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WANDERING VECTORS FOR IRRATIONAL ROTATION UNITARY SYSTEMS

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ABSTRACT. An abstract characterization for those irrational rotation unitary systems with complete wandering subspaces is given. We prove that an irrational rotation unitary system has a complete wandering vector if and only if the von Neumann algebra generated by the unitary system is finite and shares a cyclic vector with its commutant. We solve a factorization problem of Dai and Larson negatively for wandering vector multipliers, and strengthen this by showing that for an irrational rotation unitary system $\mathcal{U}$, every unitary operator in $w^*(\mathcal{U})$ is a wandering vector multiplier. Moreover, we show that there is a class of wandering vector multipliers, induced in a natural way by pairs of characters of the integer group $\mathbb{Z}$, which fail to factor even as the product of a unitary in $\mathcal{U}'$ and a unitary in $w^*(\mathcal{U})$. Incomplete maximal wandering subspaces are also considered, and some questions are raised.

An important class of operator algebras is the class of irrational rotation C*-algebras, which has been systematically studied over the past 15 years. These algebras have several equivalent definitions (see [10]). One is that they are exactly the C*-algebras $\mathcal{A}_\theta$ generated by a pair of unitary elements u and v which satisfy the relation $uv = \lambda vu$, where $\lambda = exp(2\pi i\theta)$ and $\theta \in (0, 1)$ is an irrational number. We will call the set $\mathcal{U} = \{u^n v^m : (n, m) \in \mathbb{Z} \times \mathbb{Z}\}$ an *(abstract) irrational rotation unitary system*, where $\mathbb{Z}$ is the set of all integers. It is a *proper* subset of the group generated by u and v. If $\mathcal{B}$ is a C*-algebra and a, b are two elements in $\mathcal{B}$ satisfying the relation $ab = exp(2\pi i\theta)ba$, then it is known that there is a faithful *-isomorphism π from $\mathcal{A}_\theta$ into $\mathcal{B}$ satisfying $\pi(u) = a$ and $\pi(v) = b$ (see [4] or [9]).

Following Dai and Larson [2], a *unitary system* $\mathcal{U}$ is a subset of the unitary operators acting on a separable Hilbert space H which contains the identity operator. A norm one element $\psi \in H$ is called a *wandering vector* for $\mathcal{U}$ if $\mathcal{U}\psi = \{U\psi : U \in \mathcal{U}\}$ is an orthonormal set; that is, $\langle U\psi, V\psi\rangle = 0$ if $U, V \in \mathcal{U}$ and $U \neq V$. If $\mathcal{U}\psi$ is an orthonormal basis for H, then ψ is called a *complete wandering vector* for $\mathcal{U}$. The set of all complete wandering vectors for $\mathcal{U}$ is denoted by $\mathcal{W}(\mathcal{U})$. More generally, a closed subspace M of H is called a *wandering subspace* of $\mathcal{U}$ if UM and VM are orthogonal for any different U and V in $\mathcal{U}$. A wandering subspace M is called *complete* if $\overline{span}\{\mathcal{U}M\} = H$. The set of all the complete wandering subspaces for $\mathcal{U}$ is denoted by $\mathcal{WS}(\mathcal{U})$. More generally, a unital unitary subset $\mathcal{U}$ of a C*-algebra $\mathcal{A}$ is called an *abstract* unitary system. In this case, one is interested in representations π of $\mathcal{A}$ for which $\pi(\mathcal{U})$ has wandering subspaces.

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If U, V are unitary operators in $B(H)$, we write $\mathcal{U}_{U,V} = \{U^n V^m : (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$. Unitary systems of this form (but with different relations between the generators than we consider in this paper) have importance in wavelet theory. If U, V satisfy the relation $UV = \lambda VU$ with $\lambda = exp(2\pi i\theta)$ and $\theta \in (0,1)$ an irrational number, then we call $\mathcal{U}_{U,V}$ a (concrete) irrational rotation unitary system.

In this paper we are concerned with irrational rotation unitary systems and their wandering vectors, and more generally their wandering subspaces. We prove that, up to unitary equivalence, there is only one *-representation of an irrational rotation C*-algebra such that the image unitary system of the representation has a complete wandering vector. We will give an abstract characterization for those irrational rotation unitary systems which have complete wandering subspaces. We also show that Problem C in [2] has a negative answer for an irrational rotation unitary system with a complete wandering vector. (We note that an independent counterexample was given by Li, McCarthy and Timotin in [8]. They did not consider the irrational rotation C^*-algebras, but instead considered a generalization of a structure property of unitary systems given in [2]).

Let $\mathcal{S}$ be a subset of $B(H)$. We use $w^*(\mathcal{S})$ to denote the von Neumann algebra generated by $\mathcal{S}$, and as in [2] use $\mathbb{U}(\mathcal{S})$ to denote the set of all unitary operators in $\mathcal{S}$. The commutant of $\mathcal{S}$ is $\mathcal{S}' = \{T \in B(H) : TS - ST = 0, \forall S \in \mathcal{S}\}$. For a subset $\mathcal{M}$ of H, we use $[\mathcal{M}]$ to denote the closure of the linear span of $\mathcal{M}$. We use the term *coisometry* for the operator T when T^* is an isometry. Recall that two unitary systems $\mathcal{U}_i$ $(i = 1, 2)$ are *unitarily equivalent* if there is a unitary W such that $W\mathcal{U}_1 W^* = \mathcal{U}_2$. Two *-representations π_1 and π_2 of a C*-algebra $\mathcal{A}$ are called *unitarily equivalent* if there exists a corresponding unitary operator W such that $W\pi_1(a)W^* = \pi_2(a)$ for every $a \in \mathcal{A}$. If $\mathcal{U}$ is a unitary system and $\psi \in \mathcal{W}(\mathcal{U})$, the *local commutant* $C_\psi(\mathcal{U})$ at ψ is defined by $\{T \in B(H) : (TU - UT)\psi = 0, U \in \mathcal{U}\}$. A useful result is the one-to-one correspondence between the complete wandering vectors and the unitary operators in $C_\psi(\mathcal{U})$. In particular, if $\psi \in \mathcal{W}(\mathcal{U})$, then $\mathcal{W}(\mathcal{U}) = \mathbb{U}(C_\psi(\mathcal{U}))\psi = \{T\psi : T \in \mathbb{U}(C_\psi(\mathcal{U})\}$ (see [2], Proposition 1.3).

Theorem 1. *Let $\mathcal{A}_\theta$ be an irrational rotation C*-algebra with unitary generators u, v for which $uv = e^{2\pi i\theta} vu$ for some irrational number $\theta \in (0,1)$. Then, up to unitary equivalence, there exists a unique faithful *-representation π of $\mathcal{A}_\theta$ on a Hilbert space H such that the irrational rotation unitary system $\mathcal{U} = \{U^n V^m : n, m \in \mathbb{Z}\}$, where $U = \pi(u)$ and $V = \pi(v)$, has a complete wandering vector. Moreover, $\mathcal{W}(\mathcal{U})$ is a closed and connected subset of H and $\overline{span}(\mathcal{W}(\mathcal{U})) = H$.*

Proof. Let π_1 and π_2 be faithful *-representations on Hilbert spaces H_1 and H_2, respectively, such that $\mathcal{U}_{U_i,V_i}$ has a compete wandering vector ψ_i, where $U_i = \pi_i(u), V_i = \pi_i(v), i = 1, 2$. Since u, v are generators for $\mathcal{A}_\theta$, we only need to prove that there is a unitary operator W satisfying $WU_1W^* = U_2$ and $WV_1W^* = V_2$. For this purpose, write $\psi^{(i)}_{n,m} = U_i^n V_i^m \psi_i$ for $i = 1, 2$ and $n, m \in \mathbb{Z}$. Then $\{\psi^{(i)}_{n,m} : n, m \in \mathbb{Z}\}$ is an orthonormal basis for H_i. Define $W : H_1 \longrightarrow H_2$ by $W\psi^{(1)}_{n,m} = \psi^{(2)}_{n,m}$ for all n and m. Then W is a unitary operator, and we have

$$WU_1\psi^{(1)}_{n,m} = WU_1U_1^nV_1^m\psi_1 = U_2U_2^nV_2^m\psi_2 = U_2W\psi^{(1)}_{n,m}$$

and

$$\begin{aligned} WV_1\psi^{(1)}_{n,m} &= WV_1U_1^nV_1^m\psi_1 = e^{-2\pi in\theta}WU_1^nV_1^{m+1}\psi_1 \\ &= e^{-2\pi in\theta}U_2^nV_2^{m+1}\psi_2 = V_2U_2^nV_2^m\psi_2 = V_2W\psi^{(1)}_{n,m}. \end{aligned}$$

Thus $WU_1W^* = U_2$ and $WV_1W^* = V_2$, since these relations hold on an orthonormal basis for H_1. Hence π_1 and π_2 are unitarily equivalent.

For the existence of such a *-representation π, let us consider the following concrete unitary system. Let H be the Hilbert space $l^2(\mathbb{Z}\times\mathbb{Z})$, and let $e_{n,m}$ be the element of H which is 1 at (n,m) and 0 elsewhere. Define unitary operators U, V on H by $Ue_{m,n} = e_{m+1,n}$ and $Ve_{m,n} = \lambda^{-m}e_{m,n+1}$, where $\lambda = e^{2\pi i\theta}$. Then $UV = \lambda VU$ follows from

$$\begin{aligned} UVe_{m,n} &= U(\lambda^{-m}e_{m,n+1}) = \lambda^{-m}e_{m+1,n+1} \\ &= \lambda\lambda^{-(m+1)}e_{m+1,n+1} = \lambda VUe_{m,n}. \end{aligned}$$

Thus $\mathcal{U}_{U,V}$ is an irrational rotation unitary system. Let π be the faithful *-isomorphism from $\mathcal{A}_\theta$ into $B(H)$ such that $\pi(u) = U$ and $\pi(v) = V$. We will show that $\mathcal{W}(\mathcal{U})$ is a closed and connected subset of H and $\overline{span}(\mathcal{W}(\mathcal{U})) = H$, where $\mathcal{U} = \{U^nV^m : n,m\in\mathbb{Z}\}$.

We have that $\mathcal{U}e_{0,0} = \{e_{k,l} : k,l\in\mathbb{Z}\}$. So $e_{0,0}$ is a complete wandering vector for $\mathcal{U}$. Moreover, for any $m,n\in\mathbb{Z}$, we have $\mathcal{U}e_{m,n} = \{\lambda^{-ml}e_{m+k,n+l} : k,l\in\mathbb{Z}\}$, which is an orthonormal basis for H. Thus in fact $e_{m,n}\in\mathcal{W}(\mathcal{U})$ for all $n,m\in\mathbb{Z}$. So $\overline{span}\mathcal{W}(\mathcal{U}) = H$, since $\{e_{n,m} : n,m\in\mathbb{Z}\}$ is an orthonormal basis for H..

Let $A\in C_\psi(\mathcal{U})$ for some $\psi\in\mathcal{W}(\mathcal{U})$. The relation $UV = \lambda VU$ implies that $span(\mathcal{U})$ is an algebra. So for each $S,T\in\mathcal{U}$, we have $ST\in span(\mathcal{U})$. So $AS(T\psi) = (ST)A\psi = S(AT)\psi = SA(T\psi)$. Since $T\in\mathcal{U}$ is arbitrary and $\overline{span}(\mathcal{U}\psi) = H$, it follows that $AS = SA$. Thus $C_\psi(\mathcal{U})\subseteq\mathcal{U}'$. The inclusion "$\supseteq$" is trivial. Thus $C_\psi(\mathcal{U}) = \mathcal{U}'$. So $C_\psi(\mathcal{U})$ is a von Neumann algebra. Since the unitary group of a von Neumann algebra is norm connected, $\mathcal{W}(\mathcal{U}) = \mathbb{U}(\mathcal{U}')\psi$ is norm-pathwise connected.

We claim that the von Neumann algebra $w^*(\mathcal{U})$ generated by U and V is finite and so is its commutant $\mathcal{U}'$. Let $\psi\in\mathcal{W}(\mathcal{U})$ be arbitary. First we show that $\langle AB\psi,\psi\rangle = \langle BA\psi,\psi\rangle$ for all $A,B\in w^*(\mathcal{U})$. It is enough to verify that this holds for $A = U^nV^m, B = U^kV^l$ with $n,m,k,l\in\mathbb{Z}$, since the linear span of $\mathcal{U}$ is an algebra. In fact, this follows from

$$\begin{aligned} \langle U^nV^mU^kV^l\psi,\psi\rangle &= e^{-2mk\pi i\theta}\langle U^{n+k}V^{m+l}\psi,\psi\rangle \\ &= \begin{cases} 0, & (n+k,m+l)\neq(0,0), \\ e^{-2mk\pi i\theta}, & (n+k,m+l)=(0,0), \end{cases} \end{aligned}$$

and

$$\langle U^kV^lU^nV^m\psi,\psi\rangle = \begin{cases} 0, & (n+k,m+l)\neq(0,0), \\ e^{-2ln\pi i\theta}, & (n+k,m+l)=(0,0). \end{cases}$$

Thus ψ is a trace vector of $w^*(\mathcal{U})$. Note that ψ is also a cyclic vector for $w^*(\mathcal{U})$, since $\mathcal{U}\psi$ is an orthonormal basis for H. Thus, by Lemma 7.2.14 in [6], ψ is a joint cyclic trace vector for $w^*(\mathcal{U})$ and $\mathcal{U}'$. By Theorem 7.2.15 in [6], this implies that both $w^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras.

For the closedness of $\mathcal{W}(\mathcal{U})$, suppose that $\{\psi_n\}$ is a sequence in $\mathcal{W}(\mathcal{U})$ converging in norm to a vector η. Fix $\psi\in\mathcal{W}(\mathcal{U})$. Then by Proposition 1.3 in [2], since $C_\psi(\mathcal{U}) = \mathcal{U}'$, there are unitary operators $V_n\in\mathcal{U}'$ with $\psi_n = V_n\psi$. In order to show that $\eta\in\mathcal{W}(\mathcal{U})$, again by Propositon 1.3 in [2], it is enough to show that $\eta = W\psi$ for some unitary operator W in $\mathcal{U}'$.

Let $\{U_\lambda\}$ be a subnet of $\{V_n\}$ such that $U_\lambda\longrightarrow U_0$ in the weak operator topology for some operator $U_0\in\mathcal{U}'$. Then $U_\lambda\psi\to\eta$ in norm and $U_\lambda\psi\to U_0\psi$ in the weak topology on H. So $\eta = U_0\psi$. Now for any $x \in H$, we have $|\langle U_\lambda^*(U_\lambda\psi - U_0\psi),\ x\rangle| \leq \|U_\lambda\psi - U_0\psi\|\|x\| \longrightarrow 0$ and $\langle U_0\psi, U_\lambda x\rangle \longrightarrow \langle U_0\psi, U_0 x\rangle = \langle U_0^*U_0\psi,\ x\rangle$. Thus

$$\begin{aligned}\langle\psi, x\rangle &= \langle U_\lambda^*U_\lambda\psi,\ x\rangle \\ &= \langle U_\lambda^*(U_\lambda\psi - U_0\psi),\ x\rangle + \langle U_0\psi,\ U_\lambda x\rangle \\ &\longrightarrow \langle U_0^*U_0\psi,\ x\rangle,\end{aligned}$$

which implies that $U_0^*U_0\psi = \psi$.

Since ψ is cyclic for $span(\mathcal{U})$, it follows that ψ separates $\mathcal{U}'$. So since $U_0^*U_0 \in \mathcal{U}'$ and $(U_0^*U_0 - I)\psi = 0$, we get $U_0^*U_0 = I$. But $\mathcal{U}'$ is finite, so U_0 is a unitary in $\mathcal{U}'$ as required. □

In fact we have more:

Corollary 2. *Let n be a natural number or ∞. Then, up to unitary equivalence, there is only one faithful *-representation π_n of $\mathcal{A}_\theta$ such that $\pi_n(\mathcal{U})$ has a complete wandering subspace of dimension n.*

Proof. For the existence, let $\phi = \pi \otimes I_n$, acting on $H \otimes \mathbb{C}^n$ if $n < \infty$ and on $H \otimes l^2(\mathbb{Z})$ if $n = \infty$, where π is as in Theorem 1. If x is any complete wandering vector for π, then $(\mathbb{C}x) \otimes \mathbb{C}^n$ (or $(\mathbb{C}x) \otimes l^2(\mathbb{Z})$ if $n = \infty$) is an n-dimensional complete wandering subspace for ϕ. For the uniqueness, let π_n be a faithful *-representation of $\mathcal{A}_\theta$ on a Hilbert space K such that $\pi_n(\mathcal{U})$ has a complete wandering subspace M of dimension n. Let $\{\xi_i : i = 1, 2, ..., n\}$ be an orthonormal basis for M. Then $\{\pi_n(U^kV^l)\xi_i : k, l \in \mathbb{Z}, 1 \leq i \leq n\}$ is an orthonormal basis for K. Fix an orthonormal basis $\{f_i : 1 \leq i \leq n\}$ for $\mathbb{C}^n$. Define a unitary operator W by $W\pi_n(U^kV^l)\xi_i = \pi(U^kV^l)e_{0,0} \otimes f_i$. Then, as in the first part of the proof in Theorem 1, we have $W\pi_n(\cdot)W^* = \phi(\cdot)$. □

From the proof of Theorem 1, the following general result can be in fact abstracted:

Proposition 3. *Let $\mathcal{U}$ be a unitary system such that $C_\psi(\mathcal{U}) = \mathcal{U}'$ for some $\psi \in \mathcal{W}(\mathcal{U})$. Then $\mathcal{W}(\mathcal{U})$ is connected. If, in addition, $w^*(\mathcal{U})$ is a finite von Neumann algebra and $span(\mathcal{W}(\mathcal{U}))$ is dense in H, then $\mathcal{W}(\mathcal{U})$ is closed.*

Proof. The connectedness follows from the fact that $\mathcal{W}(\mathcal{U}) = \mathbb{U}(\mathcal{U}')\psi$ and the fact that $\mathbb{U}(\mathcal{U}')$ is connected in norm. For the closedness, looking at the same part as in the proof of Theorem 1, it suffices to check that $\mathcal{U}'$ is a finite von Neumann algebra. Since $\mathcal{W}(\mathcal{U}) = \mathbb{U}(\mathcal{U}')\psi$ and $span(\mathcal{W}(\mathcal{U}))$ is dense in H, we have ψ is also cyclic for $\mathcal{U}'$. Thus $\mathcal{U}'$ is finite by Lemma 9.1.1 in [6], as required. □

For a general unitary system $\mathcal{U}$, it is possible that $\mathcal{W}(\mathcal{U})$ is not closed. For example, let $\mathcal{U} = \mathcal{U}_{D,T}$, where D and T are defined by $(Tf)(t) = f(t-1)$ and $(Df)(t) = \sqrt{2}f(2t)$ for all $f \in L^2(\mathbb{R})$. Then $\mathcal{W}(\mathcal{U})$ is not closed. It is true that $span(\mathcal{W}(\mathcal{U}))$ is dense in $L^2(\mathbb{R})$, but the connectedness problem is still open for this unitary system (cf. [2]).

For a unitary system $\mathcal{U}$ such that $\mathcal{WS}(\mathcal{U})$ is not empty, we define the *index set* of $\mathcal{U}$ by $ind_s(\mathcal{U}) = \{dim M : M \in \mathcal{WS}(\mathcal{U})\}$. In many cases $ind_s(\mathcal{U})$ is a singleton set. In some other cases it is all of $\mathbb{Z}_+ \cup \{\infty\}$ (cf. [5]). (Question: Are other cases possible?). We will prove that for an irrational rotation unitary system the index set is always singleton, and give two ways to construct irrational rotation unitary systems which have no complete wandering vectors. One comes from the following result.

Lemma 4. *If $\mathcal{U}$ is an irrational rotation unitary system such that $\mathcal{W}(\mathcal{U})$ is not empty, then for any non-trivial invariant subspace M of $\mathcal{U}$, $(\mathcal{U}\mid_M)$ has no wandering vectors.*

Proof. Suppose that there is some element $x \in M$ such that $\{\mathcal{U}x\}$ is an orthonormal set. Take $\psi \in \mathcal{W}(\mathcal{U})$ and define an operator $WH \longrightarrow H$ by $WU\psi = Ux, U \in \mathcal{U}$. Then $W^*W = I$ and $W \in C_\psi(\mathcal{U}) = \mathcal{U}'$. Since $\mathcal{U}'$ is a finite von Neumann algebra, we have $WW^* = I$, which contradicts the fact that $M \neq H$. Thus we obtain that $\mathcal{U}|_M$ has no wandering vectors. □

The other comes from the following proposition.

Proposition 5. *Let $\mathcal{U}$ be an irrational rotation unitary system such that $\mathcal{WS}(\mathcal{U})$ is not empty. Then $ind_s(\mathcal{U})$ is a singleton set.*

Proof. Take $M \in \mathcal{WS}(\mathcal{U})$ such that $dimM = k \in ind_s(\mathcal{U})$ (k may be ∞) and fix an orthonormal basis $\{\xi_i\}$ for M. By defining a unitary operator $W : H \to \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes M$ such that

$$W(U^nV^m\xi_i) = e_{n,m} \otimes \xi_i \qquad \forall i = 1, 2, ..., k, \forall n, m \in \mathcal{Z},$$

we can assume that $\mathcal{U}$ has a *-representation of $\mathcal{U}_{U,V} \otimes I_k$, where I_k is the identity operator on $\mathbb{C}^{(k)}$ and U and V are defined as in the proof of Theorem 1.

Now suppose that there is $m \in ind_s(\mathcal{U})$ such that $k \neq m$. Without loss of generality, we assume that $m < k$. This implies that $\mathcal{U}$ also has a *-representation of the form $\mathcal{U}_{U,V} \otimes I_m$. Thus, By Corollary 2, there is a unitary operator

$$T = \begin{pmatrix} A \\ B \end{pmatrix} : \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes \mathbb{C}^{(m)} \to \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes \mathbb{C}^{(k)},$$

such that $T(U \otimes I_m)T^* = U \otimes I_k$ and $T(V \otimes I_m)T^* = V \otimes I_k$, where

$$A : \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes \mathbb{C}^{(m)} \to \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes \mathbb{C}^{(m)},$$
$$B : \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes \mathbb{C}^{(m)} \to \ell^2(\mathbb{Z}\times\mathbb{Z}) \otimes \mathbb{C}^{(k-m)}.$$

By $TT^* = I$, we have that $AA^* = I_{\ell^2(\mathbb{Z}\times\mathbb{Z})\otimes\mathbb{C}^{(m)}}$ and $BB^* = I_{\ell^2(\mathbb{Z}\times\mathbb{Z})\otimes\mathbb{C}^{(k-m)}}$. Since $T(U \otimes I_m)T^* = U \otimes I_k$ and $T(V \otimes I_m)T^* = V \otimes I_k$, we get that $A \in \{\mathcal{U} \otimes I_m\}' = \mathcal{U}' \otimes B(\mathbb{C}^{(m)})$. We know that $\mathcal{U}'$ is a finite von Neumann algebra. Thus $A^*A = I_{\ell^2(\mathbb{Z}\times\mathbb{Z})\otimes\mathbb{C}^{(m)}}$. So it follows from $T^*T = A^*A + B^*B = I_{\ell^2(\mathbb{Z}\times\mathbb{Z})\otimes\mathbb{C}^{(m)}}$ that $B = 0$, which contradicts the relation $BB^* = I_{\ell^2(\mathbb{Z}\times\mathbb{Z})\otimes\mathbb{C}^{(k-m)}}$. Thus $ind_s(\mathcal{U})$ is a singleton set. □

Corollary 6. *Let $\mathcal{U}$ be an irrational rotation unitary system such that $\mathcal{W}(\mathcal{U})$ is not empty. Then $\mathcal{W}(\mathcal{U} \otimes I_k)$ is empty if $k > 1$.*

By Corollary 2 and the above proposition, we also have

Corollary 7. *Let $\mathcal{U}_i$ $(i = 1, 2)$ be irrational rotation unitary systems with respect to θ_i $(i = 1, 2)$. If $\mathcal{WS}(\mathcal{U}_1)$ is not empty, then $\mathcal{U}_1$ and $\mathcal{U}_2$ are unitarily equivalent if and only if $\theta_1 = \theta_2$ and $ind_s(\mathcal{U}_1) = ind_s(\mathcal{U}_2)$.*

Let $\mathcal{U}$ be a unitary system such that $\mathcal{W}(\mathcal{U})$ is not empty. A unitary V is called a *wandering vector multiplier* if $V\mathcal{W}(\mathcal{U}) \subseteq \mathcal{W}(\mathcal{U})$. Let $M_{\mathcal{U}}$ be the set of all wandering vector multipliers. If $\mathcal{U} = \mathcal{U}\mathcal{U}_0$, with $\mathcal{U}_0$ a group, then it is clear from the definition that every operator either in $\mathcal{U}_0$ or in $\mathbb{U}(\mathcal{U}')$ is a wandering vector multiplier. It was first proved in [2] that every unitary operator in $w^*(\mathcal{U}_0)$ is also a wandering vector multiplier for $\mathcal{U}$ if $\mathcal{U}_0$ is abelian, and later this was extended to the non-abelian case (see [5], [8]). Problem C in [2] is: Does every V in $M_{\mathcal{U}}$ factor as $V = V_1V_0$ for some unitaries $V_1 \in \mathcal{U}'$ and $V_0 \in w^*(\mathcal{U}_0)$? The following provides a negative answer (see also [8]).

Proposition 8. *Let $\mathcal{U}$ be an irrational rotation unitary system with generators U and V as in Theorem 1 such that $\mathcal{W}(\mathcal{U})$ is not empty. Then $U \in M_{\mathcal{U}}$, but $U \notin \mathcal{U}'w^*(V)$.*

Proof. Let $\psi \in \mathcal{W}(\mathcal{U})$. By the relation $UV = e^{2i\pi\theta}VU$, we have

$$\{U^nV^mU\psi : n, m \in \mathbb{Z}\} = \{e^{-2mi\pi\theta}UU^nV^m\psi : n, m \in \mathbb{Z}\}.$$

This is an orthonormal basis since $\{U^nV^m\psi : n, m \in \mathbb{Z}\}$ is. Thus $U\psi \in \mathcal{W}(\mathcal{U})$. Since ψ is arbitary, we get that $U \in M_{\mathcal{U}}$.

Assume that $U = AB$ for some $A \in \mathcal{U}'$ and $B \in w^*(V)$. Then $U \in w^*(V)'$, since $w^*(V)$ is abelian. This is a contradiction, because $UV \neq VU$. □

The above proposition can be strengthened considerably. For a general unitary system $\mathcal{U}$, not every unitary in $w^*(\mathcal{U})$ belongs to $M_{\mathcal{U}}$. For example, $D \notin M_{\mathcal{U}_{D,T}}$ since $\{D^nT^mD\psi : n, m \in \mathbb{Z}\} = \{D^{n+1}T^{2m}\psi : n, m \in \mathbb{Z}\}$ is not an orthonormal basis (although it is an orthonormal set) for any $\psi \in \mathcal{W}(\mathcal{U}_{D,T})$. But for the irrational rotation unitary system, we have

Proposition 9. *Every unitary operator in $w^*(\mathcal{U})$ is a wandering vector multiplier.*

Proof. Let $\psi \in \mathcal{W}(\mathcal{U})$. We proved in the proof of Theorem 1 that $\langle AB\psi, \psi\rangle = \langle BA\psi, \psi\rangle$ for all $A, B \in w^*(\mathcal{U})$. Suppose that $T \in w^*(\mathcal{U})$ is a unitary operator. Then for $A = U^nV^m, B = U^kV^l$ with $n, m, k, l \in \mathcal{Z}$, we have

$$\begin{aligned} \langle AT\psi,\ BT\psi\rangle &= \langle T^*B^*AT\psi,\ \psi\rangle = \langle TT^*B^*A\psi,\ \psi\rangle \\ &= \langle B^*A\psi,\ \psi\rangle = \langle A\psi, B\psi\rangle = \begin{cases} 0, & (n,m) \neq (k,l), \\ 1, & (n,m) = (k,l). \end{cases} \end{aligned}$$

Thus $\{U^nV^mT\psi : n, m \in \mathcal{Z}\}$ is an orthonormal set. Define an operator S by $SU^nV^m\psi = U^nV^mT\psi$ for all $n, m \in \mathcal{Z}$. Then $S \in C_\psi(\mathcal{U}) = \mathcal{U}'$ and $S^*S = I$. Thus S is unitary, since $\mathcal{U}'$ is a finite von Neumann algebra. Therefore $T\psi \in \mathcal{W}(\mathcal{U})$, which implies that $T \in M_{\mathcal{U}}$. □

Since $M_{\mathcal{U}}$ is a semigroup, we have that $M_{\mathcal{U}} \supseteq \mathbb{U}(\mathcal{U}')\mathbb{U}(w^*(\mathcal{U}))$ by the above proposition. We claim that the containment is proper. To prove this, we need some notations. Let $\widehat{\mathbb{Z}}$ be the dual group of $\mathbb{Z}$ and let $\sigma, \tau \in \mathbb{Z}$. Let ψ be a fixed complete wandering vector for $\mathcal{U}$ and define a unitary operator $A_{\sigma,\tau} \in B(H)$ by

$$A_{\sigma,\tau}U^nV^m\psi = \sigma(n)\tau(m)U^nV^m\psi$$

for all $n, m \in \mathcal{W}(\mathcal{U})$. The following result tells us that we have a negative answer even for a weaker factorization problem.

Theorem 10. *For any σ and τ, $A_{\sigma,\tau}$ is a wandering vector multiplier, and $A_{\sigma,\tau}$ belongs to $\mathcal{U}'w^*(\mathcal{U})$ if and only if both $\sigma(1)$ and $\tau(1)$ are in $\{e^{2n\pi i\theta} : n \in \mathbb{Z}\}$. In particular, $M_{\mathcal{U}}$ is not equal to $\mathbb{U}(\mathcal{U}')\mathbb{U}(w^*(\mathcal{U}))$.*

Proof. Let $\eta \in \mathcal{W}(\mathcal{U})$ and suppose that

$$\eta = \sum_{n,m\in\mathbb{Z}} \lambda_{nm} U^n V^m \psi.$$

Then

$$\begin{aligned} U^k V^l A_{\sigma,\tau}\eta &= \sum_{n,m\in\mathbb{Z}} \lambda_{n,m}\sigma(n)\tau(m) U^k V^l U^n V^m \psi \\ &= \sum_{n,m\in\mathbb{Z}} \lambda_{n,m}\sigma(n)\tau(m) e^{-2ln\pi i\theta} U^{n+k} V^{m+l}\psi \\ &= \sum_{n,m\in\mathbb{Z}} \lambda_{n-k,m-l}\sigma(n-k)\tau(m-l) e^{2l(n-k)\pi i\theta} U^n V^m \psi, \end{aligned}$$

and similarly we have

$$U^r V^s \eta = \sum_{n,m\in\mathbb{Z}} \lambda_{n-r,m-s}\sigma(n-r)\tau(m-s) e^{2s(n-r)\pi i\theta} U^n V^m \psi.$$

Thus, by the orthonormality of $\{U^n V^m \psi\}$ and the equality

$$\sigma(n-k)\tau(m-l)\overline{\sigma(n-r)\tau(m-s)} \;=\; \sigma(r-k)\tau(s-l),$$

we obtain that

$$\begin{aligned} \langle U^k V^l A_{\sigma,\tau}\eta, U^r V^s A_{\sigma,\tau}\eta\rangle &= \sigma(r-k)\tau(s-l)\langle U^k V^l \eta, U^r V^s \eta\rangle \\ &= \begin{cases} 1, & (k,l) = (r,s), \\ 0, & (k,l) \neq (r,s). \end{cases} \end{aligned}$$

It follows that $\{U^n V^m A_{\sigma,\tau}\eta\}$ is an orthonormal set. By the similar argument as in the proof of Proposition 9, we get $A_{\sigma,\tau}\eta \in \mathcal{W}(\mathcal{U})$. Hence $A_{\sigma,\tau} \in M_{\mathcal{U}}$.

Now we prove that if either $\sigma(1) \notin \{e^{2n\pi i\theta} : n \in \mathbb{Z}\}$ or $\tau(1) \notin \{e^{2n\pi i\theta} : n \in \mathbb{Z}\}$, then $A_{\sigma,\tau} \notin \mathbb{U}(\mathcal{U}')\mathbb{U}(w^*(\mathcal{U}))$. Assume, for the contrary, then there exists a unitary operator $T \in \mathcal{U}'$ such that $A_{\sigma,\tau}T$ belongs to $w^*(\mathcal{U})$. Fix any $k, l \in \mathbb{Z}$ and let $\eta = U^k V^l \psi$. Then $\eta \in \mathcal{W}(\mathcal{U})$ by Proposition 9. And hence there is a unitary operator W in $\mathcal{U}'$ such that $W\psi = \eta$ Therefore we have $A_{\sigma,\tau}TW\psi = WA_{\sigma,\tau}T\psi$. Let

$$T\psi = \sum_{n,m\in\mathbb{Z}} \lambda_{nm} U^n V^m \psi.$$

Then

$$\begin{aligned} A_{\sigma,\tau}TW\psi &= A_{\sigma,\tau}TU^k V^l \psi = A_{\sigma,\tau}U^k V^l T\psi \\ &= \sum \lambda_{nm}\sigma(n+k)\tau(m+l) e^{-2ln\pi i\theta} U^{n+k} V^{m+l}\psi \end{aligned}$$

and

$$\begin{aligned} WA_{\sigma,\tau}T\psi &= W\sum \lambda_{nm}\sigma(n)\tau(m)U^nVm\psi \\ &= \sum \lambda_{nm}\sigma(n)\tau(m)U^nV^mW\psi \\ &= \sum \lambda_{nm}\sigma(n)\tau(m)e^{-2mk\pi i\theta}U^{n+k}V^{m+l}\psi. \end{aligned}$$

It follows that

$$\lambda_{nm}(\sigma(k)\tau(l)e^{-2ln\pi i\theta} - e^{-2mk\pi i\theta}) = 0$$

for all $n, m, k, l \in \mathbb{Z}$. Suppose that $\lambda_{n_0m_0} \neq 0$. Then we have that $\sigma(k)\tau(l) = e^{-2(m_0k-ln_0)\pi i\theta}$ for all $k, l \in \mathbb{Z}$, which implies $\sigma(k) = e^{-2km_0\pi i\theta}, \tau(l) = e^{2ln_0\pi i\theta}$ for all $k, l \in \mathbb{Z}$. This contradicts the assumption on σ and τ.

Now assume that $\sigma(1) = e^{-2m_0\pi i\theta}$ and $\tau(1) = e^{2n_0\pi i\theta}$. Let us define an unitary operator T by $TU^nV^m\psi = e^{-2mn_0\pi i\theta}U^{n+n_0}V^{m+m_0}\psi$. Then we have $TU^nV^m\psi = U^nV^mT\psi$ for all $n, m \in \mathbb{Z}$, which implies that $T \in C_\psi(\mathcal{U}) = \mathcal{U}'$. Let $W \in \mathcal{U}'$ be a unitary operator and let $W\psi = \sum \lambda_{nm}U^nV^m\psi$. For any pair (k, l), we have

$$\begin{aligned} A_{\sigma,\tau}TWU^kV^l\psi &= A_{\sigma,\tau}U^kV^lTW\psi \\ &= A_{\sigma,\tau}U^kV^l\sum \lambda_{nm}U^nV^mT\psi \\ &= WA_{\sigma,\tau}U^kV^l\sum \lambda_{nm}e^{-2\pi imn_0\theta}U^{n+n_0}V^{m+M_0}\psi \\ &= \sum s(n, m, k, l)U^{n+n_0+k}V^{m+m_0+l}\psi \end{aligned}$$

and

$$\begin{aligned} WA_{\sigma,\tau}TU^kV^l\psi &= WA_{\sigma,\tau}U^kV^lT\psi \\ &= WA_{\sigma,\tau}e^{-2\pi ln_0i\theta}U^{k+n_0}V^{l+m_0}\psi \\ &= \sigma(k+n_0)\tau(l+m_0)e^{-2\pi ln_0i\theta}U^{k+n_0}V^{l+m_0}W\psi \\ &= \sum t(n, m, k, l)U^{n+n_0+k}V^{m+m_0+l}\psi, \end{aligned}$$

where

$$s(n, m, k, l) = \lambda_{nm}\sigma(n+n_0+k)\tau(m+m_0+l)e^{-2\pi(mn_0+ln_0+ln)i\theta}$$

and

$$t(n, m, k, l) = \lambda_{nm}\sigma(n_0+k)\tau(m_0+l)e^{-2\pi(ln_0+ln+m_0n)i\theta}.$$

Note that $\sigma(n)\tau(m) = e^{2\pi(nm_0-mn_0)i\theta}$; we have $s(n, m, k, l) = t(n, m, k, l)$. Thus $A_{\sigma,\tau}TW = WA_{\sigma,\tau}T$. Since W is arbitary in $\mathcal{U}'$, we get $A_{\sigma,\tau}T \in w^*(\mathcal{U})$. □

Remark. In a separate paper joint with D. Larson, we will study some properties of wandering subspaces and wandering vector multipliers, which are closely related to the classical wavelet theory, for the general unitary group case. In fact, we prove that $M_\mathcal{U}$ is a group for irrational rotation unitary systems and most interesting unitary group systems including abelian groups and free groups. For an irrational rotation unitary system $\mathcal{U}_{U,V}$ with a fixed complete wandering vector ψ, given a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$, we can define a unitary operator B_f by $B_fU^nV^m\psi = f(n, m)U^nV^m\psi$ for all $n, m \in \mathbb{Z}$. We also prove that B_f is a wandering vector multiplier if and only if there exist two characters σ and τ of $\mathbb{Z}$ and a modulus one number z satisfying $f(n, m) = z\sigma(n)\tau(m)$ for all $(n, m) \in \mathbb{Z} \otimes \mathbb{Z}$. However, the concrete structure of $M_\mathcal{U}$ still seems complicated, even when $\mathcal{U}$ is an irrational rotation unitary system, or just simply a unitary group. For example, let $H = L^2(\mathbb{T})$ and let $\mathcal{U} = \{M_z^n : n \in \mathbb{Z}\}$, where $\mathbb{T}$ is the unit circle and M_z is the unitary operator of multiplication by z. Then the constant function 1 is a complete wandering vector for $\mathcal{U}$, and in fact $\mathcal{W}(\mathcal{U})$ is the set of all unimodular functions on $\mathbb{T}$. So characterizing all the wandering vector multipliers is equivalent to characterizing all the unitary operators on $L^2(\mathbb{T})$ which send unimodular functions to unimodular functions. All the unitary operators of multiplication M_f by a unimodular function f belongs to $M_{\mathcal{U}}$. There are others. Let σ be a measure preserving bijective mapping from $\mathbb{T}$ to itself. Define a unitary operator A_σ on $L^2(\mathbb{T})$ by $(A_\sigma f)(z) = f(\sigma^{-1}(z))$ for all $f \in L^2(\mathbb{T})$. Then $A_\sigma \in M_{\mathcal{U}}$. It is not hard to check that the group generated by all the M_f and all the A_σ has the standard form $\{A_\sigma M_f\}$, thus is contained in the wandering vector multiplier set. In fact, equality can be proven.

We now turn our attention to giving an abstract characterization for those irrational rotation unitary systems which have complete wandering subspaces.

Lemma 11. *Let $\mathcal{U}$ be an irrational rotation unitary system. If $\mathcal{U}$ has a complete wandering subspace, then $w^*(\mathcal{U})$ is a finite von Neumann algebra.*

Proof. Suppose that $M \in \mathcal{WS}(\mathcal{U})$ with $dim(M) = n$ (n may be ∞) and $\mathcal{U}_1$ is an irrational rotation unitary system with the same irrational as $\mathcal{U}$ such that $\psi \in \mathcal{W}(\mathcal{U}_1)$. Let $\{x_i\}$ be an orthonormal basis for M and $\{e_i\}$ is an orthonormal basis for $\mathbb{C}^{(n)}$. By defining $WU^kV^lx_i = U_1^kV_1^l\psi \otimes e_i$ for all $k, l \in \mathbb{Z}$ and all i, we can obtain that $\mathcal{U}$ is unitarily equivalent to $\mathcal{U}_1 \otimes I_n$, where I_n is the identity operator on $\mathbb{C}^{(n)}$. Hence $w^*(\mathcal{U})$ is finite since $w^*(\mathcal{U}_1) \otimes I_n$ is. □

Let $\mathcal{A}$ be a C*-algebra. Recall that two representations π_1 and π_2 of $\mathcal{A}$ are called quasi-equivalent if there exists a *-isomorphism α from $w^*(\pi_1(\mathcal{A}))$ onto $w^*(\pi_2(\mathcal{A}))$ such that $\alpha(\pi_1(a)) = \pi_2(a)$ for all $a \in \mathcal{A}$. It was proved in [6] that if a C*-algebra $\mathcal{A}$ admits at most one trace, then all finite representaions of $\mathcal{A}$ are quasi-equivalent. Let $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space H and let E be a projection in $\mathcal{R}$. A vector x is said to be a generating vector for E if $[\mathcal{R}'x] = EH$. If $\mathcal{R}$ and $\mathcal{R}'$ are finite, denote by τ and τ' the center-valued traces on $\mathcal{R}$ and $\mathcal{R}'$, respectively. It is known (see 9.6.7 in [7]) that there is a unique invertible element C in the algebra of operators affiliated to $\mathcal{R} \cap \mathcal{R}'$ with the following property: if F and F' are projections in $\mathcal{R}$ and $\mathcal{R}'$, then $\tau(F) = C\tau(F')$ if and only if F and F' have a common generating vector. C_0 is called the *coupling operator* of $\mathcal{R}$.

Theorem 12. *Let $\mathcal{U}$ be an irrational rotation unitary system. Then it has a complete wandering vector if and only if $w^*(\mathcal{U})$ is finite and has a common cyclic vector with $\mathcal{U}'$.*

Proof. $\Longrightarrow$. By the above lemma, $w^*(\mathcal{U})$ is finite. Now let $\psi \in \mathcal{W}(\mathcal{U})$. Then it is clear that ψ is a cyclic vector for $w^*(\mathcal{U})$. By proposition 1.3 in [2], $\mathcal{W}(\mathcal{U}) = \{T\psi : T \in \mathcal{U}'$ is unitary$\}$. Hence ψ is cyclic for $\mathcal{U}'$ by Theorem 1.

$\Longleftarrow$. Let $\mathcal{U}_1$ be an irrational rotation unitary system with the same irrational number as $\mathcal{U}$ and $\mathcal{W}(\mathcal{U}_1)$ non-empty. It is known (see [4]) that there is a *-isomorphism π from $C^*(\mathcal{U})$ onto $C^*(\mathcal{U}_1)$ such that $\pi(U) = U_1$ and $\pi(V) = V_1$.

Since $w^*(\mathcal{U})$ and $w^*(\mathcal{U}_1)$ are finite, and the C*-algebras $C^*(\mathcal{U})$ and $C^*(\mathcal{U}_1)$ admit unique traces ([1],10.11.6), there exists a *-isomorphism α from $w^*(\mathcal{U})$ onto $w^*(\mathcal{U}_1)$ such that

$$\alpha(a) = \pi(a) \qquad \text{for all } a \in C^*(\mathcal{U}).$$

By Proposition 12.1.2 in [7], we also have that $w^*(\mathcal{U}_1)$ and $w^*(\mathcal{U})$ are finite factors. Hence $\mathcal{U}_1'$ and $\mathcal{U}'$ are finite by Proposition 9.1.2 in [7]. Let C and C_1 be the coupling operators of $w^*(\mathcal{U})$ and $w^*(\mathcal{U}_1)$, respectively. We claim that both C and C_1 are identity operators. In fact, let τ and τ' be the center-valued traces on $w^*(\mathcal{U})$ and $\mathcal{U}'$, respectively. Since $w^*(\mathcal{U})$ and $\mathcal{U}'$ have a common cyclic vector, we have $\tau(I) = C\tau'(I)$. Thus $C = I$; similarly, $C_1 = I$. Therefore $\alpha(C) = C_1$ Thus it follows from 9.6.30(iv) in [7] that α is unitarily implemented. Therefore $\mathcal{U}_1$ and $\mathcal{U}$ are unitarily equivalent, which implies that $\mathcal{W}(\mathcal{U}_1)$ is not empty. □

Remark. By Lemma 4 and the fact that if $\mathcal{R}$ is a finite von Neumann algebra and M is a invariant subspace of $\mathcal{R}$, then $\mathcal{R}\mid_M$ is also finite (see [3]), we know that even if an irrational rotation unitary system generates a finite von Neumann algebra and has a cyclic vector, its wandering subspace set may be empty.

Corollary 13. *Let $\mathcal{U}$ be an irrational rotation unitary system. Then $\mathcal{U}$ has a complete wandering subspace of dimension n if and only if $w^*(\mathcal{U})$ is finite and there exists an orthonormal set $\{x_i\}_1^n$ such that $H = \bigoplus_{i=1}^n [\mathcal{U}x_i]$ and each x_i is cyclic for $\mathcal{U}'$.*

Proof. For $\Longleftarrow$, note that if we let $M_i = [\mathcal{U}x_i]$, then x_i is also cyclic for $(\mathcal{U}\mid_{M_i})'$. Hence the conclusion follows easily from Theorem 12.

For $\Longrightarrow$, by Lemma 11, $w^*(\mathcal{U})$ is finite. By Corollary 2, we may assume that the unitary system is $\mathcal{U}\otimes I_n$ such that $\mathcal{W}(\mathcal{U})$ is not empty. Take $\psi \in \mathcal{W}(\mathcal{U})$. Then $\{\psi\otimes e_i\}$ satisfies the requirements, where $\{e_i\}$ is an orthonormal basis for $\mathbb{C}^{(n)}$. □

Since $ind_s(\mathcal{U})$ is a unitarily equivalent invariant for all irrational rotation unitary systems, there are many inequivalent irrational rotation unitary system classes for the same irrational number. A weaker equivalence condition than unitary equivalence is approximate unitary equivalence. Two irrational rotation unitary systems $\mathcal{U}_{U_1,V_1}$ and $\mathcal{U}_{U_2,V_2}$ are called *approximately unitarily equivalent* if there exist unitaries $\{W_n\}$ such that $\|W_nU_1W_n^* - U_2\| \longrightarrow 0$ and $\|W_nV_1W_n^* - V_2\| \longrightarrow 0$. It is interesting to note that for this kind of equivalence, Theorem 4.9 in [4] implies

Proposition 14. *Let $\mathcal{U}_{U_1,V_1}$ and $\mathcal{U}_{U_2,V_2}$ be two irrational rotation unitary systems on Hilbert spaces H_1 and H_2 with irrationals θ_1 and θ_2, respectively. Then $\mathcal{U}_{U_1,V_1}$ and $\mathcal{U}_{U_2,V_2}$ are approximately unitarily equivalent if and only if $\theta_1 = \theta_2$.*

Proof. For "$\Longrightarrow$", let $W_n : H_1 \longrightarrow H_2$ be unitaries such that $\|W_nU_1W_n^* - U_2\| \longrightarrow 0$ and $\|W_nV_1W_n^* - V_2\| \longrightarrow 0$. Fix a unitary $S : H_1 \longrightarrow H_2$, and define $\widetilde{U_1} = SU_1S^*$ and $\widetilde{V_1} = SV_1S^*$. Then $\widetilde{\mathcal{U}_1} = \{\widetilde{U_1}^m\widetilde{V_1}^k\}$ is an irrational rotation unitary saystem with θ_1 acting on H_2, and $\mathcal{U}_{U_2,V_2}$ and $\widetilde{\mathcal{U}}_1$ are approximately unitarily equivalent. Thus, by Theorem 4.9 in [4], we have $e^{2\pi i\theta_1} = e^{2\pi i\theta_2}$, and so $\theta_1 = \theta_2$ since $\theta_1, \theta_2 \in (0, 1)$.

"$\Longleftarrow$". This follows from the proof of (i) of Proposition 4.2 in [4] □

We conclude with some questions concerning incomplete maximal wandering subspaces.

A wandering subspace for a unitary system $\mathcal{U}$ is said to be *maximal* if it is not properly contained in any other wandering subspace for $\mathcal{U}$. Using Zorn's Lemma, every wandering subspace can be extended to a maximal one. It may happen that an irrational rotation unitary system has a incomplete maximal wandering subspace even if the unitary system also has a complete wandering subspace. To explain this, let $\mathcal{A}_\theta, u, v$ and π be as in Theorem 1. We use the notation $\pi^{(\infty)}$ to denote the infinite direct sum of copies of π, which is a *-representation of $\mathcal{A}_\theta$ on the Hilbert space $\bigoplus_{k=1}^\infty H_k$ with $H_k = H$ for all k. Then $\pi^{(\infty)}$ is unitarily equivalent to $\pi \oplus \pi^{(\infty)}$, since the unitary operator W defined by $W(h \oplus (\bigoplus_{k=1}^\infty h_k)) := \bigoplus_{k=1}^\infty g_k$ induces the unitary equivalence, where $g_1 = h, g_k = h_{k-1}$ for $k > 1$. Moreover, let M be a reducing subspace for $\pi(\mathcal{A}_\theta)$ and let π_1 and π_2 be the restriction *-representations of π on M and $M^\perp$, respectively. Then $\pi^{(\infty)}$ is unitarily equivalent to $\pi_1^{(\infty)} \oplus \pi_2^{(\infty)}$.

Proposition 15. *An irrational rotation unitary system with an infinite dimensional complete wandering subspace also has an incomplete maximal wandering subspace.*

Proof. Let $\mathcal{A}_\theta, u, v$ and π be as in Theorem 1 and let $\mathcal{U} = \{u^n v^m; n, m \in \mathbb{Z}\}$. Then $\pi^{(\infty)}$ is a faithful *-representation of $\mathcal{A}_\theta$ such that $\pi^{(\infty)}(\mathcal{U})$ has a complete wandering subspace of infinite dimension. By Corollary 2, every irrational rotation unitary system which has an infintite dimensinoal complete wandering subspace must be unitarily equivalent to $\pi^{(\infty)}(\mathcal{U})$. So it suffices to prove that $\pi^{(\infty)}(\mathcal{U})$ has an incomplete maximal wandering subspace.

Choose M to be a non-trivial invariant subspace for $w^*(\pi(\mathcal{U}))$. Then $\pi(\mathcal{U})|_M$ has no non-trivial wandering subspaces by Lemma 4. Write $\pi_1 = \pi|_M$ and $\pi_2 = \pi|_{M^\perp}$. Then π is unitarily equivalent to $\pi_1 \oplus \pi_2$. Thus $\pi^{(\infty)}$ is unitarily equivalent to $\pi_1^{(\infty)} \oplus \pi_2^{(\infty)}$, which in turn is unitarily equivalent to $\pi_1 \oplus \pi_1^{(\infty)} \oplus \pi_2^{(\infty)}$. Therefore, $\pi^{(\infty)}$ is unitarily equivalent to $\pi_1 \oplus \pi^{(\infty)}$.

Since $\pi_1(\mathcal{U})$ has no wandering vectors and $\pi^{(\infty)}(\mathcal{U})$ has a complete wnadering subspace P of infinite dimension, we have that $(\pi_1 \oplus \pi^{(\infty)})(\mathcal{U})$ has a maximal wandering subspace $0 \oplus P$ which is not complete. Hence $\pi^{(\infty)}(\mathcal{U})$ has a incomplete maximal wandering subspace by the unitary equivalence. □

We conjecture that the infinite dimensional condition is essential for the above result.

Questions. (i) If an irrational rotation unitary system has a complete wandering subspace of finite dimension greater than 1, must every maximal wandering subspace be complete?

(ii) More generally, for an arbitrary irrational rotation unitary system, must all the maximal wandering subspaces have the same dimension?

If (ii) has an affirmative answer, so does (i). Also, this would give a generalization of Proposition 5. If an irrational rotation unitary system $\mathcal{U}$ has a complete wandering vector, then, by Lemma 2, every wandering vector must be complete. Thus all the maximal wandering subspaces have dimension 1. (ii) is also related to the decompositions of *-representations of $\mathcal{A}_\theta$. Let $\mathcal{U}$ be an abstract irrational rotation unitary system in a C*-algebra $\mathcal{A}_\theta$, and π a faithful *-representation of $\mathcal{A}_\theta$ on a Hilbert space H. Let M be a maximal wandering subspace for $\pi(\mathcal{U})$ and let $K = [\pi(\mathcal{U})M]$. Then K reduces $\pi(\mathcal{A}_\theta)$. Let π_K and $\pi_{K^\perp}$ be the restrictions of π on K and $K^\perp$ respectively. Then π is decomposed into the direct sum of π_K and $\pi_{K^\perp}$ such that π_K has a complete wandering subspace M, but $\pi_{K^\perp}$ has no wandering vectors. Suppose $\pi = \pi_L \oplus \pi_{L^\perp}$ is another such decomposition and suppose the answer to (ii) is yes. Then the "non-trivial" parts π_K and π_L must be unitarily equivalent.

Problem. Give a complete characterization of wandering vector multipliers for irrational rotation unitary systems.

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References

1. B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, 1986. MR **88g:**46082
2. X. Dai and D.R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs A.M.S, to appear.
3. J. Dixmier, *Von Neumann algebras*, North-Holland Pub. Comp., 1981. MR **83a:**46004
4. U. Haagerup and M. Rordam, *Perturbations of the rotation C*-algebras and of the Heisenberg commutation relations*, Duke Math. J. **77** (1995), 627-656. MR **96e:**46073
5. D. Han and V. Kamat, *Operators and multiwaveles*, preprint.
6. R. V. Kadison, *Representations of matricial operator algebras*, Proc. Neptun Conf. on Op. Alg. and Group Rep. (1980), Pitman, 1984, Vol. 2, pp. 1–22. MR **85f:**46104
7. R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. II, Academic Press, 1986. MR **88d:**46106
8. W. S. Li, J. E. McCarthy and D. Timotin, *A note on wavelets for unitary systems*, preprint.
9. M. Pimsner and D. Voiculescu, *Imbedding the irrational rotation C*-algebra into an AF algebra*, J. Op. Theory **4** (1980), 201-210. MR **82d:**46086
10. M.A. Rieffel, *C*-algebras associated with irrational rotations*, Pac. J. Math **93** (1981), 415-429. MR **83b:**46087

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