\mathbf{N}))$. Hence $\tau(F(A, T, A \otimes c_0(\mathbf{N}))) = A\varphi T$ for all norm closed subspaces T of $c_0(\mathbf{N})$, and thus A has the AP if and only if A has Property S for subspaces of $c_0(\mathbf{N})$. ∎

Remark 5.9. As noted in Remark 5.7, we can view $c_0(\mathbf{N})$ as a norm closed subspace of K. Hence if A is a C^*-algebra, then if A has Property S for subspaces of K, A has Property S for subspaces of $c_0(\mathbf{N})$. It follows from this and Theorems 5.5 and 5.8 that if A has the operator space AP, then A has the AP. It seems likely that there are C^*-algebras which have the AP but not the operator space AP, but we know of no examples of such C^*-algebras.

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