Inventiones math. 56, 251-268 (1980)

Inventiones mathematicae

A Class of C^*-Algebras and Topological Markov Chains

Joachim Cuntz and Wolfgang Krieger

Universität Heidelberg, Sonderforschungsbereich 123, Stochastische mathematische Modelle, und Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 6900 Heidelberg 1, Federal Republic of Germany

1. Introduction

In this paper we present a class of C^*-algebras and point out its close relationship to topological Markov chains, whose theory is part of symbolic dynamics. The C^*-algebra construction starts from a matrix $A=(A(i,j))_{i,j\in\Sigma}$, Σ a finite set, $A(i,j)\in\{0,1\}$, and where every row and every column of A is non-zero. (That $A(i,j)\in\{0,1\}$ is assumed for convenience only. All constructions and results extend to matrices with entries in $\mathbb{Z}_+$. We comment on this in Remark 2.18.) A C^*-algebra $\mathcal{O}_A$ is then generated by partial isometries $S_i\neq 0\,(i\in\Sigma)$ that act on a Hilbert space in such a way that their support projections $Q_i=S_i^*S_i$ and their range projections $P_i=S_iS_i^*$ satisfy the relations

$$(A)\quad P_iP_j=0\ (i\neq j),\qquad Q_i=\sum_{j\in\Sigma}A(i,j)\,P_j\ (i,j\in\Sigma).$$

The algebras $\mathcal{O}_n$ that were described in [5] arise in this way from the $n\times n$ matrix all of whose entries are 1, or, equivalently, from the 1×1 matrix (n).

For a large class of matrices A, that includes all irreducible matrices that are not permutation matrices, we prove that in fact all C^*-algebras that are generated by non-zero partial isometries that satisfy the relations (A) are canonically isomorphic (A is called irreducible if for all i,j there is an $m\in\mathbb{N}$ such that $(A^m)_{ij}>0$). The proof is based on the existence of an automorphism group $(\lambda_t^A)_{t\in\mathbb{T}}$ of $\mathcal{O}_A$ where

$$\lambda_t^A(S_i)=tS_i,\qquad (i\in\Sigma,\ t\in\mathbb{T}).$$

We show that $\mathcal{O}_A$ is simple if A is irreducible. (The ideal structure of $\mathcal{O}_A$ for reducible A will be considered elsewhere.)

On the other hand the matrix A is used in symbolic dynamics as a transition matrix to construct one-sided and two-sided subshifts. The one-sided subshift σ_A acts on the compact space

$$X_A=\{(x_k)_{k\in\mathbb{N}}\in\Sigma^{\mathbb{N}}\mid A(x_k,x_{k+1})=1\ (k\in\mathbb{N})\}$$

0020-9910/80/0056/0251/$03.60

and is defined by

$$(\sigma_A x)_k = x_{k+1}, \quad (k \in \mathbb{N}, x \in X_A).$$

The two-sided subshift $\bar{\sigma}_A$ acts on the compact space

$$\bar{X}_A = \{(x_k)_{k\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}} | A(x_k, x_{k+1}) = 1 \ (k \in \mathbb{Z})\}$$

and is defined by

$$(\bar{\sigma}_A x)_k = x_{k+1}, \quad (k \in \mathbb{Z}, x \in \bar{X}_A).$$

$\bar{\sigma}_A$ is what is called a topological Markov chain. Abstractly a topological Markov chain can be defined as an expansive homeomorphism, with canonical coordinates, of a completely disconnected compactum. By means of a one-step generator every topological Markov chain can be represented as the $\bar{\sigma}_A$ of some transition matrix A. (For the theory of topological Markov chains, see e.g. [7].)

One has $\mathcal{O}_A$ together with the automorphism group $(\lambda_t^A)_{t\in\mathbb{T}}$ invariantly associated to σ_A (Proposition 2.17). In Sect. 3 we prove that $\mathcal{K} \otimes \mathcal{O}_A$ ($\mathcal{K}$ the algebra of compact operators on a separable infinite-dimensional Hilbert space) together with the automorphism group $(\mathrm{id} \otimes \lambda_t^A)_{t\in\mathbb{T}}$ is an invariant of the isomorphism type of an irreducible topological Markov chain $\bar{\sigma}_A$. In fact $\mathcal{K} \otimes \mathcal{O}_A$ arises as the crossed product by an automorphism from the AF-algebra that is furnished by the group of uniformly finite dimensional homeomorphisms [9, 10] on an unstable manifold of the chain. Moreover, as we will see in Sect. 4, $\mathcal{K} \otimes \mathcal{O}_A$ is an invariant of flow equivalence [11] of irreducible topological Markov chains.

In Sect. 5 we identify the Bowen-Franks invariant [3] $\mathbb{Z}^\Sigma/(1-A)\mathbb{Z}^\Sigma$ as the Ext-group of $\mathcal{K} \otimes \mathcal{O}_A$. Thereby we give for irreducible topological Markov chains an interpretation of this invariant in terms of the dynamics of the chain. At the same time this invariant shows that among the $\mathcal{O}_A$ there are many new simple C^*-algebras which are not stably isomorphic to any of the $\mathcal{O}_n$. (Recall that two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called stably isomorphic if $\mathcal{K} \otimes \mathcal{A}$ and $\mathcal{K} \otimes \mathcal{B}$ are isomorphic.)

2. Uniqueness of the Algebra $\mathcal{O}_A$

In this section we follow closely [5].

Let $A = (A(i,j))_{i,j\in\Sigma}$ be a square matrix with $A(i,j) \in \{0, 1\}$ and assume, as in the introduction, that no row and no column of A is zero. In the following we fix non-zero partial isometries S_i ($i \in \Sigma$) satisfying (A) and denote by $\mathcal{A}$ the C^*-algebra generated by the S_i ($i \in \Sigma$). The sum of the range projections P_i of S_i is a unit for $\mathcal{A}$ denoted by 1. If $\mu = (i_1, \ldots, i_k)$ is a multiindex with $i_j \in \Sigma$ we denote by $|\mu|$ the length k of μ and write $S_\emptyset = 1$, $S_\mu = S_{i_1} S_{i_2} \ldots S_{i_k}$ ($\emptyset$ is also considered as a multiindex). The symbols P_μ, Q_μ will stand for the range and support projections of S_μ, respectively. Each S_μ is a partial isometry and $S_\mu \neq 0$ if and only if $A(i_j, i_{j+1}) = 1$ ($j = 1, \ldots, k-1$). Let $\mathcal{M}_A$ denote the set of all multiindices μ with entries in Σ such that $S_\mu \neq 0$.

If $\mu=(i_1,\ldots,i_k)$ and $v=(j_1,\ldots,j_l)$ are multiindices we write μv for the multiindex $\mu v=(i_1,\ldots,i_k,j_1,\ldots,j_l)$ of length $k+l$.

2.1. **Lemma.** *Let $\mu, v\in\mathcal{M}_A$ be as above and assume $S_\mu^* S_v \neq 0$.*

(a) *If $|\mu|=|v|$, then $\mu=v$. In this case $S_\mu^* S_v=Q_\mu=Q_{i_k}$ ($\mu=(i_1,\ldots,i_k)$).*

(b) *If $|\mu|>|v|$, then $\mu=v\mu'$ with $|\mu'|=|\mu|-|v|$.*

(c) *If $|\mu|<|v|$, then $v=\mu v'$ with $|v'|=|v|-|\mu|$.*

Proof. (a) follows from the relations $S_i^* S_j=\delta_{ij}Q_i$ $(i,j\in\Sigma)$.

(b) Write $\mu=\alpha\mu'$ with $|\alpha|=|v|$. Since $S_\mu^* S_v=S_{\mu'}^* S_\alpha^* S_v$, we get $\alpha=v$ from (a). Q.e.d.

2.2. **Lemma.** *Every word W in S_i, S_i^* $(i\in\Sigma)$ is a linear combination of terms of the form $S_\mu P_i S_v^*$ (thus also a linear combination of terms of the form $S_{\mu'} S_{v'}^*$).*

Proof. Assume $W\neq 0$. Then after cancellation $(S_i^* S_j=\delta_{ij}Q_i)$ $W=A_1\ldots A_r B_1\ldots B_s$ where $A_i\in\{S_i, Q_i | i\in\Sigma\}$, $B_j\in\{S_i^*, Q_i | i\in\Sigma\}$. Since $Q_i S_j=A(i,j)S_j$, W is actually of the form $W=S_\mu Q_1\ldots Q_k S_v^*$ and the product $Q_1\ldots Q_k$ is a sum of finitely many of the P_i. Q.e.d.

2.3. **Proposition.** *Let $\mathcal{F}_k$ be the C^*-algebra generated by all elements of the form $S_\mu P_i S_v^*$ where $|\mu|=|v|=k$ $(k=0,1,2,\ldots)$. Then each $\mathcal{F}_k$ is a finite-dimensional C^*-algebra with unit and $\mathcal{F}_k\subset\mathcal{F}_{k+1}$.*

Proof. Given $i\in\Sigma$ and $\mu,v\in\mathcal{M}_A$ such that $|\mu|=|v|=k$ write

$$E_{\mu,v}^i=S_\mu P_i S_v^*.$$

Using 2.1 we compute

$$E_{\mu_1,v_1}^i E_{\mu_2,v_2}^j=\delta_{v_1,\mu_2} S_{\mu_1} P_i Q_{v_1} P_j S_{v_2}^*=\delta_{v_1,\mu_2}\delta_{i,j} E_{\mu_1,v_2}^i.$$

Thus the non-zero elements among the $E_{\mu,v}^i$ form a system of matrix units generating a (not necessarily simple) finite-dimensional C^*-algebra.

The identity

$$S_\mu P_i S_v^*=\sum_{j\in\Sigma} S_\mu S_i P_j S_i^* S_v^*=\sum_{j\in\Sigma} S_{\mu i} P_j S_{vi}^*$$

shows that $\mathcal{F}_k\subset\mathcal{F}_{k+1}$. Q.e.d.

We denote by $\mathcal{F}_A$ the closure of $\bigcup_{k=0}^{\infty}\mathcal{F}_k$. Then $\mathcal{F}_A$ is an inductive limit of finite-dimensional C^*-algebras (i.e. an AF-algebra).

For $i\in\Sigma$ let $\mathcal{F}_k^i$ be the (simple) C^*-subalgebra of $\mathcal{F}_k$ that is generated by the $E_{\mu,v}^i$ ($\mu,v\in\mathcal{M}_A$, $|\mu|=|v|=k$). Then $\mathcal{F}_k$ is the direct sum of the $\mathcal{F}_k^i$ $(i\in\Sigma)$, and the embedding of $\mathcal{F}_k$ in $\mathcal{F}_{k+1}$ is given by the matrix A, i.e. $\mathcal{F}_k^i$ is embedded in $\mathcal{F}_{k+1}^j$ with multiplicity $A(i,j)$ (for the definition of the multiplicities of an embedding see [2]). The criterion for simplicity of an AF-algebra, given in [2, 3. 5], shows that $\mathcal{F}_A$ is simple if A is aperiodic, i.e. if there is $m>0$ such that $(A^m)_{ij}>0$ for all $i,j\in\Sigma$. It should also be noted that, in this case, $\mathcal{F}_A$ admits a unique trace, cf. Sect. 3.

We define a positive linear map $\phi_A: \mathscr{A} \to \mathscr{A}$ by $\phi_A(X) = \sum_{i \in \Sigma} S_i X S_i^*$. Then $\phi_A^k(X) = \sum_{|\alpha|=k} S_\alpha X S_\alpha^*$ $(k=1,2,\ldots)$.

2.4. **Lemma.** *Every element in $\phi_A^{k+1}(\mathscr{A})$ commutes with every element in $\mathscr{F}_k$ $(k=0,1,2,\ldots)$.*

Proof. Take $X = \sum_{|\alpha|=k+1} S_\alpha X' S_\alpha^*$ in $\Phi_A^{k+1}(\mathscr{A})$ and $Y = S_\mu P_i S_\nu^*$ $(|\mu|=|\nu|=k,\ i \in \Sigma)$ in $\mathscr{F}_k$. Then by (2.1)

$$\begin{aligned} XY &= \Big(\underset{|\alpha|=k+1}{S} S_\alpha X' S_\alpha^* \Big) S_\mu P_i S_\nu^* = \sum_{j \in \Sigma} S_{\mu j} X' S_j^* Q_\mu P_i S_\nu^* \\ &= S_\mu \Phi_A(X') P_i S_\nu^*, \\ YX &= S_\mu P_i S_\nu^* \Big(\sum_{|\alpha|=k+1} S_\alpha X' S_\alpha^* \Big) = S_\mu P_i \Phi_A(X') S_\nu^*. \end{aligned}$$

But $P_i \Phi_A(X') = S_i X' S_i^* = \Phi_A(X') P_i$ and the assertion is proved. Q.e.d.

It follows from 2.4 that the C^*-algebra $\mathscr{D}_A$ generated by all elements of the form $\phi_A^k(P_i)$ $(i \in \Sigma,\ k \geqq 0)$ is commutative. This C^*-algebra is seen to coincide with the C^*-algebra generated by all range projections $P_\mu (\mu \in \mathscr{M}_A)$. The restriction of ϕ_A to $\mathscr{D}_A$ is an isometric endomorphism of $\mathscr{D}_A$. Consider now the algebra $\mathscr{C}(X_A)$ of all continuous complex-valued functions on X_A. This algebra is generated by all functions of the form $\chi_i \circ \sigma_A^j = \sigma_A^{*j}(\chi_i)$ $(i \in \Sigma,\ j=0,1,2,\ldots)$ where χ_i is the characteristic function of the cylinder set $Z(i) = \{x \in X_A \mid x_1 = i\}$.

2.5. **Proposition.** *There is a unique isomorphism $\omega: \mathscr{D}_A \to \mathscr{C}(X_A)$ such that $\omega(P_i) = \chi_i$ $(i \in \Sigma)$ and $\omega \phi_A \omega^{-1}(H) = \sigma_A^*(H)$ for all $H \in \mathscr{C}(X_A)$.*

Proof. One checks that the map

$$\omega: \phi_A^j(P_i) \mapsto \sigma_A^{*j}(\chi_i) \qquad (i \in \Sigma,\ j=0,1,2,\ldots)$$

extends to an isomorphism. Q.e.d.

Call a homeomorphism $u: X_A \to X_A$ uniformly finite dimensional if for some $k_0 \in \mathbb{N}$ we have $(ux)_k = x_k$ $(x \in X_A,\ k > k_0)$. The group of uniformly finite dimensional homeomorphisms of X_A is an ample group in the sense of [10]. It gives, via a crossed product, rise to an AF-algebra that is isomorphic to $\mathscr{F}_A$ by an isomorphism that extends ω. We return to this point of view in the next section.

Let Σ_0 denote the set of all $i \in \Sigma$ for which there are at least two different multiindices $\mu = (i_1, \ldots, i_r)$ and $\nu = (j_1, \ldots, j_s)$ in $\mathscr{M}_A$ such that $i_1 = i_r = j_1 = j_s = i$ $(r, s \geqq 2)$ while $i_k, j_l \neq i$ for $1 < k < r$, $1 < l < s$. From now on we will assume that A satisfies the following condition

(I) For each $i \in \Sigma$ there is $\mu = (i_1, \ldots, i_r)$ in $\mathscr{M}_A (r \geqq 1)$ such that $i_1 = i$ and $i_r \in \Sigma_0$.

The matrix A satisfies condition (I) if and only if X_A has no isolated points, i.e. X_A is a Cantor discontinuum. If A is irreducible and is not a permutation matrix, then A satisfies condition (I).

On the other hand, A does not satisfy (I) if and only if there is $\mu_0 \in \mathcal{M}_A$, $|\mu_0| \geqq 1$ such that $P_{\mu_0} = Q_{\mu_0}$. But then, depending on the choice of the $S_i (i \in \Sigma)$, the spectrum of S_{μ_0} may be any closed subset M of $\{0\} \cup \mathbb{T}$. Thus, in this case, the isomorphism class of $\mathcal{A}$ may depend (and actually does depend) on the choice of the generators $S_i (i \in \Sigma)$. For instance, for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we may choose partial isometries S_1, S_2 satisfying (A) such that the C^*-algebra generated by S_1, S_2 is isomorphic to $M_2 \otimes \mathcal{C}(M)$, for any given closed subset M of $\mathbb{T}$. We will see that such a thing can not happen if A satisfies (I).

2.6. **Lemma.** *Let A satisfy condition* (I). *Then for all $k \in \mathbb{N}$ there is a projection $Q \in \mathcal{D}_A$ such that $QP_i \neq 0$ $(i \in \Sigma)$ and such that $\phi_A^r(Q) S_\mu \phi_A^r(Q) = 0$ for all $\mu \in \mathcal{M}_A$ such that $1 \leqq |\mu| \leqq k$ and for all $r \geqq 0$.*

Proof. We can find aperiodic admissible sequences $x^{(i)} \in Z(i)$ $(i \in \Sigma)$ of elements in Σ such that no translate of $x^{(i)}$ coincides with any translate of $x^{(j)}$ $(i \neq j)$, in other words such that the finite set $Y = \{x^{(i)} \mid i \in \Sigma\}$ satisfies $\sigma_A^{-k}(Y) \cap \sigma_A^{-l}(Y) = \emptyset$ for all $k, l \in \mathbb{N}$ such that $k \neq l$.

If V is a sufficiently small closed and open set in X_A containing Y, we have $V \cap \sigma_A^{-j}(V) = \emptyset$ for $1 \leqq j \leqq k$. Let $Q \in \mathcal{D}_A$ be such that $\omega(Q)$ is the characteristic function of V. Then $Q\, \phi_A^j(Q) = 0$ for $1 \leqq j \leqq k$ (2.5) and since ϕ_A is a homomorphism on $\mathcal{D}_A$, also $\phi_A^r(Q)\, \phi_A^{r+j}(Q) = 0$ for all $r \geqq 0$ and $1 \leqq j \leqq k$, and this is what we wanted to show. Q.e.d.

2.7. **Lemma.** *Let $Q \in \mathcal{D}_A$ be a projection such that $QP_i \neq 0$ $(i \in \Sigma)$ and set $Q_k = \phi_A^k(Q)$. Then the map $X \mapsto Q_{k+1} X$ is an isomorphism of $\mathcal{F}_k$ onto $Q_{k+1} \mathcal{F}_k Q_{k+1}$.*

Proof. By 2.4 it suffices to show that $Q_{k+1} S_\mu P_i S_\mu^* \neq 0$ for all $i \in \Sigma$ and $\mu \in \mathcal{M}_A$ such that $|\mu| = k$ and $S_\mu P_i S_\mu^* \neq 0$ (note that $S_\mu P_i S_\mu^*$ is the range projection of the matrix unit $E^i_{\mu,\nu}$, cf. 2.3). This amounts to showing that $Q_{k+1} S_\mu S_\mu^* \neq 0$ for all $\mu \in \mathcal{M}_A$ such that $|\mu| = k+1$ (replace P_i by $S_i S_i^*$). But

$$Q_{k+1} S_\mu S_\mu^* = \phi_A^{k+1}(Q) S_\mu S_\mu^* = \Big(\sum_{|\alpha| = k+1} S_\alpha Q S_\alpha^*\Big) S_\mu S_\mu^* = S_\mu Q S_\mu^*$$

since $S_\alpha^* S_\mu S_\mu^* = \delta_{\alpha,\mu} S_\mu^*$ (2.1). The assumption on Q implies that $S_\mu Q S_\mu^* \neq 0$ whenever $\mu \in \mathcal{M}_A$. Q.e.d.

We denote by $\mathcal{P}$ the star algebra generated algebraically by $S_i (i \in \Sigma)$. Let $Y = S_\mu S_\nu^* \in \mathcal{P}$ $(\mu, \nu \in \mathcal{M}_A)$. If $|\mu| > |\nu|$ then $Y = S_{\mu_1}(S_{\mu_2} S_\nu^*) = S_{\mu_1} Y'$ where $\mu = \mu_1 \mu_2$, $|\mu_1| = |\mu| - |\nu|$, $|\mu_2| = |\nu|$ and $Y' \in \mathcal{F}_A$. If $|\mu| = |\nu|$, then $Y \in \mathcal{F}_A$. If $|\mu| < |\nu|$, then $Y = (S_\mu S_{\nu_2}^*) S_{\nu_1}^* = Y'' S_{\nu_1}^*$ where $\nu = \nu_1 \nu_2$, $\nu_1 = |\nu| - |\mu|$, $\nu_2 = \mu$ and $Y'' \in \mathcal{F}_A$. Since every element X of $\mathcal{P}$ is a linear combination of elements of the form $S_\mu S_\nu^* (\mu, \nu \in \mathcal{M}_A)$, X can be written as a finite sum

$$X = \sum_{|\nu| \geqq 1} X_\nu S_\nu^* + X_0 + \sum_{|\mu| \geqq 1} S_\mu X_\mu$$

where $X_\nu, X_0, X_\mu \in \mathcal{F}_A$.

2.8. **Proposition.** *The element $X_0 \in \mathscr{F}_A$ in this representation of X is uniquely determined by X, and $\|X_0\| \leqq \|X\|$.*

Proof. Given $k>0$, let $Q'_k \in \mathscr{D}_A$ be a projection satisfying the conditions of 2.6 and let $Q_k = \phi_A^k(Q'_k)$. Then the sequence $\{Q_k\}_{k\in\mathbb{N}}$ satisfies

$$\text{(a) } Q_k F - F Q_k \to 0 \quad \text{for all } F \in \mathscr{F}_A \tag{2.4}$$

$$\text{(b) } \|Q_k F\| \to \|F\| \quad \text{for all } F \in \mathscr{F}_A \tag{2.7}$$

$$\text{(c) } Q_k S_\mu Q_k,\ Q_k S_\mu^* Q_k \to 0 \quad \text{for all } \mu \text{ such that } |\mu| \geqq 1. \tag{2.6}$$

Therefore

$$\|X_0\| \underset{(1)}{=} \lim_{k\to\infty} \|Q_k X_0 Q_k\| \underset{(2)}{=} \lim_{k\to\infty} \|Q_k X Q_k\| \leqq \|X\|$$

where (1) follows from (b), and (2) from (a) and (c). If

$$X = \sum_{|\nu|\geqq 1} X'_\nu S_\nu^* + X'_0 + \sum_{|\mu|\geqq 1} S_\mu X'_\mu \qquad (X'_\nu, X'_0, X'_\mu \in \mathscr{F}_A)$$

is another representation of X, then

$$0 = X - X = \sum (X'_\nu - X_\nu) S_\nu^* + (X_0 - X'_0) + \sum S_\mu (X_\mu - X'_\mu)$$

is a representation of 0, and therefore $X_0 - X'_0 = 0$ by the first part of the proof. Q.e.d.

2.9. **Proposition.** *Let $X \in \mathscr{P}$. Then $X=0$ if $(X^*X)_0 = 0$ and $(XX^*)_0 = 0$.*

Proof. Let $X = \sum X_\nu S_\nu^* + X_0 + \sum S_\mu X_\mu$. Then $(X^*X)_0 = \sum_{|\nu|=|\nu'|} S_\nu X_\nu^* X_{\nu'} S_{\nu'}^* + X_0^* X_0 + \sum X_\mu^* Q_\mu X_\mu$ where each of the three terms on the right hand side is positive. Thus if $(X^*X)_0 = 0$ then all the $S_\mu X_\mu$ and X_0 are zero. In the same way $(XX^*)_0 = 0$ implies that all the $X_\nu S_\nu^*$ vanish. Q.e.d.

2.10. **Proposition.** *Let $\hat{S}_i$ $(i \in \Sigma)$ be another family of non-zero partial isometries satisfying (A) and let $\hat{\mathscr{P}}$, $\hat{\mathscr{F}}_A$ be defined as above with respect to $\hat{S}_i$ $(i \in \Sigma)$.*

(a) *The map $S_\mu S_\nu^* \mapsto \hat{S}_\mu \hat{S}_\nu^*$ $(|\mu| = |\nu|)$ extends to an isomorphism from $\mathscr{F}_A$ onto $\hat{\mathscr{F}}_A$.*

(b) *The map $S_i \mapsto \hat{S}_i$ $(i \in \Sigma)$ extends to an isomorphism from $\mathscr{P}$ onto $\hat{\mathscr{P}}$.*

Proof. (a) The map in question extends to an isomorphism of $\bigcup_{k\geqq 0} \mathscr{F}_k$ onto $\bigcup_{k\geqq 0} \hat{\mathscr{F}}_k$. Since $\bigcup_{k\geqq 0} \mathscr{F}_k$, as an inductive limit of finite-dimensional C^*-algebras, admits a unique C^*-norm, this isomorphism extends to the closure.

(b) If X is a linear combination of elements of the form $S_\mu S_\nu^*$, denote by $\hat{X}$ the corresponding linear combination of the elements $\hat{S}_\mu \hat{S}_\nu^*$. We have to show that $\hat{X} = 0$ if and only if $X = 0$. But $\hat{X} = 0$ implies $\hat{X}^* \hat{X} = \hat{X}\hat{X}^* = 0$ and $(\hat{X}^*\hat{X})_0 = (\hat{X}\hat{X}^*)_0 = 0$. Thus, by (a), $(X^*X)_0 = (XX^*)_0 = 0$ hence, by 2.9, $X = 0$. Q.e.d.

We now equip $\mathscr{P}$ with the largest C^*-norm

$$\|X\|_0 = \sup\{\|\rho(X)\| \mid \rho \text{ is a star representation of } \mathscr{P} \text{ on a separable Hilbert space}\}$$

and we denote by $\tilde{\mathscr{A}}$ the completion of $\mathscr{P}$ with respect to $\|\cdot\|_0$. By Proposition 2.8 the map $X \mapsto X_0$ extends to norm-decreasing positive linear maps $f: \mathscr{A} \to \mathscr{F}_A$ and $\tilde{f}: \tilde{\mathscr{A}} \to \mathscr{F}_A$. For every $t \in \mathbb{T}$ the partial isometries $\hat{S}_i = tS_i (i \in \Sigma)$ satisfy condition (A). Therefore the map $S_i \mapsto t S_i$ $(i \in \Sigma)$ extends by Proposition 2.10(b) to an endomorphism λ_t of $\tilde{\mathscr{A}}$. For this note that λ_t: $\mathscr{P} \to \mathscr{P}$ is necessarily norm-decreasing for $\|\cdot\|_0$ ($\|\cdot\|_0$ is the largest possible C^*-norm). Since $\lambda_{\bar{t}} \lambda_t = \lambda_t \lambda_{\bar{t}} = id$ the endomorphism λ_t is in fact an automorphism.

2.11. **Proposition.** *For* $X \in \tilde{\mathscr{A}}$, *we have* $\tilde{f}(X) = \int_{\mathbb{T}} \lambda_t(X)\, dt$ *(*dt *= normalized Haar measure). In particular,* $\tilde{f}: \tilde{\mathscr{A}} \to \mathscr{F}_A$ *is faithful,* i.e. $X \geqq 0$ *and* $\tilde{f}(X) = 0$ *implies* $X = 0$.

Proof. Let $Y = S_\mu S_\nu^*$ *be non-zero and* $r = |\mu| - |\nu|$. Then $\lambda_t(Y) = t^r Y$ and

$$\int_{\mathbb{T}} \lambda_t(Y)\, dt = \begin{cases} 0 & r \neq 0 \\ Y & r = 0 \end{cases}.$$

Since every element of $\mathscr{P}$ is a linear combination of elements like Y, we get $\tilde{f}(X) = \int_{\mathbb{T}} \lambda_t(X)\, dt$ for $X \in \mathscr{P}$. But the mappings $\tilde{f}$ and $X \mapsto \int_{\mathbb{T}} \lambda_t(X)\, dt$ are both continuous, so that the same identity holds for all $X \in \tilde{\mathscr{A}}$. Q.e.d.

2.12. **Proposition.** *The identity map* id: $\mathscr{P} \to \mathscr{P}$ *extends to an isomorphism* $\pi: \tilde{\mathscr{A}} \to \mathscr{A}$.

Proof. We only have to show that π is injective. Observe first, that $f \circ \pi = \pi \circ \tilde{f}$. If now $\pi(X) = 0$ for some $X \geqq 0$, then $f(\pi(X)) = 0$ implies $\pi(\tilde{f}(X)) = 0$ and hence $\tilde{f}(X) = 0$ (π restricted to $\tilde{\mathscr{F}}_A$ is an isomorphism). But then also $X = 0$ from 2.11. Q.e.d.

2.13. **Theorem.** *Assume that* A *satisfies* (I) *and that* $S_i (i \in \Sigma)$ *and* $\hat{S}_i (i \in \Sigma)$ *are two families of non-zero partial isometries satisfying* (A). *Then the map* $S_i \mapsto \hat{S}_i$ $(i \in \Sigma)$ *extends to an isomorphism from the* C^**-algebra* $\mathscr{A}$ *generated by* $S_i (i \in \Sigma)$ *onto the* C^**-algebra* $\hat{\mathscr{A}}$ *generated by* $\hat{S}_i (i \in \Sigma)$.

Proof. By 2.10 this map extends to an algebraic isomorphism from $\mathscr{P}$ onto $\hat{\mathscr{P}}$ and therefore also to an isomorphism from $\tilde{\mathscr{A}}$ onto $\tilde{\hat{\mathscr{A}}}$. The assertion now follows from Proposition 2.12. Q.e.d.

In the following we will write $\mathcal{O}_A$ for "the" C^*-algebra generated by partial isometries $S_i \neq 0$ $(i \in \Sigma)$ satisfying (A) (always assuming that A satisfies (I)) and $(\lambda_t^A)_{t \in \mathbb{T}}$ for the automorphism group $(\lambda_t)_{t \in \mathbb{T}}$ defined above. It is possible to determine the ideal structure of $\mathcal{O}_A$ exactly. We restrict ourselves here to show that $\mathcal{O}_A$ is simple if A is irreducible.

2.14. **Theorem.** *If* A *is irreducible then* $\mathcal{O}_A$ *is simple* (i.e. *contains no non-trivial closed ideal*).

Proof. If ρ is a star representation of $\mathcal{O}_A$ such that $\rho(S_i) \neq 0$ $(i \in \Sigma)$, then the $\rho(S_i)$ $(i \in \Sigma)$ satisfy (A) and ρ is an isomorphism by Theorem 2.13. To prove simplicity of $\mathcal{O}_A$ it suffices therefore to show that a star representation ρ of $\mathcal{O}_A$ such that $\rho(S_{i_0}) = 0$ for some $i_0 \in \Sigma$, is identically zero.

Now $\rho(S_{i_0})=0$ implies $\rho(Q_{i_0})=0$ and hence $\rho(S_j)=\rho(Q_{i_0}S_j)=0$ for all $j\in\Sigma$ such that $A(i_0,j)=1$. If A is irreducible we can continue in this way to show that $\rho(S_i)=0$ for all $i\in\Sigma$. Q.e.d.

2.15. *Remark.* Let $A=(A(i,j))_{i,j\in\Sigma}$ be a matrix where Σ is countably infinite, $A(i,j)\in\{0,1\}$, and let A satisfy the analogue of condition (I). Consider a C^*-algebra generated by non-zero partial isometries S_i $(i\in\Sigma)$ satisfying $S_i^*S_j=0$ $(i\neq j)$ and

$$S_i^*S_i=\sum_{j\in\Sigma}A(i,j)S_jS_j^* \qquad (i\in\Sigma)$$

the sum converging in the strong operator topology. One can prove the uniqueness of this C^*-algebra by the same device that was used in [5] to prove the uniqueness of $\mathcal{O}_\infty$. In fact, $\mathcal{O}_\infty$ is the algebra corresponding to the infinite matrix $A=(A(i,j))_{i,j\in\mathbb{N}}$ where $A(i,j)=1$ for all $i,j\in\mathbb{N}$. It was argued in [5] that $\mathcal{O}_\infty$ is the inductive limit of the subalgebras $\mathcal{A}_k$ $(k\in\mathbb{N})$ generated by $S_1,\dots,S_k$ and that each $\mathcal{A}_k$ admits a canonical embedding into $\mathcal{O}_{k+1}$, hence is unique. This idea carries over to more general infinite matrices A without difficulty and thus Theorem 2.13 also holds for countably infinite matrices A satisfying the analogue of (I).

2.16. *Remark.* So far we have only considered matrices with entries in $\{0,1\}$. Let now $A=(A(i,j))_{i,j\in\Sigma}$ be a matrix where $A(i,j)\in\mathbb{Z}_+$ (and Σ is again finite). Set $\Sigma'=\{(i,k,j)\mid i,j\in\Sigma;\ 1\leqq k\leqq A(i,j)\}$ and

$$A'((i_1,k_1,j_1),(i_2,k_2,j_2))=\begin{cases}1 & \text{if } j_1=i_2\\ 0 & \text{if } j_1\neq i_2\end{cases}.$$

For A satisfying the analogue of condition (I), set then $\mathcal{O}_A=\mathcal{O}_{A'}$. The algebra $\mathcal{K}\otimes\mathcal{O}_A$ can be described as follows. Start with the C^*-algebra $\bigoplus_{i\in\Sigma}\mathcal{K}_i$ where each $\mathcal{K}_i$ is isomorphic to $\mathcal{K}$. We may assume that this algebra is represented on a separable Hilbert space in such a way that each of its projections is onto an infinite-dimensional subspace. Choose then partial isometries S_i $(i\in\Sigma)$ with range projections P_i and support projections Q_i such that each P_i is a projection of dimension 1 in $\mathcal{K}_i$ while

$$Q_i=\sum_{j\in\Sigma}A(i,j)P_j \qquad (i\in\Sigma)$$

where $A(i,j)P_j$ is a formal expression meaning a projection of dimension $A(i,j)$ in $\mathcal{K}_j$. The C^*-algebra generated by $\bigoplus_{i\in\Sigma}\mathcal{K}_i$ together with S_i $(i\in\Sigma)$ is isomorphic to the C^*-algebra $\mathcal{K}\otimes\mathcal{O}_A$ as defined above.

To conclude this section we show that topologically conjugate one-sided shifts σ_A and σ_B give rise to isomorphic algebras $\mathcal{O}_A$ and $\mathcal{O}_B$. Recall that σ_A and σ_B are called topologically conjugate if there is a homeomorphism $h\colon X_A\to X_B$ such that $\sigma_B=h\sigma_Ah^{-1}$. We assume that A and B both satisfy condition (I).

2.17. **Proposition.** *If σ_A and σ_B are topologically conjugate, then there is an isomorphism of $\mathcal{O}_A$ onto $\mathcal{O}_B$ transforming $\mathcal{D}_A$ into $\mathcal{D}_B$, $\Phi_{A|\mathcal{D}_A}$ into $\Phi_{B|\mathcal{D}_B}$ and $(\lambda_t^A)_{t\in\mathbb{T}}$ into $(\lambda_t^B)_{t\in\mathbb{T}}$.*

Proof. The cylinder sets $Z(i)$ $(i\in\Sigma)$ defined in 2.5 form what is called a generator for σ_A, i.e. the characteristic functions of the sets $\sigma_A^{-k}(Z(i))$ $(k\in\mathbb{N}, i\in\Sigma)$ generate $\mathscr{C}(X_A)$. Given the map σ_A, the matrix A is determined by this generator ($A(i,j)=1$ iff $Z(i)\cap\sigma_A^{-1}(Z(j))\neq\emptyset$). For the proof we may assume that σ_A and σ_B act on the same space X and that $\sigma_A=\sigma_B=\sigma$. Let $Z(i)$ $(i\in\Sigma)$ and $Y(j)(j\in\Sigma')$ be generators for σ corresponding to A and B, respectively. Then the non-empty sets among the sets $W_{ij}=Z(i)\cap Y(j)$ $(i\in\Sigma, j\in\Sigma')$ form a new generator for σ. With respect to this generator σ has the form σ_C. The algebra $\mathcal{O}_C$ is generated by partial isometries $V_{ij}(i\in\Sigma, j\in\Sigma')$ where we set $V_{ij}=0$ if $W_{ij}=\emptyset$ and where the rest of the V_{ij} is non-zero and satisfies (C). Put $S_i=\sum_{j\in\Sigma'} V_{ij}$, $T_j=\sum_{i\in\Sigma} V_{ij}$ $(i\in\Sigma, j\in\Sigma')$. The partial isometries $S_i(i\in\Sigma)$ satisfy (A) and the partial isometries $T_j(j\in\Sigma')$ satisfy (B). By Theorem 2.13 it only remains to show that each of these sets generates the whole of $\mathcal{O}_C$. For this it suffices to show that $\mathscr{D}_C$ is contained in the C^*-algebras generated by these sets, since every V_{ij} is of the form $V_{ij}=PS_i=QT_j$ with $P,Q\in\mathscr{D}_C$. But this is an immediate consequence of 2.5. Q.e.d.

2.18. *Remark.* If A satisfies (I), then $\mathscr{D}_A$ is maximal commutative in $\mathcal{O}_A$ and there is a faithful conditional expectation $d:\mathcal{O}_A\to\mathscr{D}_A$. The map d can be constructed in analogy to the construction of f using the existence of projections like Q in 2.6. Moreover, $\mathscr{D}_A$ is regular in the sense that the normalizer

$$\mathcal{N}(\mathscr{D}_A)=\{U\in\mathcal{O}_A \text{ unitary} \mid U\mathscr{D}_A U^*=\mathscr{D}_A\}$$

generates $\mathcal{O}_A$ (cf. also [6]).

In fact, $\mathcal{O}_A$ may be considered as a kind of crossed product of $\mathscr{D}_A$ by the group of automorphisms induced by elements of $\mathcal{N}(\mathscr{D}_A)$, cf. also Sect. 3. The automorphism group $(\lambda_D)_{D\in\mathscr{D}_A^u}$ considered in [6] is associated with this decomposition of $\mathcal{O}_A$ as a crossed product.

3. Topological Markov Chains

Let T be an irreducible aperiodic topological Markov chain (for the periodic case see Remark 3.9). Using some one-step generator, whose transition matrix we denote by $A=(A(i,j))_{i,j\in\Sigma}$, we represent T in the form $\bar\sigma_A$. For an $x\in\bar X_A$ set

$$W(x,l)=\{(y_k)_{k\in\mathbb{Z}}\in\bar X_A \mid y_k=x_k\,(k\leqq l)\}, \qquad (l\in\mathbb{Z})$$

and consider the unstable manifold $W(x)$ of x,

$$W(x)=\bigcup_{l\in\mathbb{Z}} W(x,l).$$

Each of the sets $W(x,l)$ inherits from the shift space the topology of a discontinuum and we put on $W(x)$ the inductive limit topology that is produced by the inclusions of $W(x,l)$ into $W(x,l-1)$, $l\in\mathbb{Z}$. Note that neither $W(x)$ nor its topology depends on the choice of the generator of T that entered into these definitions. We denote

$$\mathscr{L}(x)=\bigcup_{m\in\mathbb{N}} \{(y_k)_{k\leqq 0}\in\Sigma^{\mathbb{Z}_-} \mid y_k=x_k(k\leqq -m);\ A(y_k,y_{k+1})=1\,(k<0)\},$$

and

$$Z(a)=\{(y_k)_{k\in\mathbb{Z}}\in W(x)\,|\,(y_k)_{k\leqq 0}=a\}, \qquad (a\in\mathcal{L}(x)).$$

3.1. **Lemma.** *For all* $x_1, x_2\in X_A$ *there exists a homeomorphism*

$$h\colon W(x_1)\to W(x_2)$$

such that

$$(h\,y)_k=y_k, \qquad (k\in\mathbb{N}). \tag{1}$$

Proof. Since A is irreducible and aperiodic we can enumerate

$$\{a\in\mathcal{L}(x_1)\,|\,a_0=j\}=\{a^{(1,j)}(m)\,|\,m\in\mathbb{N}\}$$
$$\{a\in\mathcal{L}(x_2)\,|\,a_0=j\}=\{a^{(2,j)}(m)\,|\,m\in\mathbb{N}\}, \qquad (j\in\Sigma).$$

A homeomorphism h as required is then defined by stipulating that

$$h\,Z(a^{(1,j)}(m))=Z(a^{(2,j)}(m)), \qquad (m\in\mathbb{N},\ j\in\Sigma)$$

and that (1) holds. Q.e.d.

Define a homeomorphism g of an open subset B of $W(x)$ onto another such set as uniformly finite dimensional if for some $l\in\mathbb{Z}$

$$(g\,y)_k=y_k, \qquad (k\geqq l,\ y\in B).$$

Again this definition of uniform finite dimensionality does not depend on the choice of the generator [9, 11]. The uniformly finite dimensional homeomorphism of $W(x)$ onto $W(x)$ form a group that we denote by $\mathcal{G}_T(x)$.

Let $\mathcal{R}_T(x)$ be the set of homeomorphisms r of $W(x)$ that are such that for some $l\in\mathbb{Z}$

$$(r\,y)_k=y_{k+1}, \qquad (k\geqq l).$$

3.2. **Lemma.** $\mathcal{R}_T(x)\neq\emptyset$.

Proof. By Lemma (3.1) there is a homeomorphism

$$h\colon W(\bar\sigma_A x)\to W(x)$$

such that

$$(h\,y)_k=y_k, \qquad (k\in\mathbb{N}),\ y\in W(x)).$$

Set $r=h\,\bar\sigma_A$ and have then $r\in\mathcal{R}_T(x)$. Q.e.d.

3.3. **Lemma.** *For all* $r_1, r_2\in\mathcal{R}_T(x)$, $r_1 r_2^{-1}\in\mathcal{G}_T(x)$.

Proof. If for some $l\in\mathbb{Z}$

$$(r_1\,y)_k=(r_2\,y)_k=y_{k+1}, \qquad (k\geqq l,\ y\in W(x))$$

then

$$(r_1 r_2^{-1} y)_k = y_k, \quad (k \geqq l, \ y \in W(x)). \quad \text{Q.e.d.}$$

3.4. **Lemma.** *For all $x_1, x_2 \in X_A$ there exists a homeomorphism*

$$h: W(x_1) \to W(x_2)$$

such that

$$h \mathscr{G}_T(x_1) h^{-1} = \mathscr{G}_T(x_2) \tag{2}$$

and such that

$$h \mathscr{R}_T(x_1) h^{-1} = \mathscr{R}_T(x_2). \tag{3}$$

Proof. The homeomorphism h of Lemma 3.1 satisfies (2) and (3). Q.e.d.

As a consequence of Lemma 3.4 the isomorphism type of the algebras that we are going to construct does not depend on the choice of the point $x \in X_A$. We drop therefore now the x from the notation.

$\mathscr{G}_T$ acts on the Boolean ring of compact open subsets of W, and the quotient map δ_T onto the orbit space of this action was called the future dimension function of T [9, 11]. Recall that this orbit space is the positive cone of an ordered abelian group $K_0(T^{-1}) \cong \varinjlim_{A^t} (\mathbb{Z}^\Sigma, \mathbb{Z}^\Sigma_+)$.

Let $\bar{\mathscr{D}}$ stand for the algebra of continuous complex valued functions on W that vanish at infinity, and let $\bar{\mathscr{F}}_T$ be the AF-algebra that contains $\bar{\mathscr{D}}$ as a regular maximal abelian subalgebra in such a way that $\mathscr{G}_T$ is the group of homeomorphisms of W that are given by the unitaries in the multiplier algebra of $\bar{\mathscr{F}}_T$ that normalize $\bar{\mathscr{D}}$. $\bar{\mathscr{F}}_T$ is the unique stable (i.e. $\mathscr{K} \otimes \bar{\mathscr{F}}_T \cong \bar{\mathscr{F}}_T$) AF-algebra whose dimension group is $K_0(T^{-1})$ (see [8]). For an alternative description of $\bar{\mathscr{F}}_T$ consider the crossed product (see e.g. [14]) $\mathscr{A}$ of $\bar{\mathscr{D}}$ by the automorphism group that is induced by $\mathscr{G}_T$ on $\bar{\mathscr{D}}$. To every $u \in \mathscr{G}_T$ there corresponds a unitary $\hat{u}$ in the multiplier algebra of $\mathscr{A}$. Let $\mathscr{J}$ be the closed ideal of $\mathscr{A}$ generated by all elements of the form $\hat{u} P_B - \hat{v} P_B$ where u and v are uniformly finite dimensional homeomorphisms such that the restrictions of u and v to the compact open set $B \subset W$ coincide, and where P_B denotes the characteristic function of B. Then $\bar{\mathscr{F}}_T$ is the quotient $\mathscr{A}/\mathscr{J}$. The characteristic function of a compact open set $B \subset W$ defines a projection $P_B \in \bar{\mathscr{D}} \subset \bar{\mathscr{F}}_T$. Moreover, every uniformly finite dimensional homeomorphism u of a compact open subset B of W onto a compact open subset C of W has an image $\tilde{u}$ in $\bar{\mathscr{F}}_T$ that is a partial isometry with range projection P_C and support projection P_B. We remark that under the assumption of aperiodicity and irreducibility the algebra $\bar{\mathscr{F}}_T$ is simple (see [2]) and has a unique trace. A formula for the trace of $\bar{\mathscr{F}}_T$ can be read off from the formula for the measure of maximal entropy for T (see e.g. [7]). Every $r \in \mathscr{R}_T$ as an element of the normalizer of $\mathscr{G}_T$ induces an automorphism of $\bar{\mathscr{F}}_T$ and one sees that this automorphism scales the trace by the maximal real eigenvalue of A. We form now the crossed product $\bar{\mathcal{O}}_T$ of $\bar{\mathscr{F}}_T$ by such an automorphism. This automorphism gives then rise to a unitary $\tilde{r}$ in the multiplier algebra of $\bar{\mathcal{O}}_T$. Moreover, we have an automorphism group $(\bar{\lambda}_t^T)_{t\in\mathbb{T}}$ of $\bar{\mathcal{O}}_T$ where $\bar{\lambda}_t^T$ leaves $\bar{\mathcal{F}}_T$ elementwise fixed, and where

$$\bar{\lambda}_t^T \tilde{r} = t\tilde{r} \qquad (t\in\mathbb{T}).$$

In view of Lemma 3.3 neither $\bar{\mathcal{O}}_T$ nor $(\bar{\lambda}_t^T)_{t\in\mathbb{T}}$ depends on the choice of the $r\in\mathcal{R}_T$ that was used to construct it. We have now the algebra $\bar{\mathcal{O}}_T$ as well as the group $(\bar{\lambda}_t^T)_{t\in\mathbb{T}}$ invariantly associated to the topological Markov chain T. We compute these invariants in terms of the matrix A.

3.5. **Lemma.** *Let B, B', C be compact open subsets of W, set $C'=rB'$, and let*

$$u: B\to B', \qquad v: C'\to C,$$

be uniformly finite dimensional homeomorphisms. Then $\bar{\mathcal{O}}_T$ is generated by $\bar{\mathcal{F}}_T$ and $\tilde{v}\tilde{r}\tilde{u}$.

Proof. It suffices to prove that for all compact open sets $D\subset W$ such that $\delta_T(D)\leqq\delta_T(B)$ one has $\tilde{r}P_D$ in the algebra that is generated by $\bar{\mathcal{F}}_T$ and $\tilde{v}\tilde{r}\tilde{u}$. For this choose uniformly finite dimensional homeomorphisms

$$u_1: D\to D_1\subset B, \qquad v_1: vruu_1D\to rD.$$

Then

$$w=r^{-1}v_1vruu_1$$

is a uniformly finite dimensional homeomorphism of D, and it is

$$\tilde{r}P_D=\tilde{v}_1\tilde{v}\tilde{r}\tilde{u}\tilde{u}_1\tilde{w}^{-1}P_D. \quad \text{Q.e.d.}$$

3.6. **Lemma.** *For all compact open sets $B\subset W$, $P_B\bar{\mathcal{O}}_TP_B$ together with $\bar{\mathcal{F}}_T$ generates $\bar{\mathcal{O}}_T$.*

Proof. Choose a compact open set $B_0\subset B$ such that

$$\delta_T(rB_0)\leqq\delta_T(B)$$

and choose a uniformly finite dimensional homeomorphism

$$u: rB_0\to C\subset B.$$

Then have

$$\tilde{u}\tilde{r}P_{B_0}\in P_B\bar{\mathcal{O}}_TP_B$$

and by Lemma 3.5 $\bar{\mathcal{O}}_T$ is generated by $\tilde{u}\tilde{r}P_{B_0}$ and $\bar{\mathcal{F}}_T$. Q.e.d.

Let $\mathcal{C}$ be a maximal commutative C^*-subalgebra of $\mathcal{K}$.

3.7. **Lemma.** *For all compact open sets $B\subset W$*

$$(\bar{\mathcal{O}}_T,\bar{\mathcal{D}})\sim(\mathcal{K}\otimes P_B\bar{\mathcal{O}}_TP_B,\mathcal{C}\otimes P_B\bar{\mathcal{D}}).$$

Proof. With $B_1 = B$ let $\{B_l \mid l \in \mathbb{N}\}$ be a partition of W into compact open sets such that

$$\delta_T(B_l) = \delta_T(B), \qquad (l \in \mathbb{N}).$$

Let

$$u_l \colon B \to B_l, \qquad (l \in \mathbb{N})$$

be uniformly finite dimensional homeomorphisms. One has then a system $E_{kl}(k, l \in \mathbb{N})$ of matrix units in $\bar{\mathcal{O}}_T$,

$$E_{l,l} = P_{B_l},$$
$$E_{l,1} = \tilde{u}_l, \qquad (l \in \mathbb{N})$$

and one can identify $\mathcal{K} \otimes P_B \bar{\mathcal{O}}_T P_B$ with the subalgebra of $\bar{\mathcal{O}}_T$ that is generated by $\{E_{l,1} \bar{\mathcal{O}}_T E_{1,1} \mid l \in \mathbb{N}\}$, at the same time identifying $\mathcal{C} \otimes P_B \mathcal{D}$ with the algebra that is generated by $\{E_{l,l} \mathcal{D} \mid l \in \mathbb{N}\}$. One has $\bar{\mathcal{F}}_T$ generated by the $E_{l,k}$ $(l, k \in \mathbb{N})$ and by $E_{1,1} \bar{\mathcal{F}}_T E_{1,1}$. Therefore it remains only to note that by Lemma 3.5 the algebra that is generated by the $E_{l,k}$ $(l, k \in \mathbb{N})$ and by $E_{1,1} \bar{\mathcal{O}}_T E_{1,1}$ is all of $\bar{\mathcal{O}}_T$. Q.e.d.

Given $a \in \mathcal{L}(x)$, $\mu = (i_1, \ldots, i_k) \in \mathcal{M}_A$, such that $A(a_0, i_1) = 1$ set

$$Z(a, \mu) = \{y \in Z(a) \mid (y_1, \ldots, y_k) = \mu\}.$$

Further if $a' \in \mathcal{L}(x)$ and $\nu = (j_1, \ldots, j_k)$ is a second block in $\mathcal{M}_A$ of length k such that $A(a'_0, j_1) = 1$ and $i_k = j_k$, let

$$u(a', \nu, a, \mu) \colon Z(a, \mu) \to Z(a', \nu)$$

be the uniformly finite dimensional homeomorphism that is defined by

$$(u(a', \nu, a, \mu)\, y)_l = y_l \qquad (l \geqq k,\ y \in Z(a, \mu)).$$

3.8. **Theorem.** *There is an isomorphism*

$$\psi \colon (\bar{\mathcal{O}}_T, \mathcal{D}) \to (\mathcal{K} \otimes \mathcal{O}_A, \mathcal{C} \otimes \mathcal{D}_A)$$

such that $\psi \bar{\lambda}_t^T \psi^{-1} = \mathrm{id} \otimes \lambda_t^A \qquad (t \in \mathbb{T}).$

Proof. For all $i \in \Sigma$ choose an $a(i) \in \mathcal{L}(x)$ such that $A(a(i)_0, i) = 1$ and set

$$P = \sum_{i \in \Sigma} P_{Z(a(i), i)}.$$

To obtain $\bar{\mathcal{O}}_T$ use an $r \in \mathcal{R}_T(x)$ such that $(r\, y)_k = y_{k+1}$ $(k \in \mathbb{N})$. For all $i \in \Sigma$ and for all $j \in \Sigma$ such that $A(i, j) = 1$ choose an $a'_{ij} \in \mathcal{L}(x)$ such that $(a'_{ij})_0 = i$, and then let $a''_{ij} \in \mathscr{L}(x)$ be such that $A((a''_{ij})_0, i)=1$,

$$r^{-1} Z(a'_{ij}, j) = Z(a''_{ij}, (i,j)).$$

Then

$$u(a(i),(i,j);\, a''_{ij}, (i,j))\, r^{-1}\, u(a'_{ij}, j;\, a(j), j)\, Z(a(j), j) = Z(a(i), (i,j)).$$

Setting

$$S_i = \sum_{\{j \mid A(i,j)=1\}} \tilde{u}(a(i),(i,j);\, a''_{ij}(i,j))\, \tilde{r}^{-1}\, \tilde{u}(a'_{ij}, i;\, a(j), j)$$

we obtain therefore partial isometries $S_i \in P\bar{\mathcal{O}}_T P$ that satisfy the relations (A). Moreover, one computes that one has for $\mu=(i_1,\dots,i_k)$, $\nu=(j_1,\dots,j_k) \in \mathscr{M}_A$, $i_k = j_k$ that

$$S_\mu S_\nu^* = \tilde{u}(a(i_1), \mu;\, a(j_1), \nu)$$

and it follows that the S_i generate an algebra that contains $P\bar{\mathscr{F}}_T P$. The proof is then concluded by appealing to Lemma 3.5 and by noting that

$$\bar{\lambda}_t^T S_i = t S_i, \qquad (i \in \Sigma,\ t \in \mathbb{T}). \quad \text{Q.e.d.}$$

3.9. **Remark.** For an irreducible topological Markov chain T with period $p>0$ use a union

$$\bigcup_{0 \leq q < p} W(\bar{\sigma}_A^q x)\,(x \in \bar{X}_A)$$

of unstable manifolds, and consider the group $\mathscr{G}_T$ of uniformly finite dimensional homeomorphisms together with $\mathscr{R}_T$ on this union. Denote by $(T^p)_q$ the irreducible components of T^p. One has

$$\bar{\mathscr{F}}_T = \bigoplus_{0 \leq q < p} \bar{\mathscr{F}}_{(T^p)_q}, \qquad \bar{\mathscr{F}}_{(T^p)_q} \cong \bar{\mathscr{F}}_{(T^p)_q}, \qquad 1 \leq q < p$$

where the summands arise from the group $\mathscr{G}_{T^p}$ restricted to the $W(\bar{\sigma}_A^q x)$, $0 \leq q < p$. To obtain $\bar{\mathcal{O}}_T$ one can use an $r \in \mathscr{R}_T$ such that r on the $W(\bar{\sigma}_A^q x)$, $0 \leq q < p-1$, equals $\bar{\sigma}_A$ and such that r^p restricted to $W(x)$ gives an $r_0 \in \mathscr{R}_{(T^p)_0}$. Then there is an isomorphism of $\bar{\mathcal{O}}_T$ onto $M_p \otimes \bar{\mathcal{O}}_{(T^p)_0}$ that carries $\tilde{r}$ into

$$\Big(\sum_{0 \leq q < p-1} E_{q+1,q} \otimes 1\Big) + E_{0,p-1} \otimes \tilde{r}_0$$

and transforms the group $(\bar{\lambda}_t^T)_{t \in \mathbb{T}}$ into the group $(\lambda_t)_{t \in \mathbb{T}}$ where

$$\lambda_t(E_{q+1,q} \otimes X) = t(E_{q+1,q} \otimes X) \qquad (0 \leq q < p-1),$$

$$\lambda_t(E_{0,p-1} \otimes \tilde{r}_0 X) = t(E_{0,p-1} \otimes \tilde{r}_0 X) \qquad (X \in \bar{\mathscr{F}}_{(T^p)_0}).$$

Thus while $\bar{\mathcal{O}}_T$ is isomorphic to $\bar{\mathcal{O}}_{(T^p)_0}$, the groups $(\bar{\lambda}_t^T)_{t \in T}$ and $(\bar{\lambda}_t^{(T^p)_0})_{t \in \mathbb{T}}$ are different.

4. Flow Equivalence

Topological Markov chains are said to be flow equivalent if their suspension flows act on spaces that are homeomorphic under homeomorphisms that respect the orientation of the orbits [11]. Equivalently they are flow equivalent if they induce isomorphic chains on some closed open subset, that is, if they are Kakutani equivalent. Parry and Sullivan have given a description of flow equivalence in terms of a matrix operation [11]. This description leads to a sort of instant computational proof of the invariance of the pair $(\bar{\mathcal{O}}_T, \bar{\mathcal{D}})$ under flow equivalence. We want to give this proof here. We point out, however, that a conceptual proof of this fact is also possible if one exploits the circumstance that $\bar{\mathcal{O}}_T$ arises as a crossed product.

4.1. **Theorem.** *If T_1 and T_2 are flow equivalent then*

$$(\bar{\mathcal{O}}_{T_1}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{T_2}, \bar{\mathcal{D}}).$$

Proof. From the transition matrix $A=(a_{ij})_{1 \leqq i,j \leqq n}$, form the transition matrix

$$\tilde{A} = \begin{pmatrix} 0 & a_{11} & \dots & a_{1n} \\ 1 & 0 & \dots & 0 \\ 0 & a_{21} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

According to Parry and Sullivan, to prove the theorem it is enough to prove that

$$(\bar{\mathcal{O}}_{\sigma_A}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{\sigma_{\tilde{A}}}, \bar{\mathcal{D}}).$$

The algebra $\mathcal{O}_{\tilde{A}}$ is generated by $n+1$ partial isometries $S_0, \dots, S_n$ satisfying $(\tilde{A})$. By definition of $\tilde{A}$ the partial isometries $S'_1 = S_1 S_0$, $S'_2 = S_2, \dots, S'_n = S_n$ satisfy (A). Note that $S_i S_0 \neq 0$ if and only if $i=1$ and that $S_1 S_j \neq 0$ if and only if $j=0$. Thus every $S_\mu (\mu \in \mathcal{M}_{\tilde{A}})$ is of the form $S_\mu = S'_\alpha$ for some $\alpha \in \mathcal{M}_A$ or of the form $S_\mu = S_0 S'_\beta$ for some $\beta \in \mathcal{M}_A$ or it is $S_\mu = S_1$.

Set $P = S_1 S_1^* + \dots + S_n S_n^*$. If $P S_\mu S_\nu^* P \neq 0$ for some $\mu, \nu \in \mathcal{M}_{\tilde{A}}$, then using $S_1 S_1^* = S_1 S_0 S_0^* S_1 = S'_1 S'^*_1$ we see that $S_\mu = S'_\alpha$, $S_\nu = S'_\beta$ for some $\alpha, \beta \in \mathcal{M}_A$. This shows that $P \mathcal{O}_{\tilde{A}} P$ is generated by $S'_1, \dots, S'_n$, and thus is isomorphic to $\mathcal{O}_A$. Since for every range projection $S_\mu S_\mu^*$, $(\mu \in \mathcal{M}_{\tilde{A}})$ the product $S_\mu S_\mu^* P$ is either 0 or of the form $S'_\alpha S'^*_\alpha$ $(\alpha \in \mathcal{M}_A)$, we see at the same time that $\mathcal{D}_{\tilde{A}} P = \mathcal{D}_A$. The theorem follows now from Lemma 3.7. Q.e.d.

5. The Ext-Group for $\mathcal{O}_A$

Let H be a separable infinite-dimensional Hilbert space, let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H, $\mathcal{K}(H) \subset \mathcal{L}(H)$ the algebra of compact operators and let $\pi: \mathcal{L}(H) \to \mathcal{Q}$ be the quotient map onto the Calkin algebra $\mathcal{Q} = \mathcal{L}(H)/\mathcal{K}(H)$. An extension of a separable C^*-algebra $\mathcal{A}$ is a star monomorphism $\sigma: \mathcal{A} \to \mathcal{Q}$. Two extensions ρ, σ are called weakly equivalent, if there is a partial isometry $U \in \mathcal{Q}$ such that $\rho(X) = U\sigma(X)U^*$ and $\sigma(X) = U^* p(X) U$ $(X \in \mathcal{A})$. The set of weak equivalence classes of extensions is denoted by Ext $\mathcal{A}$.

On Ext $\mathcal{A}$ one defines a semigroup structure as follows. Call two extensions ρ, σ orthogonal ($\rho \perp \sigma$) if there are projections $E, F \in \mathcal{Q}$ such that $EF = 0$ and $\rho(\mathcal{A}) \subset E\mathcal{Q}E$, $\sigma(\mathcal{A}) \subset F\mathcal{Q}F$. If $[\rho], [\sigma]$ are in Ext $\mathcal{A}$ define $[\rho] + [\sigma] = [\rho_1 + \sigma_1]$ where $\rho_1 \in [\rho]$, $\sigma_1 \in [\sigma]$ and $\rho_1 \perp \sigma_1$. (Here $\rho_1 + \sigma_1$ is defined by $(\rho_1 + \sigma_1)(X) = \rho_1(X) + \sigma_1(X)$ $(X \in \mathcal{A})$.) Note that the definition of $[\rho] + [\sigma]$ does not depend on the choice of ρ_1, σ_1.

An extension $\tau: \mathcal{A} \to \mathcal{Q}$ is called trivial if it admits a lifting $\tilde{\tau}$, i.e. a star monomorphism $\tilde{\tau}: \mathcal{A} \to \mathcal{L}(H)$ such that $\tau = \pi \circ \tilde{\tau}$. Voiculescu's theorem ([17] cf. also [1]) says that all trivial extensions are equivalent and that their equivalence class is the neutral element in Ext $\mathcal{A}$. Ext $\mathcal{O}_n$ has been computed by Pimsner-Popa [15] and by Paschke-Salinas [13]. Here we follow the approach of [15].

Let $E \in \mathcal{Q}$ be a projection and $E' \in \mathcal{L}(H)$ a projection such that $\pi(E') = E$. If X is an element of $\mathcal{Q}$ such that EXE is invertible in $E\mathcal{Q}E$ we denote by $\operatorname{ind}_E X$ the Fredholm index of $E'X'E'$ in $E'(H)$ where $X' \in \mathcal{L}(H)$ is such that $\pi(X') = X$. Since the Fredholm index is invariant under compact perturbations, this definition does not depend on the choice of E' and X'. The following lemma we assume as well known.

5.1. **Lemma.** (a) *Let $E, F \in \mathcal{Q}$ be orthogonal projections, and X an element of $\mathcal{Q}$ such that EXE and FXF are invertible in $E\mathcal{Q}E$ and $F\mathcal{Q}F$ and such that X commutes with E and F. Then* $\operatorname{ind}_{E+F}(X) = \operatorname{ind}_E(X) + \operatorname{ind}_F(X)$

(b) *If $X, Y \in E\mathcal{Q}E$ are invertible in $E\mathcal{Q}E$, then*

$$\operatorname{ind}_E XY = \operatorname{ind}_E X + \operatorname{ind}_E Y.$$

Let now $A = (a_{ij})_{1 \leq i,j \leq n}$, $a_{ij} \in \{0, 1\}$, satisfy condition (I) and let $\sigma: \mathcal{O}_A \to \mathcal{Q}$ be an extension of $\mathcal{O}_A$ and $E_i = \sigma(P_i)$. There are trivial extensions τ of $\mathcal{O}_A$ such that $\tau(P_i) = E_i$. Define $d_i = \operatorname{ind}_{E_i} \sigma(S_i)\tau(S_i^*)$ $(i = 1, \ldots, n)$ and $d_{\sigma,\tau} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$.

5.2. **Proposition.** *If τ, τ' are trivial extensions of $\mathcal{O}_A$ satisfying $\tau(P_i) = \tau'(P_i) = E_i$, then $d_{\sigma,\tau} - d_{\sigma,\tau'} \in (1 - A)(\mathbb{Z}^n)$.*

Proof. By Voiculescu's theorem there is a partial isometry $U \in \mathcal{Q}$ such that $\tau'(X) = U\tau(X)U^*$ and $\tau(X) = U^*\tau'(X)U$ $(X \in \mathcal{O}_A)$.

Write $k_i = \operatorname{ind}_{E_i} U$ and $d_{\sigma,\tau'} = (d'_1, \ldots, d'_n)$. Then

$$\begin{aligned} d'_i &= \operatorname{ind}_{E_i} \sigma(S_i)\tau'(S_i^*) \\ &= \operatorname{ind}_{E_i} \sigma(S_i) U \tau(S_i^*) U^* \\ &= \operatorname{ind}_{E_i} \sigma(S_i) \left(\sum_{j=1}^{n} a_{ij} E_j U E_j\right) \tau(S_i^*)(E_i U^* E_i) \\ &= \operatorname{ind}_{E_i} \sigma(S_i)\tau(s_i^*) \left\{\tau(S_i) \sum_{j=1}^{n} a_{ij} E_j U E_j \tau(S_i^*)\right\} E_i U^* E_i \\ &= d_i - \left(k_i - \sum_{j=1}^{n} a_{ij} k_j\right). \end{aligned}$$

This computation uses the fact that U commutes with every E_i, and Lemma 5.1. Q.e.d.

Thus with every extension σ of $\mathcal{O}_A$ we can, in a unique way, associate an element d_σ of the quotient group $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$. If ρ, σ are two equivalent extensions of $\mathcal{O}_A$, then $d_\rho = d_\sigma$. Moreover, Lemma 5.1 (a) shows that the map $[\sigma] \mapsto d_\sigma$ is additive.

5.3. **Theorem.** d: Ext $\mathcal{O}_A \to \mathbb{Z}^n/(1-A)\mathbb{Z}^n$ *is an isomorphism.*

Proof. One checks that d is surjective. Let us show that d is also injective.

Let σ be an extension such that $d_\sigma = 0$. Let τ be a trivial extension of $\mathcal{O}_A$ satisfying $\tau(P_i) = \sigma(P_i)$ and let $d_{\sigma,\tau} = (d_1, \ldots, d_n)$ be defined as above. Write $E_i = \tau(P_i)$ and $E_i' = \tilde{\tau}(P_i)$ $(i = 1, \ldots, n)$ where $\tilde{\tau}$ is a lifting for τ. By assumption, there is an element $k = (k_1, \ldots, k_n)$ of $\mathbb{Z}^n$ such that $d_{\sigma,\tau} = (1-A)k$.

Choose isometries or coisometries V_i $(i = 1, \ldots, n)$ in $\mathcal{L}(E_i'(H))$ such that ind $V_i = -k_i$ and set $U = \sum_{i=1}^n V_i$. Then one has, using Lemma 5.1,

$$\begin{aligned}
&\operatorname{ind}_{E_i} \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*) \\
&= \operatorname{ind}_{E_i} \pi(V_i)\sigma(S_i)\pi\left(\sum_{j=1}^n a_{ij} V_j^*\right)\tau(S_i^*) \\
&= \operatorname{ind}_{E_i} \pi(V_i)\sigma(S_i)\tau(S_i^*)\left\{\tau(S_i)\pi\left(\sum_{j=1}^n a_{ij} V_j^*\right)\tau(S_i^*)\right\} \\
&= d_i - \left(k_i + \sum_{j=1}^n a_{ij} k_j\right) = 0.
\end{aligned}$$

Therefore there is a unitary $X_i \in \mathcal{L}(E_i'(H))$ such that $\pi(X_i) = \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*)$. Setting $T_i = X_i \tilde{\tau}(S_i)$, we have lifted each element $\pi(U)\sigma(S_i)\pi(U^*)$ to a partial isometry $T_i \in \mathcal{L}(H)$ satisfying $T_i T_i^* = E_i'$ and $T_i^* T_i = \sum_{j=1}^n a_{ij} E_j'$. Now Theorem 2.13 shows that the map $S_i \mapsto T_i$ extends to a star monomorhism from $\mathcal{O}_A$ into $\mathcal{L}(H)$, and therefore the map $S_i \mapsto \pi(T_i) = \pi(U)\sigma(S_i)\pi(U^*)$ extends to a trivial extension of $\mathcal{O}_A$. In other words σ is equivalent to a trivial extension and hence itself trivial. Q.e.d.

By the Elementarteilersatz (see e.g. [16, § 85]) the endomorphism $B = 1 - A$ of $\mathbb{Z}^n$ can be written in the form $B = JB'$ where J is an isomorphism of $\mathbb{Z}^n$, and where B' has a diagonal matrix with entries in $\mathbb{Z}_+$ with respect to some basis of $\mathbb{Z}^n$. If the eigenvalues of B' are $b_1, \ldots, b_n$ then

$$\mathbb{Z}^n/(1-A)\mathbb{Z}^n \cong \mathbb{Z}^n/B'\mathbb{Z}^n \cong \mathbb{Z}/b_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/b_n\mathbb{Z}.$$

In particular, if $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ is finite, its order is $\det B' = |\det(1-A)|$. Also the group $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ is infinite if and only if $\det(1-A) = 0$.

Remark 3.4. Ext being a stable isomorphism invariant [4] we have now examples of simple C^*-algebras that are not stably isomorphic to any of the $\mathcal{O}_n$. Look at a handful of irreducible 3×3 matrices:

$$A_1=\begin{pmatrix}0&0&1\\1&0&1\\1&1&1\end{pmatrix},\quad \operatorname{Ext}\mathcal{O}_{A_1}=\mathbb{Z}_3,$$

$$A_2=\begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix},\quad \operatorname{Ext}\mathcal{O}_{A_2}=\mathbb{Z}_4,$$

$$A_3=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix},\quad \operatorname{Ext}\mathcal{O}_{A_3}=\mathbb{Z}_2\oplus\mathbb{Z}_2,$$

$$A_4=\begin{pmatrix}1&0&1\\0&1&1\\1&1&1\end{pmatrix},\quad \operatorname{Ext}\mathcal{O}_{A_4}=\mathbb{Z}.$$

$\mathcal{O}_{A_1}$ is isomorphic to $\mathcal{O}_4$ and $\mathcal{O}_{A_2}$ is isomorphic to $M_2\otimes\mathcal{O}_5$, but neither $\mathcal{O}_{A_3}$ nor $\mathcal{O}_{A_4}$ is stably isomorphic to any of the $\mathcal{O}_n$.

That $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ is an invariant of flow equivalence of topological Markov chains $\bar{\sigma}_A$ was discovered by Bowen and Franks [3]. Compare here Theorems 4.1 and 5.3.

References

1. Arveson, W.: Notes on extensions of C^*-algebras. Duke Math. J. **44**, 329–355 (1977)
2. Bratteli, O.: Inductive limits of finite-dimensional C^*-algebras. Trans. A.M.S. **171**, 195–234 (1972)
3. Bowen, R., Franks, J.: Homology for zero-dimensional nonwandering sets. Ann. Math. **106**, 73–92 (1977)
4. Brown, L.G.: Stable isomorphism of hereditary subalgebras of C^*-algebras. Pac. J. Math. **71**, 335–348 (1977)
5. Cuntz, J.: Simple C^*-algebras generated by isometries. Commun. Math. Phys. **57**, 173–185 (1977)
6. Cuntz, J.: Automorphisms of certain simple C^*-algebras. Proc. of "Bielefeld Encounters in Mathematics and Physics II" in press (1979)
7. Denker, M., Grillenberger, Ch., Sigmund, K.: Ergodic Theory on compact spaces. Lecture Notes in Mathematics, Vol. 527. Berlin-Heidelberg-New York: Springer 1976
8. Elliott, G.A.: On the classification of inductive limits of sequences of semi-simple finite-dimensional algebras. J. Algebra **38**, 29–44 (1976)
9. Krieger, W.: On topological Markov chains, Dynamical Systems II. Warsaw June 27–July 2, 1977, Soc. Math. de France, Astérisque **50**, 193–196 (1977)
10. Krieger, W.: On a dimension for a class of homeomorphism groups. Preprint
11. Krieger, W.: On dimension functions and topological Markov chains. Invent. math. **56**, 239–250 (1980)
12. Parry, B., Sullivan, D.: A topological invariant of flows on O-dimensional spaces. Topology **14**, 297–299 (1975)
13. Paschke, W., Salinas, N.: Matrix algebras over $\mathcal{O}_n$. Preprint
14. Pedersen, G.K.: C^*-algebras and their automorphism groups. London, New York, San Francisco: Academic Press 1979
15. Pimsner, M., Popa, S.: The Ext-groups of some C^*-algebras considered by J. Cuntz. Rev. Roum. Math. Pures et Appl. **23**, 1096–1076 (1978)
16. van der Waerden, B.L.: Algebra, Zweiter Teil, 5. Aufl. Berlin-Heidelberg-New York: Springer 1967
17. Voiculescu, D.: A non-commutative Weyl-von Neumann theorem. Rev. Roum. Pures et appl. **21**, 97–113 (1976)
18. Williams, R.F.: Classification of subshifts of finite type. Ann. of Math. **98**, 120–153 (1973) Errata, Ann. of Math. **99**, 380–381 (1974)

Received October 15, 1979