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REFLEXIVITY OF PAIRS OF SHIFTS

MAREK PTAK

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ABSTRACT. In the present paper the reflexivity of a WOT-closed algebra generated by certain pairs of commuting shifts, which are not necessarily doubly commuting, is proved.

In what follows $B(\mathbf{H})$ denotes the algebra of all (linear bounded) operators on a separable complex Hilbert space $\mathbf{H}$. Throughout the paper we simply write a subspace for a closed subspace of $\mathbf{H}$ and an algebra (of operators) on $\mathbf{H}$ for a subalgebra of $B(\mathbf{H})$ with the unit (the identity on $\mathbf{H}$). If $\mathcal{S}$ is a subset of $B(\mathbf{H})$ then $\mathcal{U}(\mathcal{S})$ stands for the WOT (Weak Operator Topology)-closed algebra generated by $\mathcal{S}$ while $\text{Lat } \mathcal{S}$ denotes the lattice of all invariant subspaces for all operators from $\mathcal{S}$. For operators $T_1, \dots, T_n$ on $\mathbf{H}$ we write in the sequel $\mathcal{U}(T_1, \dots, T_n)$, $\text{Lat}(T_1, \dots, T_n)$, respectively, instead of $\mathcal{U}(\{T_1, \dots, T_n\})$ and $\text{Lat}(\{T_1, \dots, T_n\})$. $\text{AlgLat } \mathcal{S}$ stands for the algebra of all operators on $\mathbf{H}$ which leave invariant all subspaces from $\text{Lat } \mathcal{S}$. An algebra $\mathcal{A}$ is called *reflexive* if and only if $\mathcal{A} = \text{AlgLat } \mathcal{A}$. A family $\mathcal{S} \subset B(\mathbf{H})$ is called *reflexive* if the algebra $\mathcal{U}(\mathcal{S})$ is reflexive. An operator $T \in B(\mathbf{H})$ is called *reflexive* if $\mathcal{S} = \{T\}$.

Deddens [1] proved the reflexivity of an isometry using the Wold decomposition and a model for a shift. If a pair of commuting isometries is considered then a Wold-type decomposition need not exist [7]. Another difficulty is a lack of any model for a pair of commuting shifts in a general situation. In [3] the reflexivity of a pair $\{V_1, V_2\}$ of *doubly commuting* isometries (i.e. V_1, V_2 commute and so do V_1, V_2^*) was proved. Thus a pair of doubly commuting shifts is also reflexive. In this paper we prove, using a model from [3], the reflexivity of certain pairs of shifts, which are not necessarily doubly commuting.

Let S denote a set of all pairs of nonnegative integers and G be a set of all pairs of integers. Following [3], we introduce some notations and definitions. Namely, a subset $X \subset G$ is called a *diagram* if $\phi \in X, s \in S$ imply $\phi + s \in X$. The set of all diagrams is denoted by $\mathbf{X}$. For $\phi \in G$ we define $E_\phi = \{X \in \mathbf{X}: \phi \in X\}$. It is obvious that $E_\phi \subset E_{\phi+s}$ ($\phi \in G, s \in S$). Let $\mathcal{B}$ be a σ -algebra

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generated by all E_ϕ ($\phi \in G$) and μ be a positive finite measure on $(\mathbf{X}, \mathcal{B})$.

Let us denote $\mathcal{H} = \oplus_{\phi \in G} H_\phi$ where $H_\phi = \mathbf{H}$ and consider a space $\mathbf{K}$ of all measurable functions $f: \mathbf{X} \rightarrow \mathcal{H}$ such that $\int \|f(X)\|^2 d\mu < \infty$ and $f(X) \in \oplus_{\phi \in G} H_\phi$ (we identify functions equal μ a.e.). $\mathbf{K}$ is a Hilbert space with an inner product $(f, g) = \int (f(X), g(X))_{\mathcal{H}} d\mu(X)$. We note that each element of $\mathbf{K}$ can be written as follows: $f = \sum_{\phi \in G} f_\phi(\cdot) e_\phi$, where $f_\phi: \mathbf{X} \rightarrow \mathbf{H}$, $f_\phi \in L^2(\mathbf{X}, \mathcal{B}, \mu, \mathbf{H})$ and $\sum_{\phi \in G} \|f_\phi\|^2 < \infty$; also, $f_\phi(X) = 0$ μ a.e. on $\mathbf{X} - E_\phi$, e_ϕ means that $f_\phi(X) e_\phi$ is an element of H_ϕ for $X \in \mathbf{X}$. Now, we can define operators T_s on $\mathbf{K}$, $s \in S$. If $f = \sum_{\phi \in G} f_\phi(\cdot) e_\phi$, then

$$(1) \quad T_s f = \sum_{\phi \in G} f_\phi(\cdot) e_{\phi+s}.$$

Since $\mathbf{X}$ is the set of all diagrams, $T_s f \in \mathbf{K}$ and, therefore, T_s are well-defined.

Our main result is the following:

Theorem. *A WOT-closed algebra generated by T_s , $s \in S$ is reflexive.*

We denote the above algebra by $\mathcal{A}$. Let us note that $\mathcal{A} = \mathcal{U}(T_{(1,0)}, T_{(0,1)})$, thus, the Theorem implies that the pair $(T_{(1,0)}, T_{(0,1)})$ is reflexive. Now we consider the following examples.

Example 1. If $\mu = \delta_S$ (the point mass at S), then the pair $(T_{(1,0)}, T_{(0,1)})$ is a model for a pair of doubly commuting shifts [Theorem 1, 7]. The reflexivity of such pairs was proved in [4] and [5].

Example 2. [Example 1, 7]. Let $X_0 = \{(i, j): i \geq 0 \text{ or } j \geq 0\}$. If $\mu = \delta_{X_0}$ then the pair $(T_{(1,0)}, T_{(0,1)})$ is not doubly commuting. The Theorem implies that such a pair is reflexive.

Gaspar and Suciu proved [Theorem 6, 2] that each isometric semi-group is an orthogonal sum of a unitary semi-group, a left evanescent one, and a modified unilateral translation (for definitions, see [2]). A consequence of our theorem is the following:

Corollary 3. *A modified unilateral translation on the semi-group of pairs of non-negative integers is reflexive.*

Proof. The T_s , $s \in S$, for a semi-group. If we consider the measure $\mu = \delta_{X_0}$ (as in Example 2), then T_s , $s \in S$, is a modified natural unilateral translation of the defect space $\mathbf{H}$ on the above semi-group S ([2]). A modified unilateral translation is unitary equivalent to a modified natural unilateral translation; thus, it is reflexive.

Let us start proving our theorem. The following remarks and lemmas are needed.

If $\phi \in G$, then we denote $S_\phi = \phi + S$ and $L_\phi = \{f \in \mathbf{K}: f(X) \in \oplus_{\psi \in S_\phi} H_\psi, \mu \text{ a.e., } f(X) = 0 \mu \text{ a.e. on } \mathbf{X} - E_\phi\}$.

Remark 4. $\mathbf{K} = \overline{\text{span}\{L_\phi : \phi \in G\}}$

Proof. If $f \in \mathbf{K}$, then $f = \sum_{\phi \in G} f_\phi(\cdot)e_\phi$, where $f_\phi(X) = 0$ μ a.e. on $\mathbf{X} - E_\phi$ and $f_\phi(X)e_\phi \in H_\phi \subset \oplus_{\psi \in S_\phi} H_\psi$ for $X \in \mathbf{X}$; thus, $f_\phi(X)e_\phi \in L_\phi$. So the proof is finished.

Lemma 5. The subspace L_ϕ is invariant for T_s , if $s \in S$, $\phi \in G$.

Proof. Let $f \in L_\phi$ and $f = \sum_{\psi \in S_\phi} f_\psi(\cdot)e_\psi$ then, since $s \geq 0$, $(T_s f)(X) = \sum_{\psi \in S_\phi} f_\psi(X)e_{\psi+s} \subset \oplus_{\psi \in S_\phi} H_\psi$, for $X \in \mathbf{X}$. We know that $f(X) = 0$ μ a.e. on $\mathbf{X} - E_\phi$, so, $f_\psi(X) = 0$ μ a.e. on $\mathbf{X} - E_\phi$. Thus, $(T_s f)(X) = 0$ μ a.e. on $\mathbf{X} - E_\phi$.

The subspace L_ϕ is unitary equivalent to a space $H^2(\Gamma^2) \otimes L^2(E_\phi, \mu|_{E_\phi}, \mathbf{H})$, where $H^2(\Gamma^2)$ is the Hardy space (Γ denotes a unit circle). Let U_ϕ define this unitary equivalence, i.e.,

$$\left(U_\phi \sum_{t \in S_\phi} f_t(\cdot)e_t \right) = \sum_{t \in S_\phi} z^{t-\phi} \otimes f_t(\cdot)$$

We have also:

$$\begin{aligned} U_\phi T_s \sum_{t \in S_\phi} f_t(\cdot)e_t &= U_\phi \sum_{t \in S_\phi} f_t(\cdot)e_{s+t} = \sum_{t \in S_\phi} z^{s+t-\phi} \otimes f_t(\cdot) \\ &= \sum_{t \in S_\phi} M_{z^s} z^{t-\phi} \otimes f_t(\cdot) = (M_{z^s} \otimes I_\phi) \sum_{t \in S_\phi} z^{t-\phi} \otimes f_t(\cdot) \\ &= (M_{z^s} \otimes I_\phi) U_\phi \sum_{t \in S_\phi} f_t(\cdot)e_t, \end{aligned}$$

where M_{z^s} denotes a multiplication operator by $z^s = z_1^{s_1} z_2^{s_2}$ (here z_1, z_2 denote independent variables and $s = (s_1, s_2)$) in the space $H^2(\Gamma^2)$ and I_ϕ denotes identity operator in $L^2(E_\phi, \mu|_{E_\phi}, \mathbf{H})$. Thus,

$$(2) \quad U_\phi T_s U_\phi^* = M_{z^s} \otimes I_\phi \quad \text{for } \phi \in G, s \in S.$$

Lemma 6. If $\phi \leq \psi$, $\phi, \psi \in G$, then $U_\psi T_{\psi-\phi}|_{L_\phi} = U_\phi$.

Proof. Let $f \in L_\phi$ and $f = \sum_{t \in S_\phi} f_t(\cdot)e_t$ then

$$\begin{aligned} U_\psi T_{\psi-\phi} f &= U_\psi \sum_{t \in S_\phi} f_t(\cdot)e_{t+\psi-\phi} = \sum_{t \in S_\phi} z^{t+\psi-\phi-\psi} \otimes f_t(\cdot) \\ &= \sum_{t \in S_\phi} z^{t-\phi} \otimes f_t(\cdot) = U_\phi \sum_{t \in S_\phi} f_t(\cdot)e_t = U_\phi f. \end{aligned}$$

Lemma 7. If $\phi \leq \psi$, $\phi, \psi \in G$, then

$$U_\phi U_\psi^*|_{H^2(\Gamma^2) \otimes L^2(E_\phi, \mu|_{E_\phi}, \mathbf{H})} = M_{z^{\psi-\phi}} \otimes I_\phi.$$

Proof. Let $f = \sum_{t \in S} z^t \otimes f_t(\cdot)$, where $f_t \in L^2(E_\phi, \mu|_{E_\phi}, \mathbf{H})$. Then

$$\begin{aligned} U_\phi U_\psi^* f &= U_\phi \sum_{t \in S} f_t(\cdot) e_{t+\psi} = \sum_{t \in S} z^{t+\psi-\phi} \otimes f_t(\cdot) \\ &= M_{z^{\psi-\phi}} \otimes I_\phi f. \end{aligned}$$

Lemma 8. *If $\text{Lat}\{T_s : s \in S\} \subset \text{Lat } A$, then $AT_s = T_s A$ for $s \in S$.*

Proof. Let $\phi \in G$. Then from Lemma 5, $L_\phi \in \text{Lat}\{T_s : s \in S\} \subset \text{Lat } A$, thus $\text{Lat}\{T_s|_{L_\phi} : s \in S\} \subset \text{Lat } A|_{L_\phi}$. Now, (2) implies $\text{Lat}\{M_{z^s} \otimes I_\phi : s \in S\} = \text{Lat}\{U_\phi T_s U_\phi^* : s \in S\} \subset \text{Lat } U_\phi A|_{L_\phi} U_\phi^*$. Since the algebra $\mathcal{U}(M_{z^s} : s \in S)$ is reflexive [Corollary 8.(1),4], [5], the algebra $\mathcal{U}(M_{z^s} : s \in S) \otimes \mathbb{C}I_\phi = \mathcal{U}(M_{z^s} \otimes I_\phi : s \in S)$ is also reflexive [Theorem 9.18, Corollary 9.19, 6]. Hence, $U_\phi A|_{L_\phi} U_\phi^* \in \mathcal{U}(M_{z^s} \otimes I_\phi : s \in S)$, so $A|_{L_\phi} \in \mathcal{U}(T_s|_{L_\phi} : s \in S)$, thus, $A|_{L_\phi} T_s|_{L_\phi} = T_s|_{L_\phi} A|_{L_\phi}$ and Remark 4 concludes the proof.

Now for $\phi \in G$ consider subspaces $M_\phi = \{f \in \mathbf{K} : f(X) = 0, \mu \text{ a.e. on } \mathbf{X} - E_\phi\}$. A sequence $\{\phi_n\} \subset G$ is called *strongly increasing* if and only if $\phi_n^{(i)} < \phi_{n+1}^{(i)}$, $i = 1, 2$ and $\phi_k^{(i)}$ are co-ordinates of ϕ_k .

Lemma 9. *If a sequence $\{\phi_n\} \subset G$ is strongly increasing then $M_{\phi_n} \subset M_{\phi_k}$ for $n < k$ and $\overline{\bigcup_{n \in \mathbb{N}} M_{\phi_n}} = \mathbf{K}$.*

Proof. The first part of the assertion is an instantaneous consequence of the inclusion $E_{\phi_k} \subset E_{\phi_n}$. To prove the other part, let $f = \sum_{\psi \in G} f_\psi(\cdot) e_\psi$ and $f_\psi(X) = 0$ for $X \notin E_\psi$. If $\psi \leq \phi_n$, then $E_\psi \subset E_{\phi_n}$. Thus, $\mathcal{X} - E_\psi \supset \mathbf{X} - E_{\phi_n}$; but, $f_\psi(X) = 0$ for $X \in \mathcal{X} - E_\psi$; so, $f_\psi(X) = 0$ for $X \in \mathbf{X} - E_{\phi_n}$. Hence, if $\psi \leq \phi_n$, then $f_\psi(\cdot) e_\psi \in M_{\phi_n}$. The sequence $\{\phi_n\}$ is strongly increasing, thus, for any $\psi \in G$ there is an n such that $\psi \leq \phi_n$.

Proof of the theorem. Let $A \in B(\mathbf{K})$ and $\text{Lat}\{T_s : s \in S\} \subset \text{Lat } A$. Lemma 5 shows that $L_\phi \in \text{Lat}\{T_s : s \in S\} \subset \text{Lat } A$ for $\phi \in G$. Thus, as in Lemma 8, $\text{Lat}\{M_{z^s} \otimes I_\phi : s \in S\} = \text{Lat}\{U_\phi T_s U_\phi^* : s \in S\} \subset \text{Lat } U_\phi A|_{L_\phi} U_\phi^*$. The algebra $\mathcal{U}(M_{z^s} \otimes I_\phi : s \in S)$ is reflexive [4,5], hence there is a sequence of operators $\hat{\sigma}_n^\phi \otimes I_\phi$ WOT-converging to $U_\phi A|_{L_\phi} U_\phi^*$, where $\hat{\sigma}_n^\phi$ are polynomials of the multiplication operators M_{z_1}, M_{z_2} in $H^2(\Gamma^2)$. Let us note that $\hat{\sigma}_n^\phi$ may be chosen jointly bounded (it will be needed in what follows). For example, $\hat{\sigma}_n^\phi$ may be chosen as in the proof of the Theorem in [5]. The bidisc fulfills assumptions of this Theorem, because it is starlike with C^2 -boundary. According to that proof, an estimation is the following:

$$\|\hat{\sigma}_n^\phi\| \leq \|A|_{L_\phi}\| + 1 \leq \|A\| + 1 \stackrel{df}{=} C.$$

The above convergence shows also that $U_\phi A|_{L_\phi} U_\phi^* = A_\phi \otimes I_\phi$, where A_ϕ is chosen adequately.

Remark 4 implies that there is ϕ_0 such that $L_{\phi_0} \neq \{0\}$ and we denote $\hat{\sigma}_n = \hat{\sigma}_n^{\phi_0}$. The sequence $\hat{\sigma}_n \otimes I_{\phi_0}$ WOT-converges to $U_{\phi_0} A|_{L_{\phi_0}}^0 U_{\phi_0}^* = A_{\phi_0} \otimes I_{\phi_0}$, thus the sequence $U_{\phi_0}^* (\hat{\sigma}_n \otimes I_{\phi_0}) U_{\phi_0}$ WOT-converges to $A|_{L_{\phi_0}}$. Let $\psi \geq \phi_0$. Then $T_{\psi-\phi_0} L_{\phi_0} \subset L_{\phi_0}$ (Lemma 5) and, in particular, $U_{\phi_0}^* (\hat{\sigma}_n \otimes I_{\phi_0}) U_{\phi_0}$ WOT-converges to $A|_{L_{\phi_0}}$ on $T_{\psi-\phi_0} L_{\phi_0}$. Also, $T_{\psi-\phi_0} L_{\phi_0} \subset L_{\psi}$ (Lemma 6); thus, $U_{\psi} U_{\phi_0}^* (\hat{\sigma}_n \otimes I_{\phi_0}) U_{\phi_0} U_{\psi}^*$ WOT-converges to $U_{\psi} A|_{L_{\psi}} U_{\psi}^*$ on $U_{\psi} T_{\psi-\phi_0} L_{\phi_0}$. By Lemma 6, $U_{\psi} T_{\psi-\phi_0} L_{\phi_0} = U_{\phi_0} L_{\phi_0} = H^2(\Gamma^2) \otimes L^2(E_{\phi_0}, \mu|_{E_{\phi_0}}, \mathbf{H})$. Lemma 7 implies that $U_{\phi_0} U_{\psi}^* = M_{z^{\psi-\phi_0}} \otimes I_{\phi_0}$ on $H^2(\Gamma^2) \otimes L^2(E_{\phi_0}, \mu|_{E_{\phi_0}}, H)$. Thus $U_{\phi_0} U_{\psi}^*$ commutes with $\hat{\sigma}_n \otimes I_{\phi_0}$. Hence, $\hat{\sigma}_n \otimes I_{\phi_0}$ WOT-converges to $U_{\psi} A|_{L_{\psi}} U_{\psi}^* = A_{\psi} \otimes I_{\psi}$ on $H^2(\Gamma^2) \otimes L^2(E_{\phi_0}, \mu|_{E_{\phi_0}}, \mathbf{H})$. Therefore, $\hat{\sigma}_n$ WOT-converges to A_{ψ} on $H^2(\Gamma^2)$ and so $\hat{\sigma}_n \otimes I_{\psi}$ WOT-converges to $A_{\psi} \otimes I_{\psi} = U_{\psi} A|_{L_{\psi}} U_{\psi}^*$ on $H^2(\Gamma^2) \otimes L^2(E_{\psi}, \mu|_{E_{\psi}}, \mathbf{H})$.

Let $\{\phi_m\} \subset G$ be a strongly increasing sequence with first element equal to ϕ_0 . For any m let us define a sequence of operators

$$(3) \quad \eta_n^m = U_{\phi_m} (\hat{\sigma}_n \otimes I_{\phi_m}) U_{\phi_m}^*.$$

Hence, η_n^m WOT-converges to $A|_{L_{\phi_m}}$ ($n \rightarrow \infty$) and we also have $\|\eta_n^m\| \leq C$. Let us extend η_n^m to the whole M_{ϕ_m} in the following way: we consider spaces $M_{\phi_m}^l = \{f \in M_{\phi_m} : f(X) \in \oplus_{\alpha \in S_l} H_{\alpha} \mu \text{ a.e.}\}$ for all nonnegative integers l and $\mathbf{l} = (l, l)$. It is easy to see that $M_{\phi_m} = \bigcup_l M_{\phi_m}^l$. We define

$$(4) \quad \eta_n^m f = T_{\mathbf{l}+\phi_m}^* \eta_n^m T_{\mathbf{l}+\phi_m} f$$

$f \in M_{\phi_m}^l$. Let us note that $T_{\mathbf{l}+\phi_m} f \in L_{\phi_m}$ for $f \in M_{\phi_m}^l$. Since $f(X) \in \oplus_{\alpha \in S_{-1}} H_{\alpha} \mu$ a.e., thus $(T_{\mathbf{l}+\phi_m} f)(X) \in \oplus_{\alpha \in S_{\phi_m}} H_{\alpha} \mu$ a.e. We also have $f \in M_{\phi_m}$; thus, $f(X) = 0 \mu$ a.e. on $\mathbf{X} - E_{\phi_m}$. If $(X) = \sum_{\alpha \in G} f_{\alpha}(X) e_{\alpha}$, then $f_{\alpha}(X) = 0 \mu$ a.e. on $\mathbf{X} - E_{\phi_m}$ and

$$(T_{\mathbf{l}+\phi_m} f)(X) = \sum_{\alpha \in G} f_{\alpha}(X) e_{\alpha+\mathbf{l}+\phi_m} = \sum_{\alpha \in G} f_{\alpha-\mathbf{l}-\phi_m}(X) e_{\alpha} = 0$$

μ a.e. on $\mathbf{X} - E_{\phi_m}$. Hence $T_{\mathbf{l}+\phi_m} f \in L_{\phi_m}$ and definition (4) is correct. Operators η_n^m commute with $T_S|_{L_{\phi_m}}$. Hence definition (4) gives well-defined extension (independent from (I)) of operators η_n^m to $\bigcup_l M_{\phi_m}^l$. There is also $\|\eta_n^m\| \leq C$ on L_{ϕ_m} ; thus, from (4), $\|\eta_n^m\| \leq C$ on $M_{\phi_m}^l$ for all l . Now, we can extend η_n^m to the whole M_{ϕ_m} with $\|\eta_n^m\| \leq C$.

The following lemma is needed:

Lemma 10. *If $m \leq k$, $n \in \mathbb{N}$, then $\eta_n^m = \eta_n^k|_{M_{\phi_m}}$.*

Proof. First we show that $\eta_n^m f = \eta_n^k f$ for $f \in L_{\phi_m}$.

$$\eta_n^k f = T_{\phi_k - \phi_m}^* \eta_n^k T_{\phi_k - \phi_m} f = T_{\phi_k - \phi_m}^* U_{\phi_k}^* (\hat{\sigma}_n \otimes I_{\phi_k}) U_{\phi_k} T_{\phi_k - \phi_m} f,$$

and by Lemma 6 we have

$$\eta_n^k f = U_{\phi_m}^* (\hat{\sigma}_n \otimes I_{\phi_k}) U_{\phi_m} f = U_{\phi_m}^* (\hat{\sigma}_n \otimes I_{\phi_m}) U_{\phi_m} f = \eta_n^m f,$$

since the range of U_{ϕ_m} is included in $H^2(\Gamma^2) \otimes L^2(E_{\phi_m}, \mu|_{E_{\phi_m}}, \mathbf{H})$. (4) implies that $\eta_n^m = \eta_n^k$ on $M_{\phi_m}^l$ for all l ; thus, the equality is satisfied on the whole M_{ϕ_m} , because η_n^m, η_n^k are bounded.

Lemma 10 implies that we can define operators $\cup_m \eta_n^m$ on $\cup_m M_{\phi_m}$ and $\|\cup_m \eta_n^m\| \leq C$. Thus, there is sequence of operators $\beta_n \stackrel{df}{=} \overline{\cup_m \eta_n^m}$ on $\mathbf{K} = \overline{\cup_m M_{\phi_m}}$ (Lemma 9) and $\|\beta_n\| \leq C$. Operators η_n^m on L_{ϕ_m} are polynomials in $T_{(1,0)}|_{L_{\phi_m}}, T_{(0,1)}|_{L_{\phi_m}}$, so, by extension (4), η_n^m on M_{ϕ_m} are also polynomials in $T_{(1,0)}|_{M_{\phi_m}}, T_{(0,1)}|_{M_{\phi_m}}$. Hence, Lemma 6 and properties of the closure imply that β_n are polynomials in $T_{(1,0)}, T_{(0,1)}$. Thus, to finish the proof we show that β_n WOT-converges to A on $\mathbf{K}$.

Let $f, g \in M_{\phi_m}^l$ for any l, m . Then commutativity of A and $T_s, s \in S$, (Lemma 8) implies

$$\begin{aligned} (\beta_n f, g) - (A f, g) &= (\eta_n^m f, g) - (A f, g) \\ &= (T_{l+\phi_m}^* \eta_n^m T_{l+\phi_m} f, g) - (T_{l+\phi_m}^* A T_{l+\phi_m} f, g) \\ &= (T_{l+\phi_m}^* (\eta_n^m - A) T_{l+\phi_m} f, g) \\ &= ((\eta_n^m - A) T_{l+\phi_m} f, T_{l+\phi_m} g). \end{aligned}$$

Since $T_{l+\phi_m} f, T_{l+\phi_m} g \in L_{\phi_m}$, and we have shown that η_n^m WOT-converges to A on L_{ϕ_m} , then

$$(\beta_n f, g) - (A f, g) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The operators β_n are jointly bounded; hence, β_n WOT-converges to A on M_{ϕ_m} for all m and also on $\mathbf{K}$ from Lemma 9.

Remark 11. The Theorem (with the same proof) holds true, when S is the set of all finite sequences of non-negative integers.

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