A note on maximality of analytic crossed products

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Abstract

Let $G$ be a compact abelian group with the totally ordered dual group $\hat{G}$ which admits the positive semigroup $\hat{G}^+$. Let $N$ be a von Neumann algebra and $\alpha = \{\alpha_g\}_{g \in \hat{G}}$ be an automorphism group of $\hat{G}$ on $N$. We denote $N \rtimes_\alpha \hat{G}^+$ to the analytic crossed product determined by $N$ and $\alpha$. We show that if $N \rtimes_\alpha \hat{G}^+$ is a maximal $\sigma$-weakly closed subalgebra of $N \rtimes_\alpha \hat{G}$, then $\hat{G}^+$ induces an archimedean order in $\hat{G}$.

Keywords: von Neumann algebra; Analytic crossed product; Maximality; Semigroup

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1. Introduction

Let \( G \) be a compact abelian group with the totally ordered dual group \( \hat{G} \) which has a positive semigroup \( \hat{G}^+ \). Let \( N \) be a von Neumann algebra and \( \alpha = \{ \alpha_g \}_{g \in G} \) be an automorphism group of \( G \) on \( N \). We are interested in the maximality of certain subalgebra of crossed product \( N \rtimes_\alpha \hat{G} \) determined by \( N \) and \( \alpha \). This subalgebra is called an analytic crossed product. Roughly speaking, the analytic crossed products stand in the same relation to the crossed products as the Hardy algebras \( H^\infty(G) \), the space of all functions of analytic type which belongs to \( L^\infty(G) \), stand in relation to \( L^\infty(G) \). In the case that \( G = \mathbb{T} \) and \( \hat{G} = \mathbb{Z} \), it is well known that \( H^\infty(G) \) is a maximal weak * closed subalgebra of \( L^\infty(G) \). Viewing the analytic crossed products as “noncommutative \( H^\infty \) algebras” raise the following:

**Question.** When is the analytic crossed product \( N \rtimes_\alpha \hat{G}^+ \) maximal among \( \sigma \)-weakly closed subalgebras of the crossed product \( N \rtimes_\alpha \hat{G} \)?

McAsey, Muhly and the fourth author in [3] showed that, in the case that \( G = \mathbb{T} \), \( \hat{G} = \mathbb{Z} \) and \( \hat{G}^+ = \mathbb{Z}^+ \), if \( N \) is a finite von Neumann algebra, then \( N \) is a factor if and only if \( N \rtimes_\alpha \mathbb{Z}^+ \) is maximal as a \( \sigma \)-weakly closed subalgebra of \( N \rtimes_\alpha \mathbb{Z} \). They also proved in [4] the same result in the case when \( N \) is an arbitrary (\( \sigma \)-finite) von Neumann algebra. Moreover the fourth author in [7] showed that if \( N \) is a factor, then \( N \rtimes_\alpha \hat{G}^+ \) is maximal where \( G \) is a compact abelian group with an archimedean totally ordered dual \( \hat{G} \). Motivated by these facts, we consider the following problem:

**Problem.** Let \( N \) be a factor. When is the analytic crossed product \( N \rtimes_\alpha \hat{G}^+ \) maximal among \( \sigma \)-weakly closed subalgebras of \( N \rtimes_\alpha \hat{G}^+ \)?

Our aim in this paper is to give the answer for this problem as follows:

**Theorem 1.1.** Let \( N \) be a factor. Then an analytic crossed product \( N \rtimes_\alpha \hat{G}^+ \) is a maximal \( \sigma \)-weakly closed subalgebra of \( N \rtimes_\alpha \hat{G} \) if and only if \( \hat{G}^+ \) induces an archimedean order in \( \hat{G} \).

In the next section we establish the notions of spectral subspaces, crossed products and its subalgebras. In Section 3, we study the structure of analytic subalgebras in a crossed product. And we shall prove Theorem 1.1, that is, we shall show that if \( N \rtimes_\alpha \hat{G}^+ \) is a maximal \( \sigma \)-weakly closed subalgebra of \( N \rtimes_\alpha \hat{G} \), then \( \hat{G}^+ \) induces an archimedean order in \( \hat{G} \) (Theorem 3.7). In Section 4, we will pay attention to the properties of semigroups in \( \hat{G} \) and reconsider the maximality of analytic subalgebras.

2. Preliminaries

Throughout this paper, \( G \) will denote a compact abelian group with the operation written additively. Elements of \( G \) will be denoted by lowercase Roman letters and the normalized Haar measure on \( G \) will be denoted by \( m \). The dual of \( G \) will be written \( \hat{G} \) and the elements of \( \hat{G} \) will be distinguished from those of \( G \) by a caret. The pairing between \( G \) and \( \hat{G} \) will be written \( (g, \hat{h}) \) \((\forall g \in G, \forall \hat{h} \in \hat{G})\) and the Fourier transform will take this form:

\[
\hat{f}(\hat{h}) = \int_G (g, \hat{h}) f(g) \, dm(g) \quad (\forall f \in L^1(G), \forall \hat{h} \in \hat{G}).
\]
Suppose that $\hat{G}$ has a positive semigroup $\hat{G}_+$, that is, $\hat{G}_+$ satisfies the conditions

(i) $\hat{G}_+ \cap (-\hat{G}_+) = \{0\}$,

(ii) $\hat{G}_+ \cup (-\hat{G}_+) = \hat{G}$.

Under these conditions, $\hat{G}_+$ induces an order in $\hat{G}$. That is, if we define $\hat{g} \geq \hat{h}$ to mean that $\hat{g} - \hat{h} \in \hat{G}_+$, then $\hat{G}$ is a totally ordered set with the order $\geq$. The most important example is the case that $G = \mathbb{T}$, $\hat{G} = \mathbb{Z}$ and $\hat{G}_+ = \{0, 1, 2, \ldots\}$. A given group $\hat{G}$ may have many different orders. An order is said to be archimedean if it has the following property: to every pair of elements $\hat{g}, \hat{h}$ of $\hat{G}$ such that $\hat{g} > \hat{0}$ and $\hat{h} > \hat{0}$, there corresponds a positive integer $n$ such that $n\hat{g} \geq \hat{h}$. For example, it is clear that $\mathbb{Z}_+$ has an archimedean order in $\mathbb{Z}$. As a non-archimedean order, there is a lexicographic order. For example, in the case that $G = \mathbb{T}$ and $\hat{G} = \mathbb{Z}^2$, we consider the positive semigroup $\hat{G}_+$ as $\{(m, n) \in \mathbb{Z}^2 \mid m > 0$, or $m = 0$ and $n \geq 0\}$. Then $\hat{G}_+$ induces the lexicographic order $\leq_L$ in $\hat{G}$. In this case, $(1, 0)$ and $(0, 1)$ belong to $\hat{G}_+$, but, for any positive integer $n$, $n(0, 1) - (1, 0) = (-1, n) \notin \hat{G}_+$, that is, $(1, 0) \not\leq_L n(0, 1)$. Therefore this order is non-archimedean.

Let $M$ be a von Neumann algebra and $\beta = \{\beta_g\}_{g \in G}$ be an automorphism group of $G$ on $M$. For each $f \in L^1(G)$, we define $\beta(f)$ by the integral

$$\beta(f)(x) = \int \limits_G f(g) \beta_g (x) \, dm(g) \quad (\forall x \in M).$$

For each fixed $x \in M$, the set

$$I_\beta(x) = \{ f \in L^1(G) \mid \beta(f)(x) = 0 \}$$

is a closed ideal of $L^1(G)$. The Arveson spectrum $\text{Sp}_\beta(x)$ of $x$ with respect to the automorphism group $\{\beta_g\}_{g \in G}$ is defined to be the hull of $I_\beta(x)$ as follows:

$$\text{Sp}_\beta(x) = \{ \hat{h} \in \hat{G} \mid \hat{f}(\hat{h}) = 0 \ (\forall f \in I_\beta(x)) \},$$

where $\hat{f}$ is the Fourier transform of $f$. For each subset $E \subset \hat{G}$, the spectral subspace $M^\beta(E)$ is defined to be the set

$$M^\beta(E) = \{ x \in M \mid \text{Sp}_\beta(x) \subseteq E \}. $$

It is known that, for each subset $E$ of $\hat{G}$, $M^\beta(E)$ is $\beta$-invariant, that is, $\beta_g(M^\beta(E)) = M^\beta(E)$ ($\forall g \in G$).

We shall next define crossed products and analytic crossed products. Let $N$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\alpha = \{\alpha_g\}_{\hat{g} \in \hat{G}}$ be an automorphism group of $\hat{G}$ on $N$. The crossed product $N \rtimes_\alpha \hat{G}$ of $N$ by $\alpha$ is the von Neumann algebra acting on a Hilbert space $\ell^2(\hat{G}, \mathcal{H}) = \{ \xi : \hat{G} \to \mathcal{H} \mid \sum_{\hat{h} \in \hat{G}} \|\xi(\hat{h})\|^2 < \infty \}$ generated by the operators $\pi_\alpha(x)$ and $\lambda(\hat{g})$ defined by the equations, for each $x \in N$,

$$\{ \pi_\alpha(x) \xi \}(\hat{g}) = \alpha_{-\hat{g}}(x) \xi(\hat{g}) \quad (\forall \xi \in \ell^2(\hat{G}, \mathcal{H}), \ \forall \hat{g} \in \hat{G})$$

and

$$\{ \lambda(\hat{g}) \xi \}(\hat{h}) = \xi(\hat{h} - \hat{g}) \quad (\forall \xi \in \ell^2(\hat{G}, \mathcal{H}), \ \forall \hat{h}, \hat{g} \in \hat{G}).$$

For simplicity, we write $M = N \rtimes_\alpha \hat{G}$ and $K = \ell^2(\hat{G}, \mathcal{H})$. The analytic crossed product $N \rtimes_{\sigma} \hat{G}_+$ determined by $N$ and $\alpha$ is defined to be the $\sigma$-weakly closed subalgebra of $M$ generated by $\pi_\alpha(N)$ and $\{ \lambda(\hat{g}) \}_{\hat{g} \in \hat{G}_+}$, that is,

$$N \rtimes_{\sigma} \hat{G}_+ = \text{alg}\{ \pi_\alpha(N), \{ \lambda(\hat{h}) \}_{\hat{h} \in \hat{G}_+} \}^\sigma.$$
For each $g \in G$, we define
\[
(W_g \xi)(\hat{h}) = \langle g, \hat{h} \rangle \xi(\hat{h}) \quad (\forall \xi \in \mathcal{K}, \forall \hat{h} \in \hat{G}).
\]
The automorphism group \{\tilde{\alpha}_g\}_{g \in G} of $G$ on $M$ which is dual to \{\alpha_{\hat{h}}\}_{\hat{h} \in \hat{G}} in the sense of Take-saki [11] is implemented by the unitary operator $W_g$, that is,
\[
\tilde{\alpha}_g(x) = W_g x W_g^* \quad (\forall x \in M, \forall g \in G).
\]
It is elementary to check that the spectral resolution of \{\tilde{W}_g\}_{g \in G} is given by the formula
\[
\tilde{W}_g = \sum_{\hat{h} \in \hat{G}} \langle g, \hat{h} \rangle E_{\hat{h}},
\]
where $E_{\hat{h}}$ is the projection on $\mathcal{K}$ defined by the formula
\[
\left( E_{\hat{h}} \xi \right)(\hat{k}) = \begin{cases} 
\xi(\hat{h}) & \hat{h} = \hat{k}, \\
0 & \hat{h} \neq \hat{k} \end{cases} \quad (\forall \xi \in \mathcal{K}).
\]
Then it is easy to check that the projection $E_{\hat{h}}$ can be calculated as the (Bochner) integral
\[
E_{\hat{h}}(\xi) = \int_{\hat{G}} \overline{\langle g, \hat{h} \rangle W_g \xi} \, dm(g) \quad (\forall \xi \in \mathcal{K}).
\]
Moreover, for each $\hat{h} \in \hat{G}$, we define a $\sigma$-weakly continuous linear map $\varepsilon_{\hat{h}}$ on $M$ by the integral
\[
\varepsilon_{\hat{h}}(x) = \int_{\hat{G}} \overline{\langle g, \hat{h} \rangle \tilde{\alpha}_g(x)} \, dm(g) \quad (\forall x \in M).
\]

3. Proof of Theorem 1.1

In this section, we concentrate to prove Theorem 1.1.

**Lemma 3.1.** Keep the notation as in Section 2. Let $M = N \rtimes_{\alpha} \hat{G}$ and $\Gamma$ be a subset in $\hat{G}$. Then $\Gamma$ is a semigroup if and only if $M^\alpha(\Gamma)$ is a $\sigma$-weakly closed subalgebra of $M$.

**Proof.** Applying [10, Proposition 15.3], for each semigroup $\Gamma$ in $\hat{G}$, we have
\[
M^\alpha(\Gamma)M^\alpha(\Gamma) \subseteq M^\alpha(\Gamma + \Gamma) \subseteq M^\alpha(\Gamma).
\]
Conversely, we assume that $\Gamma$ is not a semigroup. Then there exist $\hat{h}, \hat{k} \in \Gamma$ such that $\hat{h} + \hat{k} \notin \Gamma$. Since, for each $g \in G$, $\tilde{\alpha}_g(\lambda(\hat{h})) = (g, \hat{h})\lambda(\hat{h})$ and $\tilde{\alpha}_g(\lambda(\hat{k})) = (g, \hat{k})\lambda(\hat{k})$, we see that $\lambda(\hat{h})$ and $\lambda(\hat{k})$ belong to $M^\alpha(\{\hat{h}\})$ and $M^\alpha(\{\hat{k}\})$, respectively, and so $\lambda(\hat{h})$ and $\lambda(\hat{k})$ belong to $M^\alpha(\Gamma)$. Since $M^\alpha(\Gamma)$ is a subalgebra of $M$, $\lambda(\hat{h})\lambda(\hat{k})$ is in $M^\alpha(\Gamma)$. This implies that $\text{Sp}_{\tilde{\alpha}_g}(\lambda(\hat{h})\lambda(\hat{k})) \subseteq \Gamma$. However, by [10, Proposition 15.3], we have
\[
\text{Sp}_{\tilde{\alpha}_g}(\lambda(\hat{h})\lambda(\hat{k})) \subseteq \text{Sp}_{\tilde{\alpha}_g}(\lambda(\hat{h})) + \text{Sp}_{\tilde{\alpha}_g}(\lambda(\hat{k})) \subseteq \{\hat{h} + \hat{k}\} \notin \Gamma.
\]
This contradicts the assumption and hence $\Gamma$ is a semigroup. This completes the proof. \qed

Using [2, Corollary 4.3.2], we have the following:
Proposition 3.2. Let $\Gamma$ be a semigroup in $\hat{G}$ which contains $\hat{0}$. Then the $\sigma$-weakly closed subalgebra of $M$ generated by $\pi_\alpha(N)$ and $\{\lambda(h)\}_{h \in \Gamma}$ coincides with $M^\alpha(\Gamma)$. In particular, $N \rtimes_\alpha \hat{G}_+ = M^\alpha(\hat{G}_+)$.  

Lemma 3.3. Keep the notation as in Section 2. Then, for each $x \in M$, we have 

$$\text{Sp}_{\tilde{\alpha}}(x) = \{\hat{h} \in \hat{G} \mid \varepsilon_{\hat{h}}(x) \neq 0\}.$$ 

Moreover, for each semigroup $\Gamma$ in $\hat{G}$, we have 

$$M^{\tilde{\alpha}}(\Gamma) = \{x \in M \mid \varepsilon_{\hat{h}}(x) = 0 \ (\forall \hat{h} \notin \Gamma)\}.$$ 

Proof. Take any $x \in M$. By a simple calculation, we note that, for each $\hat{h} \in \Gamma$ and $f \in L^1(G)$, the following equation holds: 

$$\varepsilon_{\hat{h}}(\tilde{\alpha}(f)(x)) = \hat{f}(\hat{h})\varepsilon_{\hat{h}}(x). \tag{3.1}$$ 

If $\hat{h} \in \Gamma$ satisfies $\varepsilon_{\hat{h}}(x) \neq 0$, then we have 

$$\hat{f}(\hat{h})\varepsilon_{\hat{h}}(x) = \varepsilon_{\hat{h}}(\tilde{\alpha}(f)(x)) = 0 \ (\forall f \in I_{\tilde{\alpha}}(x)).$$ 

Since $\varepsilon_{\hat{h}}(x) \neq 0$, we have $\hat{f}(\hat{h}) = 0$. This implies that $\hat{h} \in \text{Sp}_{\tilde{\alpha}}(x)$.

Conversely, we assume that $\hat{k} \in \hat{G}$ such that $\varepsilon_{\hat{k}}(x) = 0$. Putting $\hat{p}_k(g) = (g, \hat{k})$ $(\forall g \in G)$, then it is clear that $\hat{p}_k \in L^1(G)$ and

$$\hat{p}_k(d) = \int_G \langle g, d \rangle p_k(g) dm(g) = \begin{cases} 1 & (d = \hat{k}) \\ 0 & (d \neq \hat{k}). \end{cases}$$ 

By Eq. (3.1), we have

$$\varepsilon_{\hat{d}}(\tilde{\alpha}(p_k)(x)) = \hat{p}_k(d)\varepsilon_{\hat{d}}(x) = 0 \ (\forall \hat{d} \in \hat{G}).$$ 

Then we obtain $\tilde{\alpha}(p_k)(x) = 0$. Since $\hat{p}_k(\hat{k}) = 1 \neq 0$, we have $\hat{k} \notin \text{Sp}_{\tilde{\alpha}}(x)$. This completes the proof. \qed

The next result follows immediately from Lemma 3.3.

Lemma 3.4. Let $M = N \rtimes_\alpha \hat{G}$. For each semigroups $\Sigma$ and $\Gamma$ in $\hat{G}$, $\Gamma$ contains $\Sigma$ properly if and only if the subalgebra $M^{\tilde{\alpha}}(\Gamma)$ contains $M^{\tilde{\alpha}}(\Sigma)$ properly.

In the case of $M = N \rtimes_\alpha \hat{G}$, the dual action $\tilde{\alpha}$ and the generators $\{\lambda(h)\}_{h \in \hat{G}}$ of $M$ satisfy the equation 

$$\tilde{\alpha}_g(\lambda(h)) = (g, \hat{h})\lambda(h) \quad (\forall g \in G, \ \forall h \in \hat{G}).$$ 

This equation is satisfied precisely when $\text{Sp}_{\tilde{\alpha}}(\lambda(h)) = \{\hat{h}\}$ $(\forall \hat{h} \in \hat{G})$. Hence the assertion of Lemma 3.4 is natural. But, in general, Lemma 3.4 is not necessary.

Example 3.5. Let $M$ be a von Neumann algebra and $P_n$ $(n = 0, 1)$ be orthogonal projections in $M$ such that $P_0 + P_1 = I$. Putting $u_t = P_0 + e^{2\pi i t} P_1$ $(\forall t \in \mathbb{R})$, then $\{u_t\}_{t \in \mathbb{R}}$ is a strongly
continuous unitary group of \( M \) and so we can define the automorphism group \( \{ \alpha_t \}_{t \in \mathbb{R}} \) of \( \mathbb{R} \) on \( M \) which is implemented by \( u_t \). Then, for each \( x \in M \), we have
\[
\alpha_t(x) = P_0xP_0 + P_1xP_1 + e^{-2\pi i 2t} P_0xP_1 + e^{2\pi i 2t} P_1xP_0.
\]
We note that \( \alpha_t(x) = e^{2\pi inte} x \) if and only if \( \text{Sp}_{\alpha}(x) = \{ n \} \). Thus if we put \( \Gamma = \{ 2n \mid n \in \mathbb{Z}_+ \} \), then \( \Gamma \) is a semigroup in \( \mathbb{Z} \) which is contained in \( \mathbb{Z}_+ \) properly. However it easily see that
\[
M^\alpha(\Gamma) = \{ x \in M \mid P_0xP_1 = 0 \} = M^\alpha(\mathbb{Z}_+).
\]

The fourth author in [7] studied the structure of invariant subspaces and cocycles for \( N \rtimes_\alpha \hat{G}_+ \) when \( \hat{G}_+ \) induces an archimedean order in \( \hat{G} \), and showed the following:

**Theorem 3.6** [7, Theorem 6.3]. If \( N \) is a factor, then \( N \rtimes_\alpha \hat{G}_+ \) is a maximal \( \sigma \)-weakly closed subalgebra of \( M \).

We note that this result was obtained under the assumption that \( N \) admits a trace. However, considering a non-commutative \( L^2 \)-space in the sense of Haagerup [1], we may rewrite it without this assumption.

To prove Theorem 1.1, we need the converse assertion. Indeed, we shall show it without the assumption that \( N \) is a factor. Let us say that \( \Gamma \) is a maximal semigroup in \( \hat{G}_+ \) if \( \hat{G}_+ \) is the only semigroup in \( \hat{G} \) which contains \( \Gamma \) as a proper subset.

**Theorem 3.7.** Let \( M = N \rtimes_\alpha \hat{G} \). If \( M^\alpha \left( \hat{G}_+ \right) = N \rtimes_\alpha \hat{G}_+ \) is a maximal \( \sigma \)-weakly closed subalgebra of \( M \), then \( \hat{G}_+ \) induces an archimedean order in \( \hat{G} \).

**Proof.** Assume that \( M^\alpha \left( \hat{G}_+ \right) \) is a maximal \( \sigma \)-weakly closed subalgebra of \( M \). If \( \hat{G}_+ \) is not maximal, then there exists a semigroup \( \Sigma \) in \( \hat{G} \) such that \( \hat{G}_+ \subseteq \Sigma \subseteq \hat{G} \). By Lemmas 3.1 and 3.3, \( M^\alpha (\Sigma) \) is a \( \sigma \)-weakly closed subalgebra of \( M \) satisfying
\[
M^\alpha \left( \hat{G}_+ \right) \subsetneq M^\alpha (\Sigma) \subsetneq M^\alpha (\hat{G}) = M.
\]
This contradiction shows that \( \hat{G}_+ \) is a maximal semigroup in \( \hat{G} \). Moreover, \( \hat{G}_+ \) satisfies \( \hat{G}_+ \cap (-\hat{G}_+) = \{ 0 \} \) and \( \hat{G}_+ \cup (-\hat{G}_+) = \hat{G} \). Thus, by [6, Theorem 8.1.3], \( \hat{G}_+ \) induces an archimedean order in \( \hat{G} \). This completes the proof. \( \Box \)

4. Some remarks on maximality

In this section, we shall give some remarks on maximality of \( M^\alpha (\Gamma) \), where \( \Gamma \) is a semigroup of \( \hat{G} \). First we modify Theorem 3.7 as follows:

**Theorem 4.1.** Let \( M = N \rtimes_\alpha \hat{G} \) and \( \Gamma \) be a semigroup in \( \hat{G} \). If \( M^\alpha (\Gamma) \) is a maximal \( \sigma \)-weakly closed subalgebra of \( M \), then \( \Gamma \) is a maximal semigroup in \( \hat{G} \). Moreover, let \( \Gamma \) be a semigroup which satisfies the conditions
\[
\Gamma \cap (-\Gamma) = \{ 0 \} \quad \text{and} \quad \Gamma \neq \{ 0 \}.
\] (4.1)
If \( N \) is a factor, then \( M^\alpha (\Gamma) \) is maximal if and only if \( \Gamma \) is maximal. In this case \( \Gamma \) induces an archimedean order in \( \hat{G} \).
Theorem 4.1 suggests that the condition (4.1) plays an important role in the maximality of $\hat{M}^\alpha(\Gamma)$. If $\Sigma$ is an archimedean ordered semigroup in $\hat{G}$ such that $\Sigma \cup (-\Sigma) = \hat{G}$ and $\Sigma \cap (-\Sigma) \neq \{0\}$, then $\hat{M}^\alpha(\Sigma)$ does not need to be a maximal $\sigma$-weakly closed subalgebra of $M$. Indeed, we shall give an example which satisfies such a situation as follows:

**Example 4.3.** Let $G = \mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ be the two dimensional torus (the bitorus) and $L^2(\mathbb{T}^2)$ be the usual Lebesgue space with respect to the Haar measure on $\mathbb{T}^2$. Let $M$ be the von Neumann algebra acting on $L^2(\mathbb{T})$ generated by $L_1$ and $L_2$, where $L_1$ (respectively $L_2$) is the multiplication operator with the projection map on the first coordinate (respectively second coordinate). We note that $M$ is usually known as the algebra $L^\infty(\mathbb{T}^2)$, that is $M$ is spatially isomorphic to the crossed product $C \rtimes_{id} \mathbb{Z}^2$ acting on the Hilbert space $l^2(\mathbb{Z}^2) = l^2(\mathbb{Z}, C)$. Thus we may identify $M$ and $L^2(\mathbb{T}^2)$ with $C \rtimes_{id} \mathbb{Z}^2$ and $l^2(\mathbb{Z}^2)$, respectively. For each $s, t \in \mathbb{R}$, we define

$$ (W_{s,t}\xi)(m,n) = e^{-2\pi i ms}e^{-2\pi i nt} \xi(m,n) \quad (\forall \xi \in \mathcal{H}, \forall (m,n) \in \mathbb{Z}^2). $$

The automorphism group $\tilde{\alpha}$ of $\mathbb{R}^2$ on $M$ is implemented by the unitary operator $W_{s,t}$, that is,

$$ \tilde{\alpha}_{s,t}(x) = W_{s,t}x W_{s,t}^* \quad (\forall x \in M, \forall (s, t) \in \mathbb{R}^2). $$

Let $\Gamma$ be a semigroup of $\mathbb{Z}^2$ which satisfies $\Gamma \cap (-\Gamma) = \{0\}$ and $\Gamma \cup (-\Gamma) = \hat{G}$ as follows:

$$ \Gamma = \{(k, l) \in \mathbb{Z}^2 \mid k = 0 \text{ and } l \geq 0, \text{ or } k > 0\}. $$

Then $\Gamma$ induces a lexicographic order in $\mathbb{Z}^2$. For this semigroup, we can define the analytic crossed product $\hat{M}^\alpha(\Gamma)$ of $M$ with the diagonal which is a factor as follows:

$$ \hat{M}^\alpha(\Gamma) = \text{alg}\{L_1^n L_2^l \mid (m,n) \in \Gamma\}^{\sigma \omega}. $$

By Theorem 1.1, we see that the analytic crossed product $\hat{M}^\alpha(\Gamma)$ is not maximal. Indeed, if we put $\Sigma = \{(k, l) \in \mathbb{Z}^2 \mid k \geq 0\}$, then $\Sigma$ is the archimedean ordered semigroup of $\hat{G}$ which satisfies
\[ \Sigma \cup (-\Sigma) = \hat{G}, \Sigma \cap (-\Sigma) \neq \{0\} \text{ and } \Gamma \subseteq \Sigma \subsetneq \hat{G}. \]

By Lemma 4.2, we note that \( \Sigma \) is a maximal semigroup in \( \hat{G} \). By Lemmas 3.1 and 3.4, \( M^\hat{\alpha}(\Sigma) \) is a \( \sigma \)-weakly closed subalgebra of \( M \) which contains \( M^\hat{\alpha}(\Gamma) \) properly. Moreover, by [5, Section 3, Example (1)], there is not a \( \sigma \)-weakly closed subalgebra of \( M \) which contains \( M^\hat{\alpha}(\Gamma) \) and is maximal among the proper \( \sigma \)-weakly closed subalgebras of \( M \). That is, \( M^\hat{\alpha}(\Sigma) \) is not a maximal \( \sigma \)-weakly closed subalgebra of \( M \), in spite of the facts that \( \Sigma \) is maximal and the diagonal of \( M^\hat{\alpha}(\Sigma) \) is a factor.

Motivated by this fact, we shall characterize \( M^\hat{\alpha}(\Sigma) \) with some maximality in general case.

**Proposition 4.4.** Let \( N \) be a factor and \( M = N \rtimes_\alpha \hat{G} \). Let \( \Sigma \) be an archimedean ordered semigroup of \( \hat{G} \) which satisfies \( \Sigma \cup (-\Sigma) = \hat{G} \). If \( \mathfrak{A} \) is an \( \hat{\alpha} \)-invariant \( \sigma \)-weakly closed subalgebra of \( M \) which contains \( M^\hat{\alpha}(\Sigma) \), then \( \mathfrak{A} = M^\hat{\alpha}(\Sigma) \) or \( \mathfrak{A} = M \). That is, \( M^\hat{\alpha}(\Sigma) \) is maximal among \( \hat{\alpha} \)-invariant \( \sigma \)-weakly closed subalgebras of \( M \).

**Proof.** We may assume that \( M^\hat{\alpha}(\Sigma) \subsetneq \mathfrak{A} \). Since \( \mathfrak{A} \) is \( \hat{\alpha} \)-invariant, there is an element \( x \in \mathfrak{A} \) and \( h \notin \Sigma \) such that \( 0 \neq \varepsilon_h(x) \in \mathfrak{A} \). Since \( Sp_\mathfrak{A}(\varepsilon_h(x)) = \{h\} \), there exists \( y \in \pi_\alpha(N) \) such that \( \varepsilon_h(x) = y\lambda(\hat{h}) \). Thus we have

\[ \pi_\alpha(N)y\pi_\alpha(N)\lambda(\hat{h}) = \pi_\alpha(N)\varepsilon_h(x)\pi_\alpha(N) \subseteq \mathfrak{A}. \]

Since \( N \) is a factor and \( \pi_\alpha(N)y\pi_\alpha(N) \) is a two-sided ideal of \( \pi_\alpha(N) \), the \( \sigma \)-weakly closure of \( \pi_\alpha(N)y\pi_\alpha(N) \) coincides with \( \pi_\alpha(N) \), and so \( \lambda(\hat{h}) \) lies in \( \mathfrak{A} \). Let \( S \) be the semigroup generated by \( \Sigma \) and \( \hat{h} \). Since \( \mathfrak{A} \) is the algebra which contains \( M^\hat{\alpha}(\Sigma) \) and \( \lambda(\hat{h}) \), we have \( M^\hat{\alpha}(S) \subseteq \mathfrak{A} \). However, by Lemma 4.2, \( \Sigma \) is maximal and hence \( S = \hat{G} \). Therefore, we have

\[ M = M^\hat{\alpha}(\hat{G}) = M^\hat{\alpha}(S) \subseteq \mathfrak{A} \subseteq M. \]

This completes the proof. \( \square \)

**Corollary 4.5.** Keep the notation as in Example 4.3. Then every \( \sigma \)-weakly closed subalgebra of \( L^\infty(T^2) \) which contains \( M^\hat{\alpha}(\Sigma) \) properly is not \( \hat{\alpha} \)-invariant.

**Remark 4.6.** The invariance of subalgebras for an automorphism group is an interesting problem because this property is convenient to study the structure of subalgebras. For example, let \( G = T \). Then \( \hat{G} = \mathbb{Z} \) and so \( \hat{G}^+ = \mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\} \). Let \( M \) be an arbitrary finite von Neumann algebra with a faithful, normal, tracial state \( \tau \) and \( \alpha = \{\alpha_t\}_{t \in \mathbb{T}} \) be an automorphism group of \( \mathbb{T} \) on \( M \) such that \( \tau \circ \alpha_t = \tau \) (\( \forall t \in \mathbb{T} \)). Then Solel showed in [8, Corollary 4.4] that every \( \sigma \)-weakly closed subalgebra of \( M \) which contains \( M^\alpha(\mathbb{Z}^+) \) is \( \alpha \)-invariant. He also proved the same result when \( G = \hat{G} = \mathbb{R} \) and \( \hat{G}^+ = \mathbb{R}^+ \). That is, let \( M \) be a \( \sigma \)-finite von Neumann algebra and \( \alpha = \{\alpha_t\}_{t \in \mathbb{R}} \) be an automorphism group of \( \mathbb{R} \) on \( M \). Then every \( \sigma \)-weakly closed subalgebra of \( M \) that contains \( M^\alpha(\mathbb{R}^+) \) is \( \alpha \)-invariant [9, Proposition 2.1].

**Remark 4.7.** Lemma 3.1 raises the question; is every \( \sigma \)-weakly closed subalgebra of \( M \) able to describe as a subalgebra \( M^\hat{\alpha}(\Gamma) \) for some semigroup \( \Gamma \) in \( \hat{G} \)? Example 4.3 gives a counter-example for this question. In general, let \( G^+ \) be a positive semigroup of \( G \) which induces an archimedean order in \( \hat{G} \) and let \( N \) be not a factor. Then every \( \sigma \)-weakly closed subalgebra of \( M = N \rtimes_\alpha \hat{G} \) which contains \( M^\hat{\alpha}(G^+) \) never has the form \( M^\hat{\alpha}(\Lambda) \) for some semigroup \( \Lambda \) of \( \hat{G} \). However we are interested in another case. More precisely, when is a \( \sigma \)-weakly closed subalgebra of \( M^\hat{\alpha}(G^+) \) of the form \( M^\hat{\alpha}(\Lambda) \) for some semigroup \( \Lambda \) of \( \hat{G} \)?
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