JOURNAL OF FUNCTIONAL ANALYSIS **59**, 498–534 (1984)

Semi-crossed Products of C^*-Algebras

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Communicated by the Editors

Received November 25, 1983

Given a C^*-algebra $\mathfrak{A}$ and endomorphism α, there is an associated nonselfadjoint operator algebra $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$, called the semi-crossed product of $\mathfrak{A}$ with α. If α is an automorphism, $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ can be identified with a subalgebra of the C^*-crossed product $\mathbb{Z} \times_\alpha \mathfrak{A}$. If $\mathfrak{A}$ is commutative and α is an automorphism satisfying certain conditions, $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ is an operator algebra of the type studied by Arveson and Josephson. Suppose S is a locally compact Hausdorff space, $\phi: S \to S$ is a continuous and proper map, and α is the endomorphism of $\mathfrak{A} = C_0(S)$ given by $\alpha(f) = f \circ \phi$. Necessary and sufficient conditions on the map ϕ are given to insure that the semi-crossed product $\mathbb{Z}^+ \times_\alpha C_0(S)$ is (i) semiprime; (ii) semisimple; (ii) strongly semisimple. © 1984 Academic Press, Inc.

INTRODUCTION

In this paper a class of Banach algebras is studied which we call semi-crossed products of C^*-algebras. These are nonselfadjoint norm closed algebras of operators on hilbert space. They include certain nonselfadjoint subalgebras of C^*-crossed products, and in particular they include the class of operator algebras considered by Arveson and Josephson in [1].

In constructing a semi-crossed product we begin with a pair $(\mathfrak{A}, \alpha)$ where $\mathfrak{A}$ is a C^*-algebra and α is a star endomorphism of $\mathfrak{A}$. An appropriate substitute for the notion of covariant representation is needed in which unitaries are replaced by isometries. This is done as follows: we call a pair (ρ, V) an isometric covariant representation of $(\mathfrak{A}, \alpha)$ if ρ is a representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ and V is an isometry on $\mathscr{H}$ such that $V\rho(\alpha x) = \rho(x)V$, $x \in \mathfrak{A}$. Such a pair (ρ, V) yields a representation of the Banach algebra $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$, and an operator norm on this algebra is defined by taking the supremum over all such pairs. The completion of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ in this enveloping norm is called the semi-crossed product of $\mathfrak{A}$ with α, and is denoted $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$. If α is an injective endomorphism of $\mathfrak{A}$, there is a C^*-algebra $\mathscr{B}$ containing $\mathfrak{A}$ as a subalgebra and an automorphism β of $\mathscr{B}$ such that $\beta(x) = \alpha(x)$, $x \in \mathfrak{A}$. In that case, $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic with a nonselfadjoint subalgebra of the C^*-crossed product $\mathbb{Z} \times_\beta \mathscr{B}$.

0022-1236/84 $3.00

For commutative C^*-algebras $\mathfrak{A} = C_0(S)$, S locally compact hausdorff, each continuous and proper mapping $\phi: S \to S$ defines an endomorphism α of $C_0(S)$ by $\alpha(f) = f \circ \phi$, $f \in C_0(S)$. It is natural to wonder how the ring-theoretic properties of the semi-crossed product $\mathbb{Z}^+ \times_\alpha C_0(S)$ reflect properties of the mapping ϕ, and conversely. For example, what are necessary and sufficient conditions on the dynamical system (S, ϕ) for the semi-crossed product $\mathbb{Z}^+ \times_\alpha C_0(S)$ to be (i) semiprime; (ii) semisimple; (iii) strongly semisimple? In fact, the question of the semisimplicity of the Arveson–Josephson algebras was already raised in [1], although apparently no results in this direction were obtained until recently, when a sufficient condition was given in [12]. These questions are answered here, and, more generally, an explicit description of each of the three radicals (i.e., the prime radical, Jacobson radical, and the strong radical) is given. In special cases we can determine the strong structure space of $\mathbb{Z}^+ \times_\alpha C_0(S)$: this is done when S has only periodic points, and, for the opposite extreme, when S has no periodic points. (In the periodic case, our results can be compared with those of [11], in the W^*-algebra setting.) For the C^*-crossed product of a commutative C^*-algebra $C_0(S)$ with a freely acting automorphism α, the primitive ideal space of $\mathbb{Z} \times_\alpha C_0(S)$ corresponds to the orbit closures of S under ϕ [5, 8]. The situation for semi-crossed products appears to be more complex, and even in the semisimple case we do not know how to describe the primitive dual.

If S_i is locally compact hausdorff and $\phi_i: S_i \to S_i$ continuous and proper, $i = 1, 2$, (S_1, ϕ_1), (S_2, ϕ_2) are said to be conjugate if there is a homeomorphism $\Theta: S_2 \to S_1$ such that $\Theta \circ \phi_2 = \phi_1 \circ \Theta$. If (S_1, ϕ_1), (S_2, ϕ_2) are conjugate it is not hard to see that the semi-crossed products $\mathbb{Z}^+ \times_{\alpha_1} C_0(S_1)$, $\mathbb{Z}^+ \times_{\alpha_2} C_0(S_2)$ are isomorphic (II.12). The converse proposition is proved under the additional assumptions that S_i is compact and ϕ_i has no periodic points, $i = 1, 2$. In the context of the Arveson–Josephson algebras, these results are similar to, though not identical with, those obtained in [1].

Section I covers some elementary representation theory for a C^*-algebra with endomorphism. Semi-crossed products are defined and basic properties are explored in Section II. From II.6 on, only semi-crossed products with commutative C^*-algebras are considered. Section III deals with pairs (S, ϕ) for which each $s \in S$ is periodic. In Section IV, the prime radical, Jacobson radical, and strong radical of $\mathbb{Z}^+ \times_\alpha C_0(S)$ are described. Finally, the question of the isomorphism of semi-crossed products implying conjugacy is taken up in Section V.

I.

If α is an automorphism of a C^*-algebra $\mathfrak{A}$, then $(\rho, V, \mathscr{H})$ is called a covariant representation of $(\mathfrak{A}, \alpha)$ if ρ is a representation of $\mathfrak{A}$ in the hilbert space $\mathscr{H}$ and V is unitary on $\mathscr{H}$ satisfying $\rho(\alpha x) = V\rho(x) V^*$ for all x in $\mathfrak{A}$. But suppose α is a star endomorphism of $\mathfrak{A}$. If $(\rho, V, \mathscr{H})$ is to be a covariant representation of $(\mathfrak{A}, \alpha)$ in the above sense, then $\ker \rho \supseteq \ker \alpha$. If we weaken the requirement that V be unitary and ask only that it be an isometry, then there are two ways in which V could intertwine the representations ρ and $\rho \circ \alpha$:

(i) $\rho(\alpha x)V = V\rho(x)$, or

(ii) $V\rho(\alpha x) = \rho(x)V$, $x \in \mathfrak{A}$.

Notice that (i) forces $\ker \rho \supseteq \ker \alpha$. Relation (ii), however, imposes no such requirement, and we will see that there is always a faithful representation $(\rho, \mathscr{H})$ and an isometry V on $\mathscr{H}$ such that (ii) holds. But first notice that there can be representations $(\rho, \mathscr{H})$ which admit an isometry V on $\mathscr{H}$ satisfying (i) but not (ii).

I.1. Example. Let S be a compact hausdorff space and $\phi: S \to S$ be continuous and onto. Let μ be a regular borel measure on S such that $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ, and let $\delta = d\mu \circ \phi^{-1}/d\mu$ be the Radon–Nikodym derivative. Define an isometry V on $L^2(S, \mu)$ by $Vf(s) = 1/\sqrt{\delta \circ \phi(s)}\, f \circ \phi(s)$. If ϕ is constant on a set of positive μ-measure, V is not unitary. If $\rho(f)g = fg$ for $f \in C(S)$, $g \in L^2(S, \mu)$ and $\alpha: C(S) \to C(S)$ is given by $\alpha(f)(s) = f(\phi(s))$, α is an (injective) endomorphism of $C(S)$. Now

$$\begin{aligned}(\rho(\alpha f)V)\, g(s) &= (\alpha f)(s)(Vg)(s) \\ &= \frac{1}{\sqrt{\delta \circ \phi(s)}} f \circ \phi(s) g \circ \phi(s) \\ &= V(fg)(s) \\ &= V(\rho(f))\, g(s).\end{aligned}$$

Thus $\rho(\alpha f)V = V\rho(f)$. Suppose there were an isometry U satisfying $U\rho(\alpha f) = \rho(f)U$ for all $f \in C(S)$. Then $\rho(f)\, UV = U\rho(\alpha f)V = UV\rho(f)$, so $UV \in \rho(C(S))' = L^\infty(S, \mu)$. UV is an isometry, and since $L^\infty(S, \mu)$ is commutative, UV must be unitary. But that would imply $I = UV(UV)^* = UVV^*U^*$, which is impossible if V is a proper isometry.

I.2. Example. Let $\mathfrak{A}$ be an arbitrary C^*-algebra, and α a star endomorphism of $\mathfrak{A}$. Let $(\pi, \mathscr{K})$ be a representation of $\mathfrak{A}$, and let

$\mathscr{H} = H^2(\mathscr{K})$ denote the hilbert space of all sequences $(\xi_0, \xi_1, \xi_2, \ldots)$ with $\xi_n \in \mathscr{K}$ $(n \geqslant 0)$ and $\sum_{n\geqslant 0} \|\xi_n\|^2 < \infty$. Define a representation ρ of $\mathfrak{A}$ in $\mathscr{H}$ by

$$\rho(x)(\xi_0, \xi_1, \xi_2, \ldots) = (\pi(x)\,\xi_0, \pi(\alpha x)\,\xi_1, \pi(\alpha^2 x)\,\xi_2, \ldots).$$

If U_+ is the unilateral shift on $\mathscr{H}$,

$$U_+(\xi_0, \xi_1, \xi_2, \ldots) = (0, \xi_0, \xi_1, \xi_2, \ldots),$$

then $U_+ \rho(\alpha x) = \rho(x)\, U_+$.

I.3. *Notation.* We will write $\rho = \tilde{\pi}$ for the representation constructed in I.2.

Suppose now that α is an endomorphism of a C^*-algebra $\mathfrak{A}$, and ρ is a representation of $\mathfrak{A}$ on a hilbert space $\mathscr{H}$ such that $V\rho(\alpha x) = \rho(x)V$ for an isometry V on $\mathscr{H}$. Set $\mathscr{H}_1 = \bigcap_{n\geqslant 0} V^n\mathscr{H}$. $\mathscr{H}_1$ is a $\rho(\mathfrak{A})$-invariant subspace: let $\eta \in \mathscr{H}_1$; there exist $\eta_n \in \mathscr{H}$ such that $\eta = V^n\eta_n$, $n = 0, 1, 2, \ldots$. Then $\rho(x)\eta = \rho(x)\, V^n\eta_n = V^n\rho(\alpha^n x)\,\eta_n$, so $\rho(x)\eta \in \text{Range } V^n$, $n \geqslant 0$, hence $\rho(x)\eta \in \mathscr{H}_1$. Thus ρ may be decomposed as $\rho_1 \oplus \rho_2$, where $\rho_1(x) = \rho(x) | \mathscr{H}_1$, $\rho_2(x) = \rho(x) | \mathscr{H}_2$, $\mathscr{H}_2 = \mathscr{H}_1^{\perp}$. Also V may be decomposed as $V_1 \oplus V_2$, where $V_1 = V | \mathscr{H}_1$ is unitary (if it is nonzero) and $V_2 = V | \mathscr{H}_2$ is a pure isometry (if it is nonzero).

Furthermore, $V_i\rho_i(\alpha x) = \rho_i(x)\, V_i$, $i = 1, 2$. Suppose $V_2 \neq 0$. Change notation and replace ρ_2 by ρ, $\mathscr{H}_2$ by $\mathscr{H}$, V_2 by V. Thus we assume V is a pure isometry such that $V\rho(\alpha x) = \rho(x)V$. Set $\mathscr{K} = (V\mathscr{H})^{\perp} = \ker V^*$ and $W\colon H^2(\mathscr{K}) \to \mathscr{H}$ by

$$W(\xi_0 . \xi_1, \xi_2, \ldots) = \sum_{n\geqslant 0} V^n\xi_n.$$

Then $VW = WU_+$ [6, pp. 15–16], where U_+ is the unilateral shift on $H^2(\mathscr{K})$. Notice $\mathscr{K}$ is a $\rho(\mathfrak{A})$-invariant subspace. From the relation $V\rho(\alpha x) = \rho(x)V$ we obtain $\rho(\alpha x)^*\, V^* = V^*\rho(x)^*$, or, using the fact that ρ is a $*$-representation and replacing x^* by x, $\rho(\alpha x)\, V^* = V^*\rho(x)$. Thus, if $\xi \in \mathscr{K}$, $V^*\rho(x)\xi = \rho(\alpha x)\, V^*\xi = 0$.

Let π be the restriction of ρ to $\mathscr{K}$; i.e., $\pi(x) = \rho(x) | \mathscr{K}$. Let $\tilde{\pi}$ be the representation constructed from π on $H^2(\mathscr{K})$. (See notation I.3.) Then

$$\begin{aligned} W\tilde{\pi}(x)(\xi_0, \xi_1, \xi_2, \ldots) &= W(\pi(x)\,\xi_0, \pi(\alpha x)\,\xi_1, \pi(\alpha^2 x)\,\xi_2, \ldots) \\ &= \sum_{n\geqslant 0} V^n\pi(\alpha^n x)\,\xi_n \\ &= \sum_{n\geqslant 0} V^n\rho(\alpha^n x)\,\xi_n \\ &= \sum_{n\geqslant 0} \rho(x)\, V^n\xi_n \\ &= \rho(x)\, W(\xi_0, \xi_1, \xi_2, \ldots). \end{aligned}$$

Hence $W\tilde{\pi}(x) = \rho(x)W$. We collect the forgoing facts in

I.4. PROPOSITION. *Let α be an endomorphism of a C^*-algebra $\mathfrak{A}$, and $(\rho, \mathscr{H})$ a representation of $\mathfrak{A}$ such that for some isometry V of $\mathscr{H}$*

$$V\rho(\alpha x) = \rho(x)V \qquad (x \in \mathfrak{A}).$$

Then $\mathscr{H}$ can be decomposed as $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ where $\mathscr{H}_i$ is invariant under V and $\rho(x)(x \in \mathfrak{A})$, $i = 1, 2$, and $\mathscr{H}_1 = \bigcap_{n \geqslant 0} V^n \mathscr{H}$. If $V_i = V \mid \mathscr{H}_i$, $\rho_i(x) = \rho(x) \mid \mathscr{H}_i$, $i = 1, 2$, then if $\mathscr{H}_1 \neq (0)$, V_1 is unitary, if $\mathscr{H}_2 \neq (0)$, V_2 is a pure isometry, and $V_i \rho_i(\alpha x) = \rho_i(x) V_i (x \in \mathfrak{A})$. Furthermore, there exists a representation $(\pi, \mathscr{K})$ of $\mathfrak{A}$ and an isometry W from $H^2(\mathscr{K})$ onto $\mathscr{H}_2$ such that $V_2 W = WU_+$ and $\rho_2(x)W = W\tilde{\pi}(x)$, where U_+ is the unilateral shift on $H^2(\mathscr{K})$.

In each of the next two propositions, $\mathfrak{A}$ denotes a C^*-algebra with endomorphism α.

I.5. PROPOSITION. *Suppose that ρ is a representation of $\mathfrak{A}$ in $\mathscr{L}(\mathscr{H})$ such that $\rho(\mathfrak{A})''$ is maximal abelian. If there does not a unitary operator U in $\mathscr{L}(\mathscr{H})$ such that $\rho(\alpha x) = U\rho(x) U^*$, then there does not exist a pair V_1, V_2 of isometries such that both $\rho(\alpha x) V_1 = V_1 \rho(x)$ and $V_2 \rho(\alpha x) = \rho(x) V_2$ $(x \in \mathfrak{A})$.*

Proof. The proof is essentially an adaptation of what we did in I.1. Suppose such isometries V_1, V_2 exist. Then $V_2 V_1 \in \rho(\mathfrak{A})' = \rho(\mathfrak{A})''$, since $\rho(\mathfrak{A})''$ is maximal abelian. But then $(V_2 V_1)^* (V_2 V_1) = I$, whereas $(V_2 V_1)(V_2 V_1)^*$ is a proper projection, contradicting the fact that $V_2 V_1$, as an element of a commutative W^*-algebra, commutes with its adjoint.

I.6. PROPOSITION. *Let ρ be an irreducible representation of $\mathfrak{A}$ in $\mathscr{L}(\mathscr{H})$, and let $\tilde{\rho}$ be the representation of $\mathfrak{A}$ in $\mathscr{L}(H^2(\mathscr{H}))$ contructed from ρ (see* I.3).

(i) *Then the commutant of the star algebra generated by $\{\tilde{\rho}(\mathfrak{A}), U_+\}$ in $\mathscr{L}(H^2(\mathscr{H}))$ consists of scalars.*

(ii) *Suppose also that there does not exist an isometry V satisfying $\rho(\alpha^n x)V = V\rho(x)$ $(x \in \mathfrak{A})$ for any positive integer n. Then the commutant of the algebra generated by $\{\tilde{\rho}(\mathfrak{A}), U_+\}$ in $\mathscr{L}(H^2(\mathscr{H}))$ consists of scalars.*

Proof. (i) If T commutes with U_+ and U_+^*, then $T = \operatorname{diag}(T_0, T_0, \ldots)$. If T commutes with $\tilde{\rho}(\mathfrak{A})$, then T_0 commutes with $\rho(\mathfrak{A})$; hence T_0 and T are scalars.

(ii) Identify $H^2(\mathscr{H})$ with the set of all power series of the form $\xi(z) = \sum_{n \geqslant 0} \xi_n z^n$, with $\sum_{n \geqslant 0} \|\xi_n\|^2 < \infty$ and $|z| < 1$. If $T \in \mathscr{L}(H^2(\mathscr{H}))$ is an operator commuting with the unilateral shift, then T is of the form $T(z) = \sum_{n \geqslant 0} z^n T_n$, where $T_n \in \mathscr{L}(\mathscr{H})$ and $\|T_n\| \leqslant \|T\|$ [6, p. 47]. We compute

$$\begin{aligned}(T\tilde{\rho}(x)\xi)(z) &= T(z)(\tilde{\rho}(x)\xi)(z)\\ &= \sum_{n\geqslant 0} T(z)\, z^n \rho(\alpha^n x)\,\xi_n\\ &= \sum_{n\geqslant 0}\sum_{k\geqslant 0} T_k z^k z^n \rho(\alpha^n x)\,\xi_n\\ &= \sum_{m\geqslant 0} z^m \left(\sum_{k=0}^{m} T_k \rho(\alpha^{m-k}x)\,\xi_{m-k}\right).\end{aligned}$$

Also, $(\tilde{\rho}(x)\, T\xi)(z) = \sum_{m\geqslant 0} z^m \rho(\alpha^m x)(T\xi)_m$. But

$$\begin{aligned}(T\xi)(z) = T(z)\,\xi(z) &= \left(\sum_{k\geqslant 0} z^k T_k\right)\left(\sum_{n\geqslant 0} z^n \xi_n\right)\\ &= \sum_{m\geqslant 0} z^m \left(\sum_{k=0}^{m} T_k \xi_{m-k}\right)\\ &= \sum_{m\geqslant 0} z^m (T\xi)_m.\end{aligned}$$

Thus,

$$\begin{aligned}(\tilde{\rho}(x)\, T\xi)(z) &= \sum_{m\geqslant 0} z^m \rho(\alpha^m x)\left(\sum_{k=0}^{m} T_k \xi_{m-k}\right)\\ &= \sum_{m\geqslant 0} z^m \left(\sum_{k=0}^{m} \rho(\alpha^m x)\, T_k \xi_{m-k}\right).\end{aligned}$$

If T commutes with $\tilde{\rho}(x)$, we have

$$\sum_{k=0}^{m} T_k \rho(\alpha^{m-k}x)\,\xi_{m-k} = \sum_{k=0}^{m} \rho(\alpha^m x)\, T_k \xi_{m-k}$$

for all $\xi(z) = \sum_{n\geqslant 0} z^n \xi_n \in H^2(\mathscr{H})$, $x \in \mathfrak{A}$, and $m = 0, 1, 2,\ldots$. These equations yield

$$T_m \rho(x) = \rho(\alpha^m x)\, T_m, \qquad m = 0, 1, 2\ldots.$$

Since ρ was irreducible, it follows (see, e.g., [7, p. 160]) that $T_m = \lambda_m V_m$, where $\lambda_m \geqslant 0$ and V_m is an isometry. But then $V_m \rho(x) = \rho(\alpha^m x)\, V_m$, $m = 0, 1, 2,\ldots$, so by our hypothesis $T_m = 0$, $m \geqslant 1$. Also $T_0 = cI$. Thus $T = cI_{H^2(\mathscr{H})}$.

I.7. If α is a continuous endomorphism of a Banach algebra $\mathfrak{A}$, the ideals $\{\ker \alpha^n\}_{n=0}^{\infty}$ form an increasing chain, and we define

$R_\alpha = \overline{\bigcup_{n\geq 0} \ker \alpha^n}$ to be the α-radical of $\mathfrak{A}$. Observe that $\alpha(R_\alpha) \subseteq R_\alpha$, $\alpha^{-1}(R_\alpha) = R_\alpha$. Also α induces an injective endomorphism α' on $\mathfrak{A}/R_\alpha$ by $\alpha'(x + R_\alpha) = \alpha(x) + R_\alpha$.

Often it will be convenient to assume that $\mathfrak{A}$ has an identity 1 with $\alpha(1) = 1$. If this is not the case, we can embed $\mathfrak{A}$ in $\mathfrak{A} \oplus \mathbb{C} \equiv \mathfrak{A}_1$ in the usual way and set $\alpha_1(x, \lambda) = (\alpha(x), \lambda)$, so $(\mathfrak{A}_1, \alpha_1)$ has the desired property.

I.8. In the next proposition we consider whether it is always possible to "extend" an injective star endomorphism of a C^*-algebra to an automorphism (of a larger algebra).

PROPOSITION. *Given an injective star endomorphism α of a C^*-algebra $\mathfrak{A}$, there exists a C^*-algebra $\mathscr{B}$ containing $\mathfrak{A}$ as a subalgebra and an automorphism β of $\mathscr{B}$ such that $\alpha(x) = \beta(x)$ for all $x \in \mathfrak{A}$.*

Proof. If $\mathfrak{A}$ is not unital or if $\mathfrak{A}$ is unital but $\alpha(1) \neq 1$, we can embed $\mathfrak{A}$ in $\mathfrak{A}_1 = \mathfrak{A} \oplus \mathbb{C}$ and extend α to an endomorphism α_1 of $\mathfrak{A}_1$ such that $\alpha_1(1) = 1$. Thus we may as well assume $\mathfrak{A}$ is unital and $\alpha(1) = 1$.

We construct $\mathscr{B}$ as the inductive limit of a system of C^*-algebras $\{\mathfrak{A}_n, j_n\}_{n=0}^{\infty}$. To begin, let $\mathfrak{A}_1$ be any C^*-algebra isomorphic to $\mathfrak{A}$, and $\psi: \mathfrak{A} \to \mathfrak{A}_1$ be an isomorphism onto $\mathfrak{A}_1$. Then $(\psi \circ \alpha \circ \psi^{-1})(\psi \circ \alpha(x)) = \psi \circ \alpha(x))$. Changing notation, let $\mathfrak{A}_0 = \mathfrak{A}$, $\beta_0 = \alpha$, $j_1 = \psi \circ \alpha$, and $\beta_1 = \psi \circ \alpha \circ \psi^{-1}$. Then the diagram

$$\begin{tikzcd} \mathfrak{A}_0 \arrow[r, "\beta_0"] \arrow[d, "j_1"'] & \mathfrak{A}_0 \arrow[d, "j_1"'] \\ \mathfrak{A}_1 \arrow[r, "\beta_1"] & \mathfrak{A}_1 \end{tikzcd}$$

commutes and $\beta_1(\mathfrak{A}_1) = j_1(\mathfrak{A}_0)$. Repeat the above construction with $(\mathfrak{A}_1, \beta_1)$ in the role of $(\mathfrak{A}_0, \beta_0)$ and obtain a C^*-algebra $\mathfrak{A}_2$, an injection $j_2: \mathfrak{A}_1 \to \mathfrak{A}_2$, and as injective endomorphism β_2 of $\mathfrak{A}_2$ such that

$$\begin{tikzcd} \mathfrak{A}_1 \arrow[r, "\beta_1"] \arrow[d, "j_2"'] & \mathfrak{A}_1 \arrow[d, "j_2"'] \\ \mathfrak{A}_2 \arrow[r, "\beta_2"] & \mathfrak{A}_2 \end{tikzcd}$$

commutes and $\beta_2(\mathfrak{A}_2) = j_2(\mathfrak{A}_1)$. Continuing in this way, find an inductive $(\mathfrak{A}_n, j_n)$ of C^*-algebras with injective endomorphism β_n of $\mathfrak{A}_n$ such that the diagram

$$\begin{tikzcd}
\vdots \arrow[d, "j_n"'] & \vdots \arrow[d, "j_n"'] \\
\mathfrak{A}_n \arrow[r, "\beta_n"] \arrow[d, "j_{n+1}"'] & \mathfrak{A}_n \arrow[d, "j_{n+1}"'] \\
\mathfrak{A}_{n+1} \arrow[r, "\beta_{n+1}"] \arrow[d, "j_{n+2}"'] & \mathfrak{A}_{n+1} \arrow[d, "j_{n+2}"'] \\
\vdots & \vdots
\end{tikzcd}$$

commutes and $\beta_{n+1}(\mathfrak{A}_{n+1}) = j_{n+1}(\mathfrak{A}_n)$, $n = 0, 1, 2, \ldots$. Let $\mathscr{B}$ be the C^*-inductive limit of $(\mathfrak{A}_n, j_n)$. Each $\mathfrak{A}_n$ corresponds naturally to a subalgebra of $\mathscr{B}$, and for simplicity of notation call that subalgebra $\mathfrak{A}_n$. (See [15, 1.23.2].) Thus, $\mathscr{B} = \overline{\bigcup_{n\geqslant 0} \mathfrak{A}_n}$. Since β_n extends β_{n-1}, $n \geqslant 1$, we can define an injective endomorphism $\beta\colon \bigcup_{n\geqslant 0} \mathfrak{A}_n \to \bigcup_{n\geqslant 0} \mathfrak{A}_n$. Since $\beta(\mathfrak{A}_n) = \beta_n(\mathfrak{A}_n) = \mathfrak{A}_{n-1}$, β is onto. As each β_n is of norm 1, so is β, hence β admits a unique extension, which we again denote by β, to $\mathscr{B}$. Also, the extended map is one to one and onto. Thus β is an automorphism of $\mathscr{B}$. Finally, by construction, β extends each β_n, and in particular β extends $\beta_0 = \alpha$; i.e., $\beta(x) = \alpha(x)$, $x \in \mathfrak{A}$.

II.

II.1. Definition. Let α be a star endomorphism of a C^*-algebra $\mathfrak{A}$, ρ a representation of $\mathfrak{A}$ in $\mathscr{H}$, and V an isometry of $\mathscr{H}$. We say that (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$ if $V\rho(\alpha x) = \rho(x)V$, for all $x \in \mathfrak{A}$.

Suppose that β is an automorphism of a C^*-algebra $\mathscr{B}$; if δ_n denotes the Kronecker delta on $\mathbb{Z}$, the algebra $l^1(\mathbb{Z}, \mathscr{B}, \beta)$ consists of all formal sums $\sum_{-\infty}^{\infty} \delta_n \otimes x_n$ with $x_n \in \mathfrak{A}$, $\sum_{-\infty}^{\infty} \|x_n\| < \infty$. The adjoint is given (on simple tensors) by $(\delta_n \otimes x)^* = \delta_{-n} \otimes \beta^{-n}(x^*)$, and the multiplication by $(\delta_n \otimes x)(\delta_m \otimes y) = \delta_{n+m} \otimes x\beta^n(y)$ [13, 7.6.1]. A multiplication could also be defined by letting the group act on the left side: $(\delta_n \otimes x)(\delta_m \otimes y) = \delta_{n+m} \otimes \beta^m(x)y$. If the Banach space $l^1(\mathbb{Z}, \mathscr{B}, \beta)$ is provided with this alternative multiplication, and the adjoint is left unchanged, we obtain a new Banach algebra, which we will call $l^1(\mathbb{Z}, \mathscr{B}, \beta)^{0p}$. The Banach algebras $l^1(\mathbb{Z}, \mathscr{B}, \beta)$ and $l^1(\mathbb{Z}, \mathscr{B}, \beta)^{0p}$ are isomorphic. Indeed, define $\Psi\colon l^1(\mathbb{Z}, \mathscr{B}, \beta) \to l^1(\mathbb{Z}, \mathscr{B}, \beta)^{0p}$, $\Psi(\sum_{-\infty}^{\infty} \delta_n \otimes x_n) = \sum_{-\infty}^{\infty} \delta_{-n} \otimes \beta^{-n}(x_n)$. Clearly Ψ is a Banach space isomorphism. Furthermore, if $\delta_n \otimes x$, $\delta_m \otimes y \in l^1(\mathbb{Z}, \mathscr{B}, \beta)$, one checks that $\Psi[(\delta_n \otimes x)(\delta_m \otimes y)] = \Psi(\delta_n \otimes x)\ \ \Psi(\delta_m \otimes y)$ and $\Psi[(\delta_n \otimes x)^*] = \Psi(\delta_n \otimes x)^*$.

If α is an endomorphism of a C^*-algebra $\mathfrak{A}$, the absence of an inverse for

α prevents us from mimicking the contruction of $l^1(\mathbb{Z}, \mathscr{B}, \beta)$ with $(\mathfrak{A}, \alpha)$ in place $(\mathscr{B}, \beta)$. If, however, we replace $\mathbb{Z}$ by the semigroup $\mathbb{Z}^+$ of nonnegative integers, we can define a Banach algebra $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$. The elements will be of the form $F = \sum_{n \geqslant 0} \delta_n \otimes x_n$, with $x_n \in \mathfrak{A}$, and $\|F\|_1 = \sum_{n \geqslant 0} \|x_n\| < \infty$. In analogy with the above paragraph, the multiplication could be defined by letting the semigroup act on the right or on the left. For technical reasons, we choose the left action

$$(\delta_n \otimes x)(\delta_m \otimes y) = \delta_{n+m} \otimes \alpha^m(x) y$$

as the implication in $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$. Thus $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ is a Banach algebra without adjoint.

Let (ρ, V) be an isometric covariant representation of $(\mathfrak{A}, \alpha)$. Then there is a representation Π of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ in $\mathscr{L}(\mathscr{H})$,

$$\Pi\left(\sum_{n \geqslant 0} \delta_n \otimes x_n\right) = \sum_{n \geqslant 0} V^n \rho(x_n).$$

Notice that

$$\begin{aligned} \Pi(\delta_n \otimes x)\, \Pi(\delta_m \otimes y) &= V^n \rho(x)\, V^m \rho(y) \\ &= V^n V^m \rho(\alpha^m(x))\, \rho(y) \\ &= V^{n+m} \rho(\alpha^m(x) y) \\ &= \Pi(\delta_{n+m} \otimes \alpha^m(x) y) \\ &= \Pi[(\delta_n \otimes x)(\delta_m \otimes y)]. \end{aligned}$$

We denote the representation Π by $V \times \rho$.

Next we wish to define an operator enveloping norm on $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$. First note it is possible to embed $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ faithfully in some $\mathscr{L}(\mathscr{H})$, for if ρ is a faithful representation of $\mathfrak{A}$ in $\mathscr{L}(\mathscr{K})$, $U_+ \times \tilde{\rho}$ will be a faithful representation of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ in $\mathscr{L}(\mathscr{H})$, $\mathscr{H} = H^2(\mathscr{K})$.

II.2. Definition. For $F \in l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ set $\|F\| = \sup\{\|(V \times \rho)(F)\|$: (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)\}$. Define $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ to be the completion of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ with respect to this norm. $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ will be called the semi-crossed product of $\mathfrak{A}$ with α.

II.3. Note that $\mathfrak{A}$ can be embedded isometrically in $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ as a (star) subalgebra by $x \to^j \delta_0 \otimes x$.

Proposition. *Let $\mathfrak{A}$ be a unital C^*-algebra with endomorphism α. Then Π is a hilbert space representation of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ satisfying*

(i) *$\Pi(\delta_1 \otimes 1)$ is an isometry, and*

(ii) *$\Pi|_{j(\mathfrak{A})}$ is a C^*-algebra representation* *if and only if* $\Pi = V \times \rho$, *where* (ρ, V) *is an isometric covariant representation of* $(\mathfrak{A}, \alpha)$.

Proof. One direction is trivial. On the other hand, if Π satisfies (i) and (ii), we can set $V = \Pi(\delta_1 \otimes 1)$ and $\rho(x) = \Pi(\delta_0 \otimes x)$, $x \in \mathfrak{A}$. To show that (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$, write the element

$$\delta_1 \otimes \alpha(x) = (\delta_1 \otimes 1)(\delta_0 \otimes \alpha(x)),$$

and as

$$= (\delta_0 \otimes x)(\delta_1 \otimes 1).$$

We have

$$\begin{aligned} \Pi(\delta_1 \otimes \alpha(x)) &= \Pi(\delta_1 \otimes 1)\,\Pi(\delta_0 \otimes \alpha(x)) \\ &= V\rho(\alpha(x)). \end{aligned}$$

Also,

$$\begin{aligned} \Pi(\delta_1 \otimes \alpha(x)) &= \Pi(\delta_0 \otimes x)\,\Pi(\delta_1 \otimes 1) \\ &= \rho(x)V. \end{aligned}$$

Thus $V\rho(\alpha(x)) = \rho(x)V$, as desired.

II.4. PROPOSITION. *Let* α *be an injective endomorphism of a* C^**-algebra* $\mathfrak{A}$. *Let* $\mathscr{B}$ *be a* C^**-algebra containing* $\mathfrak{A}$ *as a subalgebra and* β *an automorphism of* $\mathscr{B}$ *such that* $\alpha(x) = \beta(x)$, $x \in \mathfrak{A}$ *(as in* I.8). *Then* $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ *is isomorphic with a nonselfadjoint subalgebra of the* C^**-crossed product* $\mathbb{Z} \times_\beta \mathscr{B}$.

Proof. Since $\mathfrak{A} \subseteq \mathscr{B}$, and $\beta \mid \mathfrak{A} = \alpha$, $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ can be considered as a subalgebra of $l^1(\mathbb{Z}, \mathscr{B}, \beta)^{0p}$ in a natural way. Using the notation introduced following II.1,

$$l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha) \to l^1(\mathbb{Z}, \mathscr{B}, \beta)^{0p} \xrightarrow{\Psi} l^1(\mathbb{Z}, \mathscr{B}, \beta) \to \mathbb{Z} \times_\beta \mathscr{B}$$

yields on embedding of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ in the crossed product $\mathbb{Z} \times_\beta \mathscr{B}$, which we call ι. If, for $F \in l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$, the norm $\|F\|$ as defined in II.2 is the same as the norm of $\iota(F)$ in $\mathbb{Z} \times_\beta \mathscr{B}$, then ι can be extended to an isometric isomorphism $\hat{\iota}$: $\mathbb{Z}^+ \times_\alpha \mathfrak{A} \to \mathbb{Z} \times_\beta \mathscr{B}$ so that the diagram

$$\begin{tikzcd} l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha) \arrow[d] \arrow[dr, "\iota"] & \\ \mathbb{Z}^+ \times_\alpha \mathfrak{A} \arrow[r, "\hat{\iota}"] & \mathbb{Z} \times_\beta \mathscr{B} \end{tikzcd}$$

commutes. In other words, $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$ can be viewed as a nonselfadjoint subalgebra of $\mathbb{Z} \times_\beta \mathscr{B}$. Thus it remains to show that $\|F\|$ and the norm of $\imath(F)$ are identical. Since every covariant representation (ρ, V) of $(\mathscr{B}, \beta)$ restricts to an isometric covariant representation of $(\mathfrak{A}, \alpha)$, it follows that $\|F\| \geqslant \|\imath(F)\|$. Suppose now that (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$. Since by I.4 $\rho = \rho_1 \oplus \rho_2$, $V = V_1 \oplus V_2$, with (ρ_i, V_i) an isometric covariant representation of $(\mathfrak{A}, \alpha)$ (if it is nonzero), $i = 1, 2$, such that V_1 is unitary and V_2 a pure isometry, we may treat these cases separately. So assume tat (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$ with V unitary. By the construction of $\mathscr{B}$ in I.8, the subalgebra generated by $\langle \mathfrak{A}, \beta^{-1}\mathfrak{A}, \beta^{-2}\mathfrak{A}, \ldots \rangle$ is dense in $\mathscr{B}$. Extend ρ to $\langle \mathfrak{A}, \beta^{-1}\mathfrak{A}, \beta^{-2}\mathfrak{A}, \ldots \rangle$ by $\bar{\rho}(x) = V^n \rho(\beta^n x) V^{*n}$ if $\beta^n(x) \in \mathfrak{A}$. [Note that if x is in this subalgebra, $\beta^n(x) \in \mathfrak{A}$ for some $n \geqslant 0$.] Since $\|\bar{\rho}(x)\| = \|\rho(\beta^n x)\| \leqslant \|\beta^n(x)\| = \|x\|$, $\bar{\rho}$ extends to a representation of $\mathscr{B}$, and in fact $(\bar{\rho}, V)$ is a covariant representation of $(\mathscr{B}, \beta)$.

Next if (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$ with V a pure isometry, then by I.4 (ρ, V) is equivalent to some $(\tilde{\pi}, U_+)$. If $\mathscr{K}$ is the hilbert space of the representation π of $\mathfrak{A}$, there is a hilbert space $\mathscr{H} \supset \mathscr{K}$ and a representation τ of $\mathscr{B}$ such that $\pi(x) = \tau(x)|_{\mathscr{K}}$, $x \in \mathfrak{A}$ [3, 2.10.2]. If $\hat{\tau}$ denotes the representation of $\mathscr{B}$ on $l^2(\mathscr{H})$ given by

$$\begin{aligned}\hat{\tau}(x)(\ldots, \xi_{-1}, \xi_0, \xi_2, \ldots) \\ = (\ldots, \tau(\beta^{-1}x)\, \xi_{-1}, \tau(x)\, \xi_0, \tau(\beta x)\, \xi_1, \tau(\beta^2 x)\, \xi_2, \ldots)\end{aligned}$$

and U the bilaterial shift on $l^2(\mathscr{H})$, then

$$\hat{\tau}(x) U = U \hat{\tau}(\beta x), \qquad x \in \mathscr{B}.$$

Now $l^2_+(\mathscr{K})$ (or $H^2(\mathscr{K})$) is the subspace of $l^2(\mathscr{H})$ consisting of all $\xi = (\ldots, \xi_{-1}, \xi_0, \xi_1, \xi_2, \ldots) \in l^2(\mathscr{H})$ with $\xi_{-n} = 0$, $n = 1, 2, \ldots$, and $\xi_n \in \mathscr{K}$, $n = 0, 1, 2, \ldots$. Furthermore, $\tilde{\pi}(x) = \hat{\tau}(x)|_{l^2_+(\mathscr{K})}$ $(x \in \mathfrak{A})$, and $U_+ = U|_{l^2_+(\mathscr{K})}$. In other words, the isometric covariant representation $(\tilde{\pi}, U_+)$ is the restriction of the covariant representation $(\hat{\tau}, U)$ to the (invariant) subspace $l^2_+(\mathscr{K})$. It follows that $\|\imath(F)\| \geqslant \|F\|$, and hence $\|\imath(F)\| = \|F\|$.

II.5. Proposition. *Let α be an endomorphism of a C^*-algebra $\mathfrak{A}$. For $F \in \mathbb{Z}^+ \times_\alpha \mathfrak{A}$ we have $\|F\| = \sup\{\|(V \times \rho)(F)\|$: (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$ with V a pure isometry$\}$.*

Proof. First we make the obvious point that if (ρ, V) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$, then by the definition of the norm on $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$, $V \times \rho$ extends to a bounded hilbert space representation of $\mathbb{Z}^+ \times_\alpha \mathfrak{A}$.

By I.4 every isometric covariant representation (ρ, V) of $(\mathfrak{A}, \alpha)$ can be written as $\rho = \rho_1 \oplus \rho_2$, $V = V_1 \oplus V_2$, where (ρ_i, V_i) is an isometric covariant representation of $(\mathfrak{A}, \alpha)$ and V_1 is unitary (if it is nonzero) and V_2 is a pure isometry (if it is nonzero). Thus to prove the proposition it is sufficient to show that if (ρ, V) is an isometric covariant representation with V unitary, then for every $\varepsilon > 0$ there exists an isometric covariant representation (ω, U_+) with U_+ a pure isometry such that $\|(V \times \rho)(F)\| \leqslant \|(U_+ \times \omega)(F)\| + \varepsilon$.

Since V is unitary with $\rho(\alpha x) = V^* \rho(x) V$, it follows that the kernel of ρ contains the α-radical R_α, so (ρ, V) can be lifted to an isometric covariant representation (ρ', V) of $(\mathfrak{A}/R_\alpha, \alpha')$, where α' is the (injective) endomorphism of $\mathfrak{A}/R_\alpha$ induced by α (see I.7). As in I.8, there exists a C^*-algebra $\mathscr{B}$ containing $\mathfrak{A}/R_\alpha$ as a subalgebra and an automorphism β of $\mathscr{B}$ such that $\beta(x) = \alpha'(x)$, $x \in \mathfrak{A}/R_\alpha$. Essentially as in II.4 we can embed $\mathbb{Z}^+ \times_{\alpha'} (\mathfrak{A}/R_\alpha)$ in $(\mathbb{Z} \times_\beta \mathscr{B})^{0p}$ (except here it is convenient to use $(\mathbb{Z} \times_\beta \mathscr{B})^{0p}$ in place of $\mathbb{Z} \times_\beta \mathscr{B}$). By [13, 7.7.5] and the amenability of $\mathbb{Z}$, if $(\pi, \mathscr{H})$ is any faithful representation of $\mathscr{B}$, then

$$\|G\| = \|(U \times \hat{\pi})(G)\|, \qquad G \in (\mathbb{Z} \times_\beta \mathscr{B})^{0p},$$

where

$$\hat{\pi}(x)(..., \xi_{-1}, \xi_0, \xi_1, \xi_2, ...)$$
$$= (..., \pi(\beta^{-1}x)\, \xi_{-1}, \pi(x)\, \xi_0, \pi(\beta x)\, \xi_2, \pi(\beta^2 x)\, \xi_2, ...)$$

and U is the bilateral shift on $l^2(\mathbb{Z})$.

Let $F' \in \mathbb{Z}^+ \times_{\alpha'} (\mathfrak{A}/R_\alpha)$ be the image of F under the canonical map $\mathbb{Z}^+ \times_\alpha \mathfrak{A} \to \mathbb{Z}^+ \times_{\alpha'} (\mathfrak{A}/R_\alpha)$. Then $\|(V \times \rho)(F)\| = \|(V \times \rho')(F')\| \leqslant \|F'\| = \|(U \times \hat{\pi})(F')\|$. Now the space $l^2(\mathscr{H})$ has a chain of subspaces $l^2_{+n}(\mathscr{H})$, invariant under $(U \times \hat{\pi})(\mathbb{Z}^+ \times_{\alpha'} (\mathfrak{A}/R_\alpha))$ and such that $\bigcup_{n \geqslant 0} l^2_{+n}(\mathscr{H})$ is dense in $l^2(\mathscr{H})$. Indeed, take $l^2_{+n}(\mathscr{H}) = \{\xi : \xi = (\xi_k)_{k=-\infty}^{\infty} \in l^2(\mathscr{H}),$ with $\xi_k = 0,\ k < -n\}$. If n is chosen so that $\|(U \times \hat{\pi})(F')|_{l^2_{+n}}\| > \|F'\| - \varepsilon$, and if U_+ is the restriction $U|_{l^2_{+n}(\mathscr{H})}$, and if $\omega(x) = \hat{\pi}(x)|_{l^2_{+n}(\mathscr{H})}$, $x \in \mathfrak{A}$, then $\|(V \times \rho)(F)\| \leqslant \|U_+ \times \omega)(F)\| + \varepsilon$.

II.6. We now turn our attention to commutative C^*-algebras, $\mathfrak{A} = C_0(S)$, where S is locally compact hausdorff. In this context, ϕ will be used to denote the continuous and proper mapping, $\phi: S \to S$, such that $\alpha(f) = f \circ \phi$, $f \in C_0(S)$. Thus if f is a continuous function with compact support (resp. vanishing at infinity) $f \circ \phi$ has the same property. Although not every endomorphism of $C_0(S)$ is given by composition by such a ϕ, we will henceforth only consider endomorphism of $C_0(S)$ of this form. [Notice that if α is any endomorphism of $C_0(S)$ and $S_1 = S \cup \{\infty\}$ is the one-point compactification, and if α is extended to an endomorphism α_1 of $C(S_1)$ so that $\alpha_1(1)=1$, then there is a continuous map $\phi_1: S_1 \to S_1$ such that $\alpha_1(f) = f \circ \phi_1, f \in C(S_1)$.]

II.7. PROPOSITION. *Let $F \in l^1(\mathbb{Z}^+, C_0(S), \alpha)$, $F = \sum_{n \geqslant 0} \delta_n \otimes f_n$. Then the semi-crossed product norm*

$$\|F\| = \sup_{s \in S} \sup_{\|\xi\|_2 = 1} \left[\sum_{m \geqslant 0} \left| \sum_{l=0}^{m} f_{m-l}(\phi^l(s))\, \xi_l \right|^2 \right]^{1/2},$$

where $\xi = \{\xi_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with $\|\xi\|_2 = (\sum_{n \geqslant 0} |\xi_n|^2)^{1/2} = 1$.

Proof. For $s \in S$, consider the one-dimensional representation π_s of $C_0(S)$, $\pi_s(f) = f(s)$. Then $\tilde{\pi}_s$ is a representation of $C_0(S)$ on H^2, the classical Hardy space, which we view as functions $\xi(z) = \sum_{n \geqslant 0} \xi_n z^n$ holomorphic in the open unit disk D with $\|\xi\|_2 = (\sum_{n \geqslant 0} |\xi_n|^2)^{1/2}$. The unilateral shift on H^2 becomes the operator M_z of multiplication by z. We compute the norm of $(M_z \times \tilde{\pi}_s)(F)$, $F = \sum_{n \geqslant 0} \delta_n \otimes f_n \in l^1(\mathbb{Z}^+, C_0(S), \alpha)$, as an operator on H^2. Since $(\tilde{\pi}_s, M_z)$ is an isometric covariant representation of $(C_0(S), \alpha)$, it follows from the definition of the semi-crossed product norm that $\|(M_z \times \tilde{\pi}_s)(F)\| \leqslant \|F\|$.

Now

$$\begin{aligned} [(M_z \times \tilde{\pi}_s)(\delta_n \otimes f_n)]\, \xi(z) &= M_{z^n} \tilde{\pi}_s(f_n) \left(\sum_{k \geqslant 0} \xi_k z^k \right) \\ &= M_{z^n} \sum_{k \geqslant 0} \pi_s(\alpha^k(f_n))\, \xi_k z^k \\ &= \sum_{k \geqslant 0} f_n \circ \phi^k(s)\, \xi_k z^{n+k}. \end{aligned}$$

Thus

$$\begin{aligned} [(M_z \times \tilde{\pi}_s)(F)]\, \xi(z) &= \sum_{n \geqslant 0} \sum_{k \geqslant 0} f_n \circ \phi^k(s)\, \xi_k z^{n+k} \\ &= \sum_{m \geqslant 0} \left[\sum_{k=0}^{m} f_{m-k} \circ \phi^k(s)\, \xi_k \right] z^m. \end{aligned}$$

It follows that

$$\|(M_z \times \tilde{\pi}_s)(F)\| = \sup_{\|\xi\|_2 = 1} \left(\sum_{m \geqslant 0} \left| \sum_{k=0}^{m} f_{m-k} \circ \phi^k(s)\, \xi_k \right|^2 \right)^{1/2}.$$

To complete the proof, we must show that

$$\|F\| \leqslant \sup_s \|(M_z \times \tilde{\pi}_s)(F)\|.$$

Now $\|F\| = \sup_{(\rho,V)} \|(V \times \rho)(F)\|$, where by II.5 the supremum may be taken over all isometric covariant representations (ρ, V) of $(\mathfrak{A}, \alpha)$ such that V is a pure isometry. By I.3 and I.4, any such pair (ρ, V) is unitarily equivalent to one of the form $(\tilde{\pi}, U_+)$, where π is a representation of $C_0(S)$ on a hilbert space $\mathscr{K}$ and U_+ is the unilateral shift on $H^2(\mathscr{K})$. But any cyclic representation π of $C_0(S)$ is unitarily equivalent to one of the form π_μ on $L^2(S,\mu)$ where μ is a positive regular Borel measure on S, and $\pi_\mu(f)\, g(s) = f(s)\, g(s)$, $g \in L^2(S,\mu)$. The argument which follows is a standard one using the direct integral, so it is just outlined here. Let $H_s^2 = H^2$ denote the representation space of $\tilde{\pi}_s$, and M_z^s the unilateral shift (multiplication by z) on H_s^2. Interchanging the operations of direct sum and direct integral gives a natural isomorphism

$$H^2(L^2(S,\mu)) \cong \int^{\oplus} H_s^2\, d\mu(s),$$

and correspondingly of the operators

$$\tilde{\pi}_\mu(f) \cong \int^{\oplus} \tilde{\pi}_s(f)\, d\mu(s), \qquad F \in C_0(S),$$

and

$$M_z^\mu \cong \int^{\oplus} M_z^s\, d\mu(s).$$

Here $H^2(L^2(S,\mu))$ is viewed as the space of $L^2(S,\mu)$ valued functions ξ on the open unit disk D,

$$\xi(z) = \sum_{n \geqslant 0} \xi_n z^n, \qquad z \in D,\ \xi_n \in L^2(S,\mu),$$

with $\|\xi\|_2 = (\sum \|\xi_n\|_2^2)^{1/2}$; in this context M_z^μ is defined by $(M_z^\mu \xi)(z) = z\xi(z)$, $\xi \in H^2(L^2(S,\mu))$. If $F = \sum_{n \geqslant 0} \delta_n \otimes f_n \in l^1(\mathbb{Z}^+, C_0(S), \alpha)$, then

$$\begin{aligned}(M_z^\mu \times \tilde{\pi}_\mu)(F) &= \sum_{n \geqslant 0} M_{z^n}^\mu \tilde{\pi}_\mu(f_n) \\ &= \int^{\oplus} \left(\sum_{n \geqslant 0} M_{z^n}^s \tilde{\pi}_s(f_n) \right) d\mu(s),\end{aligned}$$

and so

$$\begin{aligned}\|(M_z^\mu \times \tilde{\pi}_\mu)(F)\| &= \operatorname*{ess\,sup}_{s \in S} \left\| \sum_{n \geqslant 0} M_{z^n}^s \tilde{\pi}_s(f_n) \right\| \\ &= \operatorname*{ess\,sup}_{s \in S} \|(M_z^s \times \tilde{\pi}_s)(F)\| \\ &\leqslant \sup_{s \in S} \|(M_z^s \times \tilde{\pi}_s)(F)\|.\end{aligned}$$

This completes the proof.

II.8. Corollary. *With notation as in* II.7,

$$\|F\| = \sup_{s \in S} \|M_z^s \times \tilde{\pi}_s(F)\|.$$

Proof. This follows from the proof of II.7.

II.9. *Notation.* As before, let S be locally compact hausdorff and $\phi: S \to S$ a continuous and proper map defining an endomorphism α of $C_0(S)$ by $\alpha(f) = f \circ \phi$. If $F \in l^1(\mathbb{Z}^+, C_0(S), \alpha)$, $F = \sum_{n \geqslant 0} \delta_n \otimes f_n$, then $\|F\| \geqslant \sup_{s \in S} (\sum_{n \geqslant 0} |f_n(s)|^2)^{1/2}$, as can be seen by taking $\xi_0 = 1$, $\xi_n = 0$, $n > 0$, in II.7. In particular, $\sup_n \|f_n\| \leqslant \|F\|$. Thus if $\{F^{(k)}\}_{k=1}^{\infty} \subset l^1(\mathbb{Z}^+, C_0(S), \alpha)$ is a Cauchy sequence with respect to the semi-crossed product norm, say, $F^{(k)} = \sum_{n \geqslant 0} \delta_n \otimes f_n^{(k)}$, it follows that for each n, $\{f_n^{(k)}\}_{k=1}^{\infty}$ is Cauchy in $C_0(S)$. This means that each element F of the completion $\mathbb{Z}^+ \times_\alpha C_0(S)$ is described by a unique sequence $\{f_n\}_{n=0}^{\infty}, f_n \in C_0(S)$. If $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$ corresponds to the sequence $\{f_n\}_{n=0}^{\infty}$, we will write $F = \sum_{n \geqslant 0} M_{z^n} f_n$.

Let $K(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ ($\mathfrak{A} = C_0(S)$) be the subalgebra of $l^1(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ consisting of functions $F: \mathbb{Z}^+ \to \mathfrak{A}$ with finite support. Clearly, $K(\mathbb{Z}^+, \mathfrak{A}, \alpha)$ is dense in $\mathbb{Z}^+ \times_\alpha C_0(S)$. This does not imply, however, that if $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$ has the representation $F = \sum_{n \geqslant 0} M_{z^n} f_n$, then F is the limit of the elements $F^{(n)} = \sum_{k=0}^{n} M_{z^k} f_k$. Indeed, this can be seen when S is a single point (and ϕ the identity map). In that case $\mathbb{Z}^+ \times_\alpha C_0(S)$ is the disk algebra $\mathscr{A}$, and not every function in the disk algebra is the uniform limit of the partial sums of its fourier series. [However, see the remark following IV.2 for a discussion of summability.]

II.10. Retain the notation of II.9. If $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$, $F = \sum_{n \geqslant 0} M_{z^n} f_n$ (hence $\|f_n\| \leqslant \|F\|$, $n \in \mathbb{Z}^+$) we can associate with each $s \in S$ the holomorphic function $F(s)$ in the unit disk, given by $F(s)(z) \equiv F(s, z) = \sum_{n \geqslant 0} f_n(s) z^n$.

Corollary. *If* $f_n \circ \phi = f_n$, $n \geqslant 0$, *then* $\|F\| = \sup_{s \in S} \sup_{|z| < 1} |F(s, z)|$.

Proof. By II.8, $\|F\| = \sup_{s \in S} \|(M_z \times \tilde{\pi}_s)(F)\|$. As in II.7,

$$[(M_z \times \pi_s)(F)]\, \xi(z) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} \pi_s(f_n \circ \phi^k)\, \xi_k z^{n+k}$$

$$= \left(\sum_{n \geqslant 0} f_n(s)\, z^n\right)\left(\sum_{k \geqslant 0} \xi_k z^k\right),$$

using that $f_n \circ \phi = f_n$. Thus

$$[(M_z \times \tilde{\pi}_s)(F)]\, \xi(z) = F(s, z)\, \xi(z).$$

It follows that $\|(M_z \times \tilde{\pi}_s)(F)\| = \sup_{|z| < 1} |F(s, z)|$, and hence $\|F\| = \sup_{s \in S} \sup_{|z| < 1} |F(s, z)|$.

II.11. PROPOSITION. *Keep the notations of* II.9.

(i) *The map* $s \to \|(M_z \times \tilde{\pi}_s)(F)\|$ *is lower semicontinuous for each* $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$.

(ii) *Suppose that the family* $\{\phi^n\}_{n=0}^{\infty}$ *is uniformly equicontinuous, and regard each* $M_z \times \tilde{\pi}_s$ $(s \in S)$ *as mapping into the same space* (H^2). *Then for each* $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$, *the map* $s \to M_z \times \tilde{\pi}_s(F)$ *is continuous in the norm topology in* $\mathscr{L}(H^2)$.

Proof. (i) If $F = \sum_{n=0}^{\infty} M_{z^n} f_n$, by II.7, $\|M_z \times \tilde{\pi}_s(F)\| = \sup_{\|\xi\|_2=1} [\sum_{m \geqslant 0} |\sum_{l=0}^{m} f_{m-l}(\phi^l(s))\, \xi_l|^2]^{1/2}$. If $\|M_z \times \tilde{\pi}_{s_0}(F)\| > \alpha$, then there is a $\xi \in H^2$, $\|\xi\|_2 = 1$, and a positive integer N such that $g(s_0) > \alpha$, where $g(s) \equiv [\sum_{m=0}^{N} |\sum_{l=0}^{m} f_{m-l}(\phi^l(s))\, \xi_l]^{1/2}$. Since g is clearly continuous, $g(s_0) > \alpha$ implies $g(s) > \alpha$ for all s in some neighborhood U of s_0. But $\|M_z \times \tilde{\pi}_s(F)\| \geqslant g(s)$ for all s, so $\|M_z \times \tilde{\pi}_s(F)\| > \alpha$ for all $s \in U$.

(ii) Given $F = \mathbb{Z}^+ \times_\alpha C_0(S)$ and $\varepsilon > 0$, there is an $F' \in K(\mathbb{Z}^+, C_0(S), \alpha)$, $F' = \sum_{n=0}^{N} M_{z^n} f_n$, such that $\|F - F'\| < \varepsilon/3$. Now if $\xi \in H^2$, $\|\xi\|_2 = 1$,

$$\begin{aligned}
&[(M_z \times \tilde{\pi}_s)(F') - (M_z \times \tilde{\pi}_{s'})(F')]\, \xi(z) \\
&\quad = \sum_{m \geqslant 0} z^m \sum_{k=0}^{m} \xi_k [f_{m-k}(\phi^k(s)) - f_{m-k}(\phi^k(s'))] \\
&\quad = \sum_{m \geqslant 0} z^m \sum_{k=\max\{0,m-N\}}^{m} \xi_k [f_{m-k}(\phi^k(s)) - f_{m-k}(\phi^k(s'))].
\end{aligned}$$

Let $\eta = \varepsilon/3(N+1)$, and let U be a neighborhood of s such that for $s' \in U$, $|f_j(\phi^k(s)) - f_j(\phi^k(s'))| < \eta$, $j = 0, 1, \ldots, N$, $k = 0, 1, 2, \ldots$. It follows that

$$\begin{aligned}
&\|[(M_z \times \tilde{\pi}_s)(F') = (M_z \times \tilde{\pi}_{s'})(F')]\, \xi(z)\|^2 \\
&\quad < \sum_{m \geqslant 0} \left| \sum_{k=\max\{0,m-N\}}^{m} |\xi_k| \cdot \eta \right|^2 \\
&\quad \leqslant |\xi_0|^2 \eta^2 + 2(|\xi_0|^2 + |\xi_1|^2)\, \eta^2 + \cdots \\
&\qquad + N(|\xi_0|^2 + |\xi_1|^2 + \cdots + |\xi_{N-1}|^2)\, \eta^2 \\
&\qquad + \sum_{m \geqslant N} (N+1)(|\xi_{m-N}|^2 + |\xi_{m-N+1}|^2 + \cdots + |\xi_m|^2)\, \eta^2 \\
&\quad \leqslant (N+1)^2 \eta^2 \sum_{m \geqslant 0} |\xi_m|^2 = (N+1)^2 \eta^2 \\
&\quad \leqslant \frac{\varepsilon^2}{9} \qquad (s' \in U).
\end{aligned}$$

An application of the triangle inequality yields

$$\|(M_z \times \tilde{\pi}_s)(F) - (M_z \times \tilde{\pi}_{s'})(F)\| < \varepsilon \qquad \text{for} \quad s' \in U.$$

II.12. PROPOSITION. *Let S_i be locally compact hausdorff and $\phi_i: S_i \to S_i$ continuous and proper, $i = 1, 2$. Let $q: (S_1, \phi_1) \to (S_2, \phi_2)$ be a continuous equivariant and proper map; i.e., $q: S_1 \to S_2$ is continuous, $q^{-1}(K)$ is compact for every compact set $K \subseteq S_2$, and the diagram commutes.*

$$\begin{array}{ccc} S_1 & \xrightarrow{q} & S_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ S_1 & \xrightarrow{q} & S_2 \end{array}$$

Then there is a continuous homomorphism $\tilde{q}: \mathbb{Z}^+ \times_{\alpha_2} C_0(S_2) \to \mathbb{Z}^+ \times_{\alpha_1} C_0(S_1)$. If $F \in \mathbb{Z}^+ \times_{\alpha_2} C_0(S_2)$, $F = \sum_{n \geqslant 0} M_{z^n} f_n$, then $\tilde{q}(F) = \sum_{n \geqslant 0} M_{z^n} f_n \circ q$. Finally, if q is surjective, $\tilde{q}$ is injective. If q is injective, the image of $\tilde{q}$ is dense; if q is injective and $q(S_1)$ is open in S_2, then $\tilde{q}$ is surjective.

Proof. Define $\tilde{q}$ on $K(\mathbb{Z}^+, C_0(S_2), \alpha_2)$ by $\tilde{q}(\sum_{n=0}^N M_{z^n} f_n) = \sum_{n=0}^N M_{z^n} f_n \circ q$. Note that $f_n \circ q \in C_0(S_1)$ since q is proper. This is linear; to see that it is a homomorphism, note

$$\begin{aligned} \tilde{q}(M_{z^n} f M_{z^m} q) &= \tilde{q}(M_{z^{n+m}} f \circ \phi_2^m\, g) \\ &= M_{z^{n+m}} f \circ \phi_2^m \circ q\, g \circ g \\ &= M_{z^{n+m}} f \circ q \circ \phi_1^m g \circ q \\ &= (M_{z^n} f \circ q)(M_{z^m}\, g \circ q) \\ &= \tilde{q}(M_{z^n} f)\, \tilde{q}(M_{z^m} q). \end{aligned}$$

If $F = \sum_{n=0}^N M_{z^n} f_n \in K(\mathbb{Z}^+, C_0(S_2), \alpha_2)$, a straightforward calculation shows that

$$M_z \times \tilde{\pi}_{s_1}(\tilde{q}(F)) = M_z \times \tilde{\pi}_{q(s_1)}(F), \qquad s_1 \in S_1.$$

Thus,

$$\begin{aligned} \|\tilde{q}(F)\| &= \sup_{s_1 \in S_1} \|M_z \times \tilde{\pi}_{s_1}(\tilde{q}(F))\| \\ &= \sup_{s_1 \in S_1} \|M_z \times \tilde{\pi}_{q(s_1)}(F)\| \\ &\leqslant \sup_{s_2 \in S_2} \|M_z \times \tilde{\pi}_{s_2}(F)\| \\ &\leqslant \|F\|. \end{aligned} \tag{$*$}$$

It follows that $\tilde{q}$ extends to a continuous homomorphism $\mathbb{Z}^+ \times_{\alpha_2} C_0(S_2) \to \mathbb{Z}^+ \times_{\alpha_1} C_0(S_1)$. Since the formula $(M_z \times \tilde{\pi}_s)(\tilde{q}(F)) = M_z \times \tilde{\pi}_{q(s)}(F)$ is valid for $F \in K(\mathbb{Z}^+, C_0(S_2), \alpha_2)$, an obvious approximation argument yields the result for all $F \in \mathbb{Z}^+ \times_{\alpha_2} C_0(S_2)$.

If q is surjective, then the inequality $(*)$ is replaced by an equality, so $\|\tilde{q}(F)\| = \|F\|$, and $\tilde{q}$ is injective.

Suppose q is injective; for simplicity of notation regard S_1 as a subset of S_2. Observe that the image of $\tilde{q}$ contains $K(\mathbb{Z}^+, K(S_1), \alpha_1)$, where $K(S_1)$ is the space of continuous functions with compact support on S_1. For if $G = \sum_{n=0}^{N} M_{z^n} g_n \in K(\mathbb{Z}^+, K(S_1), \alpha_1)$ and if $f_n \in K(S_2)$ is any function such that $f_n|_{S_1} = g_n$, $0 \leqslant n \leqslant N$, then $\tilde{q}(\sum_{n=0}^{N} M_{z^n} f_n) = G$. This shows that the image of $\tilde{q}$ is dense. If $S_1 \subset S_2$ is open, the function f_n above must coincide with g_n, $0 \leqslant n \leqslant N$. From II.7 it is easy to see that the norm of F will be $\sup_{s \in S_2} \|M_z \times \tilde{\pi}_s(F)\| = \|G\|$. Since $\tilde{q}$ is isometric with dense image, it is surjective.

Corollary. *If $\Theta: S_2 \to S_1$ is a homeomorphism such that $\Theta \circ \phi_2 = \phi_1 \circ \Theta$, then $\mathbb{Z}^+ \times_{\alpha_1} C_0(S_1)$, $\mathbb{Z}^+ \times_{\alpha_2} C_0(S_2)$ are isometrically isomorphic.*

II.13. Corollary. *Let S be locally compact, $\phi: S \to S$ continuous and proper. Let $S_0 \subseteq S$ be open and assume $\phi^{-1}(S_0) = S_0$; let $\phi_0 = \phi|_{S_0}$ and $\alpha_0(f) = f \circ \phi_0$, $f \in C_0(S_0)$. Then $\mathbb{Z}^+ \times_{\alpha_0} C_0(S_0)$ is naturally identified with the closed ideal $\mathscr{J} = \{F \in \mathbb{Z}^+ \times_\alpha C_0(S),\ F = \sum_{n \geqslant 0} M_{z^n} f_n: \{s: f_n(s) \neq 0\} \subseteq S_0,\ n \geqslant 0\}$.*

Proof. Let $S \cup \{\infty\}$, $S_0 \cup \{\infty_0\}$ be the one-point compactifications of S, S_0, respectively. Let $q: S \cup \{\infty\} \to S_0 \cup \{\infty_0\}$ be given by

$$q(s) = \begin{cases} s, & s \in S_0, \\ \infty_0, & s \in S \cup \{\infty\} \backslash S_0; \end{cases}$$

then q is equivariant and surjective, so by II.12 $\tilde{q}: \mathbb{Z}^+ \times_{\alpha_0} C(S_0 \cup \{\infty_0\}) \to \mathbb{Z}^+ \times_\alpha C(S \cup \{\infty\})$ is injective. Now $\tilde{q}$ maps $K(\mathbb{Z}^+, C_0(S_0), \alpha_0)$, viewed as a subalgebra of $K(\mathbb{Z}^+, C_0(S_0 \cup \{\infty_0\}), \alpha_0)$, onto $K(\mathbb{Z}^+, C_0(S), \alpha) \cap \mathscr{J}$, and we claim this is dense in $\mathscr{J}$: For if $F = \sum_{n \geqslant 0} M_{z^n} f_n \in \mathscr{J}$, and if F_n is the nth arithmetic mean of the series $\sum_{n \geqslant 0} M_{z^n} f_n$, then $\|F_n - F\| \to 0$ (cf. Remark following IV.2) and $F_n \in K(\mathbb{Z}^+, C_0(S), \alpha) \cap \mathscr{J}$. Since $\tilde{q}$ is isometric, $\tilde{q}(\mathbb{Z}^+ \times_{\alpha_0} C_0(S_0)) = \mathscr{J}$. To see that $\mathscr{J}$ is an ideal, notice that $f\mathscr{J} \subseteq \mathscr{J}$, $f \subseteq \mathscr{J}$, $f \in C_0(S)$, and $M_z \mathscr{J} \subseteq \mathscr{J}$, $M_z \mathscr{J} \subseteq \mathscr{J}$; thus $\mathscr{J}$ is invariant under the subalgebra $K(\mathbb{Z}^+, C_0(S), \alpha)$ generated by $\{f \in C_0(S), M_z\}$. Since $\mathscr{J}$ is closed, it is invariant under left and right multiplication from $\mathbb{Z}^+ \times_\alpha C_0(S)$. Alternatively, we could observe that $\mathscr{J} = \bigcap_{s \in S \backslash S_0} \ker(M_z^s \times \tilde{\pi}_s)$.

II.14. Again, let $\phi: S \to S$ be continuous and proper, and let $S_0 \subseteq S$ be open. If, instead of assuming $\phi^{-1}(S_0) = S_0$ as before, we only assume $\phi^{-1}(S_0) \subseteq S_0$, it makes no sense to write $\phi|_{S_0}$. Still, if we set $\mathscr{F} = \{F \in \mathbb{Z}^+ \times_\alpha C_0(S),\ F = \sum_{n \geqslant 0} M_{z^n} f_n: \{s: f_n(s) \neq 0\} \subseteq S_0,\ n \geqslant 0\}$, essentially the same argument as in II.13 shows $\mathscr{F}$ is an ideal: $\mathscr{F}$ is a linear subspace such that $M_z \mathscr{F} \subseteq \mathscr{F}$, $\mathscr{F} M_z \subseteq \mathscr{F}$ and $f\mathscr{F}$, $\mathscr{F} f \subseteq \mathscr{F}$, $f \in C_0(S)$. Also, note that $\{F \in K(\mathbb{Z}^+, C_0(S), \alpha),\ F = \sum_{n=0}^{p} M_{z^n} f_n: p \in \mathbb{Z}^+, f_n$ has compact support in $S_0\}$ is dense in $\mathscr{F}$.

III.

III.1. Let $\mathscr{A}$ denote the disk algebra; that is, the commutative Banach algebra of continuous functions on the closed unit disk which are holomorphic in the interior. Fix a positive integer k_0, and let $\mathscr{B}_{k_0}$ be the algebra of all k_0 by k_0 matrices of functions $[f_{ij}]_{0 \leqslant i,j \leqslant k_0 - 1}$, $f_{ij} \in \mathscr{A}$, and of the form

$$f_{ij}(z) = \sum_{n \geqslant 0} a_n^{(ij)} z^{l + nk_0},$$

where $0 \leqslant l < k_0$ and $l = i - j \pmod{k_0}$. There are various (equivalent) norms under which $\mathscr{B}_{k_0}$ is a Banach algebra. We describe one such norm.

Let H^2 refer to the classical Hardy space of holomorphic functions $\xi(z) = \sum_{n \geqslant 0} \xi_n z^n$ in the unit disk having nontangential L^2 boundary values with

$$\|\xi\|^2 = \sum_{n \geqslant 0} |\xi_n|^2 = \frac{1}{2\pi} \int_{|z|=1} |\xi(z)|^2 \, dz,$$

and inner product

$$(\xi, \eta) = \frac{1}{2\pi} \int_{|z|=1} \xi(z)\, \eta(z)^- \, dz \qquad (\xi, \eta \in H^2).$$

Let H_j^2 $(0 \leqslant j < k_0)$ denote the subspace of functions of the form $\xi(z) = \sum_{n \geqslant 0} \xi_{j+nk_0} z^{j+nk_0}$. The subspaces H_j^2, H_k^2 are orthogonal if $j \neq k$. The mapping $P_j: H^2 \to H_j^2$, $\xi(z) = \sum_{n \geqslant 0} \xi_n z^n \to \sum_{n \geqslant 0} \xi_{j+nk_0} z^{j+nk_0}$ is the orthogonal projection of H^2 onto H_j^2, $0 \leqslant j < k_0$.

Now a matrix $[f_{ij}] \in \mathscr{B}_{k_0}$ maps a vector $[\xi^0(z), \xi^1(z), \ldots, \xi^{k_0-1}(z)] \in \bigoplus_{j=0}^{k_0-1} H_j^2$ to a vector $[\eta^0(z), \eta^1(z), \ldots, \eta^{k_0-1}(z)] \in \bigoplus_{j=0}^{k_0-1} H_j^2$ in the natural way

$$\eta^k(z) = \sum_{j=0}^{k_0-1} f_{kj}(z)\, \xi^j(z).$$

It is by means of this natural representation of $\mathscr{B}_{k_0}$ on $\bigoplus_{j=0}^{k_0-1} H_j^2$ that the norm in $\mathscr{B}_{k_0}$ is defined; it is the operator norm associated with this representation. Of course when $k_0=1$, we just obtain $\mathscr{B}_1=\mathscr{A}$. For $k_0\geqslant 2$, $\mathscr{B}_{k_0}$ is a noncommutative, nonselfadjoint operator algebra.

III.2. THEOREM. *Let k_0 be a positive integer, $S=\{s_0, s_1,..., s_{k_0-1}\}$, and $\phi: S\to S$, $\phi(s_j)=s_{j+1(\mathrm{mod}\, k_0)}$. Then the semi-crossed product $\mathbb{Z}^+\times_\alpha C(S)$ is isomorphic to $\mathscr{B}_{k_0}$.*

Proof. Let $F=\sum_{n\geqslant 0} M_{z^n} f_n\in \mathbb{Z}^+\times_\alpha C(S)$, $\xi(z)=\sum_{n\geqslant 0}\xi_n z^n\in H^2$. We compute

$$[(M_z\times\tilde{\pi}_{s_0})(F)]\,\xi(z)=\sum_{n\geqslant 0}\;\sum_{k\geqslant 0} f_n\circ\phi^k(s_0)\,\xi_k z^{n+k}.$$

Make the substitution $k\to mk_0+j$, and use $\phi^k=\phi^j$ to get

$$=\sum_{j=0}^{k_0-1}\;\sum_{n\geqslant 0} f_n\circ\phi^j(s_0)\left(\sum_{m\geqslant 0}\xi_{mk_0+j}z^{mk_0+j}\right) z^n$$

$$=\sum_{j=0}^{k_0-1}\left(\sum_{n\geqslant 0} f_n\circ\phi^j(s_0)\, z^n\right)\xi^j(z)$$

$$=\sum_{j=0}^{k_0-1} F(\phi^j(s_0), z)\,\xi^j(z),$$

where $\xi^j(z)=\sum_{m\geqslant 0}\xi_{mk_0+j}z^{mk_0+j}$, and $F(\phi^j(s_0), z)\,\xi^j(z)$ denotes the pointwise product. Next write $\sum_{j=0}^{k_0-1} F(\phi^j(s_0), z)\,\xi^j(z)=\sum_{i=0}^{k_0-1}\sum_{j=0}^{k_0-1} P_i F(\phi^j(s_0), z)\,\xi^j(z)$. The projection onto H_i^2 composed with multiplication by $F(\phi^j(s_0), z)$ restricted to H_j^2, or $P_i F(\phi^j(s_0), z)\,P_j$, is given by the multiplication operator $F_{ij}(z)\equiv P_i F(\phi^j(s_0), z)\,P_j=\sum_n f_{i-j+nk_0}(\phi^j(s_0))\, z^{i-j+nk_0}P_j$, where the sum over n begins with $n=0$ if $i\geqslant j$, and with $n=1$ if $i<j$.

Thus we see that $M_z\times\tilde{\pi}_{s_0}$ yields a faithful representation of $\mathbb{Z}^+\times_\alpha C(S)$ onto $\mathscr{B}_{k_0}$. Of course if s_0 had been replaced by any $s_k\in S$, the representation $M_z\times\tilde{\pi}_{s_k}$ would also yield an isomorphism of $\mathbb{Z}^+\times_\alpha C(S)$ with $\mathscr{B}_{k_0}$.

III.3. Next we study the maximal (two-sided) ideals of $\mathscr{B}_{k_0}$. Clearly, the kernel $M_0^{(k)}$ of the homomorphism $[f_{ij}]\to f_{kk}(0)$, $0\leqslant k\leqslant k_0-1$, is maximal, as is the kernel M_λ of the map $[f_{ij}]\to[f_{ij}(\lambda)]$, $\lambda\in\bar{D}$, $\lambda\neq 0$. As we will see, these are the only maximal ideals.

PROPOSITION. *Let M be a maximal ideal of $\mathscr{B}_{k_0}$, $M\neq M_0^{(k)}$, $0\leqslant k\leqslant k_0-1$. Then $M=M_\lambda=\{[f_{ij}]\in\mathscr{B}_{k_0}: f_{ij}(\lambda)=0,\ 0\leqslant i,\ j\leqslant k_0-1\}$, for some $\lambda\in\bar{D}$, $\lambda\neq 0$.*

Proof. Let E_{ij} be the $k_0\times k_0$ matrix with 1 in the (i,j) entry and zero elsewhere, $0\leqslant i, j\leqslant k_0-1$. Then $z^j\otimes E_{j0}$, $0\leqslant j\leqslant k_0-1$, and $z^{k_0-i}\otimes E_{0i}$,

$0 \leqslant i \leqslant k_0 - 1$, belong to $\mathscr{B}_{k_0}$. If $F = [f_{ij}] \in M$, $F \notin M_0^{(0)}$, then $E_{00}F(z^j \otimes E_{j0}) = z^j f_{0j}(z) \otimes E_{00} \in M$. Likewise $(z^{k_0-i} \otimes E_{0i})\ F(z^j \otimes E_{j0}) = z^{k_0+j-i} f_{ij}(z) \otimes E_{00} \in M$. Thus M contains all linear combinations $[\sum_{j=0}^{k_0-1} c_{0j} z^j f_{0j}(z) + \sum_{i=1}^{k_0-1} \sum_{j=0}^{k_0-1} z^{k_0+j-i} c_{ij} f_{ij}(z)] \otimes E_{00}$. Let $\mathscr{I}$ be the closed ideal in $\mathscr{A}$ generated by the set of all linear combinations $\{\sum_{j=0}^{k_0-1} c_{0j} z^j f_{0j}(z) + \sum_{i=0}^{k_0-1} \sum_{j=0}^{k_0-1} c_{ij} z^{k_0+j-i} f_{ij}(z)\}$. If $\mathscr{I}$ is proper, it is contained in an ideal of the form $\{f \in \mathscr{A} : f(\lambda) = 0\}$ for some $\lambda \in \bar{D}$ [9, Corollary, p. 87]. However, $f_{00} \in \mathscr{I}$ and $f_{00}(0) \neq 0$, since $F \notin M_0^{(0)}$; so $\mathscr{I} \not\subset \{f \in \mathscr{A} : f(0) = 0\}$. Thus if $\mathscr{I}$ is proper $\mathscr{I} \subset \{f \in \mathscr{A} : f(\lambda) = 0\}$ for some $\lambda \in \bar{D}$, $\lambda \neq 0$. But then $\lambda^j f_{0j}(\lambda) = 0$, and $\lambda^{k_0+j-i} f_{ij}(\lambda) = 0$, $0 \leqslant j \leqslant k_0 - 1$, $1 \leqslant i < k_0$; hence $f_{ij}(\lambda) = 0$, $0 \leqslant i, j \leqslant k_0 - 1$, and $F \in M_\lambda$.

If, on the other hand, $\mathscr{I} = \mathscr{A}$, then $E_{00} \in M$. But in that case an analogous argument, making use of the fact that $M \neq M_0^{(j)}$, shows that $E_{jj} \in M$, $1 \leqslant j \leqslant k_0 - 1$. Hence $I = E_{00} + E_{11} + \cdots + E_{k_0-1\,k_0-1} \in M$, so $M = \mathscr{B}_{k_0}$.

III.4. For convenience, write ψ_s for $M_z \times \tilde{\pi}_s$. Let S be a locally compact hausdorff space such that every point of S is periodic under a homeomorphism ϕ. If $S_0 = \{s_0, s_1, ..., s_{k_0-1}\}$ is the orbit of a point $s_0 \in S$, then the injection $q: S_0 \to S$ determines a surjection $\tilde{q}: \mathbb{Z}^+ \times_\alpha C_0(S) \to \mathbb{Z}^+ \times_{\alpha_0} C(S_0) \cong \mathscr{B}_{k_0}$, by II.12, III.2, and the fact that $\mathscr{B}_{k_0}$ is strongly semisimple with maximal ideals of finite codimension, so we may view ψ_{s_0} as a representation of $\mathbb{Z}^+ \times_\alpha C_0(S)$ onto $\mathscr{B}_{k_0}$. Thus the maximal ideals of $\mathscr{B}_{k_0}$, which are classified in III.3, determine maximal ideals in $\mathbb{Z}^+ \times_\alpha C_0(S)$. Say (S, ϕ) has locally bounded order if for each $s \in S$ there is a neighborhood U_s and an integer $n_s > 0$ such that $\phi^{n_s}(s') = s'$ for all $s' \in U_s$.

Proposition. *Suppose S has locally bounded order. Then every maximal modular ideal of $\mathbb{Z}^+ \times_\alpha C_0(S)$ is of the form $\psi_{s_0}^{-1}(M)$, for some $s_0 \in S$ and maximal ideal M of $\mathscr{B}_{k_0}$, where k_0 is the cardinality of the order s_0.*

Proof. Suppose $M \subset \mathbb{Z}^+ \times_\alpha C_0(S)$ is a maximal modular ideal such that $\psi_s(M)$ is not a maximal ideal in $\psi_s(\mathbb{Z}^+ \times_\alpha C_0(S))$ for any $s \in S$. Then $\psi_s(M) = \psi_s(\mathbb{Z}^+ \times_\alpha C_0(S))$, and so there exists, for each $s \in S$, an element $F^{(s)} \in M$ such that $\psi_s(F^{(s)})$ is the identity in $\psi_s(\mathbb{Z}^+ \times_\alpha C_0(S))$.

Let F be an element of $\mathbb{Z}^+ \times_\alpha C_0(S)$. We will show that F can be approximated arbitrarily closely by elements of M; since M is closed, this will imply $F \in M$ and hence $M = \mathbb{Z}^+ \times_\alpha C_0(S)$, contradicting that M is proper.

Given $\varepsilon > 0$ there exists $G \in \mathbb{Z}^+ \times_\alpha C_0(S)$ such that $G = \sum_{n=0}^{p} M_{z^n} g_n$, g_n has compact support, $0 \leqslant n \leqslant p$, and $\|F - G\| < \varepsilon$. Let $K = \bigcup_{n=0}^{p} \operatorname{supp}(g_n)$. The assumption that S has locally bounded order along with the compactness of K imply there is an open set $U \supset K$ and a positive integer N such that $\phi^N(s) = s$, $s \in U$. Taking $S_0 = \bigcup_{k=0}^{N} \phi^k(U)$, S_0 is an invariant open set containing K, and $\phi^N(s) = s$, $s \in S_0$. If $\alpha_0 = \alpha|_{S_0}$, by II.13 we can consider

$\mathbb{Z}^+ \times_{\alpha_0} C_0(S_0)$ as a closed ideal of $\mathbb{Z}^+ \times_\alpha C_0(S)$. By II.11, the map $s \in S_0 \to \psi_s(H)$ is norm continuous for each $H \in \mathbb{Z}^+ \times_{\alpha_0} C_0(S_0)$. Let $k \in C_0(S_0)$, $0 \leqslant k \leqslant 1$, such that $k(s) = 1$ for $s \in K$. Then $F^{(s)}k \in \mathbb{Z}^+ \times_{\alpha_0} C_0(S_0)$ for all $s \in S$. For $s \in K$ set $U_s = \{s' \in S_0: \|\psi_{s'}(F^{(s)}k) - I\| < \varepsilon\}$ (Recall that all ψ_s are viewed as mapping into the same space, namely, $\mathscr{B}(H^2)$.) If $\{U_i \equiv U_{s_i}: 1 \leqslant i \leqslant m\}$ is a finite sub-cover, let $\{h_i: 1 \leqslant i \leqslant m\}$ be a partition of unity on K subordinate to the $\{U_i: 1 \leqslant i \leqslant m\}$; i.e., $\operatorname{supp}(h_i) \subset U_i$, $0 \leqslant h_i \leqslant 1$, $1 \leqslant i \leqslant m$, and $\sum_{i=1}^m h_i(s) = 1$ for $s \in K$. Set $H = \sum_{i=1}^m F^{(s_i)}kh_i$; then $H \in M$. We estimate

$$\|G - HG\| = \sup_{s \in S_0} \|(I - \psi_s(H))\, \psi_s(G)\|$$

$$= \sup_{s \in K} \|(I - \psi_s(H))\, \psi_s(G)\|$$

(since $\psi_s(G) = 0$, $s \notin K$)

$$< \varepsilon \|G\|$$

$$< \varepsilon(\|F\| + \varepsilon),$$

because

$$\|I - \psi_s(H)\| = \left\| I - \psi_s\left(\sum_{i=1}^m F^{(s_i)}kh_i\right)\right\| = \left\| \sum_{i=1}^m h_i(s)[I - \psi_s(F^{(s_i)})]\right\|$$

$$\leqslant \sum_{i=1}^m h_i(s)\, \|I - \psi_s(F^{(s_i)})\|$$

$$< \sum_{i=1}^m h_i(s)\varepsilon = \varepsilon.$$

It follows that

$$\|F - HG\| \leqslant \|F - G\| + \|G - HG\| < \varepsilon + \varepsilon(\|F\| + \varepsilon).$$

This completes the proof.

III.5. Next we study the topology on the strong structure space of $\mathbb{Z}^+ \times_\alpha C_0(S)$.

If (S, ϕ) is such that every point $s \in S$ is periodic and has locally bounded order, then as in III.4 each ψ_s can be viewed as a map of $\mathbb{Z}^+ \times_\alpha C_0(S)$ onto $\mathscr{B}_{k_0}$, where $k_0 = k_0(s)$ is the cardinality of the orbit of s. Denote by $\psi_{s,\lambda}$ the composition of ψ_s with the map which sends $[f_{ij}] \in \mathscr{B}_{k_0} \to [f_{ij}(\lambda)]$, $\lambda \in \bar{D}$. From III.3 we know that the maximal ideals of $\mathscr{B}_{k_0}$ are of the form $M_0^{(j)}$, $0 \leqslant j \leqslant k_0 - 1$, and M_λ, $\lambda \in \bar{D}$, $\lambda \neq 0$. However, the M_λ's are not necessarily distinct. For let ω be a primitive k_0th root of unity. If $[F_{ij}] \in \mathscr{B}_{k_0}$, then F_{ij} satisfies $F_{ij}(\omega z) = \omega^{i-j} F_{ij}(z)$. Thus, the ideals $M_\lambda, M_{\omega\lambda}, \ldots, M_{\omega^{k_0-1}\lambda}$ are identical. On the other hand, it is clear that if $\lambda, \lambda' \in \bar{D} \backslash \{0\}$ and $(\lambda/\lambda')^{k_0} \neq 1$, then $M_\lambda, M_{\lambda'}$ are distinct. Now let (S, ϕ) be as in III.4, and let Θ_s denote the orbit of s under ϕ, and $|\Theta_s|$ the cardinality of Θ_s. From what we know about the maps $\psi_{s,\lambda}$, we can assert that $\ker \psi_{s,\lambda} = \ker \psi_{s',\lambda'}$ if and only if either

(i) $s' \in \theta_s$ and $\lambda = \lambda' = 0$; or

(ii) $s' \in \theta_s$, $|\lambda| = |\lambda'| \neq 0$, and $(\lambda'/\lambda)^{|\theta_s|} = 1$.

Also, as has been mentioned, $\ker \psi_{s,\lambda}$ $(\lambda \neq 0)$ is a maximal ideal, whereas $\ker \psi_{s,0}$ is not. If $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$, $F = \sum_{n \geq 0} M_{z^n} f_n$, then

$$\psi_{s,0}(F) = \begin{bmatrix} f_0(s) & & & \\ & f_0(\phi(s)) & & \\ & & \ddots & \\ & & & f_0(\phi^{k_0-1}(s)) \end{bmatrix}.$$

We define an equivalence relation on $S \times \bar{D}$ ($\bar{D}$ is the closed unit disk) by $(s, \lambda) \sim (s', \lambda')$ if and only if either (i) $s = s'$ and $\lambda = \lambda' = 0$, or (ii) $s' \in \theta_s$, $|\lambda| = |\lambda'| \neq 0$, and $(\lambda'/\lambda)^{|\theta_s|} = 1$. We have shown that there is a one to one correspondence between the maximal modular ideals M of $\mathbb{Z}^+ \times_\alpha C_0(S)$ and equivalence classes in $S \times \bar{D}$.

Next, recall the hull-kernel topology (Rudin topology) on the closed unit disk determined by the algebra $\mathscr{A}$. The closed sets $V \subset \bar{D}$ in this topology are of the following form:

(i) $V \cap D$ is either finite or countable; if $V \cap D$ is countable, say, $\{\lambda_1, \lambda_2, \ldots\}$, then $\sum_{n=1}^\infty (1 - |\lambda_n|) < \infty$.

(ii) $V \cap \partial D$ (i.e., the intersection of V with the unit circle) is a closed subset (in the usual topology of the circle) of Lebesgue measure zero, and contains every accumulation point of $V \cap D$ [9, p. 89].

Let the locally compact topology of S be metrizable, and $\bar{D}$ have the topology just mentioned, and the product $S \times \bar{D}$ the product topology. Let $q: S \times \bar{D} \to S \times \bar{D}/\sim = \mathscr{M}$ be the quotient map, and endow $\mathscr{M}$ with the quotient topology: the open sets $W \subset \mathscr{M}$ are precisely those for which $q^{-1}(W) \subset S \times \bar{D}$ is open.

PROPOSITION. *Let (S, ϕ) be as in* III.4. *Then the hull-kernel topology on the maximal ideal space $\mathscr{M}$ of $\mathbb{Z}^+ \times_\alpha C_0(S)$ is stronger than the quotient topology.*

Proof. An open base for the product topology on $S \times \bar{D}$ consists of sets of the form $A \times B$, where $A \subset S$ is relatively compact open and $B \subset \bar{D}$ is open (in the Rudin topology). Thus, a closed set in $S \times \bar{D}$ is the intersection of a family of closed sets of the form $(A \times B)^c$, with A, B as above. Now $(A \times B)^c = (A^c \times \bar{D}) \cup (S \times B^c)$. Since S is locally compact metric, A^c is the zero set of some continuous function with compact support: say, $A^c = Z(f)$. Also, since every closed ideal of $\mathscr{A}$ is the principal closed ideal generated by a function in $\mathscr{A}$ [9, Corollary, p. 88], there is a $g \in \mathscr{A}$ with $Z(g) = B^c$. If $g(z) = \sum_{n \geqslant 0} a_n z^n, |z| < 1$, set $F = \sum_{n \geqslant 0} M_{z^n} a_n f$. Clearly, $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$, and $Z(F) = \{(s, \lambda): F(s, \lambda) = \sum_{n \geqslant 0} \lambda^n a_n f(s) = f(s)\, g(\lambda) = 0\} = (A \times B)^c$.

We have shown that if $C \subset S \times \bar{D}$ is closed in the product topology, there is a family $\{F_\nu: F_\nu \in \mathbb{Z}^+ \times_\alpha C_0(S), \nu \in \Lambda\}$ with $C = \bigcap_{\nu \in \Lambda} Z(F_\nu)$. Suppose in addition that $C = q^{-1}(q(C))$. Then by the way we defined the quotient map, each maximal modular ideal M of $\mathbb{Z}^+ \times_\alpha C_0(S)$ in $q(C)$ contains all the F_ν, $\nu \in \Lambda$. Hence, $\ker(q(C)) \supset \{F_\nu: \nu \in \Lambda\}$, and so $\operatorname{hull}(\ker(q(C))) \subset \operatorname{hull}\{F_\nu: \nu \in \Lambda\} = q(C)$. Since $\operatorname{hull}(\ker(q(C)))$ necessarily contains $q(C)$, it must coincide with $q(C)$. Thus $q(C)$ is closed in the hull-kernel topology, which completes the proof.

III.6. *Remark.* Let (S, ϕ) be as in III.4 and suppose in addition that the zero set of each $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$ is closed in the product topology on $S \times \bar{D}$. Then, if $C \subset S \times \bar{D}$ is such that $q(C)$ is closed in the hull-kernel topology, C is the zero set of a certain collection of elements in $\mathbb{Z}^+ \times_\alpha C_0(S)$ (namely, $\ker(q(C))$), and hence C is closed in the product topology of $S \times \bar{D}$. From this observation it follows, for instance, that the quotient topology on $\mathscr{M} = S \times \bar{D}/\sim$ coincides with the hull-kernel topology if S is a finite set. In particular, this allows us to describe the hull-kernel topology on the maximal ideal space of $\mathscr{B}_{k_0}$. Let λ, λ' in the punctured closed unit disk $\bar{D} \backslash \{0\}$ be equivalent if $(\lambda/\lambda')^{k_0} = 1$. If the topology on $\bar{D}$ is the hull-kernel topology of the disk algebra, and that on $\bar{D}\backslash\{0\}$ the relative topology, endow $(\bar{D}\backslash\{0\}/\sim$ with the quotient topology. Then the maximal ideal space $\mathscr{M}$ can be identified with $(\bar{D}\backslash\{0\})/\sim \cup \{M_0^{(0)}, \ldots, M_0^{(k_0-1)}\}$. An open set in $\mathscr{M}$ is either open in $(\bar{D}\backslash\{0\})/\sim$, or else it is an open neighborhood of $M_0^{(j)}$ $(0 \leqslant j \leqslant k_0 - 1)$. The open neighborhoods of $M_0^{(j)}$ are of the form $(U\backslash\{0\}\}/\sim \cup \{M_0^{(j)}\}$, where U is an open neighborhood of 0 in $\bar{D}$.

III.7. Example. Again, let (S, ϕ) be as in III.4. We show that in general the hull-kernel topology on $\mathscr{M} = S \times \bar{D}/\sim$ is strictly stronger than the quotient topology. Following the idea of the proof of III.5, if $C \subset S \times \bar{D}$ is closed in the product topology and $(s_0, \lambda_0) \notin C$, there are open sets $A \subset S$, $B \subset \bar{D}$, with $(s_0, \lambda_0) \in A \times B \subset C^c$. Or, $C \subset (A \times B)^c$, $(s_0, \lambda_0) \notin (A \times B)^c$. Also, there is an $f \in C_0(S)$, $g \in \mathscr{A}$, $g(z) = \sum_{n \geqslant 0} a_n z^n$, with $Z(F) = (A \times B)^c$, where $F = \sum_{n \geqslant 0} M_{z^n} a_n f$.

Suppose now $S = \bar{D}$, the closed unit disk (in the Euclidean topology), and ϕ is the identity map. Let $G \in \mathbb{Z}^+ \times_\alpha C(S)$, $G(s, z) = s - z$. Since $Z(G) = \{(s, s): s \in \bar{D}\}$, this shows that the diagonal in $\bar{D} \times \bar{D}$ is closed in the hull-kernel topology. (Note that since ϕ is the identity, the maximal ideal space of $\mathbb{Z}^+ \times_\alpha C(S)$ is $\bar{D} \times \bar{D}$.) If the diagonal is closed in the product topology, there is an $F \in \mathbb{Z}^+ \times_\alpha C(S)$, $F(s,z) = f(s)\, g(z)$, not identically zero, such that $Z(F)$ contains the diagonal of $\bar{D} \times \bar{D}$. Say $F(s_0, \lambda_0) \neq 0$, so there is a neighborhood U of s_0 in $\bar{D}$, open with respect to the Euclidean topology, with $f(s) \neq 0$, $s \in U$. Since $F(s,s) = 0$ for all $s \in U$, this forces $g(s) = 0$. But then g must vanish identically, and $F = 0$. This shows that the diagonal is not closed in the product topology.

IV.

IV.1. We turn now from the special case considered in Section III, in which each $s \in S$ was periodic under ϕ, to the general case, in which we only assume $\phi: S \to S$ is continuous and proper. Unlike the periodic case, the algebras $\mathbb{Z}^+ \times_\alpha C(S)$ need not in general be strongly semisimple; indeed, they need not even be semiprime. Our goal in this section will be to give necessary and sufficient conditions on the dynamical system (S, ϕ) for $\mathbb{Z}^+ \times_\alpha C(S)$ to be (i) semiprime; (ii) semisimple; and (iii) strongly semisimple.

We recall some standard facts and terminology [2, 10, 14]. All ideals will be assumed to be two sided unless otherwise stated. An ideal P is said to be *prime* if, for any ideals, I, J, $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$; it is called *primitive* if P is the kernel of an irreducible representation of $\mathfrak{A}$, and P is called *modular* if $\mathfrak{A}/P$ has an identity. A maximal modular ideal is primitive, and a primitive ideal is prime. The *prime radical* (resp., *Jacobson radical*, *strong radical*) is the intersection of all prime ideals of $\mathfrak{A}$ (resp., all primitive ideals, all maximal modular ideals). $\mathfrak{A}$ is said to be semiprime (resp. semisimple, strongly semisimple) if the prime radical (resp., Jacobson radical, strong radical) is (0). $\mathfrak{A}$ is semiprime iff $\mathfrak{A}$ has no nontrivial nilpotent ideals. If $\mathfrak{A}$ is a Banach algebra, $\mathfrak{A}$ is semisimple iff $\mathfrak{A}$ has no nontrivial left or right ideals consisting of quasinilpotent elements. Of course there are various other characterizations of semisimplicity.

Suppose S is locally compact hausdorff and $\phi: S \to S$ is continuous and proper. We want to define the notions of *near recurrence* and *near periodicity*, which are slightly weaker than recurrence and almost periodicity, respectively. Recall that a point $s_0 \in S$ is recurrent under $\mathbb{Z}^+$ if, for every neighborhood U of s_0, $\{\phi^n(s_0)\}_{n=0}^\infty \cap U$ is infinite. We will say s_0 is *nearly recurrent* if for every neighborhood U of s_0 there exists a point $s_1 \in U$, depending on both s_0 and U, such that $U \cap \{\phi^n(s_1)\}_{n=0}^\infty$ is infinite. A set $A \subseteq \mathbb{Z}^+$ is *syndetic* if there exists a finite set $F \subseteq \mathbb{Z}^+$ such that $\mathbb{Z}^+ = A + F$. (Note that this is the same definition as in [4], except that there it is given in the context of groups.) $s_0 \in S$ is almost periodic if, given a neighborhood U

of s_0, $\{k \in \mathbb{Z}^+ : \phi^k(s_0) \in U\}$ is syndetic. We will say s_0 is *nearly periodic* if, given a neighborhood U of s_0, there exists $s_1 \in U$, which may depend on both s_0 and U, such that $\{k \in \mathbb{Z}^+ : \phi^k(s_1) \in U\}$ is syndetic. It is an easy consequence of the definitions that the subsets of nearly recurrent points and nearly periodic points are closed in S. Of course, the set of periodic points need not be closed. We are now ready to state our main result.

THEOREM. *Let S be locally compact hausdorff, and $\phi: S \to S$ continuous and proper.*

(i) *$\mathbb{Z}^+ \times_\alpha C(S)$ is semirpime if and only if every point of S is nearly recurrent*;

(ii) *$\mathbb{Z}^+ \times_\alpha C(S)$ is semisimple if and only if every point of S is nearly periodic*;

(iii) *$\mathbb{Z}^+ \times_\alpha C(S)$ is strongly semisimple if and only if the subset of S consisting of periodic points is dense.*

It is not hard to show that s_0 is a nearly recurrent point of S iff there is no neighborhood U of s_0 with $U, \phi^{-1}, \phi^{-2}(U),...$ pairwise disjoint. This is related to the term "nonwandering": if ϕ is a homeomorphism, s_0 is nonwandering if there is no neighborhood U of s_0 with $\{\phi^n(U)\}_{n \in \mathbb{Z}}$ pairwise disjoint.

In the course of establishing the Theorem we will actually provide an explicit characterization of each of the three radicals.

IV.2. Although the proofs of (i), (ii), and (iii) will be done separately, there are certain invariance properties which are shared by all three radicals, which we now discuss.

LEMMA. *$F = \sum_{n \geqslant 0} M_{z^n} f_n \in \mathbb{Z}^+ \times_\alpha C(S)$ belongs to the prime radical (resp., to the Jacobson radical, to the strong radical) if and only if $f_0 \equiv 0$ and $M_{z^n} f_n$ belongs to the prime radical (resp., to the Jacobson radical, to the strong radical) for $n \geqslant 1$.*

Proof. Since each of the radicals is a closed linear subspace, the "if" direction is clear.

Define an automorphism τ_t $(t \in \mathbb{R})$ of the dense subalgebra $K(\mathbb{Z}^+, C_0(S), \alpha)$ by $\tau_t(\sum_{n=0}^p M_{z^n} g_n) = \sum_{n=0}^p M_{z^n} e^{int} g_n$. Now τ_t is norm-preserving, for

$$(M_z \times \tilde{\pi}_s)\left(\tau_t\left(\sum_{n=0}^{p} M_{z^n} g_n\right)\right)(\xi)(z) = \sum_{m \geqslant 0}\left[\sum_{k=0}^{m} g_{m-k} \circ \phi^k(s)\, e^{i(m-k)t}\, \xi_k\right] z^m,$$

where $\xi \in H^2$, $\xi(z) = \sum_{k \geqslant 0} z^k \xi_k$, and $g_k \equiv 0$, $k > p$. If we view $(M_z \times \tilde{\pi}_s)(G)(\xi)$ (for $G \in K(\mathbb{Z}^+, C_0(S), \alpha)$ as a continuous function on the circle, then

$$(M_z \times \tilde{\pi}_s)(\tau_t(G))(\xi)(e^{i\Theta}) = (M_z \times \tilde{\pi}_s)(G)(\xi')(e^{i(\Theta + t)}),$$

where $\xi' \in H^2$, $(\xi')_k = e^{-ikt}\xi_k$. It follows that $\|(M_z \times \tilde{\pi}_s)(\tau_t(G))\| = \|(M_z \times \tilde{\pi}_s)(G)\|$, $s \in S$, and consequently $\|\tau_t(G)\| = \|G\|$, $G \in K(\mathbb{Z}^+, C_0(S), \alpha)$. Thus τ_t extends to an automorphism of the semi-crossed product $\mathbb{Z}^+ \times_\alpha C(S)$, which will also be denoted τ_t. Note that if α is injective, and hence extends to an automorphism β of a (commutative) C^*-algebra $\mathscr{B}$, say, $\mathscr{B} = C_0(S')$, then $\{\tau_t\}_{t \in \mathbb{R}}$ is just the restriction of the dual automorphism group of the C^*-crossed product $\mathbb{Z} \times_\beta C_0(S')$ to $\mathbb{Z}^+ \times_\alpha C(S)$.

Now the mapping $t \to \tau_t(F)$, $F \in \mathbb{Z}^+ \times_\alpha C(S)$, is norm continuous, and the Bochner integral $(1/2\pi) \int_0^{2\pi} e^{-int}\tau_t(F)\,dt$ converges in norm to $M_{z^n}f_n$, if $F = \sum_{n=0}^{\infty} M_{z^n}f_n$. These facts are easily seen to hold for $F \in K(\mathbb{Z}^+, C_0(S), \alpha)$, and can then be extended to $\mathbb{Z}^+ \times_\alpha C(S)$. Since any automorphism of $\mathbb{Z}^+ \times_\alpha C(S)$ maps each of the three radicals of $\mathbb{Z}^+ \times_\alpha C(S)$ onto itself, it follows that if $F = \sum_{n \geq 0} M_{z^n}f_n$ belongs to one of the radicals, so does $f_0, M_z f_1, M_{z^2} f_2, \ldots$.

Finally, observe that if $F = \sum_{n \geq 0} M_{z^n}f_n \in \mathbb{Z}^+ \times_\alpha C(S)$, the mapping $F \to \chi_s(F) = f_0(s)$ is a continuous homomorphism $\mathbb{Z}^+ \times_\alpha C(S) \to \mathbb{C}$, so $\ker \chi_s$ $(s \in S)$ is a maximal modular ideal. To complete the Lemma it is only necessary to notice that if $F \in R$, the strong radical, then $f_0 \in R$ and $f_0 \in \bigcap_{s \in S} \ker \chi_s$, so f_0 must vanish identically.

Remark. If k_n is the Fejer kernel on $[0, 2\pi]$, then $(1/2\pi) \int_0^{2\pi} k_n(t)\, \tau_t(F)\,dt$ converges to F, $F \in \mathbb{Z}^+ \times_\alpha C(S)$. Indeed, this can be easily verified for $F \in K(\mathbb{Z}^+, C_0(S), \alpha)$, and then by approximation for general F. Note that $(1/2\pi) \int_0^{2\pi} k_n(t)\, \tau_t(F)\,dt$ is just the nth arithmetic mean of the series $\sum_{n \geq 0} M_{z^n}f_n = F$. (This same observation is made in [12] in the context of the Arveson–Josephson algebras.)

IV.3. If in II.12 we take $S_1 = S_2 = S$ and $g = \phi_1 = \phi_2 = \phi$, we obtain a continuous endomorphism of $\mathbb{Z}^+ \times_\alpha C(S)$, which we denote by $\hat{\alpha}$, given by $\hat{\alpha}(\sum_{n \geq 0} M_{z^n}f_n) = \sum_{n \geq 0} M_{z^n}f_n \circ \phi$. Set $S_0 = S \setminus \bigcap_{n \geq 0} \phi^n(S)$, and let $R_{\hat{\alpha}}$ denote the $\hat{\alpha}$-radical of $\mathbb{Z}^+ \times_\alpha C(S)$ (cf. I.7).

PROPOSITION. $R_{\hat{\alpha}} = \{F \in \mathbb{Z}^+ \times_\alpha C(S),\ F = \sum_{n \geq 0} M_{z^n}f_n\colon \{s\colon f_n(s) \neq 0\} \subseteq S_0,\ n = 0, 1, 2, \ldots\}$. *In particular, if* α *is injective (equivalently,* ϕ *is surjective),* $R_{\hat{\alpha}} = (0)$, *so* $\hat{\alpha}$ *is injective.*

Proof. The argument in IV.2 can also be used to show $F = \sum_{n \geq 0} M_{z^n}f_n \in R_{\hat{\alpha}}$ iff $M_{z^n}f_n \in R_{\hat{\alpha}}$, $n = 0, 1, 2, \ldots$. But $M_{z^n}f_n \in R_{\hat{\alpha}}$ iff $f_n \in R_\alpha$, which in turn holds iff $\{s\colon f_n(s) \neq 0\} \subseteq S_0$, $n = 0, 1, 2, \ldots$.

IV.4. Let S_{nr} denote the set of nearly recurrent points of S, and set $S_0 = S \setminus S_{nr}$. Let $R = \{F \in \mathbb{Z}^+ \times_\alpha C(S),\quad F = \sum_{n \geq 1} M_{z^n}f_n\colon \{s\colon f_n(s) \neq 0\} \subseteq S_0\}$.

PROPOSITION. *R is the prime radical of $\mathbb{Z}^+ \times_\alpha C(S)$.*

Proof. Let R' denote the prime radical of $\mathbb{Z}^+ \times_\alpha C(S)$, and let $F = \sum_{n \geqslant 1} M_{z^n} f_n \in R'$. Suppose for some k_0, there exists $s_0 \in S_{nr}$ such that $f_{k_0}(s_0) \neq 0$. By IV.2, $M_{z^{k_0}} f_{k_0} \in R'$, as is $(M_{z^{k_0}} f_{k_0}) \bar{f}_{k_0}$, so we assume $f_{k_0} \geqslant 0$ and set $G = \sum_{k \geqslant k_0} M_{z^k} g_k$, where $g_k = r^k f_{k_0}$. Here r is fixed, $0 < r < 1$; thus $G \in R'$. [Note that G is the limit of the partial sums $\sum_{k=k_0}^{n} M_{z^k} g_k$, each of which is in R', and that R' is closed.] Claim that G is not nilpotent; i.e., $G^n \neq 0$, $n = 1, 2, \ldots$. Let U be a neighborhood of s_0 such that $f_{k_0}(s) > 0$, $s \in U$; then $g_k(s) > 0$, $s \in U$, $k \geqslant k_0$. Let $k_0 < k_1 < k_2 < \cdots$ be a sequence of nonnegative integers and $s_1 \in U$ be such that $\phi^{k_m}(s_1) \in U$, $m \geqslant 1$. Now G^n consists of sums of terms of the form

$$M_{z^{l_1}} g_{l_1} M_{z^{l_2}} g_{l_2} \cdots M_{z^{l_n}} g_{l_n} = M_{z^{l_1 + \cdots + l_n}} g_{l_1} \circ \phi^{l_2 + \cdots + l_n} g_{l_2} \circ \phi^{l_3 + \cdots + l_n} \cdots g_{l_n},$$

where $l_1, \ldots, l_n \geqslant k_0$, n a positive integer. Suppose in addition that $l_j + l_{j+1} + \cdots + l_n \in \{k_i\}_{i=1}^{\infty}$, $1 \leqslant j \leqslant n$. (There will always be at least one term in the expansion of G^n for which this is true.) Since

$$g_{l_1} \circ \phi^{l_2 + \cdots + l_n}(s_1)\, g_{l_2} \circ \phi^{l_3 + \cdots + l_n}(s_1) \cdots g_{l_n}(s_1)$$

is positive, $G^n \neq 0$.

A subset $\mathscr{E} \subset \mathbb{Z}^+ \times_\alpha C_0(S)$ is said to be an *m-system* (generalized multiplicative system) if $F_1, F_2 \in \mathscr{E}$ implies $F_1 F_3 F_2 \in \mathscr{E}$ for some $F_3 \in \mathbb{Z}^+ \times_\alpha C_0(S)$. An element $F \in \mathbb{Z}^+ \times_\alpha C_0(S)$ is said to have the *zero property* if every m-system that contains F also contains 0. Now the prime radical R' is the set of all elements of $\mathbb{Z}^+ \times_\alpha C_0(S)$ that have the zero property [10, Exercise 1, p. 449]. Let G be as in the previous paragraph; consider the m-system $\{G^n : n = 1, 2, \ldots\}$. We showed it does not contain zero, and hence $G \notin R'$. Thus if $F = \sum_{n \geqslant 1} M_{z^n} f_n \in R'$, $\{s : f_n(s) \neq 0\} \subseteq S_0$, $n \geqslant 1$, so $R' \subseteq R$.

To prove the reverse inclusion, let $s_0 \in S_0$ and let U be an open relatively compact neighborhood of s_0 such that for every infinite subset $A \subset \mathbb{Z}^+$, $\bigcap_{k \in A} \phi^{-k}(U) = \varnothing$. Let $0 = k_0 < k_1 < k_2 < \cdots < k_n$ be any finite sequence such that $U_m = \bigcap_{j=0}^{m} \phi^{-k_j}(U)$ is nonempty, $m = 0, \ldots, N$, and such that $U_N \cap \phi^{-k}(U) = \varnothing$ for every $k > k_N$. Set $V = U_N = \bigcap_{j=0}^{N} \phi^{-k_j}(U)$; then V is an open relatively compact neighborhood of s_0 with $V \cap \phi^{-n}(V) = \varnothing$, $n \geqslant 1$. Let $\mathscr{N}$ be the linear subspace consisting of all elements of the form $F = \sum_{n \geqslant 1} M_{z^n} f_n \in \mathbb{Z}^+ \times_\alpha C_0(S)$ such that $\operatorname{supp}(f_n) \subseteq V \cup \phi^{-1} V \cup \cdots \cup \phi^{-(n-1)} V$. It is easy to verify that $\mathscr{N}$ is in fact an ideal in $\mathbb{Z}^+ \times_\alpha C_0(S)$. (Note that $\phi^{-1}(S_0) \subseteq S_0$.) Now $\mathscr{N}^2 = (0)$: let $F = \sum_{n \geqslant 1} M_{z^n} f_n$, $G = \sum_{n \geqslant 1} M_{z^n} g_n \in \mathscr{N}$. Then $FG = H$, where $H = \sum_{n \geqslant 2} M_{z^n} h_n$, and each h_n is the sum of terms of the form $(M_{z^k} f_k)(M_{z^l} g_l)$ with $k + l = n$. But this is the same as $M_{z^n} f_k \circ \phi^l g_l$, and

$$\operatorname{supp}(f_k \circ \phi^l) \subseteq \phi^{-l}\left(\bigcup_{j=0}^{k-1} \phi^{-j}(V)\right)$$
$$\subseteq \bigcup_{j=l}^{n-1} \phi^{-j}(V),$$

and

$$\operatorname{supp}(g_l) \subseteq \bigcup_{j=1}^{l-1} \phi^{-j}(V).$$

Thus $\operatorname{supp}(f_k \circ \phi^l) \cap \operatorname{supp}(g_l) = \varnothing$, $f_k \circ \phi^l g_l = 0$, $h_n = 0$, $n \geqslant 2$, and $H = 0$.

Since the prime radical contains all nilpotent ideals, $\mathscr{N} \subseteq R'$. Let $F = \sum_{n=1}^{p} M_{z^n} f_n \in R$ and assume $\operatorname{supp}(f_n)$ is compact and contained in S_0, $1 \leqslant n \leqslant p$. Since elements of this form are dense in R, the proof will be complete if we can show $F \in R'$. Let $K = \bigcup_{n=1}^{p} \operatorname{supp}(f_n)$, and for each $s \in K$ let V_s be an open relatively compact neighborhood of s, $V_s \in S_0$, such that $V_s \cap \phi^{-n}(V_s) = \varnothing$, $n \geqslant 1$. The existence of such a V_s was demonstrated above. Let $V_i \equiv V_{s_i}$, $1 \leqslant i \leqslant m$, be a finite subcover of K, and let $\{g_i\}_{i=1}^{m}$ be a partition of unity for K subordinate to the $\{V_i\}_{i=1}^{m}$: thus, $0 \leqslant g_i \leqslant 1$, $\operatorname{supp}(g_i) \subseteq V_i$, $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} g_i(s) = 1$, $s \in K$. Then $F = \sum_{i=1}^{m} F g_i = \sum_{i=1}^{m} (\sum_{n=1}^{p} M_{z^n} f_n g_i)$. Let $\mathscr{N}_i$ be the ideal in $\mathbb{Z}^+ \times_\alpha C_0(S)$ consisting of all $H = \sum_{n \geqslant 1} M_{z^n} h_n$, with h_n supported in $\bigcup_{j=0}^{n-1} \phi^{-j}(V_i)$. Then $\mathscr{N}_i$ is an ideal, and $F g_i \in \mathscr{N}_i$, $1 \leqslant i \leqslant n$. Thus $F = \sum_{i=1}^{m} F g_i \in \mathscr{N}_1 + \cdots + \mathscr{N}_m$. But $\mathscr{N}_1, \ldots, \mathscr{N}_m$ are contained in R', so $\mathscr{N}_1 + \cdots + \mathscr{N}_m \subseteq R'$, and finally $F \in R'$.

IV.5. In order to study the Jacobson radical of semi-crossed products we will need a couple of lemmas of a combinatorial nature. As before, we assume that $\phi: S \to S$ is continuous and proper and S is locally compact hausdorff.

SUBLEMMA. Let $U \subseteq S$ be a compact neighborhood such that for every syndetic set $A \subseteq \mathbb{Z}^+$, $\bigcap_{k \in A} \phi^{-k}(U) = \varnothing$. Then given a positive integer p there exists a positive integer N such that for every syndetic set $A = \{m_j\}_{j=0}^{\infty}$ with $0 \leqslant m_0 \leqslant p$, $1 \leqslant m_j - m_{j-1} \leqslant p, j \geqslant 1$, $\bigcap_{j=0}^{N} \phi^{-m_j}(U) = \varnothing$.

Proof. Let X_j denote $\{1, 2, \ldots, p\}$, $j = 1, 2, \ldots$. Notice there is a one to one correspondence between the collection of all syndetic sets $A = \{m_j\}_{j=0}^{\infty}$ with $m_0 = 0$, $1 \leqslant m_j - m_{j-1} \leqslant p$, $j \geqslant 1$, and elements of the product space $\prod_{j=1}^{\infty} X_j$; this is given by $A = \{m_j\}_{j=0}^{\infty} \to \{m_1, m_2 - m_1, \ldots, m_n - m_{n-1}, \ldots\}$. Since $\bigcap_{k \in A} \phi^{-k}(U) = 0$ and $\phi^{-k}(U)$ is compact, there is a finite subset of A for which the intersection is empty; thus define a function $e: \prod_{j=1}^{\infty} X_j \to \mathbb{Z}^+$ by $e(\{m_j - m_{j-1}\}_{j=1}^{\infty}) = \inf\{n \in \mathbb{Z}^+ : \bigcap_{j=0}^{n} \phi^{-m_j}(U) = \varnothing\}$. Now $e^{-1}(n)$ is either empty or else consists of certain sequences in $\prod_{j=0}^{\infty} X_j$ which are arbitrary beyond the nth coordinate. If $\prod_{j=1}^{\infty} X_j$ is equipped with the product topology, it follows that e is continuous, and hence has a maximum value, N.

Let A be syndetic, $A = \{m_j\}_{j=0}^{\infty}$, with $0 \leqslant m_0 \leqslant p$, $1 \leqslant m_j - m_{j-1} \leqslant p, j \geqslant 1$. Then $\bigcap_{j=0}^{N} \phi^{-j}(U) = \phi^{-m_0}(\bigcap_{j=0}^{N} \phi^{-(m_j - m_0)}(U)) = \varnothing$.

LEMMA. *Let m be a positive integer, U_j a compact neighborhood in S such that $\bigcap_{k \in A} \phi^{-k}(U_j) = \varnothing$ for every syndetic set $A \subseteq \mathbb{Z}^+$, $1 \leqslant j \leqslant m$. Set $T_j = \{1, 2, \dots, m\}$, $1, 2, \dots$. Then given a positive integer p there exists a positive integer N such that for all syndetic sets $A = \{m_j\}_{j=0}^{\infty}$, $0 \leqslant m_0 \leqslant p$, $1 \leqslant m_j - m_{j-1} \leqslant p$, $j \geqslant 1$, and all sequences $\{i_j\}_{j=0}^{\infty} \in \prod_{j=0}^{\infty} T_j$, it follows $\bigcap_{j=0}^{N} \phi^{-m_j}(U_{i_j}) = \varnothing$.*

Proof. The proof is by induction on m. Note that for $m = 1$, the statement of the Lemma reduces to that of the sublemma.

Now suppose $m > 1$ and that the conclusion of the Lemma holds for $m - 1$ in place of m; specifically, we assume there is an integer N' such that for all syndetic sets $A = \{m_j\}_{j=0}^{\infty}$, $0 \leqslant m_0 \leqslant p$, $1 \leqslant m_j - m_{j-1} \leqslant p, j \geqslant 1$, and all sequences $\{i_j\}_{j=0}^{\infty}$ with $1 \leqslant i_j \leqslant m - 1$, for all j, $\bigcap_{j=0}^{N'} \phi^{-m_j}(U_{i_j}) = \varnothing$.

Given $\{i_j\}_{j=0}^{\infty} \in \prod_{j=0}^{\infty} T_j$, either $B \equiv \{k: i_k = m\}$ is syndetic, or else B has arbitrarily large gaps. If B is syndetic and $A = \{m_j\}_{j=0}^{\infty}$ is any syndetic set, observe that $\{m_j: j \in B\}$ is also syndetic. Thus $\bigcap_{j=0}^{\infty} \phi^{-m_j}(U_{i_j}) \subseteq \bigcap_{j \in B} \phi^{-m_j}(U_m) = \varnothing$. Suppose on the other hand that B has arbitrarily large gaps, and that $0 \leqslant m_0 \leqslant p$, $1 \leqslant m_j - m_{j-1} \leqslant p, j \geqslant 1$. Then there exist integers $j_0, j_1 \geqslant 0, j_1 - j_0 \geqslant N'$, with $\{i_j\}_{j=j_0}^{j=j_1} \subseteq \{1, 2, \dots, m-1\}$. In that case

$$\bigcap_{j=0}^{\infty} \phi^{-m_j}(U_{i_j}) \subseteq \bigcap_{j=j_0}^{j=j_1} \phi^{-m_j}(U_{i_j}) \subseteq \phi^{-m_{j_0}}\left(\bigcap_{j=0}^{j_1-j_0} \phi^{-(-m_j-m_0)}(U_{i_j})\right) = \varnothing$$

by the induction hypothesis.

With the induction hypothesis still in force, we have shown that for any $\{i_j\}_{j=0}^{\infty} \in \prod_{j=0}^{\infty} T_j$ and any syndetic set $A = \{m_j\}_{j=0}^{\infty}$, $0 \leqslant m_0 \leqslant p$, $1 \leqslant m_j - m_{j-1} \leqslant p, j \geqslant 1$, that $\bigcap_{j=0}^{\infty} \phi^{-m_j}(U_{i_j}) = \varnothing$. Set $X_j = \{1, 2, \dots, p\}, j \geqslant 1$, and define a function d: $(\prod_{j=1}^{\infty} X_j) \times (\prod_{j=0}^{\infty} T_j) \to \mathbb{Z}^+$, by $d(\{m_j - m_{j-1}\}_{j=1}^{\infty}, \{i_j\}_{j=0}^{\infty}) = \inf\{n \in \mathbb{Z}^+ : \bigcap_{j=0}^{n} \phi^{-m_j}(U_{i_j}) = \varnothing\}$. For each n, either $d^{-1}(n)$ is empty, or else consists of certain pairs of sequences which are arbitrary beyond the nth coordinate. If the domain of d is equipped with the product topology, d is continuous, and hence it has a maximum value, N. This N has the desired properties.

IV.6. PROPOSITION. *Let S_{np} denote the nearly periodic points of S, and $S_0 = S \backslash S_{np}$. Set $R = \{F = \sum_{n \geqslant 1} M_{z^n} f_n \in \mathbb{Z}^+ \times_\alpha C_0(S)$: $\{s: f_n(s) \neq 0\} \subseteq S\}$. Then R is the Jacobson radical of $\mathbb{Z}^+ \times_\alpha C(S)$.*

Proof. Let R' be the Jacobson radical of $\mathbb{Z}^+ \times_\alpha C_0(S)$. We show first that $R \subseteq R'$. Let p be a positive integer, $f_n \in C_0(S)$ with $\mathrm{supp}(f_n)$ compact and contained in S_0, $1 \leqslant n \leqslant p$, and set $F = \sum_{n=1}^{p} M_{z^n} f_n$. Since elements of this form are dense in R, and since R' is closed, it will be enough to show $F \in R'$.

Let $K = \bigcup_{j=1}^{p} \operatorname{supp}(f_j)$; for each $s \in K$, let U_s be a compact neighborhood of s with the property that for every syndetic set $A \subset \mathbb{Z}^+$, $\bigcap_{k \in A} \phi^{-k}(U_s) = \varnothing$. Since the interiors $\{U_s^0\}_{s \in K}$ form an open cover of K, there is a finite subcover $\{U_i \equiv U_{s_i}\}_{i=1}^{m}$. Let $\{g_i\}_{i=1}^{m}$ be a partition of unity for K subordinate to $\{U_i\}_{i=1}^{m}$; i.e., $0 \leqslant g_i \leqslant 1$, $\operatorname{supp}(g_i) \subseteq U_i$, $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} g_i(s) = 1$, $s \in K$. If $n \geqslant 1$, expand

$$F^n = \left(\sum_{j=1}^{p} M_{zj} f_j \right)^n = \left(\sum_{j=1}^{p} \sum_{k=1}^{m} M_{zj} f_j g_k \right)^n = \sum_{k=n}^{np} M_{z^k} h_k,$$

where each h_k is a sum of terms of the form

$$\begin{gathered} (f_{j_0} g_{k_0}) \circ \phi^{m_0} (f_{j_1} g_{k_1}) \circ \phi^{m_1} \cdots (f_{j_{n-1}} g_{k_{n-1}}) \circ \phi^{m_{n-1}}, \\ j_0, j_1, \dots, j_{n-1} \in \{1, \dots, p\},\ k_0, k_1, \dots, k_{n-1} \in \{1, \dots, m\}, \end{gathered} \tag{$*$}$$

and $m_0 = 0$, $1 \leqslant m_j - m_{j-1} \leqslant p$, $j \geqslant 1$. Since $(*)$ is supported in $\bigcap_{j=0}^{n-1} \phi^{-m_j}(U_{k_j})$, it follows from the lemma that $(*)$ is zero for $n \geqslant N+1$ (here N is as in the Lemma). Hence $F^n = 0$, $n \geqslant N+1$, so $F \in R'$.

To show the reverse inclusion, let $F = \sum_{n \geqslant 1} M_{z^n} f_n \in R'$. By IV.2, $M_{z^n} f_n \in R'$, $n \geqslant 1$; so if $F \neq 0$ there is a $k_0 > 0$ such that $f_{k_0} \neq 0$, $M_{z^{k_0}} f_{k_0} \in R'$. We must show that $\{s : f_n(s) \neq 0\} \subseteq S_0$; suppose to the contrary there is a nearly periodic point s_0 such that $f_{k_0}(s_0) \neq 0$. Since $M_{z^{k_0}} \lambda f_{k_0} \bar{f}_{k_0} \in R'$ for any constant λ, we may suppose $f_{k_0} \geqslant 0$ and $f_{k_0}(s_0) = 1$. Set $U = \{s \in S : f_{k_0}(s) \geqslant \frac{1}{2}\}$. U is a compact neighborhood of s_0, so there is a syndetic set $A = \{m_j\}_{j=0}^{\infty}$, $m_0 = 0$, $1 \leqslant m_j - m_{j-1} \leqslant p$ (for some positive integer p) and a point $s_1 \in U$ such that $\{\phi^{m_j}(s_1) : j \in \mathbb{Z}^+\} \in U$. Writing f in place of f_{k_0}, observe $M_{zj} f \in R'$, $j \geqslant k_0$, and compute $(\sum_{j=k_0}^{k_0+p} M_{zj} f)^n = \sum_{j=nk_0}^{n(k_0+p)} M_{zj} h_j$, where each h_j is a sum of terms of the form $f f \circ \phi^{j_1} \cdots f \circ \phi^{j_{n-1}}$, where $k_0 \leqslant j_i - j_{i-1} \leqslant k_0 + p$, $1 \leqslant i \leqslant n-1$ $(j_0 = 0)$. Also, given any sequence $\{j_i\}_{i=0}^{p}$ satisfying $j_0 = 0$, $k_0 \leqslant j_i - j_{i-1} \leqslant k_0 + p$, $1 \leqslant i \leqslant p$, there is an h_j with $f f \circ \phi^{j_i} \cdots f \circ \phi^{j_{n-1}}$ as a summand. Since any sequence of p consecutive integers intersects A, there is at least one choice of $\{j_i\}_{i=1}^{p} \subset A$. Furthermore, since

$$\left\| \left(\sum_{j=k_0}^{k_0+p} M_{zj} f \right)^n \right\| \geqslant \|h_j\|, \qquad nk_0 \leqslant j \leqslant n(k_0+p),$$

and since $f \geqslant 0$,

$$\begin{aligned} \left\| \left(\sum_{j=k_0}^{k_0+p} M_{zj} f \right)^n \right\| &\geqslant \| f f \circ \phi^{j_1} \cdots f \circ \phi^{j_{n-1}} \| \\ &\geqslant f(s_1) f(\phi^{j_1}(s_1)) \cdots f(\phi^{j_{n-1}}(s_1)) \\ &\geqslant (\tfrac{1}{2})^n, \qquad \text{since } \phi^{j_i}(s_1) \in U, \end{aligned}$$

$0 \leqslant i \leqslant n-1$. Thus the spectral radius of $\sum_{j=k_0}^{k_0+p} M_{z^j} f$ is $\geqslant \frac{1}{2}$, contradicting that it belongs to the Jacobson radical. Thus $R' \subseteq R$, and hence $R' = R$.

IV.7. Finally, we characterize the strong radical of $\mathbb{Z}^+ \times_\alpha C_0(S)$. Let S_p denote the periodic points of S, and set $S_0 = S \backslash \bar{S}_p$. Set $R = \{F = \sum_{n \geqslant 1} M_{z^n} f_n \in \mathbb{Z}^+ \times_\alpha C_0(S)\colon \{s\colon f_n(s) \neq 0\} \subseteq S_0\}$.

PROPOSITION. *R is the strong radical of $\mathbb{Z}^+ \times_\alpha C_0(S)$.*

Proof. Let R' denote the strong radical of $\mathbb{Z}^+ \times_\alpha C_0(S)$, and let $F = \sum_{n \geqslant 1} M_{z^n} f_n \in R'$; then by IV.2 $M_{z^n} f_n \in R'$, $n \geqslant 1$. If $F \notin R$, then there is an integer k_0 and a periodic point s_0 such that $f_{k_0}(s_0) \neq 0$. If $T = \{s_0, s_1, ..., s_{m-1}\}$ is the orbit of s_0, the inclusion $q\colon T \to S$ yields a mapping $\tilde{q}\colon \mathbb{Z}^+ \times_\alpha C_0(S) \to \mathbb{Z}^+ \times_{\alpha_0} C(T)$ by II.12 (here $\alpha_0 = \alpha|_T$) for which $\tilde{q}(M_{z^{k_0}} f_{k_0}) \neq 0$. By the results of Section III, $\mathbb{Z}^+ \times_{\alpha_0} C(T)$ is strongly semisimple; if M is a maximal ideal of $\mathbb{Z}^+ \times_{\alpha_0} C(T)$, it follows from the fact that $\tilde{q}$ has dense range and the finite codimensionality of M that $\tilde{q}^{-1}(M)$ is a maximal modular ideal of $\mathbb{Z}^+ \times_\alpha C_0(S)$. Since $\tilde{q}(M_{z^{k_0}} f_{k_0}) \neq 0$, $M_{z^{k_0}} f_{k_0}$ is not in the strong radical R'. This contradiction shows that $R' \subseteq R$.

To prove the reverse inclusion, let $\mathscr{I} = \{F \in \mathbb{Z}^+ \times_\alpha C_0(S), F = \sum_{n \geqslant 0} M_{z^n} f_n\colon \{s\colon f_n(s) \neq 0\} \subseteq S_0\}$. Although S_0 may not be invariant under ϕ, it is true that $\phi^{-1}(S_0) \subseteq S_0$, so by II.14 $\mathscr{I}$ is an ideal of $\mathbb{Z}^+ \times_\alpha C_0(S)$. Thus there is a one to one correspondence between the maximal modular ideals of $\mathbb{Z}^+ \times_\alpha C_0(S)$ not containing $\mathscr{I}$ and those of $\mathscr{I}$. By the following Lemma (IV.8), the maximal modular ideals of $\mathscr{I}$ are all of the form $\ker \chi_s$, $s \in S_0$. Thus $R' \supseteq R$, and so $R' = R$.

IV.8. Recall that $\chi_s\colon \mathbb{Z}^+ \times_\alpha C_0(S) \to \mathbb{C}$ is the (continuous) homomorphism $\chi_s(\sum_{n \geqslant 0} M_{z^n} f_n) = f_0(s)$.

LEMMA. *With notation as in* IV.7, *the maximal modular ideals of $\mathscr{I}$ are of the form* $\ker \chi_s$, $s \in S_0$.

Proof. Let $M \subseteq \mathbb{Z}^+ \times_\alpha C_0(S)$ be a maximal modular ideal of $\mathscr{I}$, and assume $M \neq \ker \chi_s$, $s \in S_0$. Then for each $s \in S_0$ there is an element $F^{(s)} \in M$ such that $\chi_s(F^{(s)}) \neq 0$, and adjusting by a scalar we may assume that $\chi_s(F^{(s)}) = 1$. Given $\varepsilon > 0$ and a compact set $K \subseteq S_0$, let U_s be the open neighborhood of s consisting of all $s' \in S_0$ with $|\chi_{s'}(F^{(s)}) - 1| < \varepsilon$. The open cover $\{U_s\colon s \in K\}$ of K has a finite subcover, say, $\{U_j \equiv U_{s_j}\colon 1 \leqslant j \leqslant m\}$. Let $\{g_i\}_{j=1}^m$ be a partition of unity for K subordinate to the cover $\{U_j\}_{j=1}^m$; i.e., $0 \leqslant g_i \leqslant 1$, $\operatorname{supp}(g_i) \subseteq U_i$, $1 \leqslant i \leqslant m$, $\sum_{i=1}^\infty g_i(s) = 1$, $s \in K$, and $\sum_{i=1}^m g_i(s) \leqslant 1$, $s \in S_0 \backslash K$.

Set $F = \sum_{j=1}^m F^{(s_j)} g_j$, and $u \equiv u_{(K,\varepsilon)} = \sum_{j=1}^m g_j$; for $s \in S_0$ we estimate

$$\begin{aligned}|\chi_s(F)-u(s)| &= \left|\chi_s\left[\sum_{j=1}^{m}(F^{(s_j)}g_j-g_j)\right]\right| \\ &= \left|\sum_{j=1}^{m}(\chi_s(F^{s_j})-1)\,g_j(s)\right| \\ &\leqslant \sum_{j=1}^{m}|\chi_s(F^{s_j})-1|\,g_j(s)<\varepsilon.\end{aligned}$$

Let $F=\sum_{n\geqslant 0}M_{z^n}f_n$ and choose $F'=\sum_{n=0}^{N}M_{z^n}f'_n\in K(\mathbb{Z}^+,C_0(S),\alpha)$ such that $\|F-F'\|<\varepsilon$ and $\operatorname{supp}(f'_n)\subseteq S_0$. Momentarily fix $s_0\in K$ and let V be a compact neighborhood of s_0; given $k>0$ we may assume $s_0\notin\phi^{-k}(V)$. Otherwise $\phi^k(s_0)$ would belong to every neighborhood of s_0, so $s_0=\phi^k(s_0)$. But that contradicts the fact that S_0 has no periodic points. Thus we may assume that $s_0\notin\phi^{-k}(V)$, $1\leqslant k\leqslant N$. Since ϕ is proper, $\phi^{-k}(V)$ is compact, and so $V\backslash\phi^{-k}(V)$ is a neighborhood of s_0. Changing notation from the first paragraph, let U_{s_0} be a compact neighborhood of s_0, $U_{s_0}\subset\bigcap_{j=1}^{N}(V\backslash\phi^{-k}(V))$. Thus, $U_{s_0},\phi^{-1}(U_{s_0}),\dots,\phi^{-N}(U_{s_0})$ are pairwise disjoint. If we do this for each point $s\in K$, we obtain an open cover of K, from which we extract a finite subcover $\{U_j\}_{j=1}^m$. Let $\{g_j\}_{j=1}^m$ be a partition of unity for K subordinate to $\{U_j\}_{j=1}^m$; in particular, if $u\equiv\sum_{j=1}^m g_j$, $u(s)=1$, $s\in K$ and $0\leqslant u(s)\leqslant 1$, $s\in S$. Let $\{h_j\}_{j=1}^m\subset C_0(S)$ have properties

(a) $h_j(s)=(f'_0(s))^{-1}$, $s\in\operatorname{supp}(g_j)\cap K$;

(b) $\operatorname{supp}(h_j)\subset U_j$, $1\leqslant j\leqslant m$, and

$$\|h_j\|=\sup\{|f'_0(s)|^{-1}: s\in\operatorname{supp}(g_j)\cap K\}.$$

Define $G\in K(\mathbb{Z}^+,C_0(S),\alpha)$ by

$$\begin{aligned}G=\sum_{j=1}^{m}h_jF'g_j &= \sum_{n=0}^{N}M_{z^n}h_1\circ\phi^nf'_n\,g_1+\sum_{n=0}^{N}M_{z^n}h_2\circ\phi^nf'_n\,g_2 \\ &\quad+\cdots+\sum_{n=0}^{N}M_{z^n}h_m\circ\phi^nf'_n\,g_m.\end{aligned}$$

We claim $\psi_s(G)=1$, $s\in K$. (Here we write ψ_s in place of $M_z^s\times\tilde{\pi}_s$.) Notice that by the construction of h_j, g_j we have

$$\begin{aligned}&h_1(s)f'_0(s)\,g_1(s)+h_2(s)f'_0(s)\,g_2(s)+\cdots+h_m(s)f'_0(s)\,g_m(s)\\ &\quad=g_1(s)+g_2(s)+\cdots+g_m(s)=1,\end{aligned}$$

for all $s\in K$. Also, for $1\leqslant k\leqslant N$ and $s\in K$, $h_j\circ\phi^k(s)f'_k(s)\,g_j(s)=0$, $1\leqslant j\leqslant m$. For clearly if $s\notin U_j$, $h_j\circ\phi^k(s)f'_k(s)g_j(s)=0$. But if $s\in U_j$, $\phi^k(s)\notin U_j$, and since $\operatorname{supp}(h_j)\subseteq U_j$, $h_j(\phi^k(s))=0$. This proves the claim.

Now for any $s \in S$, $\psi_s(G) = \psi_s(\sum_{j=1}^m h_j F' g_i) = \sum_{j=1}^m h_j(s)\, \psi_s(F')\, g_j(s)$, so $\|\psi_s(G)\| \leqslant \sum_{j=1}^m \sup\{|f_0'(s')|^{-1}\colon s' \in K\}\ \|F'\|\ g_j(s) \leqslant (1-2\varepsilon)^{-1}\ (\|F\| + \varepsilon)$. We have used that $\|f_0 - f_0'\| < \varepsilon$ and $|f_0(s) - 1| < \varepsilon$, $s \in K$. Thus $\|G\| \leqslant (1-2\varepsilon)^{-1}\ (\|F\| + \varepsilon)$.

Let $H = \sum_{j=1}^m h_j F g_j$; since $F \in M$, $H \in M$. We estimate, for any $s \in S$,

$$\begin{aligned} \|\psi_s(H-G)\| &= \left\| \psi_s \left[\sum_{j=1}^m h_j (F - F')\, g_j \right] \right\| \\ &\leqslant \sum_{j=1}^m \|h_j\|\, \|F - F'\|\ g_j(s) \\ &< \frac{\varepsilon}{1-2\varepsilon}. \end{aligned}$$

Consequently, $\|H - G\| < \varepsilon/(1-2\varepsilon)$. But then M contains an approximate identity $\{H_{(K,\varepsilon)}\}$ for $\mathscr{T}$ so M is not a proper ideal in $\mathscr{T}$, contrary to our assumption.

IV.9. COROLLARY. *The strong structure space of* $\mathbb{Z}^+ \times_\alpha C_0(S)$ *is homeomorphic to S if and only if* (S, ϕ) *has no periodic points.*

Proof. If (S, ϕ) has no periodic points, then by IV.8 the strong structure space is in one to one correspondence with S under $s \to \ker \chi_s$. But it is clear that the hull-kernel topology on S is the same as the hull-kernel topology determined by $C_0(S)$, which is the topology of S.

If $s_0 \in S$ is a periodic point with (finite) orbit T, then the inclusion $q\colon T \to S$ yields by II.12 $\tilde{q}\colon \mathbb{Z}^+ \times_\alpha C_0(S) \to \mathbb{Z}^+ \times_{\alpha_0} (T)$. (Here $\phi_0 = \phi|_T$, and $\alpha_0(f) = f \circ \phi_0$, $f \in C(T)$.) This gives rise to a continuous injection of the strong structure space of $\mathbb{Z}^+ \times_{\alpha_0} (T)$ into the strong structure space of $\mathbb{Z}^+ \times_\alpha C_0(S)$. Since the strong structure space of $\mathbb{Z}^+ \times_{\alpha_0} (T)$ is nonhausdorff by III.5, it follows that the strong structure space of $\mathbb{Z}^+ \times_\alpha C_0(S)$ is nonhausdorff, and in particular not homeomorphic with S.

IV.10. EXAMPLE. Arveson–Josephson algebras. Let S be locally compact, ϕ a homeomorphism of S, and assume there exists a separable regular Borel probablity measure m on S satisfying

(i) (quasi-invariance) $m \circ \phi$ is mutually absolutely continuous with respect to m;

(ii) $m(U) > 0$ for every nonemepty open set U;

(iii) the set of periodic points has measure zero.

Define a unitary V on $L^2(S, m)$ by $Vg = ((dm \circ \phi))/dm)^{1/2}\, g \circ \phi$, $g \in L^2(S, \phi)$, where $m \circ \phi(E) = m(\phi(E))$. Let L_f be the multiplication operator $L_f g = fg$, $f \in C_0(S)$, $g \in L^2(S, m)$. The Arveson–Josephson algebra $\mathfrak{A}(S, \phi)$ is defined as the norm closure in $\mathscr{L}(L^2(S, \phi))$ of the algebra of all finite sums

$$L_{f_0} + L_{f_1} V + \cdots + L_{f_n} V^n, \qquad f_j \in C_0(S).$$

Since $VL_f V^{-1} = L_{f \circ \phi}$, $\mathfrak{A}(S, \phi)$ is also the norm closure of the algebra of all finite sums $L_{f_0} + VL_{f_1} + \cdots + V^n L_{f_n}$, $f_1 \in C_0(S)$. If $\alpha(f) = f \circ \phi$, $f \in C_0(S)$, then (L, V) is an isometric covariant representation of $(C_0(S), \alpha)$. By II.4, $\mathbb{Z}^+ \times_\alpha C_0(S)$ isomorphic with a nonselfadjoint subalgebra of the C^*-crossed product $\mathbb{Z} \times_\alpha C_0(S)$. It follows from [1, Sect. 5] that $\mathfrak{A}(S, \phi)$ is isomorphic with $\mathbb{Z}^+ \times_\alpha C_0(S)$. In particular, all the results we have obtained regarding the prime radical, radical, and strong radical of semi-crossed products pertain to the Arveson–Josephson algebras. The authors of [1] wondered if these algebras are semisimple, since they were concerned with the boundedness of the automorphisms. If ϕ acts on $S = \mathbb{Z}$ by translation, we see from IV.4 that $\mathbb{Z}^+ \times_\alpha C_0(\mathbb{Z}) = \mathfrak{A}(\mathbb{Z}, \phi)$ is equal to its prime radical. (Nevertheless, as is mentioned in [12], the automorphisms of this algebra are all bounded.) If we consider only those algebras $\mathfrak{A}(S, \phi)$ where the measure is invariant, as is done in most of [1], it follows easily from IV.4 that these algebras are semiprime. However, the invariance of the measure m does not seem to be enough to imply that (S, ϕ) is nearly periodic (even if m is ergodic), although we have not constructed a counterexample.

V.

V.1. Let S_i be a locally compact hausdorff and $\phi_i \colon S_i \to S_i$ continuous and proper ($i = 1, 2$). As in [1], we say (S_1, ϕ_1) is conjugate to (S_2, ϕ_2) if there is an equivariant homeomorphism $\Theta \colon (S_2, \phi_2) \to (S_1, \phi_1)$; that is, the diagram

$$\begin{tikzcd} S_2 \arrow[r, "\theta"] \arrow[d, "\phi_2"] & S_1 \arrow[d, "\phi_1"'] \\ S_2 \arrow[r, "\theta"] & S_1 \end{tikzcd}$$

commutes. It follows from II.12 that if (S_1, ϕ_1), (S_2, ϕ_2) are conjugate, then $\mathbb{Z}^+ \times_{\alpha_1} C_0(S_1)$, $\mathbb{Z}^+ \times_{\alpha_2} C_0(S_2)$ are isomorphic. In this section we consider the converse proposition. In Theorem 3.11 of [1] it is proved that if ϕ_i is a homeomorphism of S_i ($i = 1, 2$) and if there exist probability measures m_i on S_i satisfying 4.10(i), (ii), and (iii) ($i = 1, 2$) with m_2 invariant and ergodic, then $\mathfrak{A}(S_1, \phi_1)$, $\mathfrak{A}(S_2, \phi_2)$ isomorphic implies that (S_1, ϕ_1), (S_2, ϕ_2) are conjugate. We, on the other hand, will assume that S_i is compact and ϕ_i has no periodic points, but make no stringent assumptions about the existence of an invariant measure. In addition, ϕ_i is only assumed to be continuous and proper.

THEOREM. *Assume that S_i is compact hausdorff and that ϕ_i has no periodic points, $i=1,2$. Then $\mathbb{Z}^+ \times_{\alpha_1} C(S_1)$ isomorphic to $\mathbb{Z}^+ \times_{\alpha_2} C(S_2)$ implies that (S_1, ϕ_1) is conjugate to (S_2, ϕ_2).*

Proof. For convenience we write $\mathfrak{A}_i$ for $\mathbb{Z}^+ \times_{\alpha_i} C(S_i)$, $i=1,2$. Let $\psi\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ be an isomorphism (not necessarily continuous); then ψ maps the maximal modular ideals of $\mathfrak{A}_2$ bijectively onto those of $\mathfrak{A}_1$, $M \to \psi^{-1}(M)$; furthermore, this mapping is a homeomorphism for the respective hull-kernel topologies. Since by Corollary IV.9 the strong structure space of $\mathfrak{A}_i$ is S_i, $i=1,2$, we obtain a homeomorphism $\Theta\colon S_2 \to S_1$ defined by $\Theta(s)=s'$ if $\psi^{-1}(\ker \chi_s) = \ker \chi_{s'}$. Now the strong radical of $\mathfrak{A}_i$ is $M_z^i \mathfrak{A}_i$, and since ψ maps the strong radical onto the strong radical, $\psi(M_z^1 \mathfrak{A}_1) = M_z^2 \mathfrak{A}_2$. Let P^i be the projection $\mathfrak{A}_i \to C(S_i)$, $P^i(\sum_{n \geqslant 0} M_z^i f_n) = f_0$ $(f_n \in C(S_i))$, $i=1,2$. P^i is a norm one homomorphism (and in this setting can be identified with the canonical mapping of $\mathfrak{A}_i$ onto the quotient of $\mathfrak{A}_i$ modulo the strong radical of $\mathfrak{A}_i$). Define a mapping $\lambda\colon C(S_1) \to C(S_2)$ by $\lambda = P^2 \circ \psi|_{C(S_1)}$. We claim λ is a $*$-algebra isomorphism. Let $f \in C(S_1)$ and write $\psi(f) = \lambda(f) + F'$, $F' \in M_z^2 \mathfrak{A}_2$. Since for $s \in S_2$, $\chi_s(F') = 0$, we have $\chi_s(\lambda(f)) = \chi_s(\lambda(f) + F') = \chi_s(\psi(f)) = \chi_{\Theta(s)}(f)$. Thus, $\lambda(f)(s) = f(\Theta(s))$, or $\lambda(f) = f \circ \Theta$.

Notice that $\psi(M_{z^2}^1 \mathfrak{A}_1) = M_{z^2}^2 \mathfrak{A}_2$; for $\psi(M_{z^2}^1 \mathfrak{A}_1) = \psi(M_z^1)\, \psi(M_z^1 \mathfrak{A}_1) \subseteq (M_z^2 \mathfrak{A}_2)(M_z^2 \mathfrak{A}_2) \subseteq M_{z^2}^2 \mathfrak{A}$. Apply the same argument with ψ^{-1} in place of ψ to get the other containment. Let $\psi(M_z^1) = M_z^2 g + G$, $G \in M_{z^2}^2 \mathfrak{A}_2$; also let $\psi^{-1}(M_z^2) = M_z^1 h + H$, $H \in M_{z^2}^1 \mathfrak{A}_1$. Then $M_z^2 = \psi(\psi^{-1}(M_z^2)) = \psi(M_z^1 h + H) = \psi(M_z^1)\, \psi(h) + \psi(H) = (M_z^2 g + G)\;\; \psi(h) + \psi(H)$. Now modulo $M_{z^2}^2 \mathfrak{A}_2$, the right-hand side is $M_z^2 gh \circ \Theta$. It follows that $gh \circ \Theta = 1$, and in particular that g, h do not vanish at any point. Next let $f \in C(S_1)$ be arbitrary, and compute

$$\begin{aligned}
\psi(fM_z^1) + M_{z^2}^2 \mathfrak{A}_2 &= \psi(M_z^1 f \circ \phi_1) + M_{z^2}^2 \mathfrak{A}_2 \\
&= \psi(M_z^1)\; \psi(f \circ \phi_1) + M_{z^2}^2 \mathfrak{A}_2 \\
&= (M_z^2 h)(f \circ \phi_1 \circ \Theta) + M_{z^2}^2 \mathfrak{A}_2 \\
&= M_z^2 (hf \circ \phi_1 \circ \Theta) + M_{z^2}^2 \mathfrak{A}_2 .
\end{aligned}$$

Also,

$$\begin{aligned}
\psi(fM_z^1) + M_{z^2}^2 \mathfrak{A}_2 &= \psi(f)\, \psi(M_z^1) + M_{z^2}^2 \mathfrak{A}_2 \\
&= (f \circ \Theta)\, M_z^2 h + M_{z^2}^2 \mathfrak{A}_2 \\
&= M_z^2 (f \circ \Theta \circ \phi_2 h) + M_{z^2}^2 \mathfrak{A}_2 .
\end{aligned}$$

It follows that $f \circ \Theta \circ \phi_2 h = hf \circ \phi_1 \circ \Theta$; since h never vanishes, $f \circ \Theta \circ \phi_2 = f \circ \phi_1 \circ \Theta$. Finally, since $f \in C(S_1)$ was arbitrary,

$$\Theta \circ \phi_2 = \phi_1 \circ \Theta.$$

V.2. It would be interesting to know if the conclusion of V.1 carries over to the periodic case as well. One very weak conclusion can be gotten from III.5: suppose S_i is locally compact hausdorff, and $\phi_i\colon S_i \to S_i$ is a homomorphism such that (S_i, ϕ_i) has locally bounded order, $i = 1, 2$. Then $\mathbb{Z}^+ \times_{\alpha_1} C_0(S_1)$, $\mathbb{Z}^+ \times_{\alpha_2} C_0(S_2)$ isomorphic implies S_1, S_2 are homeomorphic. As in V.1, we note that the strong structure spaces are homeomorphic in their respective hull-kernel topologies, and hence the complete regularization of these spaces are homeomorphic. But by the description of the topology on the strong structure space of $\mathbb{Z}^+ \times_{\alpha_i} C_0(S_i)$ given in III.5, it follows that the complete regularization of this space is homeomorphic with S_i. (See [2] for a discussion of complete regularization.)

V.3. The results of Section IV suggest that it should be possible to express other ring-theoretic properties of semi-crossed products $\mathbb{Z}^+ \times_\alpha C_0(S)$ in terms of the dynamical system (S, ϕ). Also, it would be interesting to have a description of the primitive ideal space, at least in the semisimple case.

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