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Poisson Transforms on Some C^*-Algebras Generated by Isometries

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A noncommutative Poisson transform associated to a certain class of sequences of operators on Hilbert spaces, with property (P), is defined on some universal C^*-algebras (resp. nonselfadjoint algebras) generated by isometries. Its properties are described and used to study these universal algebras and their representations. As consequences, we obtain a functional calculus, isometric (resp. unitary) dilations, and commutant lifting theorem for the class of sequences of operators with property (P). Our "geometrical" approach leads also to new and elementary proofs as well as extensions of some classical results. © 1999 Academic Press

1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. Let $T\in B(\mathscr{H})$ be a contraction, i.e., $\|T\|\leqslant 1$ and denote $\Delta(T):=I_{\mathscr{H}}-TT^*$. It is easy to see that for each $0<r<1$,

$$\sum_{n=0}^{\infty} (rT)^n\, \Delta(rT)(rT^*)^n = I_{\mathscr{H}} \tag{1.1}$$

Let S be the unilateral shift on $l^2(\mathbf{C})$ and $\{e_i\}_{i=0}^{\infty}$ be the canonical basis in $l^2(\mathbf{C})$. Let $p(S, S^*)=\sum_{m,n\geqslant 0} a_{nm} S^m S^{*n}$ be any polynomial in $C^*(S)$, the C^*-algebra generated by S. Using (1.1), an easy computation on monomials of the form $S^m S^{*n}$ shows that for any $h, k\in\mathscr{H}$,

$$\langle p(rT, rT^*)\, h, k\rangle_{\mathscr{H}} = \langle (p(S, S^*)\otimes I_{\mathscr{H}})\, K(rT)\, h, K(rT)\, k\rangle_{l^2(\mathbf{C})\otimes\mathscr{H}}, \tag{1.2}$$

where

$$K(rT)\, h=\sum_{n=0}^{\infty} e_n\otimes \Delta(rT)^{1/2}\, (rT^*)^n\, h, \qquad h\in\mathscr{H}.$$

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According to the Cauchy–Schwartz inequality and the relation (1.1), we infer that

$$|\langle p(rT, rT^*)\, h, k\rangle| \leqslant \|p(S, S^*)\| \, \|h\| \, \|k\|, \qquad \text{for any} \quad h, k \in \mathcal{H}.$$

Taking $r \to 1$ we obtain

$$\|p(T, T^*)\| \leqslant \|p(S, S^*)\|.$$

In the particular case when p is any polynomial in one variable we obtain the classical von Neumann inequality [vN]

$$\|p(T)\| \leqslant \|p(S)\|$$

(see [Pi] for a nice survey and other proofs.)

In this paper we will extend the Poisson transform (1.2) to a more general setting. Let us consider the full Fock space $F^2(H_n) = \mathbf{C}1 \oplus \bigoplus_{m \geqslant 1} H_n^{\otimes m}$, where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, ..., e_n\}$ $(n \geqslant 1)$. Let $n_1, n_2, ..., n_k \geqslant 1$ be integers. For each $i = 1, 2, ..., k$ and $j = 1, 2, ..., n_i$ let us define the operator S_{ij} on the Hilbert space $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by

$$S_{ij} = \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_j \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

where S_j is the left creation operator with e_j $(j = 1, 2, ..., n_i)$ on the full Fock space $F^2(H_{n_i})$, i.e., $S_j \xi = e_j \otimes \xi$, $\xi \in F^2(H_{n_i})$.

Let $\mathrm{Alg}(I, \{S_{ij}\})$ be the smallest closed subalgebra generated by $\{S_{ij}\}$ and the identity, and let $C^*(\{S_{ij}\})$ be the C^*-algebra generated by $\{S_{ij}\}$. We will refer to $\mathrm{Alg}(I, \{S_{ij}\})$ as the noncommutative polydisc algebra. Note that when $n_1 = n_2 = \cdots = n_k = 1$ it is isomorphic to the polydisc algebra $A(\mathbf{D}^k)$ (see [R]). On the other hand, if $k = 1$ and $n_1 = n$ we obtain the noncommutative disc algebra $\mathcal{A}_n$ (see [Po2, Po4]).

A Cauchy transform on the noncommutative polydisc algebra $\mathrm{Alg}(I, \{S_{ij}\})$ is defined in Section 2.

In Section 3 we introduce a Poisson transform on $C^*(\{S_{ij}\})$ and describe some of its properties. A Poisson kernel $K_r(\{A_{ij}\})$ $(0 < r < 1)$ is associated to any sequence of operators $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ with property (P) (see Section 3 for the definition) such that the map

$$P_r(\{A_{ij}\}) \colon C^*(\{S_{ij}\}) \to B(\mathcal{H})$$

defined by

$$P_r(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})] = \langle f(\{S_{ij}\}, \{S_{ij}^*\})\, K_r(\{A_{ij}\}), K_r(\{A_{ij}\})\rangle,$$

for any $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$, has the following properties:

(i) $P_r(\{A_{ij}\})[I] = I_{\mathcal{H}}$;

(ii) $P_r(\{A_{ij}\})$ is linear and completely contractive;

(iii) $P_r(\{A_{ij}\})|_{\mathrm{Alg}(I, \{S_{ij}\})}$ is multiplicative.

The Poisson transform of $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$ at a point $\{A_{ij}\}$ will be defined by

$$P(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})] := \lim_{\substack{r \to 1 \\ r < 1}} P_r(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})]$$

(in the uniform topology).

Let us remark that all the results of this paper hold true if we allow $n_i = \infty$ for some $i \in \{1, ..., k\}$, in a slightly adapted version.

Using the Poisson transform, we show that a sequence $\{A_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i} \subset B(\mathcal{H})$ has property (P) if and only if there is a completely contractive linear map

$$\Phi\colon C^*(\{S_{ij}\}) \to B(\mathcal{H})$$

such that $\Phi(I) = I_{\mathcal{H}}$ and

$$\Phi(S_{i_1 j_1} \cdots S_{i_p j_p} S_{\alpha_1 \beta_1}^* \cdots S_{\alpha_q \beta_q}^*) = A_{i_1 j_1} \cdots A_{i_p j_p} A_{\alpha_1 \beta_1}^* \cdots A_{\alpha_q \beta_q}^*.$$

Thus, the Poisson transform provides a functional calculus for sequences of operators with property (P).

If $U \in \mathcal{U}(H_n)$, the group of unitaries on H_n, then there is an automorphism β_U on $C^*(S_1, ..., S_n)$ (see [BEGJ]), the extension of the Cuntz algebra $\mathcal{O}_n$ by compacts [Cu], canonically generated. Similarly, one can get "canonically generated" automorphisms on $C^*(\{S_{ij}\})$. In Section 4 we show that the Poisson transform on $C^*(\{S_{ij}\})$ "commutes" with the "canonically generated" automorphisms.

In Section 5, using the results from Section 3, Stinespring's theorem [S], and Arveson's extension theorem [A] (see also [Pau]), we obtain an isometric (resp., unitary) dilation theorem and commutant lifting theorem for sequences of operators with property (P).

In Section 6 we show that the set of all characters on $\mathrm{Alg}(I, \{S_{ij}\})$ is homeomorphic to $\overline{(\mathbf{C}^{n_1})}_1 \times \overline{(\mathbf{C}^{n_2})}_1 \times \cdots \times \overline{(\mathbf{C}^{n_k})}_1$, where $\overline{(\mathbf{C}^{n_i})}_1$ is the closed unit ball of $\mathbf{C}^{n_i}$ $(i = 1, 2, ..., k)$. This helps us decide when two noncommutative polydisc algebras are not Banach isomorphic. On the other hand, the first group of cohomology of $\mathrm{Alg}(I, \{S_{ij}\}$ with coefficients in $\mathbf{C}$ is calculated showing, in particular, that the noncommutative polydisc algebras are not amenable.

In Section 7 we present some classes of sequences of operators with property (P). Using the Poisson transform, we show that $\mathrm{Alg}(I, \{S_{ij}\})$ and $C^*(\{S_{ij}\})$ are universal algebras. More precisely, we show that $C^*(\{S_{ij}\})$ is $*$-isomorphic to a tensor product $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$, of Toeplitz algebras, and $\mathrm{Alg}(I, \{S_{ij}\})$ is completely isometrically isomorphic to $\mathcal{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathcal{A}_{n_k}$ (the minimal tensor product [Pau, p. 157]), where for each $i = 1, 2, ..., k$, $\mathcal{A}_{n_i}$ is the noncommutative disc algebra [Po4] on n_i generators. The internal characterization of the matrix norm on a universal algebra [B, BP] leads to factorization theorems. On the other hand, it is proved that there is an $*$-representation $\Phi: C^*(\{S_{ij}\}) \to \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$, where $\mathcal{O}_{n_i}$ is the Cuntz algebra on n_i generators. Let us remark that this result was obtained by Cuntz [Cu] (using different techniques) for $k = 1$.

Other consequences of the Poisson transform are presented in the last two sections of this paper. In Section 8 we consider the noncommutative Poisson transform on $C^*(S_1, ..., S_n)$, the extension of the Cuntz algebra $\mathcal{O}_n$ by compacts, associated to the unit ball of $B(\mathcal{H})^n$, i.e.,

$$(B(\mathcal{H})^n)_1 = \left\{(T_1, ..., T_n) \in B(\mathcal{H})^n : \sum_{i=1}^n T_i T_i^* \leqslant I_{\mathcal{H}}\right\}.$$

This provides a new proof for the noncommutative von Neumann inequality for $(B(\mathcal{H})^n)_1$ (see [vN, Po2, Po3, Po4]) as well as an isometric (resp. unitary) dilation theorems for sequences $(T_1, ..., T_n) \in (B(\mathcal{H})^n)_1$ (see also [F, Bu, Po1]).

The last section deals with sequences of commuting operators with property (P). In the commutative case, operator-valued Poisson kernels were considered in [Pau, CV, V]. However, our "geometrical" approach leads to extensions of some results obtained in [Pau, CV, V]. We consider a Poisson transform on $C^*(M_1, ..., M_n)$, the C^*-algebra generated by the canonical unilateral shifts on $H^2(\mathbf{D}^n)$, the Hardy space on polydisc. We obtain, in particular, the following consequences: commutative von Neumann inequality for the unit ball of $B(\mathcal{H})^n$ (see [D1, D2] for a different approach), universal algebra generated by n commuting isometries (see [SzF]), and Itô's theorem [I] for a commutative family of isometries.

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2. CAUCHY TRANSFORMS

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. In the following we fix $k \in \{1, 2, ...\}$. Let $n_1, n_2, ..., n_k \geqslant 1$ be integers and let $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ be a sequence of operators such that

$$A_{i1}A_{i1}^* + A_{i2}A_{i2}^* + \cdots + A_{in_i}A_{in_i}^* \leqslant I_{\mathcal{H}} \tag{2.1}$$

for each $i = 1, 2, ..., k$, and

$$A_{ij}A_{pq} = A_{pq}A_{ij} \tag{2.2}$$

if $i, p \in \{1, 2, ..., k\}$, $i \neq p$, and $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$.

Let $\mathbf{F}_n^+$ be the unital free semigroup on n generators: $s_1, ..., s_n$, and let e be the neutral element in $\mathbf{F}_n^+$. If $\sigma \in \mathbf{F}_n^+$ the length of σ is defined by

$$|\sigma| = \begin{cases} m; & \text{if} \quad \sigma = s_{i_1} \cdots s_{i_m} \\ 0; & \text{if} \quad \sigma = e. \end{cases}$$

For any $i = \{1, 2, ..., k\}$ and $\alpha_i = s_{j_1} \cdots s_{j_m} \in \mathbf{F}_{n_i}^+$ define $A_{i,\alpha_i} := A_{ij_1} \cdots A_{ij_m}$, and if $\alpha_i = e$ then $A_{i,e} := I_{\mathcal{H}}$.

Let us consider the full Fock space $F^2(H_n) = \mathbf{C}1 \oplus \bigoplus_{m \geqslant 1} H_n^{\otimes m}$, where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, ..., e_n\}$ $(n \geqslant 1)$ (see [E]). For each $j = 1, 2, ..., n$, $S_j \in B(F^2(H_n))$ is the left creation operator with e_j, i.e., $S_j\xi = e_j \otimes \xi$, $\xi \in F^2(H_n)$. For each $\alpha = s_{j_1} \cdots s_{j_m} \in \mathbf{F}_n^+$, $j_1, ..., j_m \in \{1, 2, ..., n\}$ define $e_\alpha := e_{j_1} \otimes \cdots \otimes e_{j_m}$ and $e_\alpha = 1$ if $\alpha = e \in \mathbf{F}_n^+$. It is easy to see that $\{e_\alpha\}_{\alpha \in \mathbf{F}_n^+}$ is an orthonormal basis for the full Fock space $F^2(H_n)$.

Let $n_1, n_2, ..., n_k \geqslant 1$ be some fixed integers. For each $i = 1, 2, ..., k$ and $j = 1, 2, ..., n_i$ let us define the operator S_{ij} on the Hilbert space $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by

$$S_{ij} = \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_j \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}} \tag{2.3}$$

where S_j is the left creation operator with e_j $(j = 1, 2, ..., n_i)$ on the full Fock space $F^2(H_{n_i})$.

Let $\operatorname{Alg}(I, \{S_{ij}\})$ be the smallest closed subalgebra generated by $\{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ and the identity, and let $C^*(\{S_{ij}\})$ be the C^*-algebra generated by $\{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$. Let $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ be a sequence of operators satisfying the relation (2.1). The Cauchy kernel associated to this sequence is a family $\{C_r(\{A_{ij}\})\}_{0 \leqslant r < 1}$ of operators

$$C_r(\{A_{ij}\}) \colon \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{H}$$

defined by

$$C_r(\{A_{ij}\})\,h = \sum_{\substack{i\in\{1,2,...,k\}\\ \beta_i\in\mathbf{F}_{n_i}^+}} e_{\beta_1}\otimes\cdots\otimes e_{\beta_k}\otimes(r^{|\beta_1|+\cdots+|\beta_k|}A_{1,\beta_1}^*\cdots A_{k,\beta_k}^* h),$$

for any $h\in\mathscr{H}$. Let $\mathbf{1} := \underbrace{1\otimes\cdots\otimes 1}_{k\text{-times}} \in F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$.

THEOREM 2.1. *Let $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}\subset B(\mathscr{H})$ be a sequence of operators satisfying the relation (2.1) and $p(\{S_{ij}\})$ be any polynomial in $\{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$. If $0\leqslant r<1$ then $C_r(\{A_{ij}\})$ is a bounded operator and*

$$\langle p(\{rA_{ij}\})\,h, k\rangle = \langle (p(\{S_{ij}\})\otimes I_{\mathscr{H}})(\mathbf{1}\otimes h), C_r(\{A_{ij}\})\,k\rangle, \tag{2.4}$$

for any $h, k\in\mathscr{H}$.

Proof. Since for each $i\in\{1,2,...,k\}$, $m\in\{1,2,...\}$,

$$\Bigg\|\sum_{\substack{\beta\in\mathbf{F}_{n_i}^+\\ |\beta_i|=m}} A_{i,\beta_i}A_{i,\beta_i}^*\Bigg\| \leqslant \Bigg\|\sum_{\substack{\beta\in\mathbf{F}_{n_i}^+\\ |\beta_i|=1}} A_{i,\beta_i}A_{i,\beta_i}^*\Bigg\|^m \leqslant 1,$$

we infer that

$$\begin{aligned}
&\|C_r(\{A_{ij}\})\,h\|^2\\
&\quad= \sum_{\substack{i\in\{1,2,...,k\}\\ \beta_i\in\mathbf{F}_{n_i}^+}} \|r^{|\beta_1|+\cdots+|\beta_k|}A_{1,\beta_1}^*\cdots A_{k,\beta_k}^* h\|^2\\
&\quad= \sum_{\substack{i\in\{1,2,...,k\}\\ \beta_i\in\mathbf{F}_{n_i}^+}} \sum_{m=0}^{\infty}\left(r^{2m}\sum_{\substack{\beta_1\in\mathbf{F}_{n_1}^+\\ |\beta_1|=m}} \|A_{1,\beta_1}^* r^{\sum_{p=2}^k|\beta_p|}A_{2,\beta_2}^*\cdots A_{k,\beta_k}^* h\|^2\right)\\
&\quad\leqslant \frac{1}{1-r^2}\sum_{\substack{i\in\{2,...,k\}\\ \beta_i\in\mathbf{F}_{n_i}^+}} \|r^{|\beta_2|+\cdots+|\beta_k|}A_{2,\beta_2}^*\cdots A_{k,\beta_k}^* h\|^2\\
&\quad\leqslant \frac{1}{(1-r^2)^k}\,\|h\|^2.
\end{aligned}$$

for any $h\in\mathscr{H}$. Therefore $C_r(\{A_{ij}\})$ is a bounded operator for each $0\leqslant r<1$.

It is enough to prove (2.4) for monomials of the form $q(\{S_{ij}\}) = S_{1,\alpha_1} \cdots S_{k,\alpha_k}$, where $\alpha_i \in \mathbf{F}_{n_i}^+$ $(i = 1, 2, ..., k)$. We have

$$\begin{aligned}
&\langle (S_{1,\alpha_1} \cdots S_{k,\alpha_k} \otimes I_{\mathcal{H}})(\mathbf{1} \otimes h), C_r(\{A_{ij}\})\, h' \rangle \\
&= \Big\langle e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes h, \sum_{\substack{i \in \{1, 2, ..., k\} \\ \beta_i \in \mathbf{F}_{n_i}^+}} e_{\beta_1} \otimes \cdots \otimes e_{\beta_k} \\
&\qquad \otimes (r^{|\beta_1| + \cdots + |\beta_k|} A_{1,\beta_1}^* \cdots A_{k,\beta_k}^* h') \Big\rangle \\
&= \langle h, r^{|\alpha_1| + \cdots + |\alpha_k|} A_{1,\alpha_1}^* \cdots A_{k,\alpha_k}^* h' \rangle \\
&= \langle q(\{rA_{ij}\})\, h, h' \rangle, \qquad \text{for any} \quad h, h' \in \mathcal{H}.
\end{aligned}$$

The proof is complete. ∎

Using the results from Section 3 one can easily extend the Cauchy transform (2.4) to $\mathrm{Alg}(I, \{S_{ij}\})$. Let us remark that in the particular case when $k = 1$, $n_1 = 1$, and $T \in B(\mathcal{H})$ such that $\|T\| \leqslant 1$, the relation (2.4) is equivalent to the following operator-valued Cauchy formula

$$p(rT) = \frac{1}{2\pi} \int_0^{2\pi} p(e^{it})(1 - re^{-it}T)^{-1}\, dt.$$

3. POISSON TRANSFORMS

In this section we introduce a noncommutative Poisson transform and we describe some of its properties. We keep the notation from the previous sections.

For a given sequence of operators $\{A_{ij}\}_{i = 1, 2, ..., k,\ j = 1, 2, ..., n_i} \subset B(\mathcal{H})$, let $\Delta(\{A_{ij}\}) \in B(\mathcal{H})$ be the selfadjoint operator defined by

$$\Delta(\{A_{ij}\}) = \sum_{\substack{i \in \{1, 2, ..., k\} \\ \alpha_i \in \mathbf{F}_{n_i}^+\ |\alpha_i| \leqslant 1}} (-1)^{|\alpha_1| + \cdots + |\alpha_k|} A_{1,\alpha_1} \cdots A_{k,\alpha_k} A_{k,\alpha_k}^* \cdots A_{1,\alpha_1}^*. \tag{3.1}$$

For each $0 \leqslant r < 1$ define $\Delta_r(\{A_{ij}\}) := \Delta(\{B_{ij}\})$ where $B_{ij} = rA_{ij}$. We say that a sequence $\{A_{ij}\}_{i = 1, 2, ..., k,\ j = 1, 2, ..., n_i} \subset B(\mathcal{H})$ has property (P) if the relations (2.1), (2.2) are satisfied, and there exists $0 \leqslant \rho < 1$ such that the operator $\Delta_r(\{A_{ij}\})$ is positive for any r, $0 \leqslant \rho < r < 1$.

An important role in our investigation is played by the following.

LEMMA 3.1. *If* $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathscr{H})$ *is a sequence of operators satisfying the relations* (2.1), (2.2) *and such that* $\Delta_r(\{A_{ij}\}) \geqslant 0$ $(0<r<1)$ *then*

$$\sum_{\substack{i\in\{1,2,\ldots,k\}\\ \beta_i\in\mathbf{F}^+_{n_i}}} r^{2(|\beta_1|+\cdots+|\beta_k|)} A_{1,\beta_1}\cdots A_{k,\beta_k}\,\Delta_r(\{A_{ij}\})\,A^*_{k,\beta_k}\cdots A^*_{1,\beta_1} = I_{\mathscr{H}}, \tag{3.2}$$

where the convergence is in the strong operator topology.

Proof. If $X\in B(\mathscr{H})$ is a positive operator and $0<r<1$, then for any $h\in\mathscr{H}$, we have

$$\begin{aligned}
&\sum_{\substack{i\in\{1,2,\ldots,k\}\\ \beta_i\in\mathbf{F}^+_{n_i}}} \langle r^{2(|\beta_1|+\cdots+|\beta_k|)} A_{1,\beta_1}\cdots A_{k,\beta_k} X A^*_{k,\beta_k}\cdots A^*_{1,\beta_1}h, h\rangle\\
&\qquad\leqslant \|X\| \sum_{\substack{i\in\{1,2,\ldots,k\}\\ \beta_i\in\mathbf{F}^+_{n_i}}} \|r^{|\beta_1|+\cdots+|\beta_k|} A^*_{1,\beta_1}\cdots A^*_{k,\beta_k}h\|^2\\
&\qquad\leqslant \frac{1}{(1-r^2)^k}\,\|X\|\,\|h\|^2.
\end{aligned}$$

Therefore, the sum in (3.2) converges to a positive operator. On the other hand, we are allowed to rearrange the sum. Since

$$\Delta_r(\{A_{ij}\}) = \sum_{\substack{i\in\{1,2,\ldots,k\}\\ \alpha_i\in\mathbf{F}^+_{n_i},\ |\alpha_i|\leqslant 1}} (-r^2)^{\sum_{p=1}^k |\alpha_p|} A_{1,\alpha_1}\cdots A_{k,\alpha_k} A^*_{k,\alpha_k}\cdots A^*_{1,\alpha_1}$$

we have

$$\begin{aligned}
&\sum_{\substack{i\in\{1,2,\ldots,k\}\\ \beta_i\in\mathbf{F}^+_{n_i}}} r^{2\sum_{p=1}^k|\beta_p|} A_{1,\beta_1}\cdots A_{k,\beta_k}\Delta_r(\{A_{ij}\})\,A^*_{k,\beta_k}\cdots A^*_{1,\beta_1}\\
&\qquad= \sum_{\substack{i\in\{1,2,\ldots,k\}\\ \delta_i\in\mathbf{F}^+_{n_i}}} \left(\sum_{(\alpha)\in A_{\delta_1,\ldots,\delta_k}} (-1)^{\sum_{p=1}^k|\alpha_p|}\right)\\
&\qquad\quad\times r^{2\sum_{p=1}^k|\delta_p|} A_{1,\delta_1}\cdots A_{k,\delta_k} A^*_{k,\delta_k}\cdots A^*_{1,\delta_1}
\end{aligned}$$

where the sum

$$\sum_{(\alpha)\in A_{\delta_1,\ldots,\delta_k}} (-1)^{|\alpha_1|+\cdots+|\alpha_k|}$$

is taken over all $(\alpha) := (\alpha_1, ..., \alpha_k)$ with $\alpha_i \in \mathbf{F}_{n_i}^+$ $(i = 1, 2, ..., k)$, $|\alpha_i| \leqslant 1$ such that $\beta_i \alpha_i = \delta_i$ for some $\beta_i \in \mathbf{F}_{n_i}^+$. It is easy to see that if $\delta_1 = \delta_2 = \cdots = \delta_k = e$ then

$$\sum_{(\alpha) \in \Lambda_{e, ..., e}} (-1)^{|\alpha_1| + \cdots + |\alpha_k|} = 1.$$

On the other hand, if $\delta_1, ..., \delta_k$ are such that $\delta_j \neq e$ for some $i \in \{1, 2, ..., k\}$, then

$$\sum_{(\alpha) \in \Lambda_{\delta_1, ..., \delta_k}} (-1)^{|\alpha_1| + \cdots + |\alpha_k|} = 0.$$

The proof is complete. ∎

Throughout this section we assume that $\{A_{ij}\}_{i=1, 2, ..., k, \, j=1, 2, ..., n_i} \subset B(\mathcal{H})$ is a sequence of operators with property (P), i.e., it satisfies the relations (2.1), (2.2), and $\Delta_r(\{A_{ij}\}) \geqslant 0$ for any r such that $0 \leqslant \rho < r < 1$.

Let $\tilde{\Delta}_r(\{A_{ij}\}) \in B(F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{H})$ be defined by

$$\tilde{\Delta}_r(\{A_{ij}\}) = \underbrace{I \otimes \cdots \otimes I}_{k \text{ times}} \otimes \Delta_r(\{A_{ij}\}).$$

LEMMA 3.2. *If* $\{A_{ij}\}_{i=1, 2, ..., k, \, j=1, 2, ..., n_i} \subset B(\mathcal{H})$ *is a sequence of operators with property* (P) *then the operator*

$$K_r(\{A_{ij}\}) \colon \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{H}$$

defined by

$$K_r(\{A_{ij}\}) = \tilde{\Delta}_r(\{A_{ij}\})^{1/2} \, C_r(\{A_{ij}\}) \tag{3.3}$$

is an isometry for each r, $0 \leqslant \delta < r < 1$.

Proof. Indeed, for any $h \in \mathcal{H}$,

$$\|\tilde{\Delta}_r(\{A_{ij}\})^{1/2} C_r(\{A_{ij}\}) h\|^2 = \langle C_r(\{A_{ij}^*\}) \tilde{\Delta}_r(\{A_{ij}\}) C_r(\{A_{ij}\}) h, h \rangle$$

$$= \Big\langle \sum_{\substack{i \in \{1, 2, ..., k\} \\ \beta_i \in \mathbf{F}_{n_i}^+}} r^{2(|\beta_1| + \cdots + |\beta_k|)} A_{1, \beta_1} \cdots A_{k, \beta_k} \times \Delta_r(\{A_{ij}\}) A_{k, \beta_k}^* \cdots A_{1, \beta_1}^* h, h \Big\rangle = \|h\|^2.$$

The last equality follows from Lemma 3.1. The proof is complete. ∎

The family of operators $\{K_r(\{A_{ij}\})\}_{0\leq\delta<r<1}$ is called the Poisson kernel associated to the sequence $\{A_{ij}\}_{i=1,2,...,k,\, j=1,2,...,n_i}\subset B(\mathcal{H})$ with property (P). For each r, $0\leq\delta<r<1$ consider the map

$$P_r(\{A_{ij}\}): C^*(\{S_{ij}\})\to B(\mathcal{H})$$

be defined by

$$P_r(\{A_{ij}\})[f]=K_r(\{A_{ij}\})^*\,(f\otimes I_{\mathcal{H}})\,K_r(\{A_{ij}\}), \tag{3.4}$$

for any $f\in C^*(\{S_{ij}\})$. Let us recall that for any $\alpha_i=s_{j_1}\cdots s_{j_m}\in\mathbf{F}^+_{n_i}$, S_{i,α_i} stands for the product $S_{ij_1}\cdots S_{ij_m}$ and if $\alpha_i=e$ then $S_{i,e}:=I$ (the identity operator on $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$).

According to the relation (2.3), any polynomial in $\{S_{ij}\}_{i=1,2,...,k,\, j=1,2,...,n_i}$, $\{S^*_{ij}\}_{i=1,2,...,k,\, j=1,2,...,n_i}$ has the form

$$p(\{S_{ij}\},\{S^*_{ij}\})=\sum_{\text{finite}} a_{(\alpha),(\beta)}S_{1,\alpha_1}\cdots S_{k,\alpha_k}S^*_{1,\beta_1}\cdots S^*_{k,\beta_k} \tag{3.5}$$

where $a_{(\alpha),(\beta)}\in\mathbf{C}$ and $\alpha_i,\beta_i\in\mathbf{F}^+_{n_i}$ $(i=1,2,...,k)$. If $\{B_{ij}\}_{i=1,2,...,k,\, j=1,2,...,n_i}\subset B(\mathcal{H})$ and $p(\{S_{ij}\},\{S^*_{ij}\})$ is given by (3.5), then

$$p(\{B_{ij}\},\{B^*_{ij}\}):=\sum_{\text{finite}} a_{(\alpha),(\beta)}B_{1,\alpha_1}\cdots B_{k,\alpha_k}B^*_{1,\beta_1}\cdots B^*_{k,\beta_k}.$$

Note that $p(\{(B_{ij}\},\{B^*_{ij}\})\in B(\mathcal{H})$.

THEOREM 3.3. *If* $\{A_{ij}\}_{i=1,2,...,k,\, j=1,2,...,n_i}\subset B(\mathcal{H})$ *is a sequence of operators with property* (*P*), *then* $\{P_r(\{A_{ij}\})\}_{0\leq\delta<r<1}$ *has the following properties*:

(i) $P_r(\{A_{ij}\})$ *is a completely contractive linear map.*

(ii) *If* $p(\{S_{ij}\},\{S^*_{ij}\})$ *is any polynomial in* $\{S_{ij}\},\{S^*_{ij}\}$, *and* $0\leq\delta<r<1$ *then*

$$p(\{rA_{ij}\},\{rA^*_{ij}\})=P_r(\{A_{ij}\})[p(\{S_{ij}\},\{S^*_{ij}\})]. \tag{3.6}$$

Proof. According to the definition (3.4) and Lemma 3.2, it is easy to see that $P_r(\{A_{ij}\})$ is a completely contractive linear map. It is enough to prove the relation (3.6) for monomials of the form

$$q(\{S_{ij}\},\{S^*_{ij}\}):=S_{1,\alpha_1}\cdots S_{k,\alpha_k}S^*_{1,\beta_1}\cdots S^*_{k,\beta_k},$$

where $\alpha_i, \beta_i \in \mathbf{F}_{n_i}^+$ $(i=1, 2, ..., k)$. For any $h \in \mathscr{H}$ we have

$$\begin{aligned}\langle P_r(\{A_{ij}\})[q(\{S_{ij}\}, \{S_{ij}^*\})]\, h, h\rangle \\ = \langle (S_{1,\alpha_1} \cdots S_{k,\alpha_k} S_{1,\beta_1}^* \cdots S_{k,\beta_k}^* \otimes I_{\mathscr{H}})\, K_r(\{A_{ij}\})\, h, K_r(\{A_{ij}\})\, h\rangle.\end{aligned}$$

On the other hand, according to (3.3)

$$\begin{aligned}K_r(\{A_{ij}\})\, h = \sum_{\substack{i \in \{1, 2, ..., k\} \\ \gamma_i \in \mathbf{F}_{n_i}^+}} e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_k} \\ \otimes (r^{\sum_{p=1}^k |\gamma_p|} \Delta_r(\{A_{ij}\})^{1/2} A_{1,\gamma_1}^* \cdots A_{k,\gamma_k}^* h)\end{aligned}$$

where for each $i \in \{1, 2, ..., k\}$, $\{e_{\gamma_i}\}_{\gamma_i \in \mathbf{F}_{n_i}^+}$ is the orthonormal basis for the full Fock space $F^2(H_{n_i})$ (see Section 2).

Therefore,

$$\begin{aligned}&\langle P_r(\{A_{ij}\})[q(\{S_{ij}\}, \{S_{ij}^*\})]\, h, h\rangle \\ &\quad = \langle (S_{1,\alpha_1} \cdots S_{k,\alpha_k} S_{1,\beta_1}^* \cdots S_{k,\beta_k}^* \otimes I_{\mathscr{H}})\, x, y\rangle \\ &\quad = \langle z, y\rangle\end{aligned}$$

where

$$\begin{aligned}x = &\sum_{\substack{i \in \{1, 2, ..., k\} \\ \delta_i \in \mathbf{F}_{n_i}^+}} e_{\beta_1\delta_1} \otimes \cdots \otimes e_{\beta_k\delta_k} \\ &\otimes (r^{\sum_{p=1}^k (|\beta_p| + |\delta_p|)} \Delta_r(\{A_{ij}\})^{1/2} A_{1,\beta_1\delta_1}^* \cdots A_{k,\beta_k\delta_k}^* h), \\ y = &K_r(\{A_{ij}\}) h,\end{aligned}$$

and

$$\begin{aligned}z = &\sum_{\substack{i \in \{1, 2, ..., k\} \\ \delta_i \in \mathbf{F}_{n_i}^+}} e_{\alpha_1\delta_1} \otimes \cdots \otimes e_{\alpha_k\delta_k} \\ &\otimes (r^{\sum_{p=1}^k (|\beta_p| + |\delta_p|)} \Delta_r(\{A_{ij}\})^{1/2} A_{1,\beta_1\delta_1}^* \cdots A_{k,\beta_k\delta_k}^* h).\end{aligned}$$

Moreover, $\langle z, y\rangle = \langle z, w\rangle$ where

$$\begin{aligned}w = &\sum_{\substack{i \in \{1, 2, ..., k\} \\ \delta_i \in \mathbf{F}_{n_i}^+}} e_{\alpha_1\delta_1} \otimes \cdots \otimes e_{\alpha_k\delta_k} \\ &\otimes (r^{\sum_{p=1}^k (|\alpha_p| + |\delta_p|)} \Delta_r(\{A_{ij}\})^{1/2} A_{1,\alpha_1\delta_1}^* \cdots A_{k,\alpha_k\delta_k}^* h).\end{aligned}$$

On the other hand we have

$$\langle z, w\rangle = r^{\sum_{p=1}^{k}(|\alpha_p|+|\beta_p|)} \sum_{\substack{i\in\{1,2,\dots,k\}\\ \delta_i\in\mathbf{F}_{n_i}^+}} r^{\sum_{p=1}^{k} 2|\delta_p|}\langle \xi, \eta\rangle \tag{3.7}$$

where

$$\xi = \Delta_r(\{A_{ij}\})^{1/2} A_{1,\beta_1\delta_1}^* \cdots A_{k,\beta_k\delta_k}^* h$$

and

$$\eta = \Delta_r(\{A_{ij}\})^{1/2} A_{1,\alpha_1\delta_1}^* \cdots A_{k,\alpha_k\delta_k}^* h.$$

Using (3.7) one can infer that

$$\langle z, w\rangle = r^{\sum_{p=1}^{k}(|\alpha_p|+|\beta_p|)}\langle A_{1,\beta_1}^* \cdots A_{k,\beta_k}^* h, \lambda\rangle$$

where

$$\lambda = \left(\sum_{\substack{i\in\{1,2,\dots,k\}\\ \delta_i\in\mathbf{F}_{n_i}^+}} r^{\sum_{p=1}^{k} 2|\delta_p|} A_{1,\delta_1}\cdots A_{k,\delta_k}\Delta_r(\{A_{ij}\})\, A_{1,\delta_1}^*\cdots A_{k,\delta_k}^*\right) \times A_{1,\alpha_1}^*\cdots A_{k,\alpha_k}^* h.$$

According to Lemma 3.1 we have $\lambda = A_{1,\alpha_1}^* \cdots A_{k,\alpha_k}^* h$. Thus, we infer that

$$\langle z, w\rangle = r^{\sum_{p=1}^{k}(|\alpha_p|+|\beta_p|)}\langle A_{1,\alpha_1}\cdots A_{k,\alpha_k} A_{1,\beta_1}^*\cdots A_{k,\beta_k}^* h, h\rangle.$$

All the above equalities show that

$$\langle P_r(\{A_{ij}\})[q(\{S_{ij}\}, \{S_{ij}^*\})]\, h, h\rangle = \langle q(\{rA_{ij}\}, \{rA_{ij}^*\})]\, h, h\rangle$$

for any $h\in\mathcal{H}$. This completes the proof. ∎

One can deduce the following extension of the von Neumann inequality [vN, Po2].

Corollary 3.4. *If* $\{A_{ij}\}_{i=1,2,\dots,k,\ j=1,2,\dots,n_i}\subset B(\mathcal{H})$ *is a sequence of operators with property* (P) *and* $p(\{S_{ij}\}, \{S_{ij}^*\})$ *is any polynomial in* $\{S_{ij}\}$, $\{S_{ij}^*\}$ *then*

$$\|p(\{A_{ij}\}, \{A_{ij}^*\})\| \leqslant \|p(\{S_{ij}\}, \{S_{ij}^*\})\|. \tag{3.8}$$

Proof. According to Theorem 3.3 we have

$$\|p(\{rA_{ij}\}, \{rA_{ij}^*\})\| \leqslant \|p(\{S_{ij}\}, \{S_{ij}^*\})\|.$$

Taking $r \to 1$ the result follows. ∎

For each $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$ let us define

$$f(\{A_{ij}\}, \{A_{ij}^*\}) := \lim_{k \to \infty} q_k(\{A_{ij}\}, \{A_{ij}^*\})$$

(in the uniform topology), where $q_k(\{S_{ij}\}, \{S_{ij}^*\})$ is any sequence of polynomials in $\{S_{ij}\}, \{S_{ij}^*\}$ such that $\|f(\{S_{ij}\}, \{S_{ij}^*\}) - q_k(\{S_{ij}\}, \{S_{ij}^*\})\| \to 0$ as $k \to \infty$. According to Corollary 3.4 it is easy to see that the operator $f(\{A_{ij}\}, \{A_{ij}^*\})$ is well defined.

COROLLARY 3.5. *If* $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ *is a sequence of operators with property* (P) *and* $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$ *then*

$$\|f(\{A_{ij}\}, \{A_{ij}^*\})\| \leqslant \|f(\{S_{ij}\}, \{S_{ij}^*\})\|. \tag{3.9}$$

COROLLARY 3.6. *If* $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ *is a sequence of operators with property* (P) *and* $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$ *then*

$$f(\{rA_{ij}\}, \{rA_{ij}^*\}) = P_r(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})] \tag{3.10}$$

for any r such that $0 \leqslant \delta < r < 1$.

Proof. Let $q_k(\{S_{ij}\}, \{S_{ij}^*\})$ be a sequence of polynomials in $\{S_{ij}\}, \{S_{ij}^*\}$ such that

$$\|f(\{S_{ij}\}, \{S_{ij}^*\}) - q_k(\{S_{ij}\}, \{S_{ij}^*\})\| \to 0$$

as $k \to \infty$. We have

$$f(\{rA_{ij}\}, \{rA_{ij}^*\}) = \lim_{k \to \infty} q_k(\{rA_{ij}\}, \{rA_{ij}^*\}) = \lim_{k \to \infty} P_r(\{A_{ij}\})[q_k].$$

Since $P_r(\{A_{ij}\})$: $C^*(\{S_{ij}\}) \to B(\mathcal{H})$ is bounded according to Theorem 3.3, we infer that

$$f(\{rA_{ij}\}, \{rA_{ij}^*\}) = P_r(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})]$$

for any $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$. This completes the proof. ∎

The Poisson transform of $f \in C^*(\{S_{ij}\})$ at $\{A_{ij}\}$ is defined by

$$P(\{A_{ij}\})[f] := \lim_{\substack{r \to 1 \\ r<1}} P_r(\{A_{ij}\})[f], \tag{3.11}$$

if the limit exists in the uniform topology.

THEOREM 3.7. *If* $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{H})$ *is a sequence of operators with property* (P) *then there exists the limit*

$$\lim_{\substack{r \to 1 \\ r<1}} P_r(\{A_{ij}\})[f]$$

in the uniform topology of $B(\mathcal{H})$ *for every* $f \in C^*(\{S_{ij}\})$.

Moreover,

$$f(\{A_{ij}\}, \{A_{ij}^*\}) = P(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})] \tag{3.12}$$

for any $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$.

Proof. Let $f(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$ and let $\varepsilon > 0$. There exists $q(\{S_{ij}\}, \{S_{ij}^*\})$, a polynomial in $\{S_{ij}\}, \{S_{ij}^*\}$, such that

$$\|f(\{S_{ij}\}, \{S_{ij}^*\}) - q(\{S_{ij}\}, \{S_{ij}^*\})\| < \frac{\varepsilon}{3}. \tag{3.13}$$

According to the von Neumann inequality (3.9) and the relation (3.13), we have

$$\|f(\{A_{ij}\}, \{A_{ij}^*\}) - q(\{A_{ij}\}, \{A_{ij}^*\})\| < \frac{\varepsilon}{3} \tag{3.14}$$

and

$$\|f(\{rA_{ij}\}, \{rA_{ij}^*\}) - q(\{rA_{ij}\}, \{rA_{ij}^*\})\| < \frac{\varepsilon}{3} \tag{3.15}$$

for any r, $0 \leqslant \delta < r < 1$.

On the other hand, there exists δ_0, $0 < \delta_0 < 1$ such that

$$\|q(\{rA_{ij}\}, \{rA_{ij}^*\}) - q(\{A_{ij}\}, \{A_{ij}^*\})\| < \frac{\varepsilon}{3} \tag{3.16}$$

for any r such that $\delta_0 < r < 1$.

Using Corollary 3.6 and the relations (3.14), (3.15), (3.16), we infer that

$$\begin{aligned}
&\|f(\{A_{ij}\}, \{A_{ij}^*\}) - P_r(\{A_{ij}\})[f(\{S_{ij}\}, \{S_{ij}^*\})]\| \\
&\quad = \|f(\{A_{ij}\}, \{A_{ij}^*\}) - f(\{rA_{ij}\}, \{rA_{ij}^*\})\| \\
&\quad \leqslant \|f(\{A_{ij}\}, \{A_{ij}^*\}) - q(\{A_{ij}\}, \{A_{ij}^*\})\| \\
&\qquad + \|q(\{A_{ij}\}, \{A_{ij}^*\}) - q(\{rA_{ij}\}, \{rA_{ij}^*\})\| \\
&\qquad + \|q(\{rA_{ij}\}, \{rA_{ij}^*\}) - f(\{rA_{ij}\}, \{rA_{ij}^*\})\| < \varepsilon
\end{aligned}$$

for any r such that $\max\{\delta, \delta_0\} < r < 1$. The proof is complete. ∎

THEOREM 3.8. *Let* $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathscr{H})$ *be any sequence of operators with property* (P). *Then the Poisson transform*

$$\Phi_{\{A_{ij}\}}: C^*(\{S_{ij}\}) \to B(\mathscr{H}); \qquad \Phi_{\{A_{ij}\}}(f) := \lim_{\substack{r\to 1\\ r<1}} P_r(\{A_{ij}\})[f] \tag{3.17}$$

has the following properties:

(i) $\Phi_{\{A_{ij}\}}$ *is a completely contractive linear map*;

(ii) *for every polynomial* $p(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$,

$$\Phi_{\{A_{ij}\}}(p(\{S_{ij}\}, \{S_{ij}^*\})) = p(\{A_{ij}\}, \{A_{ij}^*\});$$

(iii) $\Phi_{\{A_{ij}\}}|_{\mathrm{Alg}(I, \{S_{ij}\})}$ *is multiplicative.*

Proof. According to Theorem 3.3, for every matrix $[f_{pq}]_{p,q=1}^n \in M_n(C^*(\{S_{ij}\}))$ we have

$$\|[P_r(\{A_{ij}\})[f_{pq}]]_{p,q=1}^n\| \leqslant \|[f_{pq}]_{p,q=1}^n\|. \tag{3.18}$$

On the other hand, Theorem 3.7 shows that $\Phi_{\{A_{ij}\}}$ is well-defined by relation (3.17). The inequality (3.18) together with the relation (3.17) shows that

$$\|[\Phi_{\{A_{ij}\}}(f_{pq})]_{p,q=1}^n\| \leqslant \|[f_{pq}]_{p,q=1}^n\|,$$

for any $[f_{pq}]_{p,q=1}^n \in M_n(C^*(\{S_{ij}\}))$. This proves part (i) of the theorem.

Part (ii) follows from Theorem 3.3 part (ii) by taking $r \to 1$. Now, it is easy to see that part (iii) of this theorem is a consequence of (i) and (ii). This completes the proof. ∎

THEOREM 3.9. *A sequence of operators* $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathscr{H})$ *has property* (P) *if and only if the map*

$$\Phi: C^*(\{S_{ij}\}) \to B(\mathscr{H}),$$

defined by

$$\Phi(p(\{S_{ij}\}, \{S_{ij}^*\})) = p(\{A_{ij}\}, \{A_{ij}^*\}) \tag{3.19}$$

for any $p(\{S_{ij}\}, \{S_{ij}^*\}) \in C^*(\{S_{ij}\})$, *is a completely contractive linear map.*

Proof. The direct implication follows from Theorem 3.8. Assume now that the map Φ defined by (3.19) is completely contractive. Since Φ is completely positive and the sequence $\{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ has property (P) (see Lemma 7.1 for a more general case), it is easy to see that the sequences $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ has also property (P). The proof is complete. ∎

4. AN INVARIANCE PROPERTY OF THE POISSON TRANSFORM

In what follows we show that the Poisson transform has an invariance property. Each k-tuple $U = (U_1, ..., U_k)$ such that $U_i \in \mathcal{U}(H_{n_i})$, the group of unitaries on $H_{n_i} (i = 1, 2, ..., k)$, generates a canonical automorphism of $C^*(\{S_{ij}\})$ defined by

$$\beta_U(S_{ij}) := \sum_{p=1}^{n_i} \lambda_{pj}^{(i)} S_{ip}, \qquad i = 1, 2, ..., k; \quad j = 1, 2, ..., n_i \tag{4.1}$$

for $U_i = [\lambda_{pq}^{(i)}]_{p,q=1}^{n_i} \in \mathcal{U}(H_{n_i})$ (see [BEGJ] for the case $k = 1$).

On the other hand, if $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ it makes sense to consider

$$\beta_U(A_{ij}) := \sum_{p=1}^{n_i} \lambda_{pj}^{(i)} A_{ip}. \tag{4.2}$$

Lemma 4.1. *If* $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ *is a sequence of operators with property* (P) *then* $\{\beta_U(A_{ij})\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ *has property* (P).

Proof. Let $U_i = [\lambda_{pq}^{(i)}]_{p,q=1}^{n_i} \in \mathcal{U}(H_{n_i})$, $i = 1, 2, ..., k$. For each $i = 1, 2, ..., k$ we have

$$\begin{aligned}\sum_{j=1}^{n_i} \beta_U(A_{ij}) \beta_U(A_{ij})^* &= \sum_{p,q=1}^{n_i} \left(\sum_{j=1}^{n_i} \lambda_{pj}^{(i)} \overline{\lambda_{qj}^{(i)}} \right) A_{ip} A_{iq}^* \\ &= \sum_{p=1}^{n_i} A_{ip} A_{ip}^* \leqslant I_{\mathcal{H}}.\end{aligned}$$

On the other hand, according to the relations (2.2) and (4.2), one can see that

$$\beta_U(A_{ij})\,\beta_U(A_{rs})=\beta_U(A_{rs})\,\beta_U(A_{ij}) \tag{4.3}$$

for any $i, r\in\{1, 2, ..., k\}$, $i\neq r$ and $j\in\{1, 2, ..., n_i\}$, $s\in\{1, 2, ..., n_r\}$.

Let us show that $\Delta(\{A_{ij}\})=\Delta(\{\beta_U(A_{ij})\})$. Denote $B_{ij}:=\beta_U(A_{ij})$. We have

$$\Delta(\{B_{ij}\})=\sum_{\substack{i\in\{1,2,...,k-1\}\\ \alpha_i\in\mathbf{F}^+_{n_i},\ |\alpha_i|\leqslant 1}}(-1)^{\sum_{p=1}^{k-1}|\alpha_p|}\times B_{1,\alpha_1}\cdots B_{k-1,\alpha_{k-1}}XB^*_{1,\alpha_1}\cdots B^*_{k-1,\alpha_{k-1}}$$

where

$$X=\sum_{\substack{\alpha_k\in\mathbf{F}^+_{n_k}\\ |\alpha_k|\leqslant 1}}(-1)^{|\alpha_k|}B_{k,\alpha_k}B^*_{k,\alpha_k}=\sum_{\substack{\alpha_k\in\mathbf{F}^+_{n_k}\\ |\alpha_k|\leqslant 1}}(-1)^{|\alpha_k|}A_{k,\alpha_k}A^*_{k,\alpha_k}.$$

Using (4.3) we obtain

$$\Delta(\{B_{ij}\})=\sum_{\substack{i\in\{1,2,...,k\}\\ \alpha_i\in\mathbf{F}^+_{n_i},\ |\alpha_i|\leqslant 1}}(-1)^{\sum_{p=1}^{k}|\alpha_p|}A_{k,\alpha_k}YY^*A^*_{k,\alpha_k}$$

where $Y=B_{1,\alpha_1}\cdots B_{k-1,\alpha_{k-1}}$. Repeating the above argument one can see that

$$\Delta(\{B_{ij}\})=\Delta(\{A_{ij}\}).$$

Therefore the sequence $\{\beta_U(A_{ij})\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ has property (P). This completes the proof. ∎

The next result establishes the invariance of the Poisson transform under the canonical automorphism of $C^*(\{S_{ij}\})$, defined by (4.1).

THEOREM 4.2. *Let* $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}\subset B(\mathcal{H})$ *be a sequence of operators with property* (P). *Then for any* $f\in C^*(\{S_{ij}\})$ *we have*

$$P(\{A_{ij}\})[\beta_U(f)]=P(\{\beta_U(A_{ij})\})[f] \tag{4.4}$$

where β_U *is any canonical automorphism of* $C^*(\{S_{ij}\})$.

Proof. It is enough to prove (4.4) for monomials of the form

$$q = S_{1,\alpha_1} \cdots S_{k,\alpha_k} S^*_{1,\beta_1} \cdots S^*_{k,\beta_k},$$

where $\alpha_i, \beta_i \in \mathbf{F}^+_{n_i}$ $(i = 1, 2, ..., k)$.

Let us denote $B_{ij} := \beta_U(A_{ij})$. According to Lemma 4.1, the sequence $\{B_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ has property (P). Using Theorem 3.7, we infer that

$$P(\{B_{ij}\})[q] = B_{1,\alpha_1} \cdots B_{k,\alpha_k} B^*_{1,\beta_1} \cdots B^*_{k,\beta_k}.$$

On the other hand, we have

$$\begin{aligned} &P(\{A_{ij}\})[\beta_U(q)] \\ &\quad = P(\{A_{ij}\})[\beta_U(S_{1,\alpha_1}) \cdots \beta_U(S_{k,\alpha_k})\, \beta_U(S^*_{1,\beta_1}) \cdots \beta_U(S^*_{k,\beta_k})] \\ &\quad = \beta_U(A_{1,\alpha_1}) \cdots \beta_U(A_{k,\alpha_k})\, \beta_U(A^*_{1,\beta_1}) \cdots \beta_U(A^*_{k,\beta_k}) \\ &\quad = B_{1,\alpha_1} \cdots B_{k,\alpha_k} B^*_{1,\beta_1} \cdots B^*_{k,\beta_k}. \end{aligned}$$

Therefore,

$$P(\{B_{ij}\})[q] = P(\{A_{ij}\})[\beta_U(q)].$$

This completes the proof. ∎

5. JOINT DILATIONS FOR SEQUENCES OF OPERATORS WITH PROPERTY (P)

Using Theorem 3.8 and Stinespring's theorem [S], one can obtain the following dilation theorem.

THEOREM 5.1. *Let $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ be any sequence of operators with property* (P) *and let $\Phi_{\{A_{ij}\}}$ be the Poisson transform associated to $\{A_{ij}\}$. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unital $*$-homomorphism $\pi\colon C^*(\{S_{ij}\}) \to B(\mathcal{K})$ such that*

$$\Phi_{\{A_{ij}\}}(f) = P_{\mathcal{H}}\pi(f)|_{\mathcal{H}}, \qquad f \in C^*(\{S_{ij}\}).$$

Let us remark that one can choose $\mathcal{K} = \bigvee_{f \in C^*(\{S_{ij}\})} \pi(f)\mathcal{H}$ in order to get a minimal Stinespring representation, which is unique up to an isomorphism.

Consider $V_{ij} := \pi(S_{ij})$. The sequence $\{V_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{K})$ has property (P) and is called the minimal isometric dilation of $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{H})$.

Let us remark that since

$$P_{\mathcal{H}} V_{ij} V_{ij}^* |_{\mathcal{H}} = P_{\mathcal{H}} V_{ij} |_{\mathcal{H}} P_{\mathcal{H}} V_{ij}^* |_{\mathcal{H}},$$

$\mathcal{H}$ is invariant subspace for each V_{ij}^* (see [P]). Therefore $A_{ij}^* = V_{ij}^* |_{\mathcal{H}}$ for any $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_i$.

Summing up we obtain the following isometric dilation theorem.

COROLLARY 5.2. *Let $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{H})$ be a sequence of operators with property* (P). *Then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a sequence of isometries $\{V_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{K})$, with property* (P) *such that*

$$A_{ij}^* = V_{ij}^* |_{\mathcal{H}} \qquad \textit{for any} \quad i = 1, 2, \ldots, k; \quad j = 1, 2, \ldots, n_i,$$

and $\mathcal{K} = \bigvee V_{i_1 j_1} \cdots V_{i_m j_m} \mathcal{H}$. Moreover, the isometric dilation $\{V_{ij}\}$ is uniquely determined up to an isomorphism.

Let us remark that the isometric dilation $\{V_{ij}\}$ has also the property that

$$V_{ij} V_{pq}^* = V_{pq}^* V_{ij}$$

if $i, p \in \{1, 2, \ldots, k\}$, $i \neq p$, and $j \in \{1, 2, \ldots, n_i\}$, $q \in \{1, 2, \ldots, n_p\}$.

We can apply [A, Theorem 1.3.1] to our setting in order to get the following commutant lifting theorem for $C^*(\{A_{ij}\})$.

COROLLARY 5.3. *Let $\{A_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{H})$ be a sequence with property* (P) *and let $\{V_{ij}\}_{i=1,2,\ldots,k,\ j=1,2,\ldots,n_i} \subset B(\mathcal{K})$ be its minimal isometric dilation. If $X \in C^*(\{A_{ij}\})'$ then there is a unique $\tilde{X} \in C^*\{V_{ij}\}' \cap \{P_{\mathcal{H}}\}'$ such that $P_{\mathcal{H}} \tilde{X}|_{\mathcal{H}} = X$, where $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. Moreover, the map $X \to \tilde{X}$ is a $*$-isomorphism.*

Let $\mathbf{F}_n$ be the free group on n-generators $s_1, \ldots, s_n$, and the Hilbert space

$$\ell^2(\mathbf{F}_n) := \left\{ f\colon \mathbf{F}_n \to \mathbf{C}\colon \sum_{\sigma \in \mathbf{F}_n} |f(\sigma)|^2 < \infty \right\}.$$

Let $\{e_\sigma\}_{\sigma \in \mathbf{F}_n}$ be the canonical basis of $\ell^2(\mathbf{F}_n)$, i.e., $e_\sigma(t) = 1$ if $t = \sigma$ and $e_\sigma(t) = 0$ otherwise. For each $j = 1, 2, \ldots, n$, $U_j \in B(\ell^2(\mathbf{F}_n))$ is the unitary operator defined by

$$U_j\left(\sum_{\sigma \in \mathbf{F}_n} \lambda_\sigma e_\sigma\right) = \sum_{\sigma \in \mathbf{F}_n} \lambda_\sigma e_{s_j \sigma}, \qquad \left(\sum_{\sigma \in \mathbf{F}_n} |\lambda_\sigma|^2 < \infty\right). \tag{5.1}$$

Note that the $C^*(U_1, ..., U_n)$ is the reduced C^*-algebra associated to $\mathbf{F}_n$ (see [Pi]).

The Hilbert space $\ell^2(\mathbf{F}_n^+)$ can be seen as a subspace of $\ell^2(\mathbf{F}_n)$. On the other hand, the full Fock space $F^2(H_n)$ can be naturally identified to $\ell^2(\mathbf{F}_n^+)$. Under this identification we have that $U_j|_{F^2(H_n)} = S_j$ $(j = 1, 2, ..., n)$ where $S_1, ..., S_n$ are the left creation operators on the Fock space $F^2(H_n)$. Now, for each $i = 1, 2, ..., k$ and $j = 1, 2, ..., n_i$ let us define the unitary operator U_{ij} on the Hilbert space $\ell^2(\mathbf{F}_{n_1}) \otimes \cdots \otimes \ell^2(\mathbf{F}_{n_k})$ by

$$U_{ij} = \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes U_j \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}} \tag{5.2}$$

where $U_j \in B(\ell^2(\mathbf{F}_{n_i}))$ was defined by (5.1). Due to our identification, one can see that

$$U_{ij}|_{F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})} = S_{ij} \tag{5.3}$$

for each $i = 1, 2, ..., k$, $j = 1, 2, ..., n_i$ (see (2.3) for the definition of S_{ij}). Let $C^*(\{U_{ij}\})$ be the C^*-algebra generated by $\{U_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i}$.

THEOREM 5.4. *Let* $\{A_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i} \in B(\mathcal{H})$ *be any sequence of operators with property* (P). *Then there exists a Hilbert space* $\mathcal{K} \supset \mathcal{H}$ *and a sequence* $\{W_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i} \subset B(\mathcal{K})$ *of unitary operators such that*

$$W_{ij} W_{pq} = W_{pq} W_{ij} \tag{5.4}$$

for $i, p \in \{1, 2, ..., k\}$, $i \neq p$, *and* $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$, *such that*

$$p(\{A_{ij}\}) = P_{\mathcal{H}}\, p(\{W_{ij}\})|_{\mathcal{H}}$$

for any polynomial $p(\{U_{ij}\}) \in \mathrm{Alg}(I, \{U_{ij}\}$.

Proof. Let $\mathcal{P}_u$ be the set of all polynomials in $\{U_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i}$. According to Theorem 3.8 and the relation (5.3) the map $\Phi : \mathcal{P}_u \to B(\mathcal{H})$ defined by

$$\Phi(p(\{U_{ij}\})) = p(\{A_{ij}\})$$

is a completely contractive homomorphism. Applying Arveson's extension theorem [A, Theorem 1.2.9] to our setting, there is a completely positive linear map

$$\Psi: C^*(\{U_{ij}\}) \to B(\mathcal{H})$$

such that $\Psi|_{\mathcal{P}_u} = \Phi$. Combining this result with Stinespring's representation theorem [S], we see that there is a $*$-representation

$$\pi\colon C^*(\{U_{ij}\}) \to B(\mathcal{K})$$

on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$\Psi(p(\{U_{ij}\})) = P_{\mathcal{H}} \pi(p(\{U_{ij}\}))|_{\mathcal{H}}$$

for any $p(\{U_{ij}\}) \in C^*(\{U_{ij}\})$. Setting $W_{ij} := \pi(U_{ij})$ $(i = 1, 2, ..., k,\ j = 1, 2, ..., n_i)$, it is clear that the sequence $\{W_{ij}\}_{i=1, 2, ..., k,\, j=1, 2, ..., n_i}$ satisfies relation (5.4), and

$$p(\{A_{ij}\}) = P_{\mathcal{H}}\, p(\{W_{ij}\})|_{\mathcal{H}}$$

for any polynomial $p(\{U_{ij}\}) \in \mathrm{Alg}(I, \{U_{ij}\})$. The proof is complete. ∎

6. CHARACTERS ON NONCOMMUTATIVE POLYDISC ALGEBRAS AND COHOMOLOGY

Let $\lambda = \{\lambda_{ij}\}_{i=1, 2, ..., k,\, j=1, 2, ..., n_i}$ be a sequence of complex numbers such that

$$|\lambda_{i1}|^2 + \cdots + |\lambda_{in_i}|^2 \leqslant 1 \qquad \text{for each} \quad i = 1, 2, ..., k,$$

and define the "evaluation" functional

$$\Phi_\lambda\colon \mathcal{P} \to \mathbf{C}; \qquad \Phi_\lambda(p(\{S_{ij}\})) = p(\{\lambda_{ij}\}),$$

where $\mathcal{P}$ is the set of all polynomials $p(\{S_{ij}\}) \in \mathrm{Alg}(I, \{S_{ij}\})$. Since the sequence $\{\lambda_{ij} I_{\mathbf{C}}\}_{i=1, 2, ..., k,\, j=1, 2, ..., n_i} \subset B(\mathbf{C})$ has property (P), the von Neumann inequality (3.8) shows that

$$|p(\{\lambda_{ij}\})| = \|p(\{\lambda_{ij} I_{\mathbf{C}}\})\| \leqslant \|p(\{S_{ij}\})\|.$$

Hence, Φ_λ has a unique extension to the polydisc algebra $\mathrm{Alg}(I, \{S_{ij}\})$. Therefore Φ_λ is a character on $\mathrm{Alg}(I, \{S_{ij}\})$. Let $M_{\mathrm{Alg}(I, \{S_{ij}\})}$ be the set of all characters of $\mathrm{Alg}(I, \{S_{ij}\})$ and let

$$\Psi\colon \overline{(\mathbf{C}^{n_1})}_1 \times \overline{(\mathbf{C}^{n_2})}_1 \times \cdots \times \overline{(\mathbf{C}^{n_k})}_1 \to M_{\mathrm{Alg}(I, \{S_{ij}\})}$$

be defined $\Psi(\lambda) = \Phi_\lambda$ where $\lambda = \{\lambda_{ij}\}_{i=1, 2, ..., k,\, j=1, 2, ..., n_i}$.

THEOREM 6.1. *The map Ψ is a homeomorphism of $\overline{(\mathbf{C}^{n_1})}_1 \times \cdots \times \overline{(\mathbf{C}^{n_k})}_1$ onto $M_{\mathrm{Alg}(I, \{S_{ij}\})}$.*

Proof. Let us show that Ψ is one-to-one. If $\lambda = \{\lambda_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ and $\mu = \{\mu_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ are in $E := \overline{(\mathbf{C}^{n_1})_1} \times \cdots \times \overline{(\mathbf{C}^{n_k})_1}$ then $\Psi(\lambda) = \Psi(\mu)$ implies that

$$\lambda_{ij} = \Phi_\lambda(S_{ij}) = \Phi_\mu(S_{ij}) = \mu_{ij}$$

for any $i = 1, 2, ...k$, $j = 1, 2, ..., n_i$. Therefore $\lambda = \mu$. Now, assume that Φ: Alg$(I, \{S_{ij}\}) \to \mathbf{C}$ is a character. Setting $\Phi(S_{ij}) = \lambda_{ij} \in \mathbf{C}$ we have

$$\Phi(p(\{S_{ij}\})) = p\{\lambda_{ij}\},$$

for any $p(\{S_{ij}\}) \in$ Alg$(I, \{S_{ij}\})$. Since Φ is a character it follows that it is completely contractive. Applying Theorem 3.9 when $A_{ij} = \lambda_{ij} I_{\mathbf{C}}$, $i = 1, 2, ..., k$, $j = 1, 2, ..., n_i$ we infer that $\{\lambda_{ij} I_{\mathbf{C}}\}$ has property (P), i.e., $\{\lambda_{ij}\} \in E$.

Moreover, the identity

$$\Phi(p(\{S_{ij}\})) = p((\{A_{ij}\})) = \Phi_\lambda(p(\{S_{ij}\}))$$

proves that Φ agrees with Φ_λ on the dense subset $\mathscr{P}$ of Alg$(I, \{S_{ij}\})$, therefore $\Phi = \Phi_\lambda$. Since both E and $M_{\mathrm{Alg}(I, \{S_{ij}\})}$ are compact Hausdorff spaces and Ψ is one-to-one and onto, to complete the proof it suffices to show that Φ is continuous.

Suppose that $\lambda^\alpha = (\lambda^\alpha_{ij})$, $(\alpha \in J)$ is net in E such that $\lim_{\alpha \in J} \lambda^\alpha = \lambda = (\lambda_{ij})$. Since $\sup_{\alpha \in J} \|\Phi_{\lambda^\alpha}\| \leqslant 1$ and $\mathscr{P}$ is dense in Alg$(I, \{S_{ij}\})$ and since

$$\lim_{\alpha \in J} \Phi_{\lambda^\alpha}(p(\{S_{ij}\})) = \lim_{\alpha \in J} p(\{\lambda_{ij}\}) = \Phi_\lambda(p(\{S_{ij}\}))$$

for every $p(\{S_{ij}\}) \in \mathscr{P}$ it follows that Ψ is continuous. The proof is complete. ∎

Let us remark that in the particular case when $n_1 = n_2 = \cdots = n_k = 1$ we get that $M_{A(\mathbf{D}^n)} = \bar{\mathbf{D}}^n$, which is a well-known result. In the particular case when $k = 1$, $n_1 = n$ we get $M_{\mathscr{A}_n} = \overline{(\mathbf{C}^n)_1}$ ($\mathscr{A}_n$ is the noncommutative disc algebra [Po2]), result that was obtained in [Po4].

The above theorem helps us see when the Banach algebras Alg$(I, \{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i})$ and Alg$(I, \{S_{ij}\}_{i=1,2,...,m,\ j=1,2,...,p_i})$ are not isomorphic.

Let A be a complex Banach algebra with unit, X be a Banach A-bimodule, and X' be the dual Banach A-bimodule (see [BD]). We need to recall from [BD] a few definitions.

A bounded X-derivation is a bounded linear mapping D of A into X such that

$$D(ab) = (Da)\,b + a(Db), \qquad \text{for any} \quad a, b \in A. \tag{6.1}$$

The set of all bounded X-derivations is denoted by $Z^1(A, X)$. For each $x \in X$ let us define $\delta_x: A \to X$ by $\delta_x(a) = ax - xa$. We call δ_x an inner X-derivation, and denote by $B^1(A, X)$ the set of all inner X-derivations. The quotient space $Z^1(A, X)/B^1(A, X)$ is called the first cohomology group of A with coefficients in X, and it is denoted by $H^1(A, X)$. A Banach algebra A is said to be amenable if $H^1(A, X') = \{0\}$ for every Banach A-bimodule X.

In what follows we shall see that the noncommutative polydisc algebra $\mathscr{A} := \mathrm{Alg}(I, \{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i})$ is not amenable.

Of course $\mathbf{C}$, the set of all complex numbers, is a Banach $\mathscr{A}$-bimodule under the module multiplication

$$\lambda \cdot f(\{S_{ij}\}) = f(\{S_{ij}\}) \cdot \lambda = \lambda f(\{0\}) \tag{6.2}$$

for each $f(\{S_{ij}\}) \in \mathscr{A}$. According to the von Neumann inequality (3.9), we infer that $|\lambda \cdot f(\{S_{ij}\})| \leqslant |\lambda|\ \|f(\{S_{ij}\})\|$, for any $\lambda \in \mathbf{C}$, $f(\{S_{ij}\}) \in \mathscr{A}$.

Since the proof of the following theorem is a straightforward extension of [Po4, Theorem 4.1], we omit it.

THEOREM 6.2. *The first cohomology group of the algebra* $\mathrm{Alg}(I, \{S_{ij}\})_{i=1,2,...,k,\ j=1,2,...,n_i}$ *with complex coefficients is isomorphic to the additive group* $\mathbf{C}^{n_1+n_2+\cdots+n_k}$.

Since $\mathbf{C}$ is a dual bimodule we have the following.

COROLLARY 6.3. *The polydisc algebra* $\mathrm{Alg}(I, \{S_{ij}\})$ *is not amenable.*

7. SEQUENCES OF OPERATORS WITH PROPERTY (P) AND UNIVERSAL ALGEBRAS

A sequence of operators $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathscr{H})$ is called with property (P*) if it satisfies the relations (2.1), (2.2), and

$$A_{ij} A_{pq}^* = A_{pq}^* A_{ij} \tag{7.1}$$

for any $i, p \in \{1, 2, ..., k\}$, $i \neq p$ and $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$. Notice that in the particular case when $n_1 = n_2 = \cdots = n_k = 1$ we obtain a sequence of double commuting contractions [SzF, Pau].

LEMMA 7.1. *Any sequence* $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathscr{H})$ *with property* (P*) *has property* (P).

Proof. Using the relations (2.2), (3.1), and (7.1) we can see that

$$\Delta_r(\{A_{ij}\}) = \prod_{i=1}^{k} (I_{\mathcal{H}} - r^2 A_{i1} A_{i1}^* - \cdots - r^2 A_{in_i} A_{in_i}^*)$$

for each $0 \leqslant r \leqslant 1$. According to (2.1), $\Delta_r(\{A_{ij}\})$ is a product of commuting positive operators. Hence, $\Delta_r(\{A_{ij}\}) \geqslant 0$. This completes the proof. ∎

In what follows we will show that $\mathrm{Alg}(I, \{S_{ij}\})$, the smallest closed algebra generated by the isometries S_{ij} ($i = 1, 2, ..., k$, $j = 1, 2, ..., n_i$) and the identity, is the universal algebra generated by the identity and a sequence $\{A_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i} \subset B(\mathcal{H})$ with property (P*), in the following sense. Given any sequence of operators $\{A_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i} \subset B(\mathcal{H})$ with property (P*) there is a completely contractive homomorphism

$$\Phi\colon \mathrm{Alg}(I, \{S_{ij}\}) \to B(\mathcal{H})$$

such that $\Phi(I) = I$ and $\Phi(S_{ij}) = A_{ij}$ for any $i = 1, 2, ..., k$, and $j = 1, 2, ..., n_i$.

Let us show that this property characterizes $\mathrm{Alg}(I, \{S_{ij}\})$ up to unital complete isometric isomorphism.

THEOREM 7.2. *Let $\{b_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i}$ be a sequence of elements in some unital C^*-algebra, with property* (P*). *If for any sequence of operators $\{A_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i}$ with property* (P*) *the map*

$$\Psi\colon \mathrm{Alg}(I, \{b_{ij}\}) \to \mathrm{Alg}(I, \{A_{ij}\})$$

defined by $\Psi(I) = I$, $\Psi(b_{ij}) = A_{ij}$, is a unital completely contractive homomorphism, then $\mathrm{Alg}(I, \{B_{ij}\})$ *is completely isometrically isomorphic to* $\mathrm{Alg}(I, \{S_{ij}\})$.

Proof. Since

$$S_{ij} S_{pq}^* = S_{pq}^* S_{ij}$$

for any $i, p \in \{1, 2, ..., k\}$, $i \neq p$ and $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$, we infer that $\{S_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i}$ has property (P*). Setting $A_{ij} = S_{ij}$ we obtain

$$\|[P_{rs}(\{S_{ij}\})]_{r, s=1}^m\| \leqslant \|[P_{rs}(\{b_{ij}\})]_{r, s=1}^m\| \tag{7.2}$$

for any matrix $[P_{rs}(\{S_{ij}\})]_{r, s=1}^m \in M_m(\mathrm{Alg}(\{S_{ij}\}))$. On the other hand, since $\{b_{ij}\}_{i=1, 2, ..., k,\ j=1, 2, ..., n_i}$ has property (P*), Lemma 7.1 shows that it has property (P). Applying Theorem 3.8 to our setting, we infer that

$$\|[P_{rs}(\{b_{ij}\})]_{r, s=1}^m\| \leqslant \|[P_{rs}(\{S_{ij}\})]_{r, s=1}^m\|.$$

This inequality together with (7.2) show that

$$\|[P_{rs}(\{b_{ij}\})]_{r,s=1}^m\| = \|[P_{rs}(\{S_{ij}\})]_{r,s=1}^m\|$$

for any $[P_{rs}(\{S_{ij}\})]_{r,s,=1}^m \in M_m(\mathrm{Alg}(\{S_{ij}\}))$. Therefore $\mathrm{Alg}(I, \{b_{ij}\})$ is completely isometrically isomorphic to $\mathrm{Alg}(I, \{S_{ij}\})$. ∎

The C^*-algebra $C^*(\{S_{ij}\})$ can be viewed as the universal C^*-algebra generated by a sequence of isometrics $\{V_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathscr{H})$ with property (P*), in the following sense.

THEOREM 7.3. *If* $\{V_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathscr{H})$ *is a sequence of isometries with property* (P*) *then there is a* $*$*-representation*

$$\pi: C^*(\{S_{ij}\}) \to C^*(\{V_{ij}\}); \qquad \pi(S_{ij}) = V_{ij}. \tag{7.3}$$

Moreover, any $*$*-representation of* $C^*(\{S_{ij}\})$ *is determined by a sequence of isometries* $\{V_{ij}\}$ *with property* (P*).

One can prove that this property characterizes $C^*(\{S_{ij}\})$ up to a $*$-isomorphism. Using Theorem 3.8, the proof is similar to that of Theorem 7.2, so we omit it.

The Toeplitz algebra $\mathscr{T}_n$ is the unique unital C^*-algebra generated by $n = 2, 3, ...$ isometries $s_1, ..., s_n$ satisfying

$$s_i^* s_j = \delta_{ij} 1, \qquad \sum_{i=1}^n s_i s_i^* < 1$$

(see [Cu2, BEGJ, Po3]). The Fock or regular representation of $\mathscr{T}_n$ on $F^2(H_n)$ is generated by the left creation operators S_i $(i = 1, 2, ..., n)$ (see Section 1). The noncommutative disc algebra $\mathscr{A}_n$ is the unique nonselfadjoint closed algebra generated by $1, s_1, ..., s_n$ (see [Po4]).

Using Theorem 7.3 one can easily prove that there is a unique C^*-cross norm on $\mathscr{T}_{n_1} \otimes \cdots \otimes \mathscr{T}_{n_k}$ $(n_1, ..., n_k \geqslant 2)$ and $C^*(\{S_{ij}\}) \simeq \mathscr{T}_{n_1} \otimes \cdots \otimes \mathscr{T}_{n_k}$. According to the definition of the min norm on tensor products of operator algebras [Pau] and since $\mathscr{A}_{n_i}$ can be seen as a subalgebra of $\mathscr{T}_{n_i}$ $(i = 1, 2, ..., k)$ (see [Po4]), we deduce the following result.

COROLLARY 7.4. $\mathrm{Alg}(I, \{S_{ij}\}) \simeq \mathscr{A}_{n_1} \otimes_{\min} \cdots \otimes_{\min} \mathscr{A}_{n_k}$.

In what follows we show that $C^*(\{S_{ij}\})$ is completely isometrically isomorphic to a free operator algebra considered by D. Blecher [B] (see also [BP]). We need a few definitions from [B].

Let Γ be a set, and let $n: \Gamma \to \mathbf{N}$ be a function with $n(\gamma) = n_\gamma$. Let Λ be a set of variables (or formal symbols) x_{ij}^γ, one variable for each $\gamma \in \Gamma$ and each i, j, $1 \leqslant i, j \leqslant n_\gamma$. Let $\mathscr{F}$ be the free associative algebra on Λ. Let $\mathscr{R}$ be a set of polynomial identities $P=0$ in the variables in Λ. Regard $\mathscr{R}$ as subset of $\mathscr{F}$. Take a quotient of $\mathscr{F}$ by the ideal generated by $\mathscr{R}$.

We define a semi-norm on $M_n(\mathscr{F})$ by

$$\|[u_{ij}]\|_\Lambda = \sup\{\|[\pi(u_{ij})]\|\}$$

where the supremum is taken over all algebra representations π of $\mathscr{F}$ on a Hilbert space satisfying the condition $\pi(\mathscr{R})=0$ and $\|[\pi(x_{ij}^\gamma)]\| \leqslant 1$ for all γ. This later matrix is indexed on rows by i and on columns by j, for all $1 \leqslant i, j \leqslant n_\gamma$.

Now, quotient by nullspace of this semi-norm to obtain an operator algebra. The completion of this space is denoted by $OA(\Lambda, \mathscr{R})$. This is called the free operator algebra on Λ with relations $\mathscr{R}$ (see [B]).

Let Λ have the identity e and also contain the ordinary variables $\{x_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$, $\{y_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$, and let $\mathscr{R}$ be the relations

$$x_{ij}x_{pq} = x_{pq}x_{ij} \qquad \text{and} \qquad x_{ij}y_{pq} = y_{pq}x_{ij}$$

if $i, p \in \{1, 2, ..., k\}$, $i \neq p$, and $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$, and $y_{ir}x_{ij} = \delta_{rj}e$ for any $i \in \{1, 2, ..., k\}$, and $r, j \in \{1, 2, ..., n_i\}$. Form the universal algebra $OA(\Lambda, \mathscr{R})$.

Using Theorem 7.3 one can extend Theorem 4.3 from [Po5] to our setting. We omit the proof which is straightforward.

THEOREM 7.5. *The universal algebra $OA(\Lambda, \mathscr{R})$ is completely isometric to $C^*(\{S_{ij}\})$.*

The internal characterization of the matrix norm on a universal algebra $OA(\Lambda, \mathscr{R})$ (see [B, BP]) leads to the following factorization theorem.

THEOREM 7.6. *If $P=[p_{rs}]_{m\times m}$ is a matrix of polynomials in I, $\{S_{ij}\}$, $\{S_{ij}^*\}$ then, $\|P\|<1$ if and only if there is a positive integer t such that*

$$P = A_0 D_1 A_1 D_2 \cdots D_t A_t,$$

where A_ℓ ($\ell = 0, 1, ..., t$) are scalar matrices (with a finite number of nonzero entries), each $\|A_\ell\|<1$, and each D_ℓ is diagonal matrix with I, S_{ij}, S_{ij}^* ($i \in \{1, 2, ..., k\}$ and $j \in \{1, 2, ..., n_i\}$) as the diagonal entries.

Let us remark that a similar result holds for matrix polynomials in I, $\{S_{ij}\}$.

Another class of sequences of operators with property (P) is consider in what follows.

LEMMA 7.7. *If* $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathscr{H})$ *is a sequence of operators such that*

$$A_{i1}A_{i1}^* + A_{i2}A_{i2}^* + \cdots + A_{in_i}A_{in_i}^* = I_{\mathscr{H}} \tag{7.4}$$

for each $i = 1, 2, ..., k$ *and*

$$A_{ij}A_{pq} = A_{pq}A_{ij} \tag{7.5}$$

if $i, p \in \{1, 2, ..., k\}$, $i \neq p$ *and* $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$, *then* $\{A_{ij}\}$ *has property* (P).

Proof. Consider the sequence of operators defined by $Y_0 = I_{\mathscr{H}}$ and

$$Y_i = Y_{i-1} - r^2 A_{i1} Y_{i-1} A_{i1}^* - \cdots - r^2 A_{in_i} Y_{i-1} A_{in_i}^*$$

for $i = 1, 2, ..., k$.

Notice that $\Delta_r(\{A_{ij}\}) = Y_k$ $(0 < r < 1)$. According to (7.4), we have $Y_1 = (1 - r^2) I_{\mathscr{H}}$. By induction, we infer that $Y_k = (1 - r^2)^k I_{\mathscr{H}} \geqslant 0$ if $0 \leqslant r \leqslant 1$. Therefore $\Delta_r(\{A_{ij}\}) \geqslant 0$ and the sequence $\{A_{ij}\}$ satisfying the conditions (7.4) and (7.5) has property (P). The proof is complete. ∎

Let us remark that if $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \in B(\mathscr{H})$ is a sequence of operators satisfying the condition (7.5) and such that $\sum_{j=1}^{n_i} A_{ij}A_{ij}^* \leqslant I_{\mathscr{H}}$ for each $i = 1, 2, ..., k$, it does not follow that it has property (P). To see this, consider Parrott's example [Pa] and use Theorem 5.4 in the particular case $k = 3$ and $n_1 = n_2 = n_3 = 1$.

LEMMA 7.8. *Let* $\{V_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathscr{H})$ *be a sequence of isometries such that*

$$V_{i1}V_{i1}^* + \cdots + V_{in_i}V_{in_i}^* = I_{\mathscr{H}} \tag{7.6}$$

for each $i = 1, 2, ..., k$, *and*

$$V_{ij}V_{pq} = V_{pq}V_{ij} \tag{7.7}$$

for any $i, p \in \{1, 2, ..., k\}$, $i \neq p$ *and* $j \in \{1, 2, ..., n_i\}$, $q \in \{1, 2, ..., n_p\}$. *Then* $\{V_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ *has property* (P*).

The proof is straightforward, so we omit it. According to Theorem 7.3 and Lemma 7.8, there is a $*$-representation

$$\pi\colon C^*(\{S_{ij}\}) \to C^*(\{V_{ij}\}); \qquad \pi(S_{ij}) = V_{ij}. \tag{7.8}$$

Let us recall that the Cuntz algebra $\mathcal{O}_n$ is uniquely defined as the C^*-algebra generated by $n = 2, 3, ...$ isometries satisfying

$$\sigma_i^* \sigma_j = \delta_{ij} 1, \qquad \sum_{j=1}^{n} \sigma_i \sigma_j^* = 1$$

[Cu]. Since the Cuntz algebra $\mathcal{O}_n$ $(n \geqslant 2)$ is nuclear [Cu] there is a unique structure of C^*-algebra on $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ $(n_1, ..., n_k \geqslant 2)$. According to (7.8), one can easily deduce the following result.

THEOREM 7.9. *There is a* $*$*-representation*

$$\Phi : C^*(\{S_{ij}\}) \to \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$$

such that $\Phi(S_{ij}) = \sigma_{ij}$, *where for each* $i = 1, 2, ..., k$

$$\sigma_{ij} = \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes \sigma_j \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-i \text{ times}},$$

and $\{\sigma_j\}_{j=1}^{n_i}$ *is a set of generators of the Cuntz algebra* $\mathcal{O}_{n_i}$.

Let us remark that this result was obtained by Cuntz [Cu] (using different techniques) for $k = 1$. On the other hand, using the short exact sequence obtained by Cuntz [Cu], one can prove that the above $*$-representation is surjective.

8. POISSON TRANSFORM ASSOCIATED TO THE UNIT BALL OF $B(\mathcal{H})^N$

In Section 3 we introduced a Poisson transform associated to sequences of operators $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ with property (P).

Let us consider the particular case when $k = 1$ and $n_1 = n \in \{1, 2, ...\}$. Any sequence $\{T_j\}_{j=1}^{n} \subset B(\mathcal{H})$ such that $T_1 T_1^* + \cdots + T_n T_n^* \leqslant I_{\mathcal{H}}$ has property (P). Indeed, in this case we have

$$\Delta(\{T_j\}) = I_{\mathcal{H}} - T_1 T_1^* - \cdots - T_n T_n^*$$

and $\Delta_r(\{T_j\}) \geqslant 0$ for any $0 \leqslant r \leqslant 1$.

For each $j = 1, 2, ..., n$, $S_j \in B(F^2(H_n))$ is the left creation operator with e_j, i.e., $S_j \xi = e_j \otimes \xi$, $\xi \in F^2(H_n)$. Let $\mathbf{F}_n^+$ be the unital free semigroup on n generators $s_1, ..., s_n$, and let e be the neutral element in $\mathbf{F}_n^+$. For each $\alpha = s_{j_1} \cdots s_{j_m} \in \mathbf{F}_n^+$, $j_1, ..., j_m \in \{1, 2, ..., n\}$ define $e_\alpha := e_{j_1} \otimes \cdots \otimes e_{j_m}$ and

$e_\alpha = 1$ if $\alpha = e \in \mathbf{F}_n^+$. It is easy to see that $\{e_\alpha\}_{\alpha \in \mathbf{F}_n^+}$ is an orthonormal basis for the full Fock space $F^2(H_n)$.

Applying Theorem 3.8 to our setting, we obtain that the Poisson transform on $C^*(S_1, ..., S_n)$, the extension of the Cuntz algebra through compacts, is the completely contractive linear map

$$P(\{T_j\}): C^*(S_1, ..., S_n) \to B(\mathcal{H})$$

defined by

$$P(\{T_j\})[f] = \lim_{\substack{r \to 1 \\ r < 1}} K_r(\{T_j\})^* \, (f \otimes I_{\mathcal{H}}) \, K_r(\{T_j\}) \tag{8.1}$$

(in the uniform topology of $B(\mathcal{H})$), where the Poisson kernel

$$K_r(\{T_j\}): \mathcal{H} \to F^2(H_n) \otimes \mathcal{H}$$

is defined by

$$K_r(\{T_j\})h = \sum_{\gamma \in \mathbf{F}_n^+} e_\gamma \otimes (r^{|\gamma|} \Delta_r(\{T_j\})^{1/2} \, T_\gamma^* h).$$

Moreover, we can deduce the following result obtained in [Po3].

THEOREM 8.1. *If* $(T_1, ..., T_n) \in (B(\mathcal{H})^n)_1$ *then the linear map*

$$\Phi: C^*(S_1, ..., S_n) \to B(\mathcal{H})$$

defined by

$$\Phi(S_{i_1} \cdots S_{i_p} S_{j_1}^* \cdots S_{j_m}^*) = T_{i_1} \cdots T_{i_p} T_{j_1}^* \cdots T_{j_m}^*$$

$1 \leqslant i_1, ..., i_p, j_1 ..., j_m \leqslant n$, *is completely contractive.*

In particular, we obtain a new and elementary proof of the main result in [Po2], i.e., the von Neumann inequality [vN, SzF], for $(B(\mathcal{H})^n)_1$ (the case $n = 1$ was considered in Section 1).

COROLLARY 8.2. *If* $(T_1, ..., T_n) \in (B(\mathcal{H})^n)_1$ *then*

$$\|p(T_1, ..., T_n)\| \leqslant \|p(S_1, ..., S_n)\|$$

for any polynomial $p(S_1, ..., S_n)$ *in* $I, S_1, ..., S_n$.

It is easy to see that applying Corollary 5.2 to our setting one can obtain a new proof of the isometric dilation theorem for sequences $(T_1, ..., T_n) \in (B(\mathcal{H})^n)_1$ (see [F, Bu, Po1]). On the other hand, Theorem 5.4 provides a unitary dilation for such sequences of operators as well as the Bozejko's version of the von Neumann inequality [Bo] to our setting.

Let us remark that, in the particular case when $(T_1, ..., T_n) \in (B(\mathcal{H})^n)_1$ and $T_iT_j = T_jT_i$, $1 \leqslant i, j \leqslant n$, Theorem 8.1 remains true if we replace the left creation operators S_i, $1 \leqslant i \leqslant n$, by their compressions to the symmetric Fock space $F_s^2(H_n) \subset F^2(H_n)$. Indeed, this follows from (8.1) if we take into account that $F_s^2(H_n)$ is invariant to S_i^*, $1 \leqslant i \leqslant n$ and the Poisson kernel $K_r(\{T_j\})$ takes values in $F_s^2(H_n) \otimes \mathcal{H}$.

We recall that the Cuntz algebra $\mathcal{O}_n$ is uniquely defined as the C^*-algebra generated by $n = 2, 3, ...$ isometries satisfying

$$\sigma_i^* \sigma_j = \delta_{ij} 1, \qquad \sum_{j=1}^{n} \sigma_i \sigma_j^* = 1$$

[Cu]. For any $f(\{S_i\}, \{S_i^*\}) \in C^*(S_1, ..., S_n)$ the Poisson formula (8.1) becomes

$$f(\{\sigma_i\}, \{\sigma_i^*\}) = \lim_{\substack{r \to 1 \\ r < 1}} (1 - r^2)\, C_r(\{\sigma_j\})^* \,(f(\{S_i\}, \{S_i^*\}) \otimes I_{\mathcal{H}})\, C_r(\{\sigma_j\})$$

where the Cauchy kernel is defined by

$$C_r(\{\sigma_j\}) = \sum_{\alpha \in \mathbf{F}_n^+} e_\alpha \otimes r^{|\alpha|} \sigma_\alpha^* h.$$

In our particular setting, Theorem 7.9 shows that there is a $*$-representation

$$\Phi: C^*(S_1, ..., S_n) \to \mathcal{O}_n$$

such that $\Phi(S_i) = \sigma_i$, $i = 1, 2, ..., k$. This is a well known result obtained (using different techniques) by [C] for $n = 1$ and [Cu] for $n \geqslant 2$.

Let us remark that if $(T_1, ..., T_n) \in (B(\mathcal{H})^n)_1$ is completely non-coisometric (see [Po3]) one can use the results from [Po3] to extend the Poisson transform (8.1) to $\overline{\mathrm{Alg}(I, S_1, ..., S_n)}^{\mathrm{so}}$ (the closure in the strong operator topology).

Now let us consider the particular case when $n = 1$. Let S be the unilateral shift on the Hardy space $H^2(\mathbf{D})$, i.e., $(Sf)(z) = zf(z)$, $z \in \mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$. Let $T \in B(\mathcal{H})$ be such that $\|T\| \leqslant 1$ and let $p(S, S^*) = \sum_{m, n \geqslant 0} a_{nm} S^m S^{*n}$ be in $C^*(S)$. The Poisson transform on $C^*(S)$, the C^*-algebra generated by S, has the following equivalent form (see Section 1).

$$p(T, T^*) = \lim_{\substack{r \to 1 \\ r < 1}} \frac{1}{2\pi} \int_0^{2\pi} (I_{\mathcal{H}} - re^{-it}T)^{-1}\, \Delta_r(T)^{1/2}\, p(S, S^*) \times \Delta_r(T)^{1/2}\, (I_{\mathcal{H}} - re^{it}T^*)^{-1}\, dt$$

(in the uniform topology of $B(\mathcal{H})$), where $\Delta_r(T) = I_{\mathcal{H}} - r^2 TT^*$.

Let us mention that this Poisson transform is an extension of [V, Corollary 3.5] and [Pau, p. 24].

9. SEQUENCES OF COMMUTING OPERATORS WITH PROPERTY (P)

Let $\{A_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i} \subset B(\mathcal{H})$ be a sequence of operators satisfying the relations (2.1), (2.2), and let $\{S_{ij}\}_{i=1,2,...,k,\ j=1,2,...,n_i}$ be the sequence of isometries defined by (2.3).

Let us consider the particular case when $n_1 = n_2 = \cdots = n_k = 1$. For each $i = 1, 2, ..., k$ denote $T_i := A_{i1}$ and $M_i := S_{i1}$. The relations (2.1), (2.2) become $\|T_i\| \leqslant 1$ and $T_i T_p = T_p T_i$, respectively. According to (3.1), we have

$$\Delta(\{T_i\}) = \sum_{\varepsilon_1, ..., \varepsilon_k \in \{0,1\}} (-1)^{\varepsilon_1 + \cdots + \varepsilon_k} T_1^{\varepsilon_1} \cdots T_k^{\varepsilon_k} (T_k^*)^{\varepsilon_k} \cdots (T_1^*)^{\varepsilon_1} \tag{9.1}$$

Using an inductive argument, we infer the following.

PROPOSITION 9.1. *If $\{T_i\}_{i=1}^k \subset B(\mathcal{H})$ is a sequence of commuting operators such that*

$$T_1 T_1^* + \cdots + T_k T_k^* \leqslant I_{\mathcal{H}}$$

then $\{T_i\}_{i=1}^k$ has property (P).

THEOREM 9.2. *Let $\{T_i\}_{i=1}^k \subset B(\mathcal{H})$ be a sequence of commuting operators such that*

$$T_1 T_1^* + \cdots + T_k T_k^* \leqslant I_{\mathcal{H}}. \tag{9.2}$$

Then, there is a completely contractive linear map

$$\Phi\colon C^*(M_1, ..., M_k) \to B(\mathcal{H})$$

such that

$$\Phi(M_{i_1} \cdots M_{i_q} M_{j_1}^* \cdots M_{j_p}^*) = T_{i_1} \cdots T_{i_q} T_{j_1}^* \cdots T_{j_p}^*$$

for any $i_1, ..., i_q, j_1, ..., j_p \in \{1, 2, ..., k\}$.

Moreover, the result holds true if one replaces M_i, $1 \leqslant i \leqslant n$, by the compressions of the left creation operators to the symmetric Fock space.

Proof. According to Proposition 9.1, the sequence $\{T_i\}_{i=1}^k$ has property (P). Applying Theorem 3.8 to our setting, the result follows. The second part of the theorem is contained in Section 8. ∎

Notice that if $\{T_i\}_{i=1}^k \subset B(\mathcal{H})$ is a sequence of double commuting contractions [SzF] then it has property (P). Therefore the first part of Theorem 9.2 holds true.

Let us also remark that an isometric (resp. unitary) dilation theorem for sequences $\{T_i\}_{i=1}^k \subset B(\mathcal{H})\}$ of commuting operators with property (9.2) (resp. double commuting contractions) can be obtained applying Corollary 5.2 (resp. Theorem 5.4) to our setting.

Let S_i ($i \in \{1, 2, ..., k\}$ be the unilateral shift on the Hardy space $H^2(\mathbf{D}^k)$, i.e., $(S_i f)(z) = z_i f(z)$ for any $z \in \mathbf{D}^k$, where

$$\mathbf{D}^k = \{(z_1, ..., z_k) : z_i \in \mathbf{C}, |z_i| < 1 \text{ for every } i = 1, 2, ..., k\}.$$

Under the canonical identification of the Hilbert space $\underbrace{F^2(\mathbf{C}) \otimes F^2(\mathbf{C}) \otimes \cdots \otimes F^2(\mathbf{C})}_{k\text{-times}}$ to the Hardy space $H^2(\mathbf{D}^k)$, the operators $M_1, ..., M_k$ are unitarily equivalent to $S_1, ..., S_k$, respectively. Let $A(\mathbf{D}^k)$ be the closure of the set of all polynomials in the uniform norm $\|\cdot\|_\infty$ defined by

$$\|p\|_\infty = \sup_{\substack{|z_i| \leqslant 1 \\ i \in \{1, 2, ..., k\}}} |p(z_1, ..., z_n)|.$$

Corollary 9.3. *Let $\{T_i\}_{i=1}^k \subset B(\mathcal{H})$ be a sequence of commuting operators such that*

$$T_1 T_1^* + \cdots + T_k T_k^* \leqslant I_{\mathcal{H}}$$

Then, there is a completely contractive homomorphism

$$\Phi : A(\mathbf{D}^k) \to B(\mathcal{H})$$

such that $\Phi(z_i) = T_i$ for $i = 1, 2, ..., k$, where $z_1, ..., z_n$ are the coordinate functions.

Let us mention that if $\{T_i\}_{i=1}^k \subset B(\mathcal{H})$ is a sequence of commuting operators satisfying (9.2) (or sequence of double commuting contractions), then the associated Poisson transform (see Theorem 3.8) has the following equivalent form, which is an extension of [CV, Theorem 2.1]. For any $p(\{S_i\}, \{S_i^*\}) \in C^*(S_1, ..., S_n)$,

$$p(\{T_i\}, \{T_i^*\}) = \lim_{\substack{r \to 1 \\ r<1}} \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} K_r(\{T_i\})^* \, p(\{S_i\}, \{S_i^*\}) \times K_r(\{T_i\}) \, dt_1 \cdots dt_k$$

(the convergence in the uniform topology), where

$$K_r(\{T_i\}) = \Delta_r(\{T_i\})^{1/2} \prod_{m=1}^{k} (I_{\mathcal{H}} - re^{it_m} T_m^*)$$

and $\Delta_r(\{T_i\})^{1/2}$ is given by (9.1).

LEMMA 9.4. *If $\{V_i\}_{i=1}^k \subset B(\mathcal{H})$ be a sequence of commuting isometries then $\{V_i^*\}_{i=1}^k$ has property* (P).

Proof. Applying Lemma 7.7 in our setting, the result follows. ∎

THEOREM 9.5. *Let $\{V_i\}_{i=1}^k \subset B(\mathcal{H})$ be a sequence of commuting isometries. Then, there is a completely contractive linear map*

$$\Psi\colon C^*(M_1, ..., M_n) \to B(\mathcal{H})$$

such that

$$\Psi(M_{i_1} \cdots M_{i_q} M_{j_1}^* \cdots M_{j_p}^*) = V_{i_1}^* \cdots V_{i_q}^* V_{j_1} \cdots V_{j_p}$$

for any $i_1, ..., i_q, j_1, ..., j_p \in \{1, 2, ..., k\}$.

Proof. Using Lemma 9.4 and applying Theorem 3.8 to our setting, the result follows. ∎

In what follows we show that the polydisc algebra $\mathrm{Alg}(I, \{S_i\})$ is the universal algebra generated by k commuting isometries and the identity.

THEOREM 9.6. *If $\{V_i\}_{i=1}^k \subset B(\mathcal{H})$ is any sequence of commuting isometries then there exists a completely contractive homomorphism*

$$\Phi : \mathrm{Alg}(I, \{S_i\}) \to B(\mathcal{H})$$

such that $\Phi(S_i) = V_i$ for $i = 1, 2, ..., k$.

Proof. According to Lemma 9.4, $\{V_i^*\}_{i=1}^k$ is a sequence with property (P). Applying Theorem 5.4 to our setting we deduce that there exists a sequence $\{W_i\}_{i=1}^k$ of commuting unitaries on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$\|p(V_1^*, ..., V_k^*)\| \leqslant \|p(W_1, ..., W_k)\| \tag{9.3}$$

for any polynomial $p(M_1, ..., M_k) \in \mathrm{Alg}(I, \{M_i\})$. This inequality shows that

$$\|q(V_1, ..., V_k)\| \leqslant \|q(W_1^*, ..., W_k^*)\| \tag{9.4}$$

for any polynomial $q(M_1, ..., M_k) \in \mathrm{Alg}(I, \{M_i\})$. According to Theorem 9.5 we infer that

$$\|q(W_1^*, ..., W_k^*)\| \leqslant \|q(M_1, ..., M_k)\| \tag{9.5}$$

The inequalities (9.4) and (9.5) show that $\|q(V_1, ..., V_k)\| \leqslant \|q(M_1, ..., M_k)\|$ for any polynomial $q(M_1, ..., M_k) \in \mathrm{Alg}(I, \{M_i\})$. Notice that all the above inequalities hold true if we pass to matrices. Using the remarks preceding Corollary 9.3, we infer that the map Φ: $\mathrm{Alg}(I, \{S_i\}) \to B(\mathcal{H})$ defined by $\Phi(S_i) = V_i$ for $i = 1, 2, ..., k$, is a completely contractive homomorphism. ∎

One can prove that the property stated in Theorem 9.6 characterizes $\mathrm{Alg}(I, \{S_i\})$ up to unital complete isometric homomorphism. The proof is similar to that of Theorem 7.2, so we omit it.

COROLLARY 9.7 (Itô). *Let $\{V_i\}_{i=1}^k \subset B(\mathcal{H})$ be a sequence of commuting isometries. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\{W_i\}_{i=1}^k \subset B(\mathcal{K})$ a sequence of commuting unitaries, that dilates $\{V_i\}_{i=1}^k$.*

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