PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2361–2368
S 0002-9939(98)04341-X

CERTAIN INVARIANT SUBSPACE STRUCTURE OF $L^2(\mathbb{T}^2)$

GUOXING JI, TOMOYOSHI OHWADA, AND KICHI-SUKE SAITO

(Communicated by Palle E. T. Jorgensen)

Abstract. In this note, we study certain structure of an invariant subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$. Considering the largest z-invariant (resp. w-invariant) subspace in the wandering subspace $\mathfrak{M} \ominus zw\mathfrak{M}$ of $\mathfrak{M}$ with respect to the shift operator zw, we give an alternative characterization of Beurling-type invariant subspaces. Furthermore, we consider a certain class of invariant subspaces.

1. Introduction

Let $\mathbb{T}^2$ be the torus that is the cartesian product of 2 unit circles in $\mathbb{C}$. Let $L^2(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$ be the usual Lebesgue and Hardy space on the torus $\mathbb{T}^2$ respectively. For $(m, n) \in \mathbb{Z}^2$ and $f \in L^2(\mathbb{T}^2)$, the Fourier coefficient of f is defined by

$$\widehat{f}(m, n) = \int_{\mathbb{T}^2} f(z, w)\overline{z}^m \overline{w}^n dm,$$

where m is the Haar measure on $\mathbb{T}^2$. We define the closed subspace $H_0^2(\mathbb{T}^2)$ of $H^2(\mathbb{T}^2)$ by

$$H_0^2(\mathbb{T}^2) = \{f \in H^2(\mathbb{T}^2) : \widehat{f}(0, 0) = 0\}.$$

A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be invariant if $z\mathfrak{M} \subseteq \mathfrak{M}$ and $w\mathfrak{M} \subseteq \mathfrak{M}$. As is well known, the form of invariant subspaces of $L^2(\mathbb{T}^2)$ or even $H^2(\mathbb{T}^2)$ is much more complicated. In general, the invariant subspaces of $L^2(\mathbb{T}^2)$ are not necessarily of the form $fH^2(\mathbb{T}^2)$ with some unimodular function f. The structure of Beurling-type invariant subspaces has been studied in recent years and, in particular, some necessary and sufficient conditions for invariant subspaces to be Beurling-type have been given(cf. [1], [2], [3], [4], [5], etc.).

In this note, we study the structure of an invariant subspace $\mathfrak{M}$ as a zw-invariant subspace. To do this, we consider the largest z-invariant (resp. w-invariant) subspace in $\mathfrak{M} \ominus zw\mathfrak{M}$. First we give an alternative approach of Beurling-type invariant subspaces. Furthermore, we study a class of invariant subspaces $\mathfrak{M}_\alpha$ which contains the class of invariant subspaces of the form $qH_0^2(\mathbb{T}^2)$, where q is a unimodular function in $L^\infty(\mathbb{T}^2)$. In particular, we give a necessary and sufficient condition for an invariant subspace to be of the form $qH_0^2(\mathbb{T}^2)$.

We define several subspaces of $L^2(\mathbb{T}^2)$ which will be used later.

Received by the editors April 9, 1996 and, in revised form, January 21, 1997.
1991 *Mathematics Subject Classification.* Primary 47A15; Secondary 46J15.
This work was supported in part by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education.

(i) $H^2(z)$ or $H^2(w)$ is the set of f (in $L^2(\mathbb{T}^2)$) with Fourier series:

$$\sum_{\substack{m\geq 0\\ n=0}} a_{mn}z^m w^n \qquad \text{or} \qquad \sum_{\substack{m=0\\ n\geq 0}} a_{mn}z^m w^n$$

respectively.

(ii) H^2_z or H^2_w is the set of f with the Fourier series:

$$\sum_{n\geq 0} a_{mn}z^m w^n \qquad \text{or} \qquad \sum_{m\geq 0} a_{mn}z^m w^n$$

respectively.

(iii) L^2_z or L^2_w is the set of f with the Fourier series:

$$\sum_{n=0} a_{mn}z^m w^n \qquad \text{or} \qquad \sum_{m=0} a_{mn}z^m w^n$$

respectively.

2. Invariant subspaces as zw-invariant subspaces

Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. Since $z^n\mathfrak{M} \supseteq z^{n+1}\mathfrak{M}$ (resp. $w^n\mathfrak{M} \supseteq w^{n+1}\mathfrak{M}$) for $n \in \mathbb{Z}_+$, $\bigcap_{n=1}^{\infty} z^n\mathfrak{M}$ (resp. $\bigcap_{n=1}^{\infty} w^n\mathfrak{M}$) is also an invariant subspace. If $\bigcap_{n=1}^{\infty} z^n\mathfrak{M} = 0$ (resp. $\bigcap_{n=1}^{\infty} w^n\mathfrak{M} = 0$), we say that $\mathfrak{M}$ is z-pure (resp. w-pure). When $z\mathfrak{M} = \mathfrak{M}$ (resp. $w\mathfrak{M} = \mathfrak{M}$), we say that $\mathfrak{M}$ is z-reducing (resp. w-reducing). The structure of z-reducing (resp. w-reducing) invarinat subspaces has been characterized in [5].

Since $\mathfrak{M}$ is an invariant subspace, $\mathfrak{M}$ is also a zw-invariant subspace and

$$(zw)^n\mathfrak{M} \supseteq (zw)^{n+1}\mathfrak{M} \quad n \in \mathbb{Z}_+.$$

If $\bigcap_{n=1}^{\infty} (zw)^n\mathfrak{M} = 0$, we say that $\mathfrak{M}$ is zw-pure. If $zw\mathfrak{M} = \mathfrak{M}$, we say that $\mathfrak{M}$ is zw-reducing. We have the following proposition.

Proposition 1. *Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. Then:*

(i) *If $\mathfrak{M}$ is either z-pure or w-pure, then $\mathfrak{M}$ is zw-pure.*

(ii) *$\mathfrak{M}$ is zw-reducing if and only if $\mathfrak{M}$ is z-reducing and w-reducing.*

(iii) *$\mathfrak{M}$ is not zw-pure if and only if $\mathfrak{M}$ is zw-reducing.*

Proof. The proof of (i) and (ii) is clear. Therefore we only prove (iii). Put $\mathfrak{M}_1 = \bigcap_{n=1}^{\infty} (zw)^n\mathfrak{M}$ and $\mathfrak{M}_2 = \mathfrak{M} \ominus \mathfrak{M}_1$. If $\mathfrak{M}_1 \neq 0$, then we easily show that both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are invariant subspaces and $\mathfrak{M}_1$ is zw-reducing. Since $z\mathfrak{M}_1 = \mathfrak{M}_1$ and $w\mathfrak{M}_1 = \mathfrak{M}_1$, as in the proof of Proposition 3 in [5], we have $z\mathfrak{M}_2 = \mathfrak{M}_2$ and $w\mathfrak{M}_2 = \mathfrak{M}_2$. This implies that $\mathfrak{M}$ is zw-reducing. This proof is complete. □

If $\mathfrak{M}$ is zw-reducing, then by [4] and [5] the form of $\mathfrak{M}$ is well-known. Throughout this note, we assume without loss of generality that $\mathfrak{M}$ is zw-pure. Put $\mathfrak{F} = \mathfrak{M} \ominus zw\mathfrak{M}$, $\mathfrak{S}_z = \mathfrak{M} \ominus z\mathfrak{M}$ and $\mathfrak{S}_w = \mathfrak{M} \ominus w\mathfrak{M}$ respectively. Note that $\mathfrak{F} = \mathfrak{S}_z \oplus z\mathfrak{S}_w = \mathfrak{S}_w \oplus w\mathfrak{S}_z$ and $\mathfrak{M} = \sum_{n=0}^{\infty} \oplus (zw)^n\mathfrak{F}$. Let $\mathfrak{F}_z$ (resp. $\mathfrak{F}_w$) be the largest

z-invariant (resp. w-invariant) subspace in $\mathfrak{F}$. It is clear that $\mathfrak{F}_z = \bigcap_{n=0}^{\infty} \overline{z}^n \mathfrak{F}$ and $\mathfrak{F}_w = \bigcap_{n=0}^{\infty} \overline{w}^n \mathfrak{F}$.

Proposition 2. *Let $\mathfrak{M}$ be a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$. Then:*

(i) *$z\mathfrak{F}_z \subsetneqq \mathfrak{F}_z$ if and only if there exists a unimodular function $\phi_z \in L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_z = \phi_z H^2(z)$.*

(ii) *$\mathfrak{F}_z = z\mathfrak{F}_z \neq 0$ if and only if $\mathfrak{M} = \chi_E q H_w^2$, where q is a unimodular function of $L^\infty(\mathbb{T}^2)$, and χ_E is the characteristic function of a Borel subset E of $\mathbb{T}^2$ with $\chi_E \in L_z^2$ and $\chi_E \neq 0$. In this case, $\mathfrak{F} = \mathfrak{F}_z$.*

Proof. (i) If $\mathfrak{F}_z = \phi_z H^2(z)$ for some unimodular function ϕ_z in $L^\infty(\mathbb{T}^2)$, then it is clear that $z\mathfrak{F}_z \subsetneqq \mathfrak{F}_z$.

Conversely, suppose that $z\mathfrak{F}_z \subsetneqq \mathfrak{F}_z$. Put $\mathfrak{F}^0 = \mathfrak{F}_z \ominus z\mathfrak{F}_z$. Take $f, g \in \mathfrak{F}^0$. Since $z\mathfrak{F}_z \subseteq \mathfrak{F}_z$, $z\mathfrak{F}^0 \perp \mathfrak{F}^0$ and $\mathfrak{F}_z \perp zw\mathfrak{M}$, we have, for $(m,n) \in \mathbb{Z}^2$ with $(m,n) \neq (0,0)$,

$$(f, z^m w^n g) = \begin{cases} (z^{n-m} f, (zw)^n g) = 0, & m \leq n, \\ (f, (zw)^n z^{m-n} g) = 0, & m > n. \end{cases}$$

It follows that $f\overline{g}$ is constant. In particular, $f\overline{f}$ is constant and $f = \lambda g$ for some $\lambda \in \mathbb{C}$. Hence the dimension of $\mathfrak{F}^0$ is 1, that is, there exists a unimodular function ϕ_z in $L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}^0 = [\phi_z]$. Let $\mathfrak{N} = \bigcap_{n=0}^{\infty} z^n \mathfrak{F}_z$. We have $z\mathfrak{N} = \mathfrak{N}$ and $\mathfrak{F}_z = \sum_{n=0}^{\infty} \oplus z^n \mathfrak{F}^0 \oplus \mathfrak{N} = \phi_z H^2(z) \oplus \mathfrak{N}$.

We next prove that $\mathfrak{N} = 0$. Let $\mathfrak{M}_1 = \sum_{n=0}^{\infty} \oplus w^n \mathfrak{N}$; then $\mathfrak{M}_1$ is an invariant subspace with $z\mathfrak{M}_1 = \mathfrak{M}_1$ and $w\mathfrak{M}_1 \subseteq \mathfrak{M}_1$. On the other hand, we have

$$\sum_{n=0}^{\infty} \oplus w^n \mathfrak{F}_z = \sum_{n=0}^{\infty} \oplus w^n \phi_z H^2(z) \oplus \sum_{n=0}^{\infty} \oplus w^n \mathfrak{N} = \phi_z H^2(\mathbb{T}^2) \oplus \mathfrak{M}_1.$$

Since $\phi_z H^2(\mathbb{T}^2)$ and $\mathfrak{M}_1$ are mutually orthogonal invariant subspaces and ϕ_z is unimodular, it is easy to see that $\mathfrak{M}_1 = 0$. Thus $\mathfrak{N} = 0$, and so $\mathfrak{F}_z = \phi_z H^2(z)$.

(ii) If $\mathfrak{M}$ is of the form $\chi_E q H_w^2$, where q is a unimodular function in $L^\infty(\mathbb{T}^2)$, and χ_E is the characteristic function of a Borel subset E of $\mathbb{T}^2$ with $\chi_E \in L_z^2$, $\chi_E \neq 0$, then it is clear that $\mathfrak{F} = \mathfrak{F}_z = z\mathfrak{F}_z \neq 0$.

Conversely, suppose $\mathfrak{F}_z = z\mathfrak{F}_z \neq 0$. It is known that

$$\begin{aligned} \mathfrak{M} &= \sum_{n=0}^{\infty} \oplus (zw)^n \mathfrak{F} = \sum_{n=0}^{\infty} \oplus (zw)^n \mathfrak{F}_z \oplus \sum_{n=0}^{\infty} \oplus (zw)^n (\mathfrak{F} \ominus \mathfrak{F}_z) \\ &= \sum_{n=0}^{\infty} \oplus w^n \mathfrak{F}_z \oplus \sum_{n=0}^{\infty} \oplus (zw)^n (\mathfrak{F} \ominus \mathfrak{F}_z). \end{aligned}$$

Put $\mathfrak{M}_1 = \sum_{n=0}^{\infty} \oplus w^n \mathfrak{F}_z$ and $\mathfrak{M}_2 = \sum_{n=0}^{\infty} \oplus (zw)^n (\mathfrak{F} \ominus \mathfrak{F}_z)$, respectively. Then $\mathfrak{M}_1$ is an invariant subspace with $z\mathfrak{M}_1 = \mathfrak{M}_1$. We next prove that $\mathfrak{M}_2$ is also an invariant subspace. In fact, since $z\mathfrak{M}_1 = \mathfrak{M}_1$, we have, for every $f \in \mathfrak{M}_1$ and $g \in \mathfrak{M}_2$,

$$(f, zg) = (\overline{z} f, g) = 0,$$

which implies that $zg \in \mathfrak{M}_2$. Hence $z\mathfrak{M}_2 \subseteq \mathfrak{M}_2$. Moreover, for $f \in \mathfrak{F}_z$, $g \in \mathfrak{F} \ominus \mathfrak{F}_z$ and $(m, n) \in \mathbb{Z}_+^2$,

$$(w^n f, w(zw)^m g) = (w^n zf, (zw)^{m+1} g) = 0$$

because $w^n zf \in \mathfrak{M}_1$ and $(zw)^{m+1} g \in \mathfrak{M}_2$. Hence $w\mathfrak{M}_2 \subseteq \mathfrak{M}_2$. It follows that $\mathfrak{M}_2$ is an invariant subspace.

Now we have $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$, and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are invariant subspaces with $z\mathfrak{M}_1 = \mathfrak{M}_1$. As in the proof of Proposition 3 in [5], we have $z\mathfrak{M}_2 = \mathfrak{M}_2$. However, we have

$$\mathfrak{M}_2 = \sum_{n=0}^{\infty} (zw)^n (\mathfrak{F} \ominus \mathfrak{F}_z) = z\mathfrak{M}_2 = \sum_{n=0}^{\infty} \oplus (zw)^n (z(\mathfrak{F} \ominus \mathfrak{F}_z)).$$

Hence we have $z(\mathfrak{F} \ominus \mathfrak{F}_z) = \mathfrak{F} \ominus \mathfrak{F}_z$. Since $\mathfrak{F}_z$ is the largest z-invariant subspace in $\mathfrak{F}$, we have $\mathfrak{F} \ominus \mathfrak{F}_z = 0$, that is, $\mathfrak{F} = \mathfrak{F}_z$ and $w\mathfrak{M} = \mathfrak{M}$. By Proposition 3 in [5]

$$\mathfrak{M} = \chi_{_E} q H_w^2,$$

where q is unimodular, and $\chi_{_E}$ is the characteristic function of a Borel subset E of $\mathbb{T}^2$ with $\chi_{_E} \in L_z^2$ and $\chi_{_E} \neq 0$. This proof is complete. □

Similarly, we have the following result about $\mathfrak{F}_w$.

Proposition 3. *Let $\mathfrak{M}$ be a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$. Then:*

(i) *$w\mathfrak{F}_w \subsetneqq \mathfrak{F}_w$ if and only if there exists a unimodular function $\phi_w \in L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_w = \phi_w H^2(w)$.*

(ii) *$\mathfrak{F}_w = w\mathfrak{F}_w \neq 0$ if and only if $\mathfrak{M} = \chi_{_E} q H_z^2$, where q is unimodular, and $\chi_{_E}$ is the characteristic function of a Borel subset E of $\mathbb{T}^2, \chi_{_E} \in L_w^2$. In this case, $\mathfrak{F} = \mathfrak{F}_w$.*

If $\mathfrak{M}$ is a zw-pure invariant subspace with $\mathfrak{F}_z \neq 0$ and $\mathfrak{F}_w \neq 0$, then we have that $z\mathfrak{F}_z \subsetneqq \mathfrak{F}_z$ and $w\mathfrak{F}_w \subsetneqq \mathfrak{F}_w$. Otherwise, for example, assume that $z\mathfrak{F}_z = \mathfrak{F}_z \neq 0$; then by (ii) of Proposition 2 we have that $\mathfrak{M} = \chi_{_E} q H_w^2$. It easily follows that $\mathfrak{F}_w = 0$. We have a contradiction. Thus there exist two unimodular functions ϕ_z and ϕ_w in $L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_z = \phi_z H^2(z)$ and $\mathfrak{F}_w = \phi_w H^2(w)$. In particular, $\phi_z H^2(\mathbb{T}^2) + \phi_w H^2(\mathbb{T}^2) \subseteq \mathfrak{M}$. Put

$$\mathfrak{M}^0 = [\phi_z H^2(\mathbb{T}^2) + \phi_w H^2(\mathbb{T}^2)].$$

It is clear that $\mathfrak{M}^0$ is a zw-pure invariant subspace of $\mathfrak{M}$. Put $\mathfrak{F}^0 = \mathfrak{M}^0 \ominus zw\mathfrak{M}^0$. Let $(\mathfrak{F}^0)_z$ (resp. $(\mathfrak{F}^0)_w$) be the largest z-invariant (resp. w-invariant) subspace in $\mathfrak{F}^0$. Thus we have the following proposition.

Proposition 4. *Keep the notations and assumptions as above. Then $\mathfrak{F}_z = (\mathfrak{F}^0)_z$ and $\mathfrak{F}_w = (\mathfrak{F}^0)_w$.*

Proof. Clearly, we have $[\mathfrak{F}_z + \mathfrak{F}_w] \subset \mathfrak{M}^0$. Since $[\mathfrak{F}_z + \mathfrak{F}_w] \perp zw\mathfrak{M}$, then $[\mathfrak{F}_z + \mathfrak{F}_w] \subset \mathfrak{F}^0$. Therefore, $\mathfrak{F}_z \subseteq (\mathfrak{F}^0)_z$. By Proposition 2, there exists a unimodular function ϕ_z^0 in $L^\infty(\mathbb{T}^2)$ such that $(\mathfrak{F}^0)_z = \phi_z^0 H^2(z)$. Thus $\phi_z H^2(z) \subseteq \phi_z^0 H^2(z)$. Let $\phi_z = \phi_z^0 h$ for some inner function $h \in H^2(z)$. Then $(z^m \phi_z^0, zwg) = (z^m \phi_z, zwhg) = 0$ for every $g \in \mathfrak{M}$ and $m \geq 0$. Thus $z^m \phi_z^0 \in \mathfrak{F}$, and hence $(\mathfrak{F}^0)_z \subseteq \mathfrak{F}_z$ because of the maximality of $\mathfrak{F}_z$. Hence, $\mathfrak{F}_z = (\mathfrak{F}^0)_z$. Similarly, we have $\mathfrak{F}_w = (\mathfrak{F}^0)_w$. This proof is complete. □

If $w\mathfrak{F}_z \subsetneqq \mathfrak{F}_w$, then $\mathfrak{F}_w = \phi_w H^2(\mathbb{T}^2)$ for some unimodular function ϕ_w of $L^\infty(\mathbb{T}^2)$. Putting $\widetilde{\mathfrak{M}} = \overline{\phi}_w \mathfrak{M}$, $\widetilde{\mathfrak{M}}$ is also an invariant subspace of $L^2(\mathbb{T}^2)$. Let $\widetilde{\mathfrak{F}} = \widetilde{\mathfrak{M}} \ominus zw\widetilde{\mathfrak{M}}$. Then $(\widetilde{\mathfrak{F}})_z$ (resp. $(\widetilde{\mathfrak{F}})_w$) is the largest z-invariant (resp. w-invariant) subspace in $\widetilde{\mathfrak{F}}$. Then we easily have the following proposition, and so omit the proof.

Proposition 5. *Keep the notations and assumptions as above.*

(i) $H^2(\mathbb{T}^2) \subseteq \widetilde{\mathfrak{M}} \subseteq H^2_w$.

(ii) $(\widetilde{\mathfrak{F}})_w = H^2(w)$. *Moreover, if* $(\widetilde{\mathfrak{F}})_z \neq 0$, *then* $(\widetilde{\mathfrak{F}})_z$ *is of the form* $qH^2(z)$ *for some unimodular function* q *which satisfies* $\widehat{q}(m,n) = 0$ *for every* $(m,n) \notin \mathbb{Z}_+ \times -\mathbb{Z}_+$.

3. Beurling-type invariant subspaces

If $\mathfrak{M}$ is a Beurling-type invariant subspace of $L^2(\mathbb{T}^2)$, then it is clear that $\mathfrak{F}_z \bigcap \mathfrak{F}_w \neq 0$. In this section, we consider whether the converse is valid. Further, we shall give an alternative characterization of Beurling-type invariant subspaces and prove that the condition is necessary and sufficient.

Theorem 6. *Let* $\mathfrak{M}$ *be an invariant subspace of* $L^2(\mathbb{T}^2)$. *Then the following assertions are equivalent.*

(i) $\mathfrak{M}$ *is of the form* $\phi H^2(\mathbb{T}^2)$ *for some unimodular function* $\phi \in L^\infty(\mathbb{T}^2)$.

(ii) $\dim(\mathfrak{F}_z \bigcap \mathfrak{F}_w) = 1$.

(iii) $\mathfrak{F}_z \bigcap \mathfrak{F}_w \neq 0$.

(iv) $z\mathfrak{F}_z \subsetneqq \mathfrak{F}_z$, $w\mathfrak{F}_w \subsetneqq \mathfrak{F}_w$ *and* $\mathfrak{F} = \mathfrak{F}_z + \mathfrak{F}_w$.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is clear.

(iii) $\Longrightarrow$ (ii). Let $\mathfrak{F}_z = \phi_z H^2(z)$ and $\mathfrak{F}_w = \phi_z H^2(w)$. Then $\mathfrak{F}_z = [\phi_z] \oplus z\mathfrak{F}_z$ and $\mathfrak{F}_w = [\phi_w] \oplus w\mathfrak{F}_w$. Let $f \in \mathfrak{F}_z \bigcap \mathfrak{F}_w$. Then there exist complex numbers a and b in $\mathbb{C}$ such that $f = a\phi_z + zg = b\phi_w + wh$ for some $g \in \mathfrak{F}_z$ and $h \in \mathfrak{F}_w$. Since $wf \in \mathfrak{F}_w$, we have

$$\begin{aligned}(g,g) &= (zg,zg) = (zg,zg) + (a\phi_z, zg) = (zg + a\phi_z, zg)\\ &= (f, zg) = (wf, zwg) = 0.\end{aligned}$$

It follows that $g = 0$. Similarly we have $h = 0$. Thus $f = a\phi_z = b\phi_w$. Hence $\dim(\mathfrak{F}_z \bigcap \mathfrak{F}_w) = 1$.

(ii) $\Longrightarrow$ (iv). Without loss of generality, we may assume that $\phi_z = \phi_w = \phi$. In this case, $\mathfrak{F}_z = \phi H^2(z)$ and $\mathfrak{F}_w = \phi H^2(w)$. Put $\mathfrak{F}_0 = \mathfrak{F}_z + \mathfrak{F}_w$ $(= \phi H^2(z) + \phi H^2(w) = \phi(H^2(z) + H^2(w)))$. Since $\mathfrak{F}_0 \subseteq \mathfrak{F}$, we have

$$\begin{aligned}\mathfrak{M} &= \sum_{n=0}^{\infty} \oplus (zw)^n \mathfrak{F}_0 \oplus \sum_{n=0}^{\infty} \oplus (zw)^n (\mathfrak{F} \ominus \mathfrak{F}_0)\\ &= \phi H^2(\mathbb{T}^2) \oplus \sum_{n=0}^{\infty} \oplus (zw)^n (\mathfrak{F} \ominus \mathfrak{F}_0).\end{aligned}$$

For every $f \in \mathfrak{F} \ominus \mathfrak{F}_0$, we know that $(zw)^n f \perp \phi H^2(\mathbb{T}^2)$ for every $n \in \mathbb{Z}$. It follows that $(zw)^n \overline{\phi} f \perp H^2(\mathbb{T}^2)$ for every $n \in \mathbb{Z}$, which implies that $\overline{\phi} f = 0$. Since ϕ is unimodular, we have $f = 0$ and so $\mathfrak{F} = \mathfrak{F}_z + \mathfrak{F}_w$.

(iv) $\Longrightarrow$ (i). Assume that $\mathfrak{F} = \mathfrak{F}_z + \mathfrak{F}_w$. Then $\mathfrak{F} = \phi_z H^2(z) + \phi_w H^2(w) = [\phi_z, \phi_w] \oplus z\mathfrak{F}_z \oplus w\mathfrak{F}_w$. It is known that $\mathfrak{F} = \mathfrak{S}_z \oplus z\mathfrak{S}_w = \mathfrak{S}_w \oplus w\mathfrak{S}_z$ and $\mathfrak{F}_z \subseteq \mathfrak{S}_w \subseteq \mathfrak{F}$. Since $\mathfrak{S}_w \perp w\mathfrak{M}$, we have $\mathfrak{S}_w \subseteq [\phi_z, \phi_w] \oplus z\mathfrak{F}_z = [\phi_w] + \mathfrak{F}_z$.

If $\mathfrak{F}_z = \mathfrak{S}_w$, then by Theorem 5 in [4], $\mathfrak{M} = \phi H^2(\mathbb{T}^2)$ for some unimodular function ϕ in $L^\infty(\mathbb{T}^2)$. Otherwise, we have $\mathfrak{S}_w = [\phi_w] + \mathfrak{F}_z$. In this case, $w\mathfrak{S}_z = w\mathfrak{F}_w$, which is equivalent to $\mathfrak{S}_z = \mathfrak{F}_w$. Again by Theorem 5 in [4] we have $\mathfrak{M} = \phi H^2(\mathbb{T}^2)$ for some unimodular function ϕ in $L^\infty(\mathbb{T}^2)$. This proof is complete. □

4. Certain classes of invariant subspaces

Keep the notations as in §2. Suppose that $\mathfrak{F}_z \neq 0$ and $\mathfrak{F}_w \neq 0$. In general, we have $\mathfrak{F}_z + \mathfrak{F}_w \subseteq [\mathfrak{S}_w + \mathfrak{S}_z] \subseteq \mathfrak{F}$. Theorem 6 says that $\mathfrak{M}$ is a Beurling-type invariant subspace if and only if $\mathfrak{F} = \mathfrak{F}_z + \mathfrak{F}_w$. In this case, it is clear that $\mathfrak{F}_z + \mathfrak{F}_w = \mathfrak{S}_w + \mathfrak{S}_z$. In this section, we study invariant subspaces with the property $\mathfrak{F}_z + \mathfrak{F}_w = \mathfrak{S}_w + \mathfrak{S}_z$. We shall study a special class of invariant subspaces with this property.

For $\alpha \in \mathbb{D}$, we define a function ψ_α by

$$\psi_\alpha(z, w) = \frac{z\overline{w} - \alpha}{1 - \overline{\alpha} z\overline{w}}.$$

Then ψ_α is a unimodular function in $L^\infty(\mathbb{T}^2)$ with $\widehat{\psi}_\alpha(m, n) = 0$ for every $(m, n) \notin \mathbb{Z}_+ \times -\mathbb{Z}_+$. Then we define an invariant subspace $\mathfrak{M}_\alpha$ of H^2_w by

$$\mathfrak{M}_\alpha = [H^2(\mathbb{T}^2) + \psi_\alpha H^2(\mathbb{T}^2)].$$

Lemma 7. *If $\mathfrak{M} = \mathfrak{M}_\alpha$, then $\mathfrak{F}_w = H^2(w), \mathfrak{F}_z = \psi_\alpha H^2(z), \mathfrak{F}_z + \mathfrak{F}_w = \mathfrak{S}_w + \mathfrak{S}_z$ and $\mathfrak{F} = \mathfrak{F}_z + \mathfrak{F}_w + [z]$.*

Proof. It is clear that $H^2(w) \subset \mathfrak{F}_w$ and $\psi_\alpha H^2(z) \subset \mathfrak{F}_z$. Thus we have $H^2(w) \subset \mathfrak{S}_z$ and $\psi_\alpha H^2(z) \subset \mathfrak{S}_w$. We next show that $\mathfrak{S}_z = H^2(w) + [\psi_\alpha]$. Since $(\psi_\alpha, zg) = 0$ for every $g \in \mathfrak{M}$, we have $H^2(w) + [\psi_\alpha] \subseteq \mathfrak{S}_z$. Let $\mathfrak{N} = H^2(w) + [\psi_\alpha] \oplus z\mathfrak{M}$. Then it is enough to show that $\mathfrak{N} = \mathfrak{M}$. Since $H^2(\mathbb{T}^2) + z\psi_\alpha H^2(\mathbb{T}^2) \subset \mathfrak{N}$, we only need to show that $\psi_\alpha H^2(w) \subset \mathfrak{N}$. In fact,

$$w\psi_\alpha = w\left(\frac{z\overline{w} - \alpha}{1 - \overline{\alpha} z\overline{w}}\right) = w(z\overline{w} - \alpha)\left(1 + \frac{\overline{\alpha} z\overline{w}}{1 - \overline{\alpha} z\overline{w}}\right) = z - \alpha w + \overline{\alpha} z\psi_\alpha.$$

Thus we have $w\psi_\alpha \in \mathfrak{N}$. Moreover, $w^n\psi_\alpha = w^{n-1}w\psi_\alpha = zw^{n-1} - \alpha w^n + \overline{\alpha} zw^{n-1}\psi_\alpha \in \mathfrak{N}$. This implies that $\mathfrak{M} = \mathfrak{N}$, and so $\mathfrak{S}_z = H^2(w) + [\psi_\alpha]$. Similarly, we have $\mathfrak{S}_w = \psi_\alpha H^2(z) + [1]$. Therefore,

$$\mathfrak{F} = \mathfrak{S}_z \oplus z\mathfrak{S}_w = (H^2(w) + [\psi_\alpha]) \oplus z(\psi_\alpha H^2(z) + [1]) = H^2(w) + \psi_\alpha H^2(z) + [z].$$

It follows that $\mathfrak{F}_z = \psi_\alpha H^2(z)$ and $\mathfrak{F}_w = H^2(w)$. This proof is complete. □

Theorem 8. *Let $\mathfrak{M}$ be a zw-pure invariant subspace. If $\mathfrak{F}_z \neq 0$ and $\mathfrak{F}_w \neq 0$, then $\mathfrak{F}_z + \mathfrak{F}_w = [\mathfrak{S}_w + \mathfrak{S}_z]$ if and only if one of the following conditions is valid.*

(i) *$\mathfrak{M} = qH^2(\mathbb{T}^2)$ for some unimodular function q in $L^\infty(\mathbb{T}^2)$.*

(ii) *$\mathfrak{M} = q\mathfrak{M}_\alpha$ for some unimodular function q in $L^\infty(\mathbb{T}^2)$ and $\alpha \in \mathbb{D}$.*

Proof. If $\mathfrak{F}_z \bigcap \mathfrak{F}_w \neq 0$, then by Theorem 6 we have $\mathfrak{M} = qH^2(\mathbb{T}^2)$ for some unimodular function q in $L^\infty(\mathbb{T}^2)$. Assume that $\mathfrak{F}_z \bigcap \mathfrak{F}_w = 0$. Without loss of generality, we may assume that $\mathfrak{F}_w = H^2(w)$ and $\mathfrak{F}_z = \phi_z H^2(z)$ for some unimodular function ϕ_z in $L^\infty(\mathbb{T}^2)$. We shall prove that $\phi_z = \theta\psi_\alpha$ for some $\theta \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. It is clear that $\mathfrak{F}_z \subsetneqq \mathfrak{S}_w$, $\mathfrak{F}_w \subsetneqq \mathfrak{S}_z$. By Proposition 5, $\widehat{\phi}_z(m, n) = 0$ if $(m, n) \notin \mathbb{Z}_+ \times -\mathbb{Z}_+$. Because $\mathfrak{F}_z + \mathfrak{F}_w = [\mathfrak{S}_w + \mathfrak{S}_z]$, we have $\mathfrak{S}_w = \mathfrak{F}_z + [1]$ and $\mathfrak{S}_z = \mathfrak{F}_w + [\phi_z]$, which implies that $[1, \phi_z] \subset \mathfrak{S}_z \bigcap \mathfrak{S}_w$. Thus

$$(z^m, \phi_z) = 0 \quad (m \geq 1) \quad \text{and} \quad (1, w^n \phi_z) = 0 \quad (n \geq 1).$$

It follows that $\widehat{\phi}_z(m,0) = \widehat{\phi}_z(0,-n) = 0$ for $m \geq 1$ and $n \geq 1$. Put $\widehat{\phi}_z(0,0) = a_{00}$ and $\phi_0 = \phi_z - a_{00}$. Since $\mathfrak{F} = \mathfrak{S}_w \oplus w\mathfrak{S}_z = \mathfrak{S}_z \oplus z\mathfrak{S}_w = \mathfrak{F}_z + [1] + w\mathfrak{F}_w + [w\phi_z] = \mathfrak{F}_z + \mathfrak{F}_w + [w\phi_z] = \mathfrak{F}_z + \mathfrak{F}_w + [z]$, we have $\dim(\mathfrak{F} \ominus [\mathfrak{F}_z + \mathfrak{F}_w]) = 1$ and $[z, w\phi_z] \subset \mathfrak{F}$. It follows that $w\phi_0 \in \mathfrak{F}$. It is clear that $w\phi_0 \perp \mathfrak{F}_w$. Moreover,

$$(w\phi_0, z^m\phi_z) = (w(\phi_z - a_{00}), z^m\phi_z) = -a_{00}(w, z^m\phi_z) = 0,$$

which implies that $w\phi_0 \perp \mathfrak{F}_z$. It follows that $w\phi_0 \perp [\mathfrak{F}_z + \mathfrak{F}_w]$ and $\mathfrak{F} = [\mathfrak{F}_z + \mathfrak{F}_w] \oplus [w\phi_0]$.

On the other hand, since $z \in \mathfrak{F}$ and $z \perp [\phi_z] + z^2\mathfrak{F}_z + \mathfrak{F}_w$, it follows that $z \in ([z\phi_z] + [w\phi_0])$. Hence

$$z = \gamma z\phi_z + \delta w\phi_0 = \gamma z\phi_z + \delta w(\phi_z - a_{00}) = \gamma z\phi_z + \delta w\phi_z - \delta a_{00} w$$

for some constants γ and δ in $\mathbb{C}$. Thus $\phi_z(\gamma z + \delta w) = z + \delta a_{00} w$. We know that ϕ_z is unimodular, and so

$$\phi_z = \frac{z + \delta a_{00} w}{\gamma z + \delta w} = \frac{z\overline{w} + \delta a_{00}}{\delta + \gamma z\overline{w}} \qquad \text{a.e.}$$

Put

$$h(\lambda) = \frac{\lambda + \delta a_{00}}{\delta + \gamma\lambda}.$$

Then $\phi_z(z,w) = h(z\overline{w})$. Since ϕ_z is not constant and $\widehat{\phi}_z(m,n) = 0$ for every $(m,n) \notin \mathbb{Z}_+ \times -\mathbb{Z}_+$, we know that h is a Blaschke product; that is,

$$h(\lambda) = \theta \frac{\lambda + \alpha}{1 + \overline{\alpha}\lambda}$$

for some constants $\theta \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. Thus $\phi_z(z,w) = h(z\overline{w}) = \theta\psi_\alpha(z,w)$, that is, $\phi_z = \theta\psi_\alpha$. Hence $\mathfrak{M} = \mathfrak{M}_\alpha$. The converse follows by Theorem 6 and Lemma 7. This proof is complete. □

Let $H_0^2(\mathbb{T}^2)$ be as before. Then we have the following corollary.

Corollary 9. *Let $\mathfrak{M}$ be a zw-pure invariant subspace such that $\mathfrak{F}_z \neq 0$ and $\mathfrak{F}_w \neq 0$. Then $\mathfrak{M} = qH_0^2(\mathbb{T}^2)$ for some unimodular function q in $L^\infty(\mathbb{T}^2)$ if and only if $\mathfrak{F}_z + \mathfrak{F}_w = [\mathfrak{S}_w + \mathfrak{S}_z]$ and $\mathfrak{F}_z \perp \mathfrak{F}_w$.*

Proof. If $\mathfrak{F}_z + \mathfrak{F}_w = [\mathfrak{S}_w + \mathfrak{S}_z]$ and $\mathfrak{F}_z \perp \mathfrak{F}_w$, then by Theorem 6 we know that $\mathfrak{M}$ is not Beurling-type. Thus by Theorem 8 we have $\mathfrak{M} = q\mathfrak{M}_\alpha$ for some unimodular function q in $L^\infty(\mathbb{T}^2)$. Without loss of generality, we may assume that $\mathfrak{M} = \mathfrak{M}_\alpha$ for some $\alpha \in \mathbb{D}$. In the proof of Theorem 8, note that $a_{00} = 0$ if $\mathfrak{F}_z \perp \mathfrak{F}_w$. In this case, $\alpha = 0$ and $\phi_0(z,w) = z\overline{w}$. Let $q = \overline{w}$; then $\mathfrak{M} = qH_0^2(\mathbb{T}^2)$. The converse is clear. This proof is complete. □

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(G. Ji and T. Ohwada) Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata, 950-21, Japan

E-mail address: `ji@dmis.gs.niigata-u.ac.jp`

E-mail address: `ohwada@dmis.gs.niigata-u.ac.jp`

(K.-S. Saito) Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-21, Japan

E-mail address: `saito@math.sc.niigata-u.ac.jp`