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Reviewed work(s):
Source: *The American Mathematical Monthly*, Vol. 81, No. 7 (Aug. - Sep., 1974), pp. 717-723
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2319559
Accessed: 20/08/2012 06:16

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THE SHIFT OPERATOR

P. A. FILLMORE

1. Introduction. By a (simple, unilateral) **shift operator** we understand a bounded linear transformation S on a separable complex Hilbert space $\mathscr{H}$ for which there exists an orthonormal basis $e_0, e_1, \cdots$ of $\mathscr{H}$ such that $Se_n = e_{n+1}$ for all $n \geqq 0$. Any two shift operators $S: \mathscr{H} \to \mathscr{H}$ and $S': \mathscr{H}' \to \mathscr{H}'$ are unitarily equivalent. If $\{e_n\}$ and $\{e_n'\}$ are the corresponding orthonormal bases, the equations $We_n = e_n'$ for all $n \geqq 0$ determine an isomorphism $W: \mathscr{H} \to \mathscr{H}'$ such that $WS = S'W$. There are two realizations of the shift operator on concrete Hilbert spaces that are particularly useful. The first, on the space l^2 of square-summable sequences of complex numbers, is defined by

$$S(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots).$$

The other is on the Hardy space H^2, consisting of all measurable complex functions f on the unit circle that are square-integrable with respect to normalized Lebesgue measure and whose Fourier coefficients of negative index all vanish:

$$\int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta = 0, \qquad n \geqq 1.$$

Here the shift operator appears as

$$(Sf)(e^{i\theta}) = e^{i\theta}f(e^{i\theta}),$$

and the corresponding orthonormal basis consists of the functions e^n, $n \geqq 0$, where e is the identity function $e^{i\theta}$.

The shift operator has been known for many years, at first as an interesting example, but more recently as a fundamental building block in the structure theory of operators on Hilbert space. The purpose of the present note is to make more widely known the modern role of the shift operator.

2. Characterizations. The shift operator evidently has the following properties:

(i) it is an **isometry:** $\|Sf\| = \|f\|$ for all $f \in \mathscr{H}$,

(ii) it is **pure:** $\bigcap_{n=0}^{\infty} S^n\mathscr{H} = \{0\}$.

Is every pure isometry a shift? For example, consider the operator T determined by the mapping $Te_n = e_{2n}$ on an orthonormal basis $\{e_n \mid n \geqq 1\}$. This is a pure isometry, but not a shift. However, it is a **direct sum of shifts** in the following sense: for each odd integer $k \geqq 1$, let $\mathscr{M}_k$ be the subspace spanned by $e_k, e_{2k}, e_{4k}, \cdots$; then these subspaces are mutually orthogonal and span the whole space, and in each, T is a shift.

THEOREM 1. *Any pure isometry is a direct sum of simple shifts.*

Proof. Let V be a pure isometry on $\mathscr{H}$, and let $\mathscr{K} = (V\mathscr{H})^{\perp}$, the orthogonal complement of the range of V. The subspaces $V\mathscr{K}$, $V^2\mathscr{K}, \cdots$ are contained in the range of V, and thus are orthogonal to $\mathscr{K}$. Any isometry has the property

$$(Vf, Vg) = (f, g), \; f, \; g \in \mathscr{H}$$

and it follows that the subspaces $\mathscr{K}, V\mathscr{K}, V^2\mathscr{K}, \cdots$ are mutually orthogonal. Moreover these subspaces span $\mathscr{H}$; indeed, $\mathscr{H}$ is spanned by the subspaces

$$\mathscr{K}, \; V\mathscr{K}, \cdots, V^{n-1}\mathscr{K} \text{ and } V^n\mathscr{H}$$

for every $n \geqq 1$, and therefore by the subspaces $\mathscr{K}$, $V\mathscr{K}$, $V^2\mathscr{K}, \cdots$ and $\bigcap_{n=0}^{\infty} V^n\mathscr{H} = \{0\}$. Now let $\{e_\alpha \mid \alpha \in A\}$ be an orthonormal basis of $\mathscr{K}$, and for each $\alpha \in A$ let $\mathscr{M}_\alpha$ be the subspace spanned by the orthonormal set $\{e_\alpha, Ve_\alpha, V^2e_\alpha, \cdots\}$. As in the example, these subspaces are mutually orthogonal and span $\mathscr{H}$, and in each, V is a shift. □

On the other hand, any pure isometry may be regarded as a shift of a suitably general type. To make this precise, let $\mathscr{K}$ be any Hilbert space, and let $l^2(\mathscr{K})$ be the Hilbert space of norm-square-summable sequences of vectors from $\mathscr{K}$. The **shift operator** on $l^2(\mathscr{K})$ is defined in the same fashion as the shift on l^2.

THEOREM 2. *Any pure isometry V on a Hilbert space $\mathscr{H}$ is unitarily equivalent to the shift operator on $l^2(\mathscr{K})$, where $\mathscr{K} = (V\mathscr{H})^\perp$.*

Proof. It was shown above that the subspaces $\mathscr{K}$, $V\mathscr{K}$, $V^2\mathscr{K}, \cdots$ are mutually orthogonal and span $\mathscr{H}$. It follows that the map

$$W: (k_0, k_1, k_2, \cdots) \to \sum_{n=0}^{\infty} V^n k_n$$

is an isomorphism of $l^2(\mathscr{K})$ with $\mathscr{H}$. Since

$$\begin{aligned} WS(k_0, k_1, \cdots) &= W(0, k_0, k_1, \cdots) \\ &= \sum_{n=0}^{\infty} V^{n+1}k_n = VW(k_0, k_1, \cdots) \end{aligned}$$

for all $(k_0, k_1, \cdots) \in l^2(\mathscr{K})$, we have $WS = VW$ as required. □

The next result describes the structure of arbitrary isometries. It was discovered by von Neumann [7] in the course of investigating extensions of symmetric operators (as will be explained in the next section). Recall that a **unitary operator** is an isometry of a Hilbert space onto itself. The structure of unitary operators is completely described by the spectral theorem [**1**, §62].

THEOREM 3. *Any isometry is uniquely the direct sum of a pure isometry and a unitary operator.*

Proof. Let V be an isometry on $\mathscr{H}$. It must be shown that there is a unique subspace $\mathscr{M}$ of $\mathscr{H}$ with the properties $V\mathscr{M} \subset \mathscr{M}$, $V(\mathscr{M}^\perp) \subset \mathscr{M}^\perp$, $V \mid \mathscr{M}$ is unitary, and

$V|\mathcal{M}^\perp$ is a pure isometry. It is easy to see that $\mathcal{M} = \bigcap_{n=0}^\infty V^n\mathcal{H}$ is the only possibility for such a subspace. To see that this one works, observe that

$$\mathcal{H} \supset V\mathcal{H} \supset V^2\mathcal{H} \supset \cdots$$

so that $V\mathcal{M} = \mathcal{M}$, and that from this the first three requirements follow. The last one results from

$$\bigcap_{n=0}^\infty V^n(\mathcal{M}^\perp) \subset \bigcap_{n=0}^\infty V^n\mathcal{H} = \mathcal{M}.$$

3. Symmetric operators. A **symmetric operator** on a Hilbert space $\mathcal{H}$ is a linear transformation A, defined on a dense linear man fold $\mathcal{D}_A$ in $\mathcal{H}$, such that $(Af, g) = (f, Ag)$ for all $f, g \in \mathcal{D}_A$. Such operators need not be bounded; the usual substitute is the requirement that A be closed (i.e., that the graph of A be closed in the Cartesian product $\mathcal{H} \times \mathcal{H}$). Symmetric operators arise naturally in the study of differential equations. A useful example is the following: let $\mathcal{H} = L^2(0, \infty)$, let $\mathcal{D}$ consist of those $f \in \mathcal{H}$ such that f is absolutely continuous, $f' \in L^2(0, \infty)$, and $f(0) = 0$, and let $Df = if'$ for all $f \in \mathcal{D}$. To see that D is symmetric, we need the fact that $f(t) \to 0$ as $t \to \infty$ for $f \in \mathcal{D}$. This follows from the formula

$$\int_0^t f\bar{f}' = |f(t)|^2 - \int_0^t f'\bar{f}$$

and the fact $|ff'|$ is integrable on $(0, \infty)$. Then

$$\begin{aligned}(Df, g) &= \lim_{t\to\infty} \int_0^t if'\bar{g} \\ &= \lim_{t\to\infty} \left(f(t)\overline{g(t)} - \int_0^t if\overline{g'}\right) \\ &= \lim_{t\to\infty} \int_0^t f(\overline{ig'}) = (f, Dg).\end{aligned}$$

It may also be shown that D is closed but not bounded.

In [7] von Neumann proved that any symmetric operator possesses maximal symmetric extensions, and described the structure of these extensions. We give a brief account of his reasoning. Let A be symmetric with domain $\mathcal{D}_A$. Then $A + iI$ is one-to-one on $\mathcal{D}_A$, so $V = (A - iI)(A + iI)^{-1}$ is a well-defined linear operator with domain $\mathcal{D}_V = (A + iI)\mathcal{D}_A$ and range $\mathcal{R}_V = (A - iI)\mathcal{D}_A$. This operator is called the **Cayley transform** of A. That this is an isometry follows from the easy relations:

$$\|(A + iI)f\|^2 = \|Af\|^2 + \|f\|^2 = \|(A - iI)f\|^2.$$

Not every (partially-defined) isometry arises in this way. In fact, with A and V as above we have

$$(I - V)\mathcal{D}_V = \mathcal{D}_A,$$

and so $(I - V)\mathscr{D}_V$ is dense. Conversely, if V is an isometry such that $(I - V)\mathscr{D}_V$ is dense, then $I - V$ is one-to-one on $\mathscr{D}_V$,

$$A = i(I + V)(I - V)^{-1}$$

defines a symmetric transformation on $\mathscr{D}_A = (I - V)\mathscr{D}_V$, and the Cayley transform of A is V.

The basis of von Neumann's argument is now clear: symmetric extensions of A correspond, via the Cayley transform, to isometric extensions of V. In particular, maximal symmetric operators correspond to maximal isometries. An isometry V is maximal if and only if either $\mathscr{D}_V = \mathscr{H}$ or $\mathscr{R}_V = \mathscr{H}$. Any isometry has such an extension, and therefore any symmetric operator has a maximal symmetric extension.

An interesting case occurs when both $\mathscr{D}_V$ and $\mathscr{R}_V$ are all of $\mathscr{H}$; i.e., V is unitary. In this case A is self-adjoint and is described by the spectral theorem [**1**, §66].

Now let A be maximal symmetric. Since $\mathscr{D}_V$ and $\mathscr{R}_V$ are interchanged when A is replaced by $-A$, it can be assumed that $\mathscr{D}_V = \mathscr{H}$, so that V is an isometry defined on all of $\mathscr{H}$. According to Theorems 1 and 3, V is a direct sum of a unitary operator and a number of copies of the simple shift. Hence A is a direct sum of a self-adjoint operator and a number of copies of the simple maximal symmetric operator (i.e., the operator with Cayley transform the simple shift).

To complete this discussion, we remark that the differential operator D introduced above is simple and maximal. In fact, if V is the Cayley transform of D and $h(t) = e^{-t}$, then $\{h, Vh, V^2h, \cdots\}$ is an orthonormal basis of $L^2(0, \infty)$, and consequently V is a simple shift [**1**, §82].

4. Models. A subspace $\mathscr{M}$ of a space $\mathscr{H}$ is **invariant** for a linear transformation T on $\mathscr{H}$ if $Tf \in \mathscr{M}$ for all $f \in \mathscr{M}$. One way to obtain new operators from old is by restricting to invariant subspaces. By a **part of an operator** T we shall mean a restriction of T to an invariant subspace.

A part of a pure isometry is itself a pure isometry, and therefore, by Theorem 2, a part of a shift is another shift. On the other hand, an astonishing variety of operators arise as parts of the adjoint of the shift. This situation, discovered by Rota [**9**], will now be described. Recall first that the **adjoint** of a bounded operator T on a Hilbert space $\mathscr{H}$ is the unique operator T^* satisfying

$$(Tf, g) = (f, T^*g), \; f, \; g \in \mathscr{H}.$$

The adjoint of the shift S on $l^2(\mathscr{H})$ is the **backward shift**, given by

$$S^*(f_0, f_1, f_2, \cdots) = (f_1, f_2, f_3, \cdots).$$

Let T be an operator on a Hilbert space $\mathscr{H}$, and consider the map

$$R: f \to (f, Tf, T^2f, \cdots), \qquad f \in \mathscr{H}.$$

We want this sequence to be in $l^2(\mathscr{H})$, and for this it is sufficient that T be a strict contraction (i.e., $\| T \| < 1$):

$$\sum_{n=0}^{\infty} \| T^n f \|^2 \leqq \sum_{n=0}^{\infty} \| T \|^{2n} \| f \|^2 = (1 - \| T \|^2)^{-1} \| f \|^2 < \infty.$$

Of course R is linear, and this computation also shows that it is bounded (by $(1 - \| T \|^2)^{-\frac{1}{2}}$). Moreover, it follows from the inequality

$$\| Rf \| \geqq \| f \|, \; f \in \mathscr{H}$$

that the range of R (call it $\mathscr{N}$) is a closed subspace of $l^2(\mathscr{H})$. Then the closed graph theorem implies that the inverse operator $R^{-1}: \mathscr{N} \to \mathscr{H}$ is bounded. Finally we have

$$\begin{aligned} RTf &= (Tf, T^2 f, T^3 f, \cdots) \\ &= S^*(f, Tf, T^2 f, \cdots) = S^* Rf \end{aligned}$$

for all f in $\mathscr{H}$. This says that R carries the action of T on $\mathscr{H}$ to that of S^* on $\mathscr{N}$. Two operators related in this fashion, by a bounded operator with a bounded inverse, are said to be **similar**. Thus we have shown that T and $S^* |\mathscr{N}$ are similar.

THEOREM 4. *Any strict contraction is similar to a part of a backward shift.*

This result has implications for the **invariant subspace problem**, which asks whether any bounded linear operator on a complex Hilbert space of dimension greater than 1 has a proper (different from $\{0\}$ and $\mathscr{H}$) invariant subspace, and which remains unsolved in spite of the efforts of many mathematicians. Since any bounded operator can be "scaled" so as to be a strict contraction, the theorem gives the following reformulation of the problem: are the minimal nonzero invariant subspaces of backward shifts one-dimensional? Invariant subspaces of shifts are considered in the next section.

Soon after Rota's result appeared, De Branges and Rovnyak [3] and Foias [4] noticed that a modification of his argument will produce a description, up to unitary equivalence, of all the parts of backward shifts. Backward shifts have the properties $\| S^* \| \leqq 1$ and $\| S^{*n} f \| \to 0$ as $n \to \infty$ for all f, as do all of their parts. These conditions are also sufficient.

THEOREM 5. *Any contraction with powers tending strongly to zero is unitarily equivalent to a part of a backward shift.*

Proof. Let T be such an operator. We want to duplicate the situation of the last proof, but with R replaced by an isometry. Now

$$\begin{aligned} ((I - T^*T)f, f) &= (f, f) - (f, T^*Tf) \\ &= \| f \|^2 - \| Tf \|^2 \geqq 0 \end{aligned}$$

since $\| T \| \leqq 1$. This means that the operator $I - T^*T$ is *positive*, and as such has a unique positive square root D [8]. Consider the map

$$W: f \to (Df, DTf, DT^2 f, \cdots), \qquad f \in \mathscr{H}.$$

Since

$$\begin{aligned}\|DT^nf\|^2 &= (T^nf, D^2T^nf)\\ &= (T^nf, T^nf) - (T^nf, (T^*T)T^nf)\\ &= \|T^nf\|^2 - \|T^{n+1}f\|^2,\end{aligned}$$

the series $\Sigma_{n=0}^{\infty} \| D\, T^nf \|^2$ is telescoping, and we have

$$\|Wf\|^2 = \|f\|^2 - \lim_{n\to\infty} \|T^nf\|^2 = \|f\|^2$$

since the powers of T tend to 0 strongly. Hence W is an isometry with range in $l^2(\mathscr{D})$, where $\mathscr{D}$ is the closure of the range of D. The rest of the argument is as before.

5. Invariant Subspaces. The result of the previous section makes the nature of the invariant subspaces of backward shifts a matter of great importance. For any operator T, the subspaces invariant for T^* are precisely the orthogonal complements of the subspaces invariant for T. Thus it will suffice to study shifts.

One of the few operators whose invariant subspace structure has been completely and satisfactorily described is the simple shift, in a fundamental paper of Beurling [**2**] (see also [**5**]). For this we use the realization of the shift as multiplication by e (the identity function $e^{i\theta}$) on the Hardy space H^2. To begin with, there are the obvious invariant subspaces e^nH^2, consisting of all e^nf, $f \in H^2$, and spanned by $\{e^n, e^{n+1}, \cdots\}$. More generally, if $\phi \in H^2$ is of unit modulus almost everywhere (such functions are said to be **inner**) then multiplication by ϕ is an isometry on H^2, so the range ϕH^2 is a closed subspace that is evidently invariant. Conversely:

THEOREM 6. *Any closed nonzero invariant subspace of the shift on H^2 is of the form ϕH^2 for a suitable inner function ϕ.*

Proof. Let $\mathscr{M}$ be invariant, and assume for the moment that the function 1 is not orthogonal to $\mathscr{M}$, so that the component ψ of 1 in $\mathscr{M}$ is not zero. Then $e^n\psi \in \mathscr{M}$ for all $n \geqq 1$, and therefore

$$\begin{aligned}0 = (e^n\psi, 1 - \psi) &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\psi(\theta)(1 - \overline{\psi(\theta)})\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\psi(\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}|\psi(\theta)|^2 d\theta\\ &= -\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}|\psi(\theta)|^2 d\theta\end{aligned}$$

since $\psi \in H^2$. By conjugation we obtain

$$\int_0^{2\pi} e^{in\theta}|\psi(\theta)|^2 d\theta = 0 \text{ for all } n \neq 0,$$

and so $|\psi(\theta)|$ is equal a.e. to a nonzero constant. Thus, a constant multiple of ψ is inner, and it will suffice to show $\mathcal{M} = \psi H^2$. It is clear that $\mathcal{M}$ contains ψH^2 (since $\mathcal{M}$ contains $e^n\psi$, $n \geqq 0$), so suppose that f is in $\mathcal{M}$ and orthogonal to ψH^2. Since $e^n\psi \in \psi H^2$ for all $n \geqq 0$, we have

$$0 = (f, e^n\psi) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\overline{\psi(\theta)}d\theta, \qquad n \geqq 0.$$

On the other hand, arguing with $(e^n f, 1 - \psi)$, just as in the first calculation above, gives

$$\int_0^{2\pi} e^{in\theta} f(\theta)\overline{\psi(\theta)}d\theta = 0, \qquad n \geqq 1.$$

Hence $f\bar{\psi} = 0$ a.e., and, since ψ has constant nonzero modulus, $f = 0$. Thus $\mathcal{M} = \psi H^2$.

Finally, if 1 is orthogonal to $\mathcal{M}$, then all functions in $\mathcal{M}$ have vanishing Fourier coefficient of order zero, so $\mathcal{M} = e\mathcal{N}$ with $\mathcal{N}$ closed and invariant. Since $\mathcal{M} \neq \{0\}$ there is a largest integer n for which $\mathcal{M} = e^n\mathcal{M}_0$ with $\mathcal{M}_0$ closed and invariant. Then by the above $\mathcal{M}_0 = \phi H^2$ with ϕ inner, so $\mathcal{M} = e^n\phi H^2$ and $e^n\phi$ is inner. $\square$

This representation of invariant subspaces in terms of inner functions is not unique; however, it is easily seen that if $\phi_1 H^2 = \phi_2 H^2$ with ϕ_1 and ϕ_2 inner, then ϕ_1/ϕ_2 is constant almost everywhere. A great deal is known about inner functions [**6**], and therefore the theorem is a useful tool for answering questions about invariant subspaces of the simple shift.

In particular, the question of the previous section can be shown to have an affirmative answer in this case: the minimal nonzero invariant subspaces of the simple backward shift are one-dimensional. The same is true of the backward shift on $l^2(\mathcal{H})$ for $\mathcal{H}$ finite-dimensional (see [**5**]), but the general question remains open.

Based on a talk at the Indiana sectional meeting, Franklin College, May 13, 1972.

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