# On $C^{*}$-algebras associated with $C^{*}$-correspondences 

Takeshi Katsura*<br>Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan

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#### Abstract

We study $C^{*}$-algebras arising from $C^{*}$-correspondences, which were introduced by the author. We prove the gauge-invariant uniqueness theorem, and obtain conditions for our $C^{*}$-algebras to be nuclear, exact, or satisfy the Universal Coefficient Theorem. We also obtain a 6 -term exact sequence of $K$-groups involving the $K$-groups of our $C^{*}$-algebras. (C) 2004 Elsevier Inc. All rights reserved.


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## 0. Introduction

In [Ka2], we introduce a method to construct $C^{*}$-algebras from $C^{*}$-correspondences. This construction is similar to the one of Cuntz-Pimsner algebras [P], and in fact these two constructions coincide when the left action of a given $C^{*}$ correspondence is injective. However, when the left action of a $C^{*}$-correspondence is not injective, our construction differs from the one in $[\mathrm{P}]$. Our construction of $C^{*}-$ algebras from $C^{*}$-correspondences whose left actions are not injective is motivated by the constructions of graph algebras of graphs with sinks in [FLR], $C^{*}$-algebras

[^0]from topological graphs in [Ka1], and crossed products by Hilbert $C^{*}$-bimodules in [AEE]. In fact, our construction generalizes all of these constructions. In our next paper [Ka3], we will explain that our $C^{*}$-algebras have a nice property which crossed products by automorphisms also have.

In this paper, we prove several theorems on our $C^{*}$-algebras, which generalize or improve known results on Cuntz-Pimsner algebras or other classes of $C^{*}$-algebras. After preliminaries of $C^{*}$-correspondences and their representations in Sections 1 and 2, we give definitions of our $C^{*}$-algebras $\mathscr{T}_{X}$ and $\mathcal{O}_{X}$ for a $C^{*}$-correspondence $X$ in Section 3. Sections 4 and 5 are preparatory sections for our main theorems. In Section 4, we review constructions of Fock spaces and Fock representations. Most of the results in this section have been already known. In Section 5, we analyze so-called cores. Main theorems can be found in Sections 6-8. In Section 6, we present selfcontained proofs of the gauge-invariant uniqueness theorems of our $C^{*}$-algebras. This theorem will play an important role in the analysis of their ideals in [Ka3]. In Section 7, we give necessary and sufficient conditions for our $C^{*}$-algebras to be nuclear or exact. In Section 8, we obtain a 6-term exact sequence of $K$-groups which seems to be helpful to compute $K$-groups of our $C^{*}$-algebras. We also give a sufficient condition for our $C^{*}$-algebras to satisfy the Universal Coefficient Theorem of [RS].

We denote by $\mathbb{N}=\{0,1,2, \ldots\}$ the set of natural numbers, and by $\mathbb{T}$ the group consisting of complex numbers whose absolute values are 1 . We use a convention that $\gamma(A, B)=\{\gamma(a, b) \in D \mid a \in A, b \in B\}$ for a map $\gamma: A \times B \longrightarrow D$ such as inner products, multiplications or representations. We denote by $\overline{\operatorname{span}}\{\cdots\}$ the closure of linear spans of $\{\cdots\}$. An ideal of a $C^{*}$-algebra means a closed two-sided ideal.

## 1. $C^{*}$-Correspondences

We use [L2] for the general reference of Hilbert $C^{*}$-modules and $C^{*}$-correspondences.

Definition 1.1. Let $A$ be a $C^{*}$-algebra. A (right) Hilbert $A$-module $X$ is a Banach space with a right action of the $C^{*}$-algebra $A$ and an $A$-valued inner product $\langle\cdot, \cdot\rangle_{X}: X \times X \longrightarrow A$ satisfying certain conditions.

Recall that a Hilbert $A$-module $X$ is said to be full if $\overline{\operatorname{span}}\langle X, X\rangle_{X}=A$. We do not assume that Hilbert $C^{*}$-modules $X$ are full. For a $C^{*}$-algebra $A, A$ itself is a Hilbert $A$-module where the inner product is defined by $\langle\xi, \eta\rangle_{A}=\xi^{*} \eta$, and the right action is multiplication.

Definition 1.2. For Hilbert $A$-modules $X, Y$, we denote by $\mathscr{L}(X, Y)$ the space of all adjointable operators from $X$ to $Y$. For $\xi \in X$ and $\eta \in Y$, the operator $\theta_{\eta, \xi} \in \mathscr{L}(X, Y)$ is defined by $\theta_{\eta, \xi}(\zeta)=\eta\langle\xi, \zeta\rangle_{X} \in Y$ for $\zeta \in X$. We
define $\mathscr{K}(X, Y) \subset \mathscr{L}(X, Y)$ by

$$
\mathscr{K}(X, Y)=\overline{\operatorname{span}}\left\{\theta_{\eta, \xi} \in \mathscr{L}(X, Y) \mid \xi \in X, \eta \in Y\right\} .
$$

For a Hilbert $A$-module $X$, we set $\mathscr{L}(X)=\mathscr{L}(X, X)$, which is a $C^{*}$-algebra, and $\mathscr{K}(X)=\mathscr{K}(X, X)$, which is an ideal of $\mathscr{L}(X)$.

Definition 1.3. For a $C^{*}$-algebra $A$, we say that $X$ is a $C^{*}$-correspondence over $A$ when $X$ is a Hilbert $A$-module and a $*$-homomorphism $\varphi_{X}: A \longrightarrow \mathscr{L}(X)$ is given.

We refer to $\varphi_{X}$ as the left action of a $C^{*}$-correspondence $X$. A $C^{*}$-correspondence $X$ over $A$ is said to be non-degenerate if $\overline{\operatorname{span}}\left(\varphi_{X}(A) X\right)=X$. We do not assume that $C^{*}$-correspondences are non-degenerate.

Let $A$ be a $C^{*}$-algebra. We can define a left action of the $C^{*}$-algebra $A$ on the Hilbert $A$-module $A$ by the multiplication. Thus we get a $C^{*}$-correspondence over $A$, which is called the identity correspondence over $A$ and denoted by $A$. Note that the left action $\varphi_{A}$ of the identity correspondence $A$ gives an isomorphism from $A$ onto $\mathscr{K}(A) \subset \mathscr{L}(X)$.

Definition 1.4. Let $X, Y$ be $C^{*}$-correspondences over a $C^{*}$-algebra $A$. We denote by $X \odot Y$ the quotient of the algebraic tensor product of $X$ and $Y$ by the subspace generated by $(\xi a) \otimes \eta-\xi \otimes\left(\varphi_{Y}(a) \eta\right)$ for $\xi \in X, \eta \in Y$ and $a \in A$. We can define an $A$ valued inner product, right and left actions of $A$ on $X \odot Y$ by

$$
\begin{aligned}
& \left\langle\xi \otimes \eta, \xi^{\prime} \otimes \eta^{\prime}\right\rangle_{X \otimes Y}=\left\langle\eta, \varphi_{Y}\left(\left\langle\xi, \xi^{\prime}\right\rangle_{X}\right) \eta^{\prime}\right\rangle_{Y}, \\
& (\xi \otimes \eta) a=\xi \otimes(\eta a), \quad \varphi_{X \otimes Y}(a)(\xi \otimes \eta)=\left(\varphi_{X}(a) \xi\right) \otimes \eta,
\end{aligned}
$$

for $\xi, \xi^{\prime} \in X, \eta, \eta^{\prime} \in Y$ and $a \in A$. One can show that these operations are well defined and extend to the completion of $X \odot Y$ with respect to the norm coming from the $A$-valued inner product defined above (see [L2, Proposition 4.5]). Thus the completion of $X \odot Y$ is a $C^{*}$-correspondence over $A$. This $C^{*}$-correspondence is called the tensor product of $X$ and $Y$, and denoted by $X \otimes Y$.

By definition, we have

$$
X \otimes Y=\overline{\operatorname{span}}\{\xi \otimes \eta \mid \xi \in X, \eta \in Y\},
$$

and $(\xi a) \otimes \eta=\xi \otimes\left(\varphi_{Y}(a) \eta\right)$ for $\xi \in X, \eta \in Y$ and $a \in A$.
Definition 1.5. For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$ and $n \in \mathbb{N}$, we define a $C^{*}$-correspondence $X^{\otimes n}$ over $A$ by $X^{\otimes 0}=A, X^{\otimes 1}=X$, and $X^{\otimes(n+1)}=X \otimes X^{\otimes n}$ for $n \geqslant 1$.

For each $n \in \mathbb{N}$, the left action $\varphi_{X^{\otimes n}}$ of the $C^{*}$-correspondence $X^{\otimes n}$ will be simply denoted by $\varphi_{n}: A \longrightarrow \mathscr{L}\left(X^{\otimes n}\right)$. For a positive integer $n$, we have

$$
X^{\otimes n}=\overline{\operatorname{span}}\left\{\xi_{1} \otimes \xi_{2} \otimes \cdots \otimes \xi_{n} \mid \xi_{1}, \xi_{2}, \ldots, \xi_{n} \in X\right\}
$$

Note that for positive integers $n, m$, there exists a natural isomorphism between $X^{\otimes n} \otimes X^{\otimes m}$ and $X^{\otimes(n+m)}$. We have such isomorphisms for $m=0$, but for $n=0$ we just get an injection $X^{\otimes 0} \otimes X^{\otimes m} \longrightarrow X^{\otimes m}$. When $X$ is non-degenerate, this injection is actually an isomorphism, but it is not surjective in general.

Definition 1.6. Let $n$ be a positive integer, and take $S \in \mathscr{L}\left(X^{\otimes n}\right)$. For each $m \in \mathbb{N}$, we define $S \otimes \operatorname{id}_{m} \in \mathscr{L}\left(X^{\otimes(n+m)}\right)$ by $\left(S \otimes \operatorname{id}_{m}\right)(\xi \otimes \eta)=S(\xi) \otimes \eta$ for $\xi \in X^{\otimes n}$ and $\eta \in X^{\otimes m}$.

We note that $S \otimes \mathrm{id}_{0}=S$. The $*$-homomorphism $\mathscr{L}\left(X^{\otimes n}\right) \ni S \mapsto S \otimes \mathrm{id}_{m} \in \mathscr{L}$ $\left(X^{\otimes(n+m)}\right)$ is injective when $\varphi_{X}$ is injective, but this is not the case in general. When $X$ is non-degenerate, we can define $S \otimes \operatorname{id}_{n} \in \mathscr{L}\left(X^{\otimes n}\right)$ for $S \in \mathscr{L}\left(X^{\otimes 0}\right)$ and $n \geqslant 1$ because $X^{\otimes 0} \otimes X^{\otimes n} \cong X^{\otimes n}$. In this case, we have $a \otimes \mathrm{id}_{n}=\varphi_{n}(a)$ for $a \in A \cong \mathscr{K}\left(X^{\otimes 0}\right)$. By abuse of notation, for $a \in A \cong \mathscr{K}\left(X^{\otimes 0}\right)$ we use the notation $a \otimes \operatorname{id}_{n}$ for denoting $\varphi_{n}(a) \in \mathscr{L}\left(X^{\otimes n}\right)$ even though $X$ is degenerate. Note that we cannot define $S \otimes \operatorname{id}_{n} \in \mathscr{L}\left(X^{\otimes n}\right)$ for $S \in \mathscr{L}\left(X^{\otimes 0}\right)$ in general. In other words, the $*-$ homomorphism $\varphi_{n}: A \longrightarrow \mathscr{L}\left(X^{\otimes n}\right)$ need not extend to a $*$-homomorphism $\mathscr{M}(A) \longrightarrow \mathscr{L}\left(X^{\otimes n}\right)$ unless $X$ is non-degenerate.

Definition 1.7. Let us take $\xi \in X^{\otimes n}$ with $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, we define an operator $\tau_{m}^{n}(\xi) \in \mathscr{L}\left(X^{\otimes m}, X^{\otimes(n+m)}\right)$ by

$$
\tau_{m}^{n}(\xi): X^{\otimes m} \ni \eta \mapsto \xi \otimes \eta \in X^{\otimes(n+m)}
$$

Note that for $a \in A=X^{\otimes 0}$, we have $\tau_{m}^{0}(a)=\varphi_{m}(a) \in \mathscr{L}\left(X^{\otimes m}\right)$ for each $m \in \mathbb{N}$. Note also that $\tau_{0}^{n}: X^{\otimes n} \longrightarrow \mathscr{L}\left(X^{\otimes 0}, X^{\otimes n}\right)$ is an isometry onto $\mathscr{K}\left(X^{\otimes 0}, X^{\otimes n}\right)$ for each $n \in \mathbb{N}$. The adjoint $\tau_{m}^{n}(\xi)^{*} \in \mathscr{L}\left(X^{\otimes(n+m)}, X^{\otimes m}\right)$ of $\tau_{m}^{n}(\xi)$ satisfies that $\tau_{m}^{n}(\xi)^{*}(\zeta \otimes \eta)=\varphi_{m}\left(\langle\xi, \zeta\rangle_{X^{\otimes n}}\right) \eta$ for $\zeta \in X^{\otimes n}, \eta \in X^{\otimes m}$. It is not difficult to see the following two lemmas.

Lemma 1.8. For $n_{1}, n_{2}, m \in \mathbb{N}$ and $\xi_{1} \in X^{\otimes n_{1}}, \xi_{2} \in X^{\otimes n_{2}}$, we have

$$
\tau_{n_{2}+m}^{n_{1}}\left(\xi_{1}\right) \tau_{m}^{n_{2}}\left(\xi_{2}\right)=\tau_{m}^{n_{1}+n_{2}}\left(\xi_{1} \otimes \xi_{2}\right) \quad \text { in } \mathscr{L}\left(X^{\otimes m}, X^{\otimes\left(n_{1}+n_{2}+m\right)}\right)
$$

Lemma 1.9. For $n, m \in \mathbb{N}, \xi, \eta \in X^{\otimes n}$ and $a \in A$, we have the following:
(i) $\tau_{m}^{n}(\xi) \tau_{m}^{n}(\eta)^{*}=\theta_{\xi, \eta} \otimes \mathrm{id}_{m}$ in $\mathscr{L}\left(X^{\otimes(n+m)}\right)$,
(ii) $\tau_{m}^{n}(\xi)^{*} \tau_{m}^{n}(\eta)=\varphi_{m}\left(\langle\xi, \eta\rangle_{X^{\otimes n}}\right)$ in $\mathscr{L}\left(X^{\otimes m}\right)$,
(iii) $\tau_{m}^{n}(\xi) \varphi_{m}(a)=\tau_{m}^{n}(\xi a)$ in $\mathscr{L}\left(X^{\otimes m}, X^{\otimes(n+m)}\right)$,
(iv) $\varphi_{n+m}(a) \tau_{m}^{n}(\xi)=\tau_{m}^{n}\left(\varphi_{n}(a) \xi\right)$ in $\mathscr{L}\left(X^{\otimes m}, X^{\otimes(n+m)}\right)$.

## 2. Representations of $C^{*}$-correspondences

Definition 2.1. A representation of a $C^{*}$-correspondence $X$ over $A$ on a $C^{*}$-algebra $B$ is a pair consisting of a $*$-homomorphism $\pi: A \longrightarrow B$ and a linear map $t: X \longrightarrow B$ satisfying
(i) $t(\xi)^{*} t(\eta)=\pi\left(\langle\xi, \eta\rangle_{X}\right)$ for $\xi, \eta \in X$,
(ii) $\pi(a) t(\xi)=t\left(\varphi_{X}(a) \xi\right)$ for $a \in A, \xi \in X$.

We denote by $C^{*}(\pi, t)$ the $C^{*}$-algebra generated by the images of $\pi$ and $t$ in $B$.
A representation of a $C^{*}$-correspondence was called an isometric covariant representation in [MS]. Note that for a representation $(\pi, t)$ of $X$, we have $t(\xi) \pi(a)=$ $t(\xi a)$ automatically because the condition (i) above, combining with the fact that $\pi$ is a *-homomorphism, implies

$$
\|t(\xi) \pi(a)-t(\xi a)\|^{2}=\left\|(t(\xi) \pi(a)-t(\xi a))^{*}(t(\xi) \pi(a)-t(\xi a))\right\|=0
$$

Note also that for $\xi \in X$, we have $\|t(\xi)\| \leqslant\|\xi\|_{X}$ because

$$
\|t(\xi)\|^{2}=\left\|t(\xi)^{*} t(\xi)\right\|=\left\|\pi\left(\langle\xi, \xi\rangle_{X}\right)\right\| \leqslant\left\|\langle\xi, \xi\rangle_{X}\right\|=\|\xi\|_{X}^{2}
$$

Definition 2.2. A representation $(\pi, t)$ is said to be injective if a $*$-homomorphism $\pi$ is injective.

By the above computation, we see that $t$ is isometric for an injective representation $(\pi, t)$.

Definition 2.3. For a representation $(\pi, t)$ of a $C^{*}$-correspondence $X$ on $B$, we define a $*$-homomorphism $\psi_{t}: \mathscr{K}(X) \longrightarrow B$ by $\psi_{t}\left(\theta_{\xi, \eta}\right)=t(\xi) t(\eta)^{*} \in B$ for $\xi, \eta \in X$.

For the well-definedness of a $*$-homomorphism $\psi_{t}$, see, for example, [KPW, Lemma 2.2]. The following lemma is easily verified.

Lemma 2.4. For a representation $(\pi, t)$ of $a C^{*}$-correspondence $X$ over $A$, we have $\pi(a) \psi_{t}(k)=\psi_{t}\left(\varphi_{X}(a) k\right)$ and $\psi_{t}(k) t(\xi)=t(k \xi)$ for $a \in A, \quad \xi \in X$ and $k \in \mathscr{K}(X)$.

By this lemma, we see that $\psi_{t}$ is injective for an injective representation $(\pi, t)$.

Definition 2.5. Let $(\pi, t)$ be a representation of $X$. We set $t^{0}=\pi$ and $t^{1}=t$. For $n=2,3, \ldots$, we define a linear map $t^{n}: X^{\otimes n} \longrightarrow C^{*}(\pi, t)$ by $t^{n}(\xi \otimes \eta)=t(\xi) t^{n-1}(\eta)$ for $\xi \in X$ and $\eta \in X^{\otimes(n-1)}$.

It is routine to see that $t^{n}$ is well defined and that $\left(\pi, t^{n}\right)$ is a representation of the $C^{*}$-correspondence $X^{\otimes n}$. Hence we can define $\psi_{t^{n}}: \mathscr{K}\left(X^{\otimes n}\right) \longrightarrow C^{*}(\pi, t)$ by $\psi_{t^{n}}\left(\theta_{\xi, \eta}\right)=t^{n}(\xi) t^{n}(\eta)^{*}$ for $\xi, \eta \in X^{\otimes n}$. Note that $t^{n}$ and $\psi_{t^{n}}$ are isometric if $(\pi, t)$ is an injective representation.

Lemma 2.6. Let $(\pi, t)$ be a representation of $X$. Take $\xi \in X^{\otimes n}$ and $\eta \in X^{\otimes m}$ for $n, m \in \mathbb{N}$ with $n \geqslant m$. Then we have $t^{m}(\eta)^{*} t^{n}(\xi)=t^{n-m}(\zeta)$ where $\zeta=$ $\tau_{n-m}^{m}(\eta)^{*} \xi \in X^{\otimes(n-m)}$.

Proof. When $m=0$, this follows from the fact that $\left(\pi, t^{n}\right)$ is a representation of the $C^{*}$-correspondence $X^{\otimes n}$. Let $m$ be a positive integer. We may assume $\xi=\eta^{\prime} \otimes \zeta^{\prime}$ for $\eta^{\prime} \in X^{\otimes m}$ and $\zeta^{\prime} \in X^{\otimes(n-m)}$ because the linear span of such elements is dense in $X^{\otimes n}$. We have

$$
\begin{aligned}
t^{m}(\eta)^{*} t^{n}(\xi) & =t^{m}(\eta)^{*} t^{m}\left(\eta^{\prime}\right) t^{n-m}\left(\zeta^{\prime}\right) \\
& =\pi\left(\left\langle\eta, \eta^{\prime}\right\rangle_{X \otimes m}\right) t^{n-m}\left(\zeta^{\prime}\right) \\
& =t^{n-m}\left(\varphi_{n-m}\left(\left\langle\eta, \eta^{\prime}\right\rangle_{X^{\otimes m}}\right) \zeta^{\prime}\right)
\end{aligned}
$$

On the other hand, we get

$$
\tau_{n-m}^{m}(\eta)^{*} \xi=\tau_{n-m}^{m}(\eta)^{*}\left(\eta^{\prime} \otimes \zeta^{\prime}\right)=\varphi_{n-m}\left(\left\langle\eta, \eta^{\prime}\right\rangle_{X^{\otimes m}}\right) \zeta^{\prime}
$$

We are done.

Proposition 2.7. For a representation ( $\pi, t$ ) of $X$, we have

$$
C^{*}(\pi, t)=\overline{\operatorname{span}}\left\{t^{n}(\xi) t^{m}(\eta)^{*} \mid \xi \in X^{\otimes n}, \eta \in X^{\otimes m}, n, m \in \mathbb{N}\right\} .
$$

Proof. Clearly, the right-hand side is a closed $*$-invariant linear space which contains the images of $\pi$ and $t$, and is contained in $C^{*}(\pi, t)$. Hence all we have to do is to check that this set is closed under the multiplication, and this follows from Lemma 2.6.

## 3. $C^{*}$-algebras associated with $C^{*}$-correspondences

In this section, we give definitions of the $C^{*}$-algebras $\mathscr{T}_{X}$ and $\mathcal{O}_{X}$ for a $C^{*}$ correspondence $X$.

Definition 3.1. Let $X$ be a $C^{*}$-correspondence over a $C^{*}$-algebra $A$. We denote by $\left(\bar{\pi}_{X}, \bar{t}_{X}\right)$ the universal representation of $X$, and set $\mathscr{T}_{X}=C^{*}\left(\bar{\pi}_{X}, \bar{t}_{X}\right)$.

The universal representation $\left(\bar{\pi}_{X}, \bar{t}_{X}\right)$ can be obtained by taking a direct sum of sufficiently many representations. By the universality, for every representation $(\pi, t)$ of $X$ we have a surjection $\rho: \mathscr{T}_{X} \longrightarrow C^{*}(\pi, t)$ with $\pi=\rho \circ \bar{\pi}_{X}$ and $t=\rho \circ \bar{t}_{X}$. This surjection will be called a natural surjection.

Definition 3.2. For a $C^{*}$-correspondence $X$ over $A$, we define an ideal $J_{X}$ of $A$ by

$$
\begin{aligned}
J_{X} & =\varphi_{X}^{-1}(\mathscr{K}(X)) \cap\left(\operatorname{ker} \varphi_{X}\right)^{\perp} \\
& =\left\{a \in A \mid \varphi_{X}(a) \in \mathscr{K}(X) \text { and } a b=0 \text { for all } b \in \operatorname{ker} \varphi_{X}\right\} .
\end{aligned}
$$

Note that $J_{X}=\varphi_{X}^{-1}(\mathscr{K}(X))$ when $\varphi_{X}$ is injective. The ideal $J_{X}$ is the largest ideal to which the restriction of $\varphi_{X}$ is an injection into $\mathscr{K}(X)$. The ideal $J_{X}$ has the following property.

Proposition 3.3. Let $X$ be a $C^{*}$-correspondence over a $C^{*}$-algebra $A$, and $(\pi, t)$ be an injective representation of $X$. If $a \in A$ satisfies $\pi(a) \in \psi_{t}(\mathscr{K}(X))$, then we have $a \in J_{X}$ and $\pi(a)=\psi_{t}\left(\varphi_{X}(a)\right)$.

Proof. Take $a \in A$ with $\pi(a) \in \psi_{t}(\mathscr{K}(X))$. Let $k \in \mathscr{K}(X)$ be an element with $\pi(a)=$ $\psi_{t}(k)$. For each $\xi \in X$, we have

$$
t\left(\varphi_{X}(a) \xi\right)=\pi(a) t(\xi)=\psi_{t}(k) t(\xi)=t(k \xi)
$$

Since $t$ is injective, we have $\varphi_{X}(a) \xi=k \xi$ for every $\xi \in X$. This implies that $\varphi_{X}(a)=$ $k \in \mathscr{K}(X)$. Thus we get $\pi(a)=\psi_{t}\left(\varphi_{X}(a)\right)$. Take $b \in \operatorname{ker} \varphi_{X}$ and we will show that $a b=0$. We get

$$
\pi(a b)=\pi(a) \pi(b)=\psi_{t}\left(\varphi_{X}(a)\right) \pi(b)=\psi_{t}\left(\varphi_{X}(a) \varphi_{X}(b)\right)=0
$$

Since $\pi$ is injective, we obtain $a b=0$ as desired. Thus $a \in J_{X}$.
The above proposition motivates the following definition.
Definition 3.4. A representation $(\pi, t)$ is said to be covariant if we have $\pi(a)=$ $\psi_{t}\left(\varphi_{X}(a)\right)$ for all $a \in J_{X}$.

Definition 3.5. For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, the $C^{*}$-algebra $\mathcal{O}_{X}$ is defined by $\mathcal{O}_{X}=C^{*}\left(\pi_{X}, t_{X}\right)$ where $\left(\pi_{X}, t_{X}\right)$ is the universal covariant representation of $X$.

By the universality, for each covariant representation $(\pi, t)$ of a $C^{*}$-correspondence $X$, there exists a natural surjection $\rho: \mathcal{O}_{X} \longrightarrow C^{*}(\pi, t)$ satisfying $\pi=\rho \circ \pi_{X}$ and $t=\rho \circ t_{X}$.

The construction of $C^{*}$-algebras $\mathcal{O}_{X}$ from $C^{*}$-correspondences $X$ generalizes both the one in $[\mathrm{P}]$ for $C^{*}$-correspondences with injective left actions and the one in [AEE] for $C^{*}$-correspondences coming from Hilbert $C^{*}$-bimodules. This is also a generalization of the construction of graph algebras [FLR,KPR,KPRR] and more generally $C^{*}$-algebras arising from topological graphs [Kal]. For the detail, see [Ka2].

## 4. The Fock representation

In this section, we construct a representation of a given $C^{*}$-correspondence, which is called the Fock representation. The Fock representation is injective, and from this we get an injective covariant representation. Most of the results in this section can be found in [MS] or [P]. We will need them in Sections 7 and 8 . For the convenience of the readers, we give complete proofs.

Definition 4.1. The Hilbert $A$-module $\mathscr{F}(X)$, obtained as the direct sum of the Hilbert $A$-modules $X^{\otimes 0}, X^{\otimes 1}, \ldots$, is called the Fock space of $X$.

We consider $X^{\otimes n}$ as a submodule of $\mathscr{F}(X)$ for each $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$, we consider the space $\mathscr{L}\left(X^{\otimes n}, X^{\otimes m}\right)$ of adjointable operators from $X^{\otimes n}$ to $X^{\otimes m}$ as a subspace of $\mathscr{L}(\mathscr{F}(X))$.

Definition 4.2. We define a $*$-homomorphism $\varphi_{\infty}: A \longrightarrow \mathscr{L}(\mathscr{F}(X))$ and a linear map $\tau_{\infty}: X \longrightarrow \mathscr{L}(\mathscr{F}(X))$ by

$$
\varphi_{\infty}(a)=\sum_{m=0}^{\infty} \varphi_{m}(a), \quad \tau_{\infty}(\xi)=\sum_{m=0}^{\infty} \tau_{m}^{1}(\xi),
$$

for $a \in A$ and $\xi \in X$, where we always use the strong topology for the infinite sum of elements in $\mathscr{L}(\mathscr{F}(X))$.

Proposition 4.3 ([P, Proposition 1.3]). The pair $\left(\varphi_{\infty}, \tau_{\infty}\right)$ is an injective representation of $X$ on $\mathscr{L}(\mathscr{F}(X))$.

Proof. By taking $n=1$ in Lemma 1.9 (ii) and (iv), we see that ( $\varphi_{\infty}, \tau_{\infty}$ ) is a representation of $X$. It is injective because $\varphi_{0}: A \longrightarrow \mathscr{L}\left(X^{\otimes 0}\right)$ is an isomorphism onto $\mathscr{K}\left(X^{\otimes 0}\right)$.

This representation $\left(\varphi_{\infty}, \tau_{\infty}\right)$ is called the Fock representation. From the Fock representation $\left(\varphi_{\infty}, \tau_{\infty}\right)$, we can define a linear map $\tau_{\infty}^{n}: X^{\otimes n} \longrightarrow \mathscr{L}(\mathscr{F}(X))$ for each
$n \in \mathbb{N}$ as in Definition 2.5. It is easy to see that $\tau_{\infty}^{n}(\xi)=\sum_{m=0}^{\infty} \tau_{m}^{n}(\xi)$ for $\xi \in X^{\otimes n}$ and $n \in \mathbb{N}$.

Proposition 4.4. For $a \in J_{X}$, we have

$$
\varphi_{\infty}(a)-\psi_{\tau_{\infty}}\left(\varphi_{X}(a)\right)=\varphi_{0}(a) \in \mathscr{L}\left(X^{\otimes 0}\right) \subset \mathscr{L}(\mathscr{F}(X)) .
$$

Proof. For $\xi, \eta \in X$, we have $\psi_{\tau_{\infty}}\left(\theta_{\xi, \eta}\right)=\sum_{m=1}^{\infty} \theta_{\xi, \eta} \otimes \mathrm{id}_{m-1}$ by Lemma 1.9(i). Hence we have $\psi_{\tau_{\infty}}(k)=\sum_{m=1}^{\infty} k \otimes \mathrm{id}_{m-1}$ for all $k \in \mathscr{K}(X)$. Therefore we obtain

$$
\varphi_{\infty}(a)-\psi_{\tau_{\infty}}\left(\varphi_{X}(a)\right)=\sum_{m=0}^{\infty} \varphi_{m}(a)-\sum_{m=1}^{\infty} \varphi_{X}(a) \otimes \operatorname{id}_{m-1}=\varphi_{0}(a)
$$

because $\varphi_{m}(a)=\varphi_{X}(a) \otimes \mathrm{id}_{m-1}$ for $m \geqslant 1$.
Corollary 4.5. If $a \in A$ satisfies $\varphi_{\infty}(a) \in \psi_{\tau_{\infty}}(\mathscr{K}(X))$, then $a=0$.
Proof. For $a \in A$ with $\varphi_{\infty}(a) \in \psi_{\tau_{\infty}}(\mathscr{K}(X))$, we have $a \in J_{X}$ and $\varphi_{\infty}(a)=\psi_{\tau_{\infty}}\left(\varphi_{X}(a)\right)$ by Proposition 3.3. By Proposition 4.4, we get $\varphi_{0}(a)=\varphi_{\infty}(a)-\psi_{\tau_{\infty}}\left(\varphi_{X}(a)\right)=0$. Thus we obtain $a=0$ because $\varphi_{0}$ is injective.

The set $\mathscr{F}(X) J_{X}$ is a Hilbert $J_{X}$-module [Ka3, Corollary 1.4], and we have

$$
\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)=\overline{\operatorname{span}}\left\{\theta_{\xi, \eta, \eta} \in \mathscr{K}(\mathscr{F}(X)) \mid \xi, \eta \in \mathscr{F}(X), a \in J_{X}\right\},
$$

which is an ideal of $\mathscr{L}(\mathscr{F}(X))$. We see that $k \in \mathscr{K}(\mathscr{F}(X))$ is in $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ if and only if $\langle\xi, k \eta\rangle \in J_{X}$ for all $\xi, \eta \in \mathscr{F}(X)$ (see [FMR, Lemma 2.6] or [Ka3, Lemma 1.6]).

Proposition 4.6. We have $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right) \subset C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right)$.
Proof. For $\xi \in X^{\otimes n}, \eta \in X^{\otimes m}$ and $a \in J_{X}$, we have

$$
\begin{aligned}
\theta_{\xi, a, \eta} & =\tau_{\infty}^{n}(\xi) \varphi_{0}(a) \tau_{\infty}^{m}(\eta)^{*} \\
& =\tau_{\infty}^{n}(\xi)\left(\varphi_{\infty}(a)-\psi_{\tau_{\infty}}\left(\varphi_{X}(a)\right)\right) \tau_{\infty}^{m}(\eta)^{*} \in C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right)
\end{aligned}
$$

by Proposition 4.4. Hence $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right) \subset C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right)$.
Let $\sigma: \mathscr{L}(\mathscr{F}(X)) \longrightarrow \mathscr{L}(\mathscr{F}(X)) / \mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ be the quotient map, and set $\varphi=$ $\sigma \circ \varphi_{\infty}$ and $\tau=\sigma \circ \tau_{\infty}$. By Proposition 4.4, $(\varphi, \tau)$ is a covariant representation of $X$ on $\mathscr{L}(\mathscr{F}(X)) / \mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$. We will see that this representation $(\varphi, \tau)$ is injective.

Lemma 4.7. For $n \geqslant 1$, the restriction of the $*$-homomorphism $\mathscr{L}\left(X^{\otimes n}\right) \ni$ $S \mapsto S \otimes \mathrm{id}_{1} \in \mathscr{L}\left(X^{\otimes(n+1)}\right)$ to $\mathscr{K}\left(X^{\otimes n} J_{X}\right)$ is injective.

Proof. Take $k \in \mathscr{K}\left(X^{\otimes n} J_{X}\right)$ with $k \otimes \operatorname{id}_{1}=0$. Then for all $\xi, \xi^{\prime} \in X^{\otimes n}$ and all $\eta, \eta^{\prime} \in X$, we have

$$
0=\left\langle\xi \otimes \eta,\left(k \otimes \mathrm{id}_{1}\right)\left(\xi^{\prime} \otimes \eta^{\prime}\right)\right\rangle_{X^{\otimes(n+1)}}=\left\langle\eta, \varphi_{X}\left(\left\langle\xi, k \xi^{\prime}\right\rangle_{X^{\otimes n}}\right) \eta^{\prime}\right\rangle_{X} .
$$

Hence we have $\varphi_{X}\left(\left\langle\xi, k \xi^{\prime}\right\rangle_{X^{\otimes n}}\right)=0$ for all $\xi, \xi^{\prime} \in X^{\otimes n}$. Since $k \in \mathscr{K}\left(X^{\otimes n} J_{X}\right)$, we have $\left\langle\xi, k \xi^{\prime}\right\rangle_{X^{\otimes n}} \in J_{X}$. Thus $\left\langle\xi, k \xi^{\prime}\right\rangle_{X^{\otimes n}}=0$ for all $\xi, \xi^{\prime} \in X^{\otimes n}$ because $\varphi_{X}$ is injective on $J_{X}$. Therefore we get $k=0$. Thus the restriction of the map $S \mapsto S \otimes \mathrm{id}_{1}$ to $\mathscr{K}\left(X^{\otimes n} J_{X}\right)$ is injective.

Lemma 4.8. For $a \in A, \varphi_{\infty}(a) \in \mathscr{K}(\mathscr{F}(X))$ implies $\lim _{n \rightarrow \infty}\left\|\varphi_{n}(a)\right\|=0$.
Proof. For each $n \in \mathbb{N}$, let $P_{n} \in \mathscr{L}(\mathscr{F}(X))$ be the projection onto the direct summand $X^{\otimes n} \subset \mathscr{F}(X)$. Since $\quad \varphi_{n}(a)=P_{n} \varphi_{\infty}(a) P_{n}$, it suffices to show that $\lim _{n \rightarrow \infty}\left\|P_{n} k P_{n}\right\|=0$ for each $k \in \mathscr{K}(\mathscr{F}(X))$. We may assume $k=\theta_{\xi, \eta}$ for $\xi, \eta \in \mathscr{F}(X)$ because the linear span of such elements is dense in $\mathscr{K}(\mathscr{F}(X))$. By the same reason, we may assume $\xi \in X^{\otimes k}$ and $\eta \in X^{\otimes l}$ for some $k, l \in \mathbb{N}$. Now it is clear that we have $\lim _{n \rightarrow \infty}\left\|P_{n} k P_{n}\right\|=0$. This completes the proof.

Proposition 4.9. The covariant representation $(\varphi, \tau)$ is injective.
Proof. Take $a \in A$ with $\varphi(a)=0$. Then we have $\varphi_{\infty}(a) \in \mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$. For each $n \in \mathbb{N}$, we have

$$
\varphi_{n}(a)=P_{n} \varphi_{\infty}(a) P_{n} \in P_{n} \mathscr{K}\left(\mathscr{F}(X) J_{X}\right) P_{n}=\mathscr{K}\left(X^{\otimes n} J_{X}\right)
$$

where $P_{n} \in \mathscr{L}(\mathscr{F}(X))$ is the projection onto the direct summand $X^{\otimes n} \subset \mathscr{F}(X)$. By taking $n=0$, we get $a \in J_{X}$. Since $\varphi_{1}=\varphi_{X}$ is injective on $J_{X}$, we have $\|a\|=\left\|\varphi_{1}(a)\right\|$. By Lemma 4.7, we have $\left\|\varphi_{n}(a)\right\|=\left\|\varphi_{n}(a) \otimes \mathrm{id}_{1}\right\|=\left\|\varphi_{n+1}(a)\right\|$ for all positive integer $n$. Therefore we get $\left\|\varphi_{n}(a)\right\|=\|a\|$ for all $n \in \mathbb{N}$. Thus we have $a=0$ by Lemma 4.8. This proves that the covariant representation $(\varphi, \tau)$ is injective.

As consequences of Corollary 4.5 and Proposition 4.9, we have the following propositions.

Proposition 4.10. The universal representation $\left(\bar{\pi}_{X}, \bar{t}_{X}\right)$ of $X$ on $\mathscr{T}_{X}$ satisfies that $\left\{a \in A \mid \bar{\pi}_{X}(a) \in \psi_{\bar{t}_{X}}(\mathscr{K}(X))\right\}=0$.

Proposition 4.11. The universal covariant representation $\left(\pi_{X}, t_{X}\right)$ of $X$ on $\mathcal{O}_{X}$ is injective.

We will see in Section 6 that the Fock representation $\left(\varphi_{\infty}, \tau_{\infty}\right)$ is the universal representation, and $(\varphi, \tau)$ is the universal covariant representation.

Note that the $C^{*}$-algebra $C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right)$ is the augmented Cuntz-Toeplitz algebra defined in $[\mathrm{P}]$, and the $C^{*}$-algebra $C^{*}(\varphi, \tau)$ is the relative Cuntz-Pimsner algebra $\mathcal{O}\left(J_{X}, X\right)$ defined in [MS, Definition 2.18].

## 5. Analysis of the cores

In this section, we investigate the so-called cores of $C^{*}$-algebras $C^{*}(\pi, t)$ for representations $(\pi, t)$ of a $C^{*}$-correspondence $X$. Fix a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, and a representation $(\pi, t)$ of $X$.

Definition 5.1. For each $n \in \mathbb{N}$, we set $B_{n}=\psi_{t^{n}}\left(\mathscr{K}\left(X^{\otimes n}\right)\right) \subset C^{*}(\pi, t)$.
Note that $B_{0}=\pi(A)$ and that $B_{n} \cong \mathscr{K}\left(X^{\otimes n}\right)$ when $(\pi, t)$ is injective. We can easily see the next lemma.

Lemma 5.2. For $n, m \in \mathbb{N}$ with $n \geqslant 1$, we have

$$
\overline{\operatorname{span}}\left(t^{n}\left(X^{\otimes n}\right) B_{m} t^{n}\left(X^{\otimes n}\right)^{*}\right)=B_{n+m}
$$

and $t^{n}\left(X^{\otimes n}\right)^{*} B_{n+m} t^{n}\left(X^{\otimes n}\right) \subset B_{m}$.
Definition 5.3. For $m, n \in \mathbb{N}$ with $m \leqslant n$, we define $B_{[m, n]} \subset C^{*}(\pi, t)$ by $B_{[m, n]}=B_{m}+$ $B_{m+1}+\cdots+B_{n}$.

We have $B_{[n, n]}=B_{n}$ for each $n \in \mathbb{N}$. By the next lemma, we see that $B_{[m, n]}$ 's are $C^{*}$ subalgebras of $C^{*}(\pi, t)$.

Lemma 5.4. For $m, n \in \mathbb{N}$ with $m \leqslant n, k \in \mathscr{K}\left(X^{\otimes m}\right)$ and $k^{\prime} \in \mathscr{K}\left(X^{\otimes n}\right)$, we have $\psi_{t^{m}}(k) \psi_{t^{n}}\left(k^{\prime}\right)=\psi_{t^{n}}\left(\left(k \otimes \mathrm{id}_{n-m}\right) k^{\prime}\right)$.

Proof. It suffices to show that $\psi_{t^{m}}(k) t^{n}(\xi)=t^{n}\left(\left(k \otimes \operatorname{id}_{n-m}\right) \xi\right)$ for $k \in \mathscr{K}\left(X^{\otimes m}\right)$ and $\xi \in X^{\otimes n}$. When $m=0$, this equation follows from the fact that $\left(\pi, t^{n}\right)$ is a representation of the $C^{*}$-correspondence $X^{\otimes n}$. Suppose $m \geqslant 1$. We may assume $k=$ $\theta_{\zeta, \eta}$ for $\zeta, \eta \in X^{\otimes m}$. We have

$$
\begin{aligned}
\psi_{t^{m}}(k) t^{n}(\xi) & =t^{m}(\zeta) t^{m}(\eta)^{*} t^{n}(\xi) \\
& =t^{m}(\zeta) t^{n-m}\left(\tau_{n-m}^{m}(\eta)^{*} \xi\right) \\
& =t^{n}\left(\zeta \otimes\left(\tau_{n-m}^{m}(\eta)^{*} \xi\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& =t^{n}\left(\left(\tau_{n-m}^{m}(\zeta) \tau_{n-m}^{m}(\eta)^{*}\right) \xi\right) \\
& =t^{n}\left(\left(k \otimes \operatorname{id}_{n-m}\right) \xi\right)
\end{aligned}
$$

by Lemma 2.6 and Lemma 1.9(i). We are done.

By the above lemma, $B_{[k, n]}$ is an ideal of $B_{[m, n]}$ for $m, k, n \in \mathbb{N}$ with $m \leqslant k \leqslant n$. In particular, $B_{n}$ is an ideal of $B_{[0, n]}$ for each $n \in \mathbb{N}$.

Definition 5.5. For $m \in \mathbb{N}$, we define a $C^{*}$-subalgebra $B_{[m, \infty]}$ of $C^{*}(\pi, t)$ by $B_{[m, \infty]}=$ $\overline{\bigcup_{n=m}^{\infty} B_{[m, n]}}$.

Note that the $C^{*}$-algebra $B_{[m, \infty]}$ is an inductive limit of the increasing sequence of $C^{*}$-algebras $\left\{B_{[m, n]}\right\}_{n=m}^{\infty}$. The $C^{*}$-algebra $B_{[0, \infty]}$ is called the core of the $C^{*}$-algebra $C^{*}(\pi, t)$. The core $B_{[0, \infty]}$ naturally arises when the $C^{*}$-algebra $C^{*}(\pi, t)$ has an action of $\mathbb{T}$ called a gauge action.

Definition 5.6. A representation $(\pi, t)$ of $X$ is said to admit a gauge action if for each $z \in \mathbb{T}$, there exists a $*$-homomorphism $\beta_{z}: C^{*}(\pi, t) \longrightarrow C^{*}(\pi, t)$ such that $\beta_{z}(\pi(a))=$ $\pi(a)$ and $\beta_{z}(t(\xi))=z t(\xi)$ for all $a \in A$ and $\xi \in X$.

If it exists, such a $*$-homomorphism $\beta_{z}$ is unique. By the assumptions in the definition above, $\beta_{z}$ is a $*$-automorphism for all $z \in \mathbb{T}$ and the map $\beta: \mathbb{T} \longrightarrow \operatorname{Aut}\left(C^{*}(\pi, t)\right)$ is automatically a strongly continuous homomorphism. By the universality, both the universal representation $\left(\bar{\pi}_{X}, \bar{t}_{X}\right)$ on $\mathscr{T}_{X}$ and the universal covariant representation $\left(\pi_{X}, t_{X}\right)$ on $\mathcal{O}_{X}$ admit gauge actions. We denote these actions by $\bar{\gamma}: \mathbb{T} \curvearrowright \mathscr{T}_{X}$ and $\gamma: \mathbb{T} \curvearrowright \mathcal{O}_{X}$. It is clear that for a representation $(\pi, t)$ admitting a gauge action $\beta$ we have $\beta_{z} \circ \rho=\rho \circ \bar{\gamma}_{z}$ for each $z \in \mathbb{T}$, where $\rho: \mathscr{T}_{X} \longrightarrow C^{*}(\pi, t)$ is the natural surjection. It is also clear that for a covariant representation $(\pi, t)$ admitting a gauge action $\beta$ we have $\beta_{z} \circ \rho=\rho \circ \gamma_{z}$ for each $z \in \mathbb{T}$, where $\rho: \mathcal{O}_{X} \longrightarrow C^{*}(\pi, t)$ is the natural surjection.

Proposition 5.7. When a representation $(\pi, t)$ admits a gauge action $\beta$, the core $B_{[0, \infty]}$ coincides with the fixed point algebra $C^{*}(\pi, t)^{\beta}$.

Proof. Since

$$
\beta_{z}\left(t^{n}(\xi) t^{m}(\eta)^{*}\right)=z^{n-m} t^{n}(\xi) t^{m}(\eta)^{*}
$$

for $\xi \in X^{\otimes n}, \eta \in X^{\otimes m}$ and $z \in \mathbb{T}$, it is clear that $B_{[0, \infty]} \subset C^{*}(\pi, t)^{\beta}$. Take $x \in C^{*}(\pi, t)^{\beta}$. By Proposition 2.7, there exists a sequence $\left\{x_{k}\right\}_{k=0}^{\infty}$ of linear sums of elements in the
form $t^{n}(\xi) t^{m}(\eta)^{*}$ such that $x=\lim _{k \rightarrow \infty} x_{k}$. Then we have

$$
x=\int_{\mathbb{T}} \beta_{z}(x) d z=\lim _{k \rightarrow \infty} \int_{\mathbb{T}} \beta_{z}\left(x_{k}\right) d z
$$

where $d z$ is the normalized Haar measure on $\mathbb{T}$. By the above computation, we get $\int_{\mathbb{U}} \beta_{z}\left(x_{k}\right) d z \in \bigcup_{n=0}^{\infty} B_{n}$ for every $k$. Thus we have $x \in B_{[0, \infty]}$. We have shown that $B_{[0, \infty]}=C^{*}(\pi, t)^{\beta}$.

We are going to compute the core $B_{[0, \infty]} \subset C^{*}(\pi, t)$. To this end, we need the following notation.

Definition 5.8. For a representation $(\pi, t)$ of $X$, we set

$$
I_{(\pi, t)^{\prime}}=\left\{a \in A \mid \pi(a) \in B_{1}=\psi_{t}(\mathscr{K}(X))\right\}
$$

which is an ideal of $A$. For each $n \in \mathbb{N}$, we define

$$
B_{n}{ }^{\prime}=\psi_{t^{n}}\left(\mathscr{K}\left(X^{\otimes n} I_{(\pi, t)}{ }^{\prime}\right)\right) \subset B_{n} \subset C^{*}(\pi, t)
$$

Proposition 5.9. For each $n \in \mathbb{N}$, we have $B_{n} \cap B_{n+1}=B_{n}{ }^{\prime}$.
Proof. The case $n=0$ follows from the definition of $I_{(\pi, t)}{ }^{\prime}$. Let $n$ be a positive integer. For $a \in I_{(\pi, t)^{\prime}}$ and $\xi, \eta \in X^{\otimes n}$, we have

$$
\psi_{t^{n}}\left(\theta_{\xi, \xi, \eta}\right)=t^{n}(\xi a) t^{n}(\eta)^{*}=t^{n}(\xi) \pi(a) t^{n}(\eta)^{*} \in B_{n+1}
$$

because $\pi(a) \in B_{1}$. Hence we get $B_{n}{ }^{\prime} \subset B_{n} \cap B_{n+1}$. Conversely take $x \in B_{n} \cap B_{n+1}$. Take $k \in \mathscr{K}\left(X^{\otimes n}\right)$ with $\psi_{t^{n}}(k)=x$. For each $\xi, \eta \in X^{\otimes n}$, we have

$$
\pi\left(\langle\xi, k \eta\rangle_{X}\right)=t^{n}(\xi)^{*} \psi_{t^{n}}(k) t^{n}(\eta)=t^{n}(\xi)^{*} x t^{n}(\eta) \in B_{1}
$$

because $x \in B_{n+1}$. This implies that $\langle\xi, k \eta\rangle_{X} \in I_{(\pi, t)}{ }^{\prime}$ for all $\xi, \eta \in X^{\otimes n}$. Hence we have $k \in \mathscr{K}\left(X^{\otimes n} I_{(\pi, t)}{ }^{\prime}\right)$. Thus we get $x=\psi_{t^{n}}(k) \in B_{n}{ }^{\prime}$. We have shown $B_{n} \cap B_{n+1}=B_{n}{ }^{\prime}$ for all $n \in \mathbb{N}$.

Lemma 5.10. Let $n$ be a positive integer. For an approximate unit $\left\{u_{\lambda}\right\}$ of $\mathscr{K}\left(X^{\otimes n}\right)$ and $k \in \mathscr{K}\left(X^{\otimes(n+1)}\right)$, we have $k=\lim _{\lambda}\left(u_{\lambda} \otimes \mathrm{id}_{1}\right) k$.

Proof. Clearly the equality holds for $k=\left(k^{\prime} \otimes \mathrm{id}_{1}\right) k^{\prime \prime} \in \mathscr{K}\left(X^{\otimes(n+1)}\right)$ where $k^{\prime} \in \mathscr{K}\left(X^{\otimes n}\right)$ and $k^{\prime \prime} \in \mathscr{K}\left(X^{\otimes(n+1)}\right)$. We will show that the linear span of such elements is dense in $\mathscr{K}\left(X^{\otimes(n+1)}\right)$. To do so, it suffices to show that the linear span of elements in the form $\left(k^{\prime} \otimes \mathrm{id}_{1}\right) \zeta$ with $k^{\prime} \in \mathscr{K}\left(X^{\otimes n}\right)$ and $\zeta \in X^{\otimes(n+1)}$ is dense in
$X^{\otimes(n+1)}$ because we have $\left(k^{\prime} \otimes \operatorname{id}_{1}\right) \theta_{\zeta, \zeta^{\prime}}=\theta_{\left(k^{\prime} \otimes \mathrm{id}_{1}\right) \zeta, \zeta^{\prime}}$. For $k^{\prime}=\theta_{\xi, \xi^{\prime}}$ and $\zeta=\eta \otimes \eta^{\prime}$ with $\xi, \xi^{\prime}, \eta \in X^{\otimes n}$ and $\eta^{\prime} \in X$, we have

$$
\begin{aligned}
\left(k^{\prime} \otimes \mathrm{id}_{1}\right) \zeta & =\tau_{1}^{n}(\xi) \tau_{1}^{n}\left(\xi^{\prime}\right)^{*}\left(\eta \otimes \eta^{\prime}\right) \\
& =\tau_{1}^{n}(\xi)\left(\varphi_{1}\left(\left\langle\xi^{\prime}, \eta\right\rangle_{X^{\otimes n}}\right) \eta^{\prime}\right) \\
& =\xi \otimes\left(\varphi_{X}\left(\left\langle\xi^{\prime}, \eta\right\rangle_{X^{\otimes n}}\right) \eta^{\prime}\right) \\
& =\xi\left\langle\xi^{\prime}, \eta\right\rangle_{X^{\otimes n}} \otimes \eta^{\prime} .
\end{aligned}
$$

Since the linear span of elements in the form $\xi\left\langle\xi^{\prime}, \eta\right\rangle_{X^{\otimes n}}$ with $\xi, \xi^{\prime}, \eta \in X^{\otimes n}$ is dense in $X^{\otimes n}$ and the linear span of elements in the form $\xi \otimes \eta^{\prime}$ with $\xi \in X^{\otimes n}$ and $\eta^{\prime} \in X$ is dense in $X^{\otimes(n+1)}$, we see that the linear span of elements in the form $\left(k^{\prime} \otimes \operatorname{id}_{1}\right) \zeta$ with $k^{\prime} \in \mathscr{K}\left(X^{\otimes n}\right)$ and $\zeta \in X^{\otimes(n+1)}$ is dense in $X^{\otimes(n+1)}$. We are done.

Proposition 5.11. For each $n \in \mathbb{N}$, we have $B_{[0, n]} \cap B_{n+1} \subset B_{n}$.
Proof. The assertion is obvious for $n=0$. We assume $n \geqslant 1$. Take $x \in B_{[0, n]} \cap B_{n+1}$. Choose $k \in \mathscr{K}\left(X^{\otimes(n+1)}\right)$ such that $x=\psi_{t^{n+1}}(k)$. For an approximate unit $\left\{u_{\lambda}\right\}$ of $\mathscr{K}\left(X^{\otimes n}\right)$, we have $k=\lim _{\lambda}\left(u_{\lambda} \otimes \operatorname{id}_{1}\right) k$ by Lemma 5.10. Since $\psi_{t^{n}}\left(u_{\lambda}\right) \psi_{t^{n+1}}(k)=$ $\psi_{t^{n+1}}\left(\left(u_{\lambda} \otimes \mathrm{id}_{1}\right) k\right)$ by Lemma 5.4, we see

$$
x=\psi_{t^{n+1}}(k)=\lim _{\lambda} \psi_{t^{n}}\left(u_{\lambda}\right) \psi_{t^{n+1}}(k)=\lim _{\lambda} \psi_{t^{n}}\left(u_{\lambda}\right) x .
$$

Since $B_{n}$ is an ideal of $B_{[0, n]}$, we have $x \in B_{n}$. Thus we obtain $B_{[0, n]} \cap B_{n+1} \subset B_{n}$.
Proposition 5.12. For each $n \in \mathbb{N}$, we have $B_{[0, n]} \cap B_{n+1}=B_{n}{ }^{\prime}$, and we get the following commutative diagram with exact rows:


Proof. The former part follows from Propositions 5.9 and 5.11. The latter part follows from the former and the fact $B_{[0, n+1]}=B_{[0, n]}+B_{n+1}$.

Proposition 5.13. For $n=1,2, \ldots, \infty$, we have the following short exact sequences:

$$
0 \longrightarrow B_{[1, n]} \longrightarrow B_{[0, n]} \longrightarrow B_{0} / B_{0}^{\prime} \longrightarrow 0
$$

Proof. We will first prove $B_{0} \cap B_{[1, n]}=B_{0}{ }^{\prime}$ by the induction with respect to $n$. The case that $n=1$ follows from Proposition 5.9. Suppose that we have proved
$B_{0} \cap B_{[1, n]}=B_{0}{ }^{\prime}$. Take $x \in B_{0} \cap B_{[1, n+1]}$. Choose $y \in B_{[1, n]}$ and $z \in B_{n+1}$ with $x=y+z$. We have $z=x-y \in B_{[0, n]} \cap B_{n+1}$. By Proposition 5.11, we have $z \in B_{n}$. Thus $x=$ $y+z \in B_{[1, n]}$. Hence we have shown $B_{0} \cap B_{[1, n+1]} \subset B_{0} \cap B_{[1, n]}$. Since the converse inclusion is obvious, we get $B_{0} \cap B_{[1, n+1]}=B_{0} \cap B_{[1, n]}=B_{0}{ }^{\prime}$. Thus we obtain $B_{0} \cap B_{[1, n]}=B_{0}{ }^{\prime}$ for all positive integer $n$. This implies the existence of the desired short exact sequences for $n=1,2, \ldots$, because $B_{[0, n]}=B_{[1, n]}+B_{0}$. By taking inductive limits, we obtain the short exact sequences for $n=\infty$.

The $C^{*}$-subalgebras of $\mathscr{T}_{X}$ and $\mathcal{O}_{X}$ corresponding to $B_{n}, B_{[m, n]}$ are denoted by $\overline{\mathscr{B}}_{n}, \overline{\mathscr{B}}_{[m, n]} \subset \mathscr{T}_{X}$ and $\mathscr{B}_{n}, \mathscr{B}_{[m, n]} \subset \mathcal{O}_{X}$. By Proposition 5.7 we have $\mathscr{T}_{X}^{\bar{\gamma}}=\overline{\mathscr{B}}_{[0, \infty]}$ and $\mathcal{O}_{X}^{\gamma}=\mathscr{B}_{[0, \infty]}$.

Proposition 5.14. There exists a short exact sequence

$$
0 \longrightarrow \overline{\mathscr{B}}_{n+1} \longrightarrow \overline{\mathscr{B}}_{[0, n+1]} \longrightarrow \overline{\mathscr{B}}_{[0, n]} \longrightarrow 0
$$

which splits by the natural inclusion $\overline{\mathscr{B}}_{[0, n]} \hookrightarrow \overline{\mathscr{B}}_{[0, n+1]}$.
Proof. This follows from Proposition 5.12 because Proposition 4.10 implies $I_{\left(\bar{\pi}_{X}, \bar{t}_{X}\right)}{ }^{\prime}=0$.

Proposition 5.15. There exists a surjection from $\mathscr{T}_{X}^{\bar{\gamma}}$ to $A$.
Proof. This follows from Proposition 5.13.
Proposition 5.16. We get the following commutative diagram with exact rows:


Proof. By noting that $\mathscr{B}_{0} \cong A$ and $\mathscr{B}_{0}{ }^{\prime} \cong J_{X}$, this follows from Proposition 5.13.
Proposition 5.17. We get the following commutative diagram with exact rows:


Proof. This follows from Proposition 5.13.

Proposition 5.18. For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, the following conditions are equivalent:
(i) the injection $\pi_{X}: A \longrightarrow \mathcal{O}_{X}^{\beta}$ is an isomorphism,
(ii) we have $\mathscr{B}_{0} \supset \mathscr{B}_{1}$,
(iii) the injection $\varphi_{X}: J_{X} \longrightarrow \mathscr{K}(X)$ is an isomorphism,
(iv) the $C^{*}$-correspondence $X$ comes from a Hilbert $A$-bimodule.

Proof. It is clear that (i) implies (ii). From the condition (ii), we obtain $\mathscr{B}_{n} \supset \mathscr{B}_{n+1}$ for all $n \in \mathbb{N}$ by Lemma 5.2. Hence (ii) implies $\mathcal{O}_{X}^{\beta}=\mathscr{B}_{0}=\pi_{X}(A)$. This shows the implication (ii) $\Rightarrow$ (i). By setting $n=0$ in Proposition 5.12, we have the following commutative diagram with exact rows:


From this diagram, we have the equivalence (ii) $\Leftrightarrow$ (iii). Finally, the equivalence (iii) $\Leftrightarrow$ (iv) was shown in [Ka2].

## 6. The gauge-invariant uniqueness theorems

In this section, we will give conditions for representations or covariant representations to be universal. The idea of the proof can be seen in [Ka1, Section 4] (and also in [P, Section 3; FMR, Section 4]). Let us take a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$.

Proposition 6.1. For a representation $(\pi, t)$ of $X$ satisfying $I_{(\pi, t)}{ }^{\prime}=0$, the restriction of $\rho: \mathscr{T}_{X} \longrightarrow C^{*}(\pi, t)$ to the fixed point algebra $\mathscr{T}_{X}^{\bar{\gamma}}$ is injective.

Proof. For $n \in \mathbb{N}$ let $B_{n}$ and $B_{[0, n]}$ be $C^{*}$-subalgebras of $C^{*}(\pi, t)$ defined in Definitions 5.1 and 5.3. From the condition $I_{(\pi, t)}{ }^{\prime}=0$, we get the following commutative diagram with exact rows:

by the same argument as in Proposition 5.14. Since the condition $I_{(\pi, t)}{ }^{\prime}=0$ implies that the representation $(\pi, t)$ is injective, we see that the restriction of $\rho$ to $\overline{\mathscr{B}}_{n}$ is injective for all $n \in \mathbb{N}$. By using this fact and the commutative diagram above, we can inductively show that the restriction of $\rho$ to $\mathscr{B}_{[0, n]}$ is injective. Hence the restriction of $\rho$ to $\mathscr{T}_{X}^{\bar{\gamma}}=\overline{\mathscr{B}}_{[0, \infty]}$ is injective.

The following is the gauge-invariant uniqueness theorem for the $C^{*}$-algebra $\mathscr{T}_{X}$.
Theorem 6.2. Let $X$ be a $C^{*}$-correspondence over a $C^{*}$-algebra $A$. For a representation $(\pi, t)$ of $X$, the surjection $\rho: \mathscr{T}_{X} \longrightarrow C^{*}(\pi, t)$ is an isomorphism if and only if $(\pi, t)$ satisfies $I_{(\pi, t)}{ }^{\prime}=0$ and admits a gauge action.

Proof. We had already seen that the two conditions are necessary. Now suppose that a representation $(\pi, t)$ admits a gauge action $\beta$, and satisfies $I_{(\pi, t)}{ }^{\prime}=0$. Take $x \in \mathscr{T}_{X}$ with $\rho(x)=0$. Then we have

$$
\rho\left(\int_{\mathbb{T}} \bar{\gamma}_{z}\left(x^{*} x\right) d z\right)=\int_{\mathbb{T}} \rho\left(\bar{\gamma}_{z}\left(x^{*} x\right)\right) d z=\int_{\mathbb{T}} \beta_{z}\left(\rho\left(x^{*} x\right)\right) d z=0,
$$

where $d z$ is the normalized Haar measure on $\mathbb{T}$. Since $\int_{\mathbb{T}} \bar{\gamma}_{z}\left(x^{*} x\right) d z \in \mathscr{T}_{X}^{\gamma}$, we have $\int_{\mathbb{T}} \bar{\gamma}_{z}\left(x^{*} x\right) d z=0$ by Proposition 6.1. This implies $x^{*} x=0$. Hence $\rho$ is injective.

Proposition 6.3. For an injective covariant representation $(\pi, t)$ of $X$, the restriction of the surjection $\rho: \mathcal{O}_{X} \longrightarrow C^{*}(\pi, t)$ to the fixed point algebra $\mathcal{O}_{X}^{\gamma}$ is injective.

Proof. For $n \in \mathbb{N}$ let $B_{n}$ and $B_{[0, n]}$ be $C^{*}$-subalgebras of $C^{*}(\pi, t)$ defined in Definitions 5.1 and 5.3. Since $\psi_{t^{n}}$ is injective, the restriction of $\rho$ to $\mathscr{B}_{n}$ is an isomorphism onto $B_{n}$. It is easy to see that the restriction of $\rho$ to $\mathscr{B}_{[0, n]}$ is a surjection onto $B_{[0, n]}$ for each $n \in \mathbb{N}$. We will show that these are injective by the induction with respect to $n$. The case that $n=0$ follows from the fact that $\pi$ is injective. Suppose that we had shown that the restriction of $\rho$ to $\mathscr{B}_{[0, n]}$ is an isomorphism onto $B_{[0, n]}$. By Proposition 3.3, we have $I_{\left(\pi_{X}, t_{X}\right)}{ }^{\prime}=I_{(\pi, t)}{ }^{\prime}=J_{X}$. Hence the restriction of $\rho$ to $\mathscr{B}_{n}{ }^{\prime}$ is an isomorphism onto $B_{n}{ }^{\prime}$. Thus we get an isomorphism $\mathscr{B}_{[0, n]} / \mathscr{B}_{n}{ }^{\prime} \longrightarrow B_{[0, n]} / B_{n}{ }^{\prime}$. By Proposition 5.12 we get the following commutative diagram with exact rows:


By the 5-lemma, we see that the surjection $\mathscr{B}_{[0, n+1]} \longrightarrow B_{[0, n+1]}$ is an isomorphism. Thus we have shown that the restriction of $\rho$ to $\mathscr{B}_{[0, n]}$ is injective for all $n \in \mathbb{N}$. Hence the restriction of $\rho$ to $\mathcal{O}_{X}^{\gamma}=\mathscr{B}_{[0, \infty]}$ is injective.

The following is the gauge-invariant uniqueness theorem for the $C^{*}$-algebra $\mathcal{O}_{X}$.
Theorem 6.4. For a covariant representation $(\pi, t)$ of a $C^{*}$-correspondence $X$, the *homomorphism $\rho: \mathcal{O}_{X} \longrightarrow C^{*}(\pi, t)$ is an isomorphism if and only if $(\pi, t)$ is injective and admits a gauge action.

Proof. The proof goes similarly as in Theorem 6.2 with the help of Proposition 6.3.

When the left actions of $C^{*}$-correspondences are injective, Theorem 6.4 is the gauge-invariant uniqueness theorem for Cuntz-Pimsner algebras which was proved in [FMR, Theorem 4.1]. In the case that $C^{*}$-correspondences are defined from graphs with or without sinks, this was already proved in [BHRS, Theorem 2.1]. For $C^{*}$-algebras arising from topological graphs, this was proved in $[\mathrm{Ka} 1$, Theorem 4.5].

We can apply the two gauge-invariant uniqueness theorems to the representations $\left(\varphi_{\infty}, \tau_{\infty}\right)$ and $(\varphi, \tau)$ in Section 4.

Proposition 6.5. Both the representation $\left(\varphi_{\infty}, \tau_{\infty}\right)$ and the covariant representation $(\varphi, \tau)$ are universal, that is, we have natural isomorphisms $C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right) \cong \mathscr{T}_{X}$ and $C^{*}(\varphi, \tau) \cong \mathcal{O}_{X}$.

Proof. To apply Theorems 6.2 and 6.4 , it suffices to see that both of the representations $\left(\varphi_{\infty}, \tau_{\infty}\right)$ and $(\varphi, \tau)$ admit gauge actions because the other conditions had already been checked in Section 4.

For each $z \in \mathbb{T}$, define a unitary $u_{z} \in \mathscr{L}(\mathscr{F}(X))$ by $u_{z}(\xi)=z^{n} \xi$ for $\xi \in X^{\otimes n} \subset \mathscr{F}(X)$ and $n \in \mathbb{N}$. It is routine to see that the automorphisms $\operatorname{Ad} u_{z}$ of $\mathscr{L}(\mathscr{F}(X))$, defined by $\operatorname{Ad} u_{z}(x)=u_{z} x u_{z}^{*}$ for $x \in \mathscr{L}(\mathscr{F}(X))$, give a gauge action for the representation $\left(\varphi_{\infty}, \tau_{\infty}\right)$. The ideal $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ of $\mathscr{L}(\mathscr{F}(X))$ is closed under the automorphisms $\operatorname{Ad} u_{z}$ for each $z \in \mathbb{T}$. Hence we can define an automorphism $\beta_{z}$ of $\mathscr{L}(\mathscr{F}(X)) / \mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ by $\beta_{z}(\sigma(x))=\sigma\left(u_{z} x u_{z}^{*}\right)$ for $x \in \mathscr{L}(\mathscr{F}(X))$ and $z \in \mathbb{T}$. It is clear that $\beta$ is a gauge action for the representation $(\varphi, \tau)$. We are done.

By Proposition 6.5, the $C^{*}$-algebra $\mathcal{O}_{X}$ is isomorphic to the relative Cuntz-Pimsner algebras $C^{*}(\varphi, \tau)=\mathcal{O}\left(J_{X}, X\right)$ introduced in [MS] (cf. [MS, Theorem 2.19]). The isomorphism $C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right) \cong \mathscr{T}_{X}$ was already proved in [P, Theorem 3.4] under small assumption on $C^{*}$-correspondences.

The $C^{*}$-algebra $\mathcal{O}_{X}$ was defined as the largest $C^{*}$-algebra among $C^{*}$-algebras $C^{*}(\pi, t)$ generated by covariant representations $(\pi, t)$ of $X$. Theorem 6.4 tells us that we have $C^{*}(\pi, t) \cong \mathcal{O}_{X}$ when a covariant representation $(\pi, t)$ satisfies two conditions; being injective and admitting a gauge action. In the next paper $[\mathrm{Ka3}]$, we will see that the $C^{*}$-algebra $\mathcal{O}_{X}$ can be defined as the smallest $C^{*}$-algebra among $C^{*}$-algebras $C^{*}(\pi, t)$ generated by representations $(\pi, t)$ of $X$ which satisfy the two conditions above; being
injective and admitting gauge actions. Thus we can define $\mathcal{O}_{X}$ without using the ideal $J_{X}$.

## 7. Nuclearity and exactness

In this section, we study when the $C^{*}$-algebras $\mathscr{T}_{X}$ and $\mathcal{O}_{X}$ become nuclear or exact. We use the facts on nuclearity and exactness appeared in Appendices A and B as well as in [W].
On the exactness of $\mathscr{T}_{X}$ and $\mathcal{O}_{X}$, we have the following which generalizes [DS, Theorem 3.1] slightly.

Theorem 7.1 (cf. [DS, Theorem 3.1]). For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, the following conditions are equivalent:
(i) $A$ is exact,
(ii) $\mathscr{T}_{X}^{\bar{\gamma}}$ is exact,
(iii) $\mathscr{T}_{X}$ is exact,
(iv) $\mathcal{O}_{X}^{\gamma}$ is exact,
(v) $\mathcal{O}_{X}$ is exact.

Proof. Suppose that $A$ is exact. By Proposition B.7, $\mathscr{K}\left(X^{\otimes n}\right)$ is exact for all $n \in \mathbb{N}$. By Proposition 5.14, we can prove inductively that $\mathscr{B}_{[0, n]} \subset \mathscr{T}_{X}^{\bar{\gamma}}$ is exact for all $n \in \mathbb{N}$ because exactness is closed under taking splitting extensions. Thus $\mathscr{T}_{X}^{\bar{\gamma}}$ is exact because it is an inductive limit of exact $C^{*}$-algebras. This proves (i) $\Rightarrow$ (ii). The equivalences (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v) follow from Proposition A.13. Since there exists a surjection $\mathscr{T}_{X} \longrightarrow \mathcal{O}_{X}$, (iii) implies (v). Finally, (v) implies (i) because $\pi_{X}(A) \subset \mathcal{O}_{X}$ is isomorphic to $A$.

On the nuclearity of $\mathscr{T}_{X}$, we have the following.
Theorem 7.2. For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, the following conditions are equivalent:
(i) $A$ is nuclear,
(ii) $\mathscr{T}_{X}^{\bar{\gamma}}$ is nuclear,
(iii) $\mathscr{T}_{X}$ is nuclear.

Proof. In a similar way to the proof of (i) $\Rightarrow$ (ii) in Theorem 7.1, we can show that (i) implies (ii). The implication (ii) $\Rightarrow$ (i) follows from Proposition 5.15. Finally, Proposition A. 13 gives the equivalence (ii) $\Leftrightarrow$ (iii).

On the nuclearity of $\mathcal{O}_{X}$, we have the following.

Theorem 7.3. For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, the following conditions are equivalent:
(i) $A / J_{X}$ is a nuclear $C^{*}$-algebra, and $\pi_{X}: J_{X} \longrightarrow \mathscr{B}_{[1, \infty]}$ is a nuclear map,
(ii) $\pi_{X}: A \longrightarrow \mathcal{O}_{X}^{\gamma}$ is a nuclear map,
(iii) $\pi_{X}: A \longrightarrow \mathcal{O}_{X}$ is a nuclear map,
(iv) $\mathcal{O}_{X}^{\gamma}$ is nuclear,
(v) $\mathcal{O}_{X}$ is nuclear.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is shown by applying Proposition A. 6 to the diagram in Proposition 5.17. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition A.12. Obviously (iv) implies (ii). Assume (ii). We see that $A / J_{X}$ is nuclear from the equivalence (i) $\Leftrightarrow$ (ii). We will prove that the embedding $\mathscr{B}_{[0, n]} \hookrightarrow \mathscr{B}_{[0, \infty]}$ is nuclear for all $n \in \mathbb{N}$ by the induction on $n$. The case $n=0$ follows from condition (ii). Suppose we have shown that $\mathscr{B}_{[0, n]} \hookrightarrow \mathscr{B}_{[0, \infty]}$ is nuclear. Let us set $Y_{n}=$ $\overline{\operatorname{span}}\left(t_{X}(X) \mathscr{B}_{[0, n]}\right)$ and $Y_{\infty}=\overline{\operatorname{span}}\left(t_{X}(X) \mathscr{B}_{[0, \infty]}\right)$. Then by Lemma 5.2, $Y_{n}$ is a Hilbert $\mathscr{B}_{[0, n]}$-module with $\mathscr{K}\left(Y_{n}\right) \cong \mathscr{B}_{[1, n+1]}$, and $Y_{\infty}$ is a Hilbert $\mathscr{B}_{[0, \infty]}$-module with $\mathscr{K}\left(Y_{\infty}\right) \cong \mathscr{B}_{[1, \infty]}$. By applying Proposition B. 8 to the inclusions $\mathscr{B}_{[0, n]} \hookrightarrow \mathscr{B}_{[0, \infty]}$ and $Y_{n} \hookrightarrow Y_{\infty}$, we see that the inclusion $\mathscr{B}_{[1, n+1]} \hookrightarrow \mathscr{B}_{[1, \infty]}$ is nuclear. Now by applying Proposition A. 6 to the diagram in Proposition 5.16, we see that $\mathscr{B}_{[0, n+1]} \hookrightarrow \mathscr{B}_{[0, \infty]}$ is nuclear. Hence we have shown that $\mathscr{B}_{[0, n]} \hookrightarrow \mathscr{B}_{[0, \infty]}$ is nuclear for all $n \in \mathbb{N}$. Since $\bigcup_{n \in \mathbb{N}} \mathscr{B}_{[0, n]}$ is dense in $\mathscr{B}_{[0, \infty]}$, we see that the identity map $\mathscr{B}_{[0, \infty]} \longrightarrow \mathscr{B}_{[0, \infty]}$ is nuclear. Thus $\mathscr{B}_{[0, \infty]}$ is a nuclear $C^{*}$-algebra. This shows that (ii) implies (iv). Finally, the equivalence (iv) $\Leftrightarrow$ (v) follows from Proposition A.13.

We give two sufficient conditions on $C^{*}$-correspondences $X$ for $\mathcal{O}_{X}$ to be nuclear, which may be useful. Both of them easily follows from Theorem 7.3.

Corollary 7.4. If $A$ is nuclear then $\mathcal{O}_{X}$ is nuclear.
Corollary 7.5. If both the $C^{*}$-algebra $A / J_{X}$ and the *-homomorphism $\varphi_{X}: J_{X} \longrightarrow \mathscr{K}(X)$ are nuclear, then $\mathcal{O}_{X}$ is nuclear.

Remark 7.6. We can prove Corollary 7.4 directly by showing that $\mathcal{O}_{X}^{\gamma}$ is nuclear when $A$ is nuclear in a similar way to the proof of (i) $\Rightarrow$ (ii) in Theorem 7.1.

The converses of Corollaries 7.4 and 7.5 are not true as the following example shows. We would like to thank Narutaka Ozawa who gave us this example.

Example 7.7. Let $B$ be a nuclear $C^{*}$-algebra, and $D$ be a non-nuclear $C^{*}$-subalgebra of $B$. For an integer $n$, we define $A_{n}$ by $A_{n}=B$ for $n>0$ and $A_{n}=D$ for $n \leqslant 0$. We set $A=\oplus_{n=-\infty}^{\infty} A_{n}$. We define an injective endomorphism $\varphi: A \longrightarrow A$ so that $\left.\varphi\right|_{A_{0}}: A_{0} \longrightarrow A_{1}$ is a natural embedding and $\left.\varphi\right|_{A_{n}}: A_{n} \longrightarrow A_{n+1}$ is an isomorphism for a non-zero integer $n$. Since $D$ is not nuclear, the injective endomorphism $\varphi$ is not
nuclear. Let $X$ be the $C^{*}$-correspondence over $A$ which is isomorphic to $A$ as Hilbert $A$-modules, and whose left action $\varphi_{X}: A \longrightarrow \mathscr{L}(X)$ is defined as the composition of $\varphi: A \longrightarrow A$ and the isomorphism $A \cong \mathscr{K}(X) \subset \mathscr{L}(X)$. Then we have $J_{X}=A$ and the $\operatorname{map} \varphi_{X}: J_{X} \longrightarrow \mathscr{K}(X)$ is not nuclear as $\varphi$ is not. Thus the $C^{*}$-correspondence $X$ does not satisfy the assumption of Corollary 7.4 nor Corollary 7.5. However, the $C^{*}$ algebra $\mathcal{O}_{X}$ is nuclear because the fixed point algebra $\mathcal{O}_{X}^{\beta}$ is isomorphic to the inductive limit $\lim (A, \varphi) \cong \oplus_{n=-\infty}^{\infty} B$, which is nuclear.

A Hilbert $A$-bimodule $X$ is naturally considered as a $C^{*}$-correspondence over $A$, and the $C^{*}$-algebra $\mathcal{O}_{X}$ is isomorphic to the crossed product $A \rtimes_{X} \mathbb{Z}$ of $A$ by $X$ defined in [AEE, Definition 2.4] (see [Ka2, Subsection 3.3]). We have a nice characterization of the nuclearity of such a $C^{*}$-algebra.

Proposition 7.8. When a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$ comes from a Hilbert A-bimodule, the $C^{*}$-algebra $\mathcal{O}_{X}$ is nuclear if and only if $A$ is nuclear.

Proof. By Proposition 5.18, we see that $\pi_{X}: A \longrightarrow \mathcal{O}_{X}^{\beta}$ is an isomorphism. Hence the conclusion follows from Theorem 7.3, or rather Proposition A.13.

## 8. $K$-groups

The purpose of this section is to obtain the 6 -term exact sequence of $K$-groups, which seems to be useful to compute the $K$-groups $K_{0}\left(\mathcal{O}_{X}\right)$ and $K_{1}\left(\mathcal{O}_{X}\right)$ of $\mathcal{O}_{X}$. Mainly we follow the arguments in [P, Section 4]. There, Pimsner used $K K$-theory to obtain his 6 -term exact sequence. For this reason, he assumed the separability of the $C^{*}$-algebras involved. Here, we work directly with $K$-theory instead of using $K K$ theory, and obtain the 6 -term exact sequence without the assumption of separability.

For a $C^{*}$-algebra $A$, we denote by $K_{*}(A)$ the $K$-group $K_{0}(A) \oplus K_{1}(A)$ of $A$ which has a $\mathbb{Z} / 2 \mathbb{Z}$-grading. By maps between $K$-groups, we mean group homomorphisms which preserve the grading. Thus for $C^{*}$-algebras $A$ and $B$, considering maps between $K$-groups $K_{*}(A) \longrightarrow K_{*}(B)$ is same as considering two homomorphisms $K_{0}(A) \longrightarrow K_{0}(B)$ and $K_{1}(A) \longrightarrow K_{1}(B)$. For a $*$-homomorphism $\rho: A \longrightarrow B$, we denote by $\rho_{*}$ the $\operatorname{map} K_{*}(A) \longrightarrow K_{*}(B)$ induced by $\rho$.

Fix a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$. Since we have $\mathscr{T}_{X} \cong C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right)$ by Proposition 6.5, there exists an embedding $j: \mathscr{K}\left(\mathscr{F}(X) J_{X}\right) \longrightarrow \mathscr{T}_{X}$ by Proposition 4.6. Since $C^{*}(\varphi, \tau) \cong \mathcal{O}_{X}$ by Proposition 6.5 , we have the following short exact sequence:

$$
0 \quad \longrightarrow \mathscr{K}\left(\mathscr{F}(X) J_{X}\right) \quad \stackrel{j}{\rightarrow} \mathscr{T}_{X} \quad \longrightarrow \mathcal{O}_{X} \quad \longrightarrow 0 .
$$

The following two propositions enable us to compute the $K$-groups of $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ and $\mathscr{T}_{X}$.

Proposition 8.1. The $*$-homomorphism $\varphi_{0}: J_{X} \longrightarrow \mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ induces an isomorph$\operatorname{ism}\left(\varphi_{0}\right)_{*}: K_{*}\left(J_{X}\right) \longrightarrow K_{*}\left(\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)\right)$.

Proof. The $*$-homomorphism $\varphi_{0}: J_{X} \longrightarrow \mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ is an isomorphism onto the $C^{*}$-subalgebra $\mathscr{K}\left(X^{\otimes 0} J_{X}\right)$ of $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$. Since $X^{\otimes 0} J_{X}$ is a full Hilbert $J_{X^{-}}$ submodule of $\mathscr{F}(X) J_{X}, \mathscr{K}\left(X^{\otimes 0} J_{X}\right)$ is a hereditary and full $C^{*}$-subalgebra of $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$. Hence $\left(\varphi_{0}\right)_{*}$ is an isomorphism by Proposition B.5.

Proposition 8.2. The $*$-homomorphism $\bar{\pi}_{X}: A \longrightarrow \mathscr{T}_{X}$ induces an isomorphism $\left(\bar{\pi}_{X}\right)_{*}: K_{*}(A) \longrightarrow K_{*}\left(\mathscr{T}_{X}\right)$.

Proof. See Appendix C.
Next, we will compute $j_{*}: K_{*}\left(\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)\right) \longrightarrow K_{*}\left(\mathscr{T}_{X}\right)$.
Definition 8.3. We denote by $1: J_{X} \hookrightarrow A$ the natural embedding. We define a map $[X]: K_{*}\left(J_{X}\right) \longrightarrow K_{*}(A)$ by the composition of the map $\left(\varphi_{X}\right)_{*}: K_{*}\left(J_{X}\right) \longrightarrow K_{*}(\mathscr{K}(X))$ induced by the restriction of $\varphi_{X}$ to $J_{X}$ and the map $X_{*}: K_{*}(\mathscr{K}(X)) \longrightarrow K_{*}(A)$ induced by the Hilbert $A$-module $X$ as in Remark B.4.

The map $[X]: K_{*}\left(J_{X}\right) \longrightarrow K_{*}(A)$ is same as the map induced by the element $\left(X, \varphi_{X}, 0\right)$ of $K K\left(J_{X}, A\right)$. When a $C^{*}$-correspondence $X$ is defined from an injective *-homomorphism $\varphi: A \longrightarrow A$, we have $J_{X}=A$ and $[X]=\varphi_{*}$. For the notation in the proof of the next lemma, consult Appendix B.

Lemma 8.4. The composition of the two maps $[X]: K_{*}\left(J_{X}\right) \longrightarrow K_{*}(A)$ and $\left(\bar{\pi}_{X}\right)_{*}: K_{*}(A) \longrightarrow K_{*}\left(\mathscr{T}_{X}\right)$ coincides with $\left(\psi_{\bar{t}_{X}} \circ \varphi_{X}\right)_{*}$.

Proof. Let $M_{2}\left(\mathscr{T}_{X}\right)$ be the $C^{*}$-algebra of two-by-two matrices with entries in $\mathscr{T}_{X}$. For $i, j \in\{0,1\}$, we denote by $l_{i j}$ the natural embedding $\mathscr{T}_{X} \longrightarrow M_{2}\left(\mathscr{T}_{X}\right)$ onto the $i, j$-component. By the definition of $K$-groups, $\left(l_{00}\right)_{*}=\left(l_{11}\right)_{*}$ is an isomorphism.

From the maps $\bar{\pi}_{X}: A \longrightarrow \mathscr{T}_{X}$ and $\bar{t}_{X}: X \longrightarrow \mathscr{T}_{X}$, we get a $*$-homomorphism $\rho: D_{X} \longrightarrow M_{2}\left(\mathscr{T}_{X}\right)$ such that $\rho \circ \iota_{A}=l_{11} \circ \bar{\pi}_{X}$ and $\rho \circ l_{X}=l_{01} \circ \bar{t}_{X}$. We have $\rho \circ \iota_{\mathscr{K}(X)}=$ $l_{00} \circ \psi_{\bar{t}_{X}}$. Since $X_{*}$ is defined as $\left(l_{A}\right)_{*}^{-1} \circ\left(l_{\mathscr{K}(X)}\right)_{*}$, we have

$$
\begin{aligned}
\left(\bar{\pi}_{X}\right)_{*} \circ X_{*} & =\left(\bar{\pi}_{X}\right)_{*} \circ\left(l_{A}\right)_{*}^{-1} \circ\left(l_{\mathscr{K}(X)}\right)_{*} \\
& =\left(l_{11}\right)_{*}^{-1} \circ \rho_{*} \circ\left(l_{\mathscr{K}(X)}\right)_{*} \\
& =\left(l_{11}\right)_{*}^{-1} \circ\left(l_{00}\right)_{*} \circ\left(\psi_{\bar{t}_{X}}\right)_{*} \\
& =\left(\psi_{\bar{t}_{X}}\right)_{*} .
\end{aligned}
$$

Hence we get

$$
\left(\bar{\pi}_{X}\right)_{*} \circ[X]=\left(\bar{\pi}_{X}\right)_{*} \circ X_{*} \circ\left(\varphi_{X}\right)_{*}=\left(\psi_{\bar{t}_{X}}\right)_{*} \circ\left(\varphi_{X}\right)_{*}=\left(\psi_{\bar{t}_{X}} \circ \varphi_{X}\right)_{*}
$$

We are done.
Lemma 8.5. The ${ }^{\text {-homomorphism }} \bar{\pi}_{X} \circ \mathfrak{l}: J_{X} \longrightarrow \mathscr{T}_{X}$ is the sum of the two ${ }^{*}$ homomorphisms $\psi_{\bar{t}_{X}} \circ \varphi_{X}$ and $j \circ \varphi_{0}$.

Proof. If we identify $\mathscr{T}_{X}$ and $C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right)$, this follows from Proposition 4.4.
By the above two lemmas, the map $j_{*}: K_{*}\left(\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)\right) \longrightarrow K_{*}\left(\mathscr{T}_{X}\right)$ is same as the map $i_{*}-[X]: K_{*}\left(J_{X}\right) \longrightarrow K_{*}(A)$ modulo the isomorphisms $\left(\varphi_{0}\right)_{*}: K_{*}\left(J_{X}\right) \longrightarrow K_{*}\left(\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)\right)$ and $\left(\bar{\pi}_{X}\right)_{*}: K_{*}(A) \longrightarrow K_{*}\left(\mathscr{T}_{X}\right):$


Thus by rewriting the 6 -term exact sequence of $K$-groups obtained from the short exact sequence

$$
0 \quad \longrightarrow \mathscr{K}\left(\mathscr{F}(X) J_{X}\right) \quad \xrightarrow{j} \mathscr{T}_{X} \quad \longrightarrow \mathcal{O}_{X} \quad \longrightarrow 0,
$$

we get the following.
Theorem 8.6 (cf. $\left[\mathrm{P}\right.$, Theorem 4.9]). For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$, we have the following exact sequence:


For a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$ and an ideal $J$ of $A$ satisfying $\varphi_{X}(J) \subset \mathscr{K}(X)$, the relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ is defined as the quotient $C^{*}\left(\varphi_{\infty}, \tau_{\infty}\right) / \mathscr{K}(\mathscr{F}(X) J)$ [MS, Definition 2.18]. Thus we can prove the following statement in the same way as the proof of Theorem 8.6.

Proposition 8.7. Let $X$ be a $C^{*}$-correspondence over a $C^{*}$-algebra $A$, and $J$ be an ideal of $A$ with $\varphi_{X}(J) \subset \mathscr{K}(X)$. Then we have the following exact sequence:

where $\imath: J \hookrightarrow A$ is the embedding, $\pi: A \longrightarrow \mathcal{O}(J, X)$ is the natural $*$-homomorphism, and $[X, J]: K_{*}(J) \longrightarrow K_{*}(A)$ is defined by $[X, J]=X_{*} \circ\left(\left.\varphi_{X}\right|_{J}\right)_{*}$.

It is not difficult to see that the two $*$-homomorphisms in Propositions 8.1 and 8.2 induce $K K$-equivalences between $J_{X}$ and $\mathscr{K}\left(\mathscr{F}(X) J_{X}\right)$ and between $A$ and $\mathscr{T}_{X}$ when the involving $C^{*}$-algebras are separable. Hence by applying "two among three principle" to the short exact sequence

$$
0 \quad \longrightarrow \mathscr{K}\left(\mathscr{F}(X) J_{X}\right) \xrightarrow{\stackrel{j}{\rightarrow}} \mathscr{T}_{X} \quad \longrightarrow \mathcal{O}_{X} \quad \longrightarrow 0,
$$

we get the following.

Proposition 8.8. Let $X$ be a separable $C^{*}$-correspondence over a separable nuclear $C^{*}$-algebra $A$. If $A$ and $J_{X}$ satisfy the Universal Coefficient Theorem of $[\mathrm{RS}]$, then so does $\mathcal{O}_{X}$.

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## Appendix A. On nuclear maps

In Appendices A and B , we gather the results on nuclear maps and linking algebras. We use these results in Sections 7 and 8. Most of them should be known among the specialists. Some results in this appendix hold with less assumption.

Definition A.1. For $C^{*}$-algebras $A$ and $D$, we denote by $A \otimes_{\min } D$ (resp. $A \otimes_{\max } D$ ) the minimal (resp. maximal) tensor product of $A$ and $D$, and by $A \ominus D$ the kernel of the natural surjection $\pi_{A, D}: A \otimes_{\max } D \longrightarrow A \otimes_{\min } D$.

Definition A.2. For a $*$-homomorphism $\varphi: A \longrightarrow B$, we can define $*$-homomorphisms $\varphi \otimes_{\min } \mathrm{id}_{D}: A \otimes_{\min } D \longrightarrow B \otimes_{\min } D$ and $\varphi \otimes_{\max } \mathrm{id}_{D}: A \otimes_{\max } D \longrightarrow B \otimes_{\max } D$ such that $\varphi \otimes \min ^{\operatorname{id}} D_{D}(a \otimes d)=\varphi \otimes_{\max ^{2}} \operatorname{id}_{D}(a \otimes d)=\varphi(a) \otimes d$ for $a \in A$ and $d \in D$. Since we have the commutative diagram:

the restriction of $\varphi \otimes_{\max } \mathrm{id}_{D}$ to $A \ominus D \subset A \otimes_{\max } D$ induces a $*$-homomorphism $\varphi \ominus \operatorname{id}_{D}: A \ominus D \longrightarrow B \ominus D$.

Definition A.3. A $*$-homomorphism $\varphi: A \longrightarrow B$ is said to be nuclear if for all $C^{*}$ -
 the surjection $\pi_{A, D}: A \otimes_{\max } D \longrightarrow A \otimes_{\min } D$ :


A $C^{*}$-algebra $A$ is said to be nuclear if $\mathrm{id}_{A}: A \longrightarrow A$ is a nuclear map.
In other words, a $*$-homomorphism $\varphi: A \longrightarrow B$ is nuclear if and only if $\varphi \ominus \operatorname{id}_{D}=0$ for all $C^{*}$-algebra $D$, and a $C^{*}$-algebra $A$ is nuclear if and only if $A \ominus D=0$ for all $C^{*}$-algebra $D$.

Remark A.4. A *-homomorphism is nuclear if and only if it has the completely positive approximation property (see [W]).

Lemma A.5. Let

$$
0 \quad \longrightarrow \quad \xrightarrow{l} A \xrightarrow{\pi} B \longrightarrow 0
$$

be a short exact sequence of $C^{*}$-algebras, and $D$ be a $C^{*}$-algebra. Then the following sequence is exact:

$$
0 \quad I \ominus D \xrightarrow{\imath \ominus \mathrm{id}_{D}} A \ominus D \xrightarrow{\pi \ominus \mathrm{id}_{D}} B \ominus D .
$$

If there exists an injective nuclear $*$-homomorphism $A \longrightarrow A^{\prime}$ for some $C^{*}$-algebra $A^{\prime}$, then $\pi \ominus \mathrm{id}_{D}$ is surjective.

Proof. The former statement follows from the fact that maximal tensor products preserve short exact sequences. If there exists an injective nuclear *-homomorphism $A \longrightarrow A^{\prime}$ for some $C^{*}$-algebra $A^{\prime}$, then $A$ is exact by [W, Proposition 7.2]. Since exact $C^{*}$-algebras have Property $\mathrm{C}[\mathrm{Ki}]$, the sequence

$$
0 \longrightarrow I \otimes_{\min } D \xrightarrow{l \otimes_{\min \text { id } D}} A \otimes_{\min } D \xrightarrow{\pi \otimes_{\min } \mathrm{id}_{D}} \quad B \otimes_{\min } D \quad \longrightarrow \quad 0
$$

is exact (see Proposition 5.2 and Remark 9.5.2 in [W]). Hence the conclusion follows from $3 \times 3$-lemma.

Proposition A.6. Suppose that we have a following commutative diagram with exact rows:


Suppose also that $\varphi$ is injective. Then $\varphi$ is nuclear if and only if both $B$ and $\varphi_{0}$ are nuclear.

Proof. Take a $C^{*}$-algebra $D$. By Lemma A. 5 we have the following commutative diagram with exact rows:


Suppose that $\varphi$ is nuclear. By Lemma A.5, the $*$-homomorphism $\pi \ominus \mathrm{id}_{D}$ is surjective. Hence we have $B \ominus D=0$ for all $C^{*}$-algebra $D$. We also have $\varphi_{0} \ominus \operatorname{id}_{D}=0$ for all $C^{*}$-algebra $D$ by the diagram above. Thus both $B$ and $\varphi_{0}$ are nuclear. Conversely, assume that both $B$ and $\varphi_{0}$ are nuclear. Then we have $\varphi \ominus \mathrm{id}_{D}=0$ for all $C^{*}$-algebra $D$ by the diagram above. Therefore $\varphi$ is nuclear. We are done.

Proposition A.7. Let $A, B$ be $C^{*}$-algebras, and $A_{0}, B_{0}$ be $C^{*}$-subalgebras of $A$ and $B$, respectively. Let $\varphi: A \longrightarrow B$ be $a *$-homomorphism with $\varphi\left(A_{0}\right) \subset B_{0}$. Let $\varphi_{0}: A_{0} \longrightarrow B_{0}$ be the restriction of $\varphi$. When $B_{0}$ is a hereditary $C^{*}$-subalgebra of $B$, the nuclearity of $\varphi$ implies the nuclearity of $\varphi_{0}$.

Proof. When $\varphi$ is nuclear, its restriction $\varphi^{\prime}: A_{0} \longrightarrow B$ is also nuclear. Hence for any $C^{*}$-algebra $D$, the map $\varphi^{\prime} \ominus \mathrm{id}_{D}: A_{0} \ominus D \longrightarrow B \ominus D$ is 0 . Since $B_{0}$ is a hereditary
$C^{*}$-subalgebra of $B$, we see that the inclusion $l: B_{0} \hookrightarrow B$ induces an injective $*-$ homomorphism $l \otimes_{\max } \mathrm{id}_{D}: B_{0} \otimes_{\max } D \longrightarrow B \otimes_{\max } D$ by [L1, Theorem 3.3]. Hence the *-homomorphism $l \ominus \mathrm{id}_{D}: B_{0} \ominus D \longrightarrow B \ominus D$ is also injective. This shows that $\varphi_{0} \ominus \operatorname{id}_{D}: A_{0} \ominus D \longrightarrow B_{0} \ominus D$ is 0 for all $C^{*}$-algebra $D$. Thus $\varphi_{0}$ is injective.

The following complements the proposition above.
Proposition A.8. With the same notation in Proposition A.7, when $A_{0}$ is a hereditary and full $C^{*}$-subalgebra of $A$, the nuclearity of $\varphi_{0}$ implies the nuclearity of $\varphi$.

Proof. Take a $C^{*}$-algebra $D$. Since $A_{0}$ is a hereditary and full $C^{*}$-subalgebra of $A$, $A_{0} \otimes_{\max } D$ is a hereditary and full $C^{*}$-subalgebra of $A \otimes_{\max } D$. Hence $A_{0} \ominus D=$ $\left(A_{0} \otimes_{\max } D\right) \cap(A \ominus D)$ is also hereditary and full in $A \ominus D$. When $\varphi_{0}$ is nuclear, the $*-$ homomorphism $\varphi \otimes_{\max } \mathrm{id}_{D}: A \otimes_{\max } D \longrightarrow B \otimes_{\max } D$ vanishes on $A_{0} \ominus D$. Thus $\varphi \otimes \max \mathrm{id}_{D}$ vanishes on $A \ominus D$. This shows that $\varphi$ is nuclear.

The following is an immediate consequence of Propositions A. 7 and A.8.
Corollary A.9. A hereditary and full $C^{*}$-subalgebra $A_{0}$ of a $C^{*}$-algebra $A$ is nuclear if and only if $A$ is nuclear.

We also have the following.
Proposition A.10. A hereditary and full $C^{*}$-subalgebra $A_{0}$ of a $C^{*}$-algebra $A$ is exact if and only if $A$ is exact.

Proof. Since a $C^{*}$-subalgebra of an exact $C^{*}$-algebra is exact, $A_{0}$ is exact if $A$ is exact. Suppose that $A_{0}$ is exact. Take a short exact sequence of $C^{*}$-algebras:

$$
0 \quad \longrightarrow \quad I \quad \xrightarrow{l} B \quad \xrightarrow{\pi} \quad D \quad \longrightarrow \quad 0 .
$$

All we have to do is to prove $\operatorname{ker}\left(\pi \otimes_{\min } \mathrm{id}_{A}\right)=I \otimes_{\min } A$. Since $A_{0}$ is full and hereditary in $A, B \otimes_{\min } A_{0}$ is full and hereditary in $B \otimes_{\min } A$. Thus $\operatorname{ker}\left(\pi \otimes_{\min } \mathrm{id}_{A}\right)$ is generated by its intersection with $B \otimes_{\min } A_{0}$, which is $I \otimes_{\min } A_{0}$ by the exactness of $A_{0}$. Hence we get $\operatorname{ker}\left(\pi \otimes_{\min } \mathrm{id}_{A}\right)=I \otimes_{\min } A$. We are done.

Remark A.11. We can prove Proposition A. 10 by using Proposition A. 8 together with the deep fact that a $C^{*}$-algebra is exact if and only if its one (or all) faithful representation is nuclear due to Kirchberg [Ki]. We can also prove Proposition A. 10 in a similar way to the proof of Proposition B.3.

The above investigation of hereditary $C^{*}$-subalgebras can be extended to other classes of $C^{*}$-subalgebras. In Section 7, we just need the following two results.

Proposition A.12. Let $\alpha: G \curvearrowright A$ be an action of a compact group $G$ on $a$ $C^{*}$-algebra $A$. Let $\varphi: D \longrightarrow A$ be $a *$-homomorphism whose image is contained in the fixed point algebra $A^{\alpha}$ of $\alpha$. Then the restriction $\varphi_{0}: D \longrightarrow A^{\alpha}$ is nuclear if and only if $\varphi$ is nuclear.

Proof. Similar as the proof of Proposition A. 7

Proposition A.13. Let $\alpha: G \curvearrowright A$ be an action of a compact group $G$ on a $C^{*}$-algebra $A$. Then $A$ is nuclear or exact if and only if the fixed point algebra $A^{\alpha}$ is also.

Proof. For nuclearity, it was proved in [DLRZ, Proposition 2]. It was pointed out by Narutaka Ozawa that the technique in [DLRZ] works for exactness. We will sketch his argument.

When $A$ is exact, $A^{\alpha}$ is exact. Assume that $A^{\alpha}$ is exact. Take a short exact sequence of $C^{*}$-algebras:

$$
0 \quad \longrightarrow \quad \longrightarrow \quad B \quad \xrightarrow{\pi} \quad D \quad \longrightarrow \quad 0 .
$$

Let us take a positive element $x$ of $\operatorname{ker}\left(\pi \otimes_{\min ^{i n}} \mathrm{id}_{A}\right)$. To derive a contradiction, we assume $x \notin I \otimes{ }_{\min } A$. Then we can find a state $\varphi$ of $B \otimes{ }_{\min } A$ such that $\varphi$ vanishes on $I \otimes_{\min } A$ and $\varphi(x)>0$. We set $x_{0}=\int_{G} \mathrm{id}_{B} \otimes_{\min } \alpha_{z}(x) d z$ where $d z$ is the normalized Haar measure of $G$. Then we see $x_{0} \in B \otimes \min A^{\alpha}$. We have

$$
\begin{aligned}
\left(\pi \otimes_{\min ^{2}} \mathrm{id}_{A^{\alpha}}\right)\left(x_{0}\right) & =\int_{G} \pi \otimes_{\min } \operatorname{id}_{A}\left(\operatorname{id}_{B} \otimes_{\min } \alpha_{z}(x)\right) d z \\
& =\int_{G} \mathrm{id}_{D} \otimes_{\min } \alpha_{z}\left(\pi \otimes_{\min } \operatorname{id}_{A}(x)\right) d z=0 .
\end{aligned}
$$

Since $A^{\alpha}$ is exact, we have $x_{0} \in I \otimes{ }_{\min } A^{\alpha}$. This leads a contradiction as

$$
0=\varphi\left(x_{0}\right)=\int_{G} \varphi\left(\mathrm{id}_{B} \otimes_{\min } \alpha_{z}(x)\right) d z>0 .
$$

Therefore we have $x \in I \otimes \min A$ for all positive element $x$ of $\operatorname{ker}\left(\pi \otimes \min \mathrm{id}_{A}\right)$. Thus we have shown $\operatorname{ker}\left(\pi \otimes_{\min ^{1 i d}}^{A}\right)=I \otimes_{\min } A$. This implies that $A$ is exact.

## Appendix B. On linking algebras

Definition B.1. Let $A$ be a $C^{*}$-algebra and $X$ be a Hilbert $A$-module. The $C^{*}$-algebra $\mathscr{K}(X \oplus A)$ is called the linking algebra of $X$, and denoted by $D_{X}$.

Since $\mathscr{K}(A, X) \cong X$ and $\mathscr{K}(A) \cong A$ naturally, we have the following matrix representation of $D_{X}$ :

$$
D_{X}=\left(\begin{array}{ll}
\mathscr{K}(X) & X \\
\widetilde{X} & A
\end{array}\right)
$$

where $\widetilde{X}=\mathscr{K}(X, A)$ is the dual left Hilbert $A$-module of $X$. The natural embeddings are denoted by

$$
l_{\mathscr{K}(X)}: \mathscr{K}(X) \hookrightarrow D_{X}, \quad l_{X}: X \hookrightarrow D_{X}, \quad \text { and } \quad \imath_{A}: A \hookrightarrow D_{X} .
$$

Both maps $i_{A}$ and $v_{\mathscr{K}(X)}$ are injective $*$-homomorphisms onto corners of $D_{X}$. The $C^{*}{ }^{-}$ subalgebra $A$ of $D_{X}$ is always full, but $\mathscr{K}(X)$ is full in $D_{X}$ only in the case that $X$ is a full Hilbert $A$-module.

Lemma B.2. Let $A$ be a $C^{*}$-algebra and $X$ be a Hilbert $A$-module. For separable subsets $A_{0} \subset A$ and $X_{0} \subset X$, there exist a separable $C^{*}$-subalgebra $A_{\infty} \subset A$ containing $A_{0}$ and a separable closed subspace $X_{\infty}$ of $X$ containing $X_{0}$ such that $X_{\infty}$ is a Hilbert $A_{\infty}$-module by restricting the operations of $X$.

Proof. Let $A_{1}$ be the $C^{*}$-algebra generated by $A_{0}+\left\langle X_{0}, X_{0}\right\rangle_{X}$. We set $X_{1}=$ $\overline{\operatorname{span}}\left(X_{0}+X_{0} A_{0}\right)$ which is a closed subspace of $X$. We inductively define families of separable $C^{*}$-subalgebras $\left\{A_{n}\right\}_{n=1}^{\infty}$ of $A$ and separable closed subspaces $\left\{X_{n}\right\}_{n=1}^{\infty}$ of $X$ so that $A_{n+1}$ is a $C^{*}$-algebra generated by $A_{n}+\left\langle X_{n}, X_{n}\right\rangle_{X}$, and that $X_{n+1}=$ $\overline{\operatorname{span}}\left(X_{n}+X_{n} A_{n}\right)$. We set $A_{\infty}=\overline{\bigcup_{n \in \mathbb{N}} A_{n}}$ and $X_{\infty}=\overline{\bigcup_{n \in \mathbb{N}} X_{n}}$. Then $A_{\infty}$ is a separable $C^{*}$-subalgebra of $A$ containing $A_{0}$, and $X_{\infty}$ is a separable closed subspace of $X$ containing $X_{0}$. By the construction, we have $X_{\infty} A_{\infty} \subset X_{\infty}$ and $\left\langle X_{\infty}, X_{\infty}\right\rangle_{X} \subset A_{\infty}$. Hence $X_{\infty}$ is a Hilbert $A_{\infty}$-module.

Proposition B.3. For a $C^{*}$-algebra $A$ and a Hilbert $A$-module $X$, the inclusion ${ }_{1}: A \longrightarrow D_{X}$ induces an isomorphism on the $K$-groups.

Proof. When both $A$ and $X$ are separable, [B, Corollary 2.6] gives us an isometry $v$ in the multiplier algebra $\mathscr{M}\left(D_{X} \otimes_{\min } \mathbb{K}\right)$ of $D_{X} \otimes_{\min } \mathbb{K}$ such that $\Phi: D_{X} \otimes_{\min }$ $\mathbb{K} \ni x \mapsto v x v^{*} \in A \otimes_{\min } \mathbb{K}$ is an isomorphism, where $\mathbb{K}$ is the $C^{*}$-algebra of the compact operators on the infinite-dimensional separable Hilbert space. Since the composition of the isomorphism $\Phi$ and the inclusion $l_{A} \otimes_{\min } \mathrm{id}_{\mathbb{K}}$ : $A \otimes_{\min } \mathbb{K} \longrightarrow D_{X} \otimes_{\min } \mathbb{K}$ induces an identity on the $K$-groups of $D_{X} \otimes_{\min } \mathbb{K}$ (see, for example, [HR, Lemma 4.6.2]), the inclusion $l_{A} \otimes_{\text {min }} \mathrm{id}_{\mathbb{K}}$ induces an isomorphism on the $K$-groups. Hence the inclusion $t_{A}: A \longrightarrow D_{X}$ also induces an isomorphism on the $K$-groups.

Now let $A$ be a general $C^{*}$-algebra and $X$ be a general Hilbert $A$-module. By Lemma B.2, the set of the pairs $\left(A_{\lambda}, X_{\lambda}\right)$ consisting of separable $C^{*}$-subalgebras $A_{\lambda}$ of $A$ and separable closed subspaces $X_{\lambda}$ of $X$ such that $X_{\lambda}$ are Hilbert $A_{\lambda}$-modules is
upward directed with respect to the inclusions, and satisfies $A=\bigcup_{\lambda} A_{\lambda}, X=\bigcup_{\lambda} X_{\lambda}$. We have $A \cong \underset{\longrightarrow}{\lim } A_{\lambda}$ and $D_{X} \cong \lim _{X_{\lambda}}$. By the first part of this proof, the inclusion ${ }_{l_{\lambda}}: A_{\lambda} \longrightarrow D_{X_{\lambda}}$ induces an isomorphism on the $K$-groups for all $\lambda$. Thus the inclusion ${ }_{\iota_{A}}: A \longrightarrow D_{X}$ also induces an isomorphism on the $K$-groups.

Remark B.4. Let $A$ be a $C^{*}$-algebra and $X$ be a Hilbert $A$-module. By Proposition B.3, we can define a map $X_{*}: K_{*}(\mathscr{K}(X)) \longrightarrow K_{*}(A)$ by the composition of the map $\left(\mathscr{\mathscr { K }}_{(X)}\right)_{*}: K_{*}(\mathscr{K}(X)) \longrightarrow K_{*}\left(D_{X}\right)$ and the inverse of the isomorphism $\left(l_{A}\right)_{*}: K_{*}(A) \longrightarrow K_{*}\left(D_{X}\right)$. This map is the same map as the one defined in [E, Definition 5.1].

Proposition B.5. Let $A, B$ be $C^{*}$-algebras, and $l: A \longrightarrow B$ be an injective $*-$ homomorphism onto a hereditary and full $C^{*}$-subalgebra of $B$. Then $l_{*}$ is an isomorphism from $K_{*}(A)$ to $K_{*}(B)$.

Proof. The proof goes the same way as the proof of $[B$, Corollary 2.10$]$ with the help of Proposition B.3.

Remark B.6. Let $A, B$ be strongly Morita equivalent $C^{*}$-algebras. Then there exists a $C^{*}$-algebra $D$ which contains $A$ and $B$ as full and hereditary $C^{*}$-subalgebras. Hence we see that the $K$-groups of $A$ and $B$ are isomorphic by Proposition B.5, and that $A$ is nuclear or exact if and only if $B$ is also by Corollary A. 9 and Proposition A. 10.

We use the two propositions below in Section 7.
Proposition B.7. Let $A$ be a $C^{*}$-algebra and $X$ be a Hilbert $A$-module. If $A$ is nuclear or exact, then $\mathscr{K}(X)$ is also.

Proof. Since $A$ is a hereditary and full $C^{*}$-subalgebra of $D_{X}$, if $A$ is nuclear or exact then $D_{X}$ is also by Corollary A. 9 and Proposition A.10. Now the conclusion follows from the fact that $\mathscr{K}(X)$ is a hereditary $C^{*}$-subalgebra of $D_{X}$.

Proposition B.8. Let $A$ and $B$ be $C^{*}$-algebras, $X$ be a Hilbert $A$-module, and $Y$ be a Hilbert $B$-module. Let $\pi: A \longrightarrow B$ be $a *$-homomorphism and $t: X \longrightarrow Y$ be a linear map satisfying $\langle t(\xi), t(\eta)\rangle_{Y}=\pi\left(\langle\xi, \eta\rangle_{X}\right)$ for $\xi, \eta \in X$. We can define $a *$-homomorphism $\psi_{t}: \mathscr{K}(X) \longrightarrow \mathscr{K}(Y)$ by $\psi_{t}\left(\theta_{\xi, \eta}\right)=\theta_{t(\xi), t(\eta)}$ for $\xi, \eta \in X$. Then the nuclearity of $\pi$ implies the nuclearity of $\psi_{t}$.

Proof. For the well-definedness of $\psi_{t}$, see [KPW, Lemma 2.2]. We can define a $*-$ homomorphism $\rho: D_{X} \longrightarrow D_{Y}$ so that $\rho \circ l_{A}=l_{B} \circ \pi, \quad \rho \circ l_{X}=l_{Y} \circ t$ and $\rho \circ l_{\mathscr{K}(X)}=$ $l_{\mathscr{K}(Y)}{ }^{\circ} \psi_{t}$. Since $A$ is a hereditary and full $C^{*}$-subalgebra of $D_{X}$, the nuclearity of $\pi$ implies the nuclearity of $\rho$ by Proposition A.8. Since $\mathscr{K}(Y)$ is a hereditary $C^{*}$ subalgebra of $D_{Y}$, the nuclearity of $\rho$ implies the nuclearity of $\psi_{t}$ by Proposition A.7. We are done.

## Appendix C. A proof of Proposition 8.2

In this appendix, we give a $K$-theoretical proof of Proposition 8.2. In [ P , Theorem 4.4], Pimsner used $K K$-theory to prove this proposition under some hypotheses, one of which is that both $A$ and $X$ are separable. What we will do here is to get rid of $K K$-theory from the proof of [ P , Theorem 4.4] so that we can prove this proposition without the assumption of separability. We first prepare some notation and results which we will need.

Definition C.1. For a $C^{*}$-algebra $A$, we define $S A=C_{0}((0,1), A)$, which we often consider as a set of functions in $C_{0}((-1,1), A)$ vanishing on $(-1,0]$. For a $*-$ homomorphism $\varphi: A \longrightarrow B$, we denote by $S \varphi: S A \longrightarrow S B$ the $*$-homomorphism defined by $S \varphi(f)(s)=\varphi(f(s))$ for $f \in S A$ and $s \in(0,1)$.

Definition C.2. For a $C^{*}$-algebra $A$ and an ideal $I$ of $A$, we define a $C^{*}$-algebra $D(I, A)$ by

$$
D(I, A)=\left\{f \in C_{0}((-1,1), A) \mid f(s)-f(-s) \in I \text { for all } s \in(-1,1)\right\}
$$

We denote by $l$ the natural embedding $S I \longrightarrow D(I, A)$.
Lemma C.3. The $*$-homomorphism $\quad l: S I \longrightarrow D(I, A)$ induces an isomorphism $l_{*}: K_{*}(S I) \longrightarrow K_{*}(D(I, A))$.

Proof. Let us define a $*$-homomorphism $\pi: D(I, A) \longrightarrow C_{0}((-1,0], A)$ by the restriction. Then $\pi$ is surjective and its kernel is $S I$. Hence we have the following short exact sequence:

$$
0 \quad \longrightarrow \quad S I \xrightarrow{l} D(I, A) \xrightarrow{\pi} C_{0}((-1,0], A) \quad \longrightarrow 0 .
$$

The conclusion follows from the 6 -term exact sequence of $K$-groups associated with this short exact sequence together with the fact $K_{*}\left(C_{0}((-1,0], A)\right)=0$.

Definition C.4. Let $A, B$ be $C^{*}$-algebras, and $I$ be an ideals of $A$. For two $*-$ homomorphisms $\rho_{+}, \rho_{-}: B \longrightarrow A$ such that $\rho_{+}(b)-\rho_{-}(b) \in I$ for all $b \in B$, we define a *-homomorphisms $\rho: S B \longrightarrow D(I, A)$ by

$$
\rho(f)(s)= \begin{cases}\rho_{+}(f(s)) & \text { if } s \geqslant 0 \\ \rho_{-}(f(-s)) & \text { if } s \leqslant 0\end{cases}
$$

for $f \in S B$.
Lemma C.5. When $\rho_{+}=\rho_{-}$, the $*$-homomorphism $\rho: S B \longrightarrow D(I, A)$ in Definition C. 4 induces 0 on $K$-groups.

Proof. When $\rho_{+}=\rho_{-}$, the $*$-homomorphism $\rho$ factors through the $*$-homomorphism $\sigma: C_{0}([0,1), A) \longrightarrow D(I, A)$ defined by

$$
\sigma(f)(s)= \begin{cases}f(s) & \text { if } s \geqslant 0 \\ f(-s) & \text { if } s \leqslant 0\end{cases}
$$

for $f \in C_{0}([0,1), A)$. Since $K_{*}\left(C_{0}([0,1), A)\right)=0$, we have $\rho_{*}=0$.
Lemma C.6. For $j=1,2$, let $A_{j}$ be a $C^{*}$-algebra, and $I_{j}$ be an ideal of $A_{j}$. For $a *-$ homomorphism $\varphi: A_{1} \longrightarrow A_{2}$ with $\varphi\left(I_{1}\right) \subset I_{2}$, we can define a *-homomorphism $D \varphi: D\left(I_{1}, A_{1}\right) \longrightarrow D\left(I_{2}, A_{2}\right)$ by $D \varphi(f)(s)=\varphi(f(s))$, and we get a commutative diagram:


Proof. Straightforward.
We go back to the proof of Proposition 8.2. We first treat the case that the $C^{*}$ correspondence $X$ is non-degenerate. Let us take a $C^{*}$-algebra $A$ and a nondegenerate $C^{*}$-correspondence $X$.

Let $\left(\varphi_{\infty}, \tau_{\infty}\right)$ be the Fock representation of $X$ on $\mathscr{L}(\mathscr{F}(X))$. We denote by $\rho_{+}: \mathscr{T}_{X} \longrightarrow \mathscr{L}(\mathscr{F}(X))$ the $*$-homomorphism such that $\rho_{+} \circ \bar{\pi}_{X}=\varphi_{\infty}$ and $\rho_{+} \circ \bar{t}_{X}=$ $\tau_{\infty}$. We define a $*$-homomorphism $\varphi_{\infty}^{-}: A \longrightarrow \mathscr{L}(\mathscr{F}(X))$ and a linear map $\tau_{\infty}^{-}: X \longrightarrow \mathscr{L}(\mathscr{F}(X))$ by

$$
\varphi_{\infty}^{-}(a)=\sum_{m=1}^{\infty} \varphi_{m}(a), \quad \tau_{\infty}^{-}(\xi)=\sum_{m=1}^{\infty} \tau_{m}^{1}(\xi) .
$$

Similarly as the proof of Proposition 4.3, we see that $\left(\varphi_{\infty}^{-}, \tau_{\infty}^{-}\right)$is a representation of $X$. Hence there exists a $*$-homomorphism $\rho_{-}: \mathscr{T}_{X} \longrightarrow \mathscr{L}(\mathscr{F}(X))$ such that $\rho_{-} \circ \bar{\pi}_{X}=$ $\varphi_{\infty}^{-}$and $\rho_{-}{ }^{\circ} \bar{t}_{X}=\tau_{\infty}^{-}$.

Lemma C. 7 ([P, Lemma 4.2]). For every $x \in \mathscr{T}_{X}$, we have $\rho_{+}(x)-$ $\rho_{-}(x) \in \mathscr{K}(\mathscr{F}(X))$.

Proof. Since $\mathscr{T}_{X}$ is generated by the image of the two maps $\bar{\pi}_{X}$ and $\bar{t}_{X}$, it suffices to show this lemma when $x \in \mathscr{T}_{X}$ is in the image of these maps. For $a \in A$, we have

$$
\rho_{+}\left(\bar{\pi}_{X}(a)\right)-\rho_{-}\left(\bar{\pi}_{X}(a)\right)=\varphi_{0}(a) \in \mathscr{K}(\mathscr{F}(X)),
$$

and for $\xi \in X$, we have

$$
\rho_{+}\left(\bar{t}_{X}(\xi)\right)-\rho_{-}\left(\bar{t}_{X}(\xi)\right)=\tau_{0}^{1}(\xi) \in \mathscr{K}(\mathscr{F}(X)) .
$$

We are done.
Let us set $D=D(\mathscr{K}(\mathscr{F}(X)), \mathscr{L}(\mathscr{F}(X)))$. By Lemma C.7, we can define a $*$ homomorphism $\rho: S \mathscr{T}_{X} \longrightarrow D$ by

$$
\rho(f)(s)= \begin{cases}\rho_{+}(f(s)) & \text { if } s \geqslant 0 \\ \rho_{-}(f(-s)) & \text { if } s \leqslant 0\end{cases}
$$

Lemma C.8. The *-homomorphism $S \varphi_{0}: S A \longrightarrow D$ induces an isomorphism on the K-groups.

Proof. This follows from the fact that $\varphi_{0}: A \longrightarrow \mathscr{K}(\mathscr{F}(X))$ is an injection onto a hereditary and full $C^{*}$-subalgebra of $\mathscr{K}(\mathscr{F}(X))$ with the help of Proposition B. 5 and Lemma C.3.

Proposition C.9. The composition of $S \bar{\pi}_{X}: S A \longrightarrow S \mathscr{T}_{X}$ and $\rho: S \mathscr{T}_{X} \longrightarrow D$ induces an isomorphism on the K-groups.

Proof. Since we have $\rho_{+} \circ \bar{\pi}_{X}=\varphi_{0}+\rho_{-} \circ \bar{\pi}_{X}$, we can see that the composition $\rho \circ S \bar{\pi}_{X}$ induces the same map as $S \varphi_{0}$ with the help of Lemma C.5. Hence the proof completes by Lemma C.8.

Proposition C. 9 implies that $\rho_{*}$ is "the left inverse" of the map $\left(S \bar{\pi}_{X}\right)_{*}: K_{*}(S A) \longrightarrow K_{*}\left(S \mathscr{T}_{X}\right)$ modulo the isomorphism $\left(S \varphi_{0}\right)_{*}$. We will show that it is also "the right inverse". To this end, we first "shift" the $*$-homomorphism $S \bar{\pi}_{X}: S A \longrightarrow S \mathscr{T}_{X}$ along the $*$-homomorphism $S \varphi_{0}: S A \longrightarrow D$ (see Lemma C.15).

Definition C.10. For each $n \in \mathbb{N}$, we set $Y_{n}=\overline{\operatorname{span}}\left(\bar{t}_{X}^{n}\left(X^{\otimes n}\right) \mathscr{T}_{X}\right) \subset \mathscr{T}_{X}$, which is naturally a Hilbert $\mathscr{T}_{X}$-module. We denote by $Y$ the direct sum of the Hilbert $\mathscr{T}_{X^{-}}$ modules $\left\{Y_{n}\right\}_{n=0}^{\infty}$.

Remark C.11. The Hilbert $\mathscr{T}_{X}$-module $Y$ is isomorphic to the interior tensor product of the Hilbert $A$-module $\mathscr{F}(X)$ and the Hilbert $\mathscr{T}_{X}$-module $\mathscr{T}_{X}$ with the ${ }_{*}$ homomorphism $\bar{\pi}_{X}: A \longrightarrow \mathscr{T}_{X}$.

The linear maps $\bar{t}_{X}^{n}: X^{\otimes n} \longrightarrow Y_{n}$ extend a linear map $\bar{t}_{X}^{\circ}: \mathscr{F}(X) \longrightarrow Y$. By the definition, we get $Y=\overline{\operatorname{span}}\left(\bar{t}_{X}^{\cdot}(\mathscr{F}(X)) \mathscr{T}_{X}\right)$. We also have $\left\langle\bar{t}_{X}^{\circ}(\xi), \bar{t}_{X}^{*}(\eta)\right\rangle_{Y}=$ $\bar{\pi}_{X}\left(\langle\xi, \eta\rangle_{\mathscr{F}(X)}\right)$ for all $\xi, \eta \in \mathscr{F}(X)$.

Definition C.12. We define a $*$-homomorphism $\Phi: \mathscr{L}(\mathscr{F}(X)) \ni T \mapsto \Phi(T) \in \mathscr{L}(Y)$ by

$$
\Phi(T)\left(\bar{t}_{X}^{*}(\xi) x\right)=\bar{t}_{X}^{*}(T(\xi)) x \quad \text { for } \xi \in \mathscr{F}(X) \text { and } x \in \mathscr{T}_{X} .
$$

It is not difficult to see that $\Phi$ is well defined.
Lemma C.13. We have $\Phi(\mathscr{K}(\mathscr{F}(X))) \subset \mathscr{K}(Y)$.
Proof. This follows from the fact that $\Phi\left(\theta_{\xi, \eta}\right)=\theta_{\bar{t}_{X}(\xi), \bar{\tau}_{X}^{*}(\eta)}$ for $\xi, \eta \in \mathscr{F}(X)$, which is easily verified.

We define $\widetilde{D}=D(\mathscr{K}(Y), \mathscr{L}(Y))$. By Lemma C.13, we can define a $*-$ homomorphism $D \Phi: D \longrightarrow \widetilde{D}$. Since we assume that $X$ is non-degenerate, we have $Y_{0}=\mathscr{T}_{X}$. Hence the natural isomorphism $\mathscr{T}_{X} \cong \mathscr{K}\left(Y_{0}\right) \subset \mathscr{K}(Y)$ gives us a ${ }^{*}$ homomorphism $\widetilde{\varphi}_{0}: \mathscr{T}_{X} \longrightarrow \mathscr{K}(Y)$.

Lemma C.14. The $*$-homomorphism $S \widetilde{\varphi}_{0}: S \mathscr{T}_{X} \longrightarrow \widetilde{D}$ induces an isomorphism on the $K$-groups.

Proof. Similar as the proof of Lemma C.8.
Lemma C.15. We have the following commutative diagram:

$$
\begin{array}{ll}
S A \xrightarrow{S \tilde{\pi}_{X}} & S \mathcal{T}_{X} \\
\downarrow_{S \varphi_{0}} & \downarrow^{S \widetilde{\varphi}_{0}} \\
D \xrightarrow{D \Phi} & \widetilde{D}
\end{array}
$$

Proof. Straightforward.
Proposition C.16. The composition of $\rho: S \mathscr{T}_{X} \longrightarrow D$ and $D \Phi: D \longrightarrow \widetilde{D}$ induces an isomorphism on the K-groups.

Proof. We set $\pi=\Phi \circ \varphi_{\infty}: A \longrightarrow \mathscr{L}(Y)$. For each $s \in[0,1]$, we define a linear map $t_{s}: X \longrightarrow \mathscr{L}(Y)$ by

$$
t_{s}(\xi)=s \widetilde{\varphi}_{0}\left(\bar{t}_{X}(\xi)\right)+\sqrt{1-s^{2}} \Phi\left(\tau_{0}^{1}(\xi)\right)+\Phi\left(\tau_{\infty}^{-}(\xi)\right)
$$

It is routine to check that the pair $\left(\pi, t_{s}\right)$ is a representation of $X$. Thus we get a $*-$ homomorphism $\rho_{s}: \mathscr{T}_{X} \longrightarrow \mathscr{L}(Y)$ such that $\rho_{s} \circ \bar{\pi}_{X}=\pi$ and $\rho_{s} \circ \bar{t}_{X}=t_{s}$ for each $s$. We have $\rho_{0}=\Phi \circ \rho_{+}$because $t_{0}=\Phi \circ \tau_{\infty}$. We also have $\rho_{1}=\widetilde{\varphi}_{0}+\Phi \circ \rho_{-}$because $t_{1}=$ $\widetilde{\varphi}_{0} \circ \bar{t}_{X}+\Phi \circ \tau_{\infty}^{-}$and $\pi=\widetilde{\varphi}_{0} \circ \bar{\pi}_{X}+\Phi \circ \varphi_{\infty}^{-}$. For $\xi \in X$ and $s \in[0,1]$, we have $t_{s}(\xi)-$ $\Phi\left(\tau_{\infty}^{-}(\xi)\right) \in \mathscr{K}(Y)$ because $\widetilde{\varphi}_{0}\left(\bar{t}_{X}(\xi)\right), \Phi\left(\tau_{0}^{1}(\xi)\right) \in \mathscr{K}(Y)$. Since we have $\pi(a)-$ $\Phi\left(\varphi_{\infty}^{-}(a)\right)=\widetilde{\varphi}_{0}\left(\bar{\pi}_{X}(a)\right) \in \mathscr{K}(Y)$, we can prove $\rho_{s}(x)-\Phi\left(\rho_{-}(x)\right) \in \mathscr{K}(Y)$ for all
$x \in \mathscr{T}_{X}$ and $s \in[0,1]$ in a similar way to the proof of Lemma C.7. Hence we can see that the composition of $D \Phi \circ \rho$ is homotopic to the $*$-homomorphism $S \mathscr{T}_{X} \longrightarrow \widetilde{D}$ defined from the two $*$-homomorphisms $S \widetilde{\varphi}_{0}+S \Phi \circ \rho_{-}$and $S \Phi \circ \rho_{-}$. By Lemma C.5, we see that $D \Phi \circ \rho$ induces the same map as $S \widetilde{\varphi}_{0}$. Hence the proof completes by Lemma C.14.

Combining all the results above, we obtain that the composition of the map $\rho_{*}: K_{*}\left(S \mathscr{T}_{X}\right) \longrightarrow K_{*}(D)$ and the isomorphism $\left(S \varphi_{0}\right)_{*}^{-1}: K_{*}(D) \longrightarrow K_{*}(S A)$ gives the inverse of the map $\left(S \bar{\pi}_{X}\right)_{*}: K_{*}(S A) \longrightarrow K_{*}\left(S \mathscr{T}_{X}\right)$. Hence we have shown that $\left(\bar{\pi}_{X}\right)_{*}: K_{*}(A) \longrightarrow K_{*}\left(\mathscr{T}_{X}\right)$ is an isomorphism when the $C^{*}$-correspondence $X$ is nondegenerate. We will see that this is the case for general $C^{*}$-correspondences.

Let us take a $C^{*}$-correspondence $X$ over a $C^{*}$-algebra $A$. We define

$$
T=\overline{\operatorname{span}}\left(\bar{\pi}_{X}(A) \mathscr{T}_{X} \bar{\pi}_{X}(A)\right),
$$

which is the hereditary $C^{*}$-subalgebra of $\mathscr{T}_{X}$ generated by $\bar{\pi}_{X}(A)$. Since the ideal generated by $\bar{\pi}_{X}(A)$ is $\mathscr{T}_{X}$, Proposition B. 5 shows that the inclusion $T \hookrightarrow \mathscr{T}_{X}$ induces an isomorphism on the $K$-groups. Hence to prove that the $*$-homomorphism $\bar{\pi}_{X}: A \longrightarrow \mathscr{T}_{X}$ induces an isomorphism on the $K$-groups, it suffices to show that the *-homomorphism $\bar{\pi}_{X}: A \longrightarrow T$ induces an isomorphism on the $K$-groups. This can be shown by applying the discussion above to the non-degenerate $C^{*}$-correspondence in the next lemma.

Lemma C.17. Let us set $X^{\prime}=\overline{\operatorname{span}}\left(\varphi_{X}(A) X\right)$ which is a non-degenerate $C^{*}$ correspondence over $A$. Then there exists an isomorphism $\rho: \mathscr{T}_{X^{\prime}} \longrightarrow T$ such that $\rho \circ \bar{\pi}_{X^{\prime}}=\bar{\pi}_{X}$.

Proof. Let us set $\pi=\bar{\pi}_{X}$ and define a linear map $t: X^{\prime} \longrightarrow \mathscr{T}_{X}$ as the restriction of $\bar{t}_{X}$ to $X^{\prime}$. It is easy to see that the pair $(\pi, t)$ is a representation of $X^{\prime}$. Hence we have a $*-$ homomorphism $\rho: \mathscr{T}_{X^{\prime}} \longrightarrow \mathscr{T}_{X}$. It is clear that the gauge action of $\mathscr{T}_{X}$ is a gauge action for the representation $(\pi, t)$. It is also clear that $\left\{a \in A \mid \pi(a) \in \psi_{t}\left(\mathscr{K}\left(X^{\prime}\right)\right)\right\}=$ 0 . Hence $\rho$ is injective by Theorem 6.2. Finally, it is not difficult to see that the image of $\rho$ is $T$.

This completes the proof of Proposition 8.2.

## References

[AEE] B. Abadie, S. Eilers, R. Exel, Morita equivalence for crossed products by Hilbert $C^{*}$-bimodules, Trans. Amer. Math. Soc. 350 (8) (1998) 3043-3054.
[BHRS] T. Bates, J. Hong, I. Raeburn, W. Szymański, The ideal structure of the $C^{*}$-algebras of infinite graphs, Illinois J. Math. 46 (4) (2002) 1159-1176.
[B] L.G. Brown, Stable isomorphism of hereditary subalgebras of $C^{*}$-algebras, Pacific J. Math. 71 (2) (1977) 335-348.
[DLRZ] S. Doplicher, R. Longo, J.E. Roberts, L. Zsido, A remark on quantum group actions and nuclearity, Rev. Math. Phys. 14 (7-8) (2002) 787-796.
[DS] K. Dykema, D. Shlyakhtenko, Exactness of Cuntz-Pimsner $C^{*}$-algebras, Proc. Edinburgh Math. Soc. 44 (2001) 425-444.
[E] R. Exel, A Fredholm operator approach to Morita equivalence, $K$-Theory 7 (3) (1993) 285-308.
[FLR] N.J. Fowler, M. Laca, I. Raeburn, The $C^{*}$-algebras of infinite graphs, Proc. Amer. Math. Soc. 128 (8) (2000) 2319-2327.
[FMR] N.J. Fowler, P.S. Muhly, I. Raeburn, Representations of Cuntz-Pimsner algebras, Indiana Univ. Math. J. 52 (3) (2003) 569-605.
[HR] N. Higson, J. Roe, Analytic $K$-homology, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, Oxford, 2000.
[KPW] T. Kajiwara, C. Pinzari, Y. Watatani, Ideal structure and simplicity of the $C^{*}$-algebras generated by Hilbert bimodules, J. Funct. Anal. 159 (2) (1998) 295-322.
[Ka1] T. Katsura, A class of $C^{*}$-algebras generalizing both graph algebras and homeomorphism $C^{*}$-algebras I, fundamental results, Trans. Amer. Math. Soc., to appear.
[Ka2] T. Katsura, A construction of $C^{*}$-algebras from $C^{*}$-correspondences, in: G.L. Price, B.M. Baker, P.E.T. Jorgensen, P.S. Muhly (Eds.), Advances in Quantum Dynamics, Contemp. Math., vol. 335, American Mathematical Society, Providence, RI, 2003, pp. 173-182.
[Ka3] T. Katsura, Ideal structure of $C^{*}$-algebras associated with $C^{*}$-correspondences, Preprint 2003, math.OA/0309294.
[Ki] E. Kirchberg, On subalgebras of the CAR-algebra, J. Funct. Anal. 129 (1) (1995) 35-63.
[KPR] A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. 184 (1) (1998) 161-174.
[KPRR] A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids, and Cuntz-Krieger algebras, J. Funct. Anal. 144 (2) (1997) 505-541.
[L1] C. Lance, On nuclear $C^{*}$-algebras, J. Funct. Anal. 12 (1973) 157-176.
[L2] E.C. Lance, Hilbert $C^{*}$-modules, A Toolkit for Operator Algebraists, London Mathematical Society Lecture Note Series, Vol. 210, Cambridge University Press, Cambridge, 1995.
[MS] P.S. Muhly, B. Solel, Tensor algebras over $C^{*}$-correspondences: representations, dilations, and $C^{*}$-envelopes, J. Funct. Anal. 158 (2) (1998) 389-457.
[P] M.V. Pimsner, A class of $C^{*}$-algebras generalizing both Cuntz-Krieger algebras and crossed products by $Z$, in: D. Voiculescu (Ed.), Free Probability Theory, Fields Inst. Communication, Vol. 12, American Mathematical Society, Providence, RI, 1997, pp. 189-212.
[RS] J. Rosenberg, C. Schochet, The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor, Duke Math. J. 55 (2) (1987) 431-474.
[W] S. Wassermann, Exact $C^{*}$-algebras and related topics, Lecture Notes Series, Vol. 19, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1994.


[^0]:    *Fax: + 81-3-5465-7011.
    E-mail address: katsu@ms.u-tokyo.ac.jp.

