Math. Ann. 304, 685–699 (1996)

Mathematische Annalen

A non-selfadjoint Russo-Dye Theorem

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Received: 26 August 1994 / Revised version: 12 April 1995

Mathematics Subject Classification (1991): 47D25

One of the well-known results in the theory of C^* algebras is the Russo-Dye Theorem [19]: given a C^* algebra $\mathscr{A}$, the closed convex hull of the unitary elements in $\mathscr{A}$ equals the closed unit ball of $\mathscr{A}$. This result was later refined by Gardner and reached its final form by Kadison and Pedersen; today it is known that every operator in a C^* algebra $\mathscr{A}$, whose norm is less than 1, is the average of unitaries from A. The Russo-Dye Theorem initiated the theory of unitary rank in selfadjoint operator algebras. If $\mathscr{A}$ is an operator algebra, the unitary rank of an element $A \in \mathscr{A}$ is defined as the smallest number for which there is a convex combination of unitaries from $\mathscr{A}$ of length $u(A)$ and equaling A. If no such decomposition exists (in particular if $\|A\| > 1$) we define $u(A) = \infty$.

The literature on unitary rank is vast. The earliest result is due to Murray and von Neumann who proved that any selfadjoint operator of norm 1 or less is the mean of two unitary operators ([12] p. 239, 1937). The first systematic study was given by R. Kadison and G. Pedersen [8] in 1984 (previous work in the field included contributions by Popa [15], Robertson [17], Gardner [6] and others). In 1986, C. Olsen and G. Pedersen [14] characterized all elements in a factor von Neumann algebra with finite unitary rank. In the general case of a C^*-algebra, a characterization was obtained by Rordam in his important paper [18]. For more details and further information on the theory of unitary rank we refer to the excellent articles of U. Haagerup [7] and M. Rordam [18].

In the first section of the present paper, we prove a Russo-Dye type Theorem for infinite multiplicity nest algebras. The techniques employed in the proof of our result are different from that of Gardner and Kadison-Pedersen. To our knowledge, this is the first result of this type, for non-selfadjoint operator algebras and clearly initiates the unitary rank theory for such algebras.

The results of the present paper were presented by the second named author during the AMS Regional Meeting at College Station (October 22, 1993)

Specifically, we prove that if $\mathcal{N}$ is a nest with no finite dimensional atoms then each element A of $\operatorname{Alg}\mathcal{N}$ is a mean of a finite number n of unitaries from $\operatorname{Alg}\mathcal{N}$, provided that $\|A\| < 1$. (We emphasize that if $\mathcal{N}$ has atoms of finite dimension, then a Russo-Dye type theorem may not be valid for $\operatorname{Alg}\mathcal{N}$; indeed, if the atoms for $\mathcal{N}$ are finite dimensional and ordered like the one-point compactification of $\mathbb{N}$, then all unitaries in $\operatorname{Alg}\mathcal{N}$ belong to $\mathcal{N}'$!). Our estimate for n depends on the distance from A to the surface of the unit ball; as expected, the nearer A is to the surface, the larger n must be, in general. A number of corollaries follow our main theorem; the most important of them shows that every element of an infinite multiplicity nest algebra $\operatorname{Alg}\mathcal{N}$ can be expressed as a sum of unitaries from $\operatorname{Alg}\mathcal{N}$.

In the second section of the paper, we show that a variety of operator algebras, associated with infinite multiplicity nest algebras, enjoy a Russo-Dye type theorem. Specifically, if $\mathcal{N}$ is a nest with no finite dimensional atoms and $\mathcal{R}$ any factor von Neumann algebra then every operator in $\operatorname{Alg}\mathcal{N} \otimes \mathcal{R}$, of norm less than 1, is a convex combination of unitaries from $\operatorname{Alg}\mathcal{N} \otimes \mathcal{R}$; a similar result is also valid for the quasitriangular algebra $\operatorname{Qtr}\mathcal{N}$.

Finally, in the last section of the paper we give counterexamples to show that, in general, an infinite multiplicity operator algebra does not necessarily enjoy a Russo-Dye type theorem. We also indicate some directions for future investigation.

At this point, we would like to establish some connections between our results and the classical theory of H^p spaces. For many years, the nest algebras, and in particular the lower triangular matrices, have been thought as a non-commutative analog of the Banach algebra H^∞. Actually, much of the development of the theory of nest algebras ought to this analogy; results like Carleson's corona theorem, Sarason's theorem on the closure of $H^\infty + C$ etc. have found the appropriate analog in the context of nest algebras (see [1], [5], [9]). The results of the present paper should be considered as non-commutative analogs of Marshall's theorem [11], i.e., the unit sphere of H^∞ is the norm closed convex hull of the inner functions. The reader may notice that a Russo-Dye theorem does not hold for the algebra of lower triangular (infinite) matrices. However, even in this particular case, we show that a non-commutative Marshall Theorem is indeed valid; we prove (Theorem 2.1) that every lower triangular contraction is the limit of convex combinations of lower triangular isometries. In addition, we characterize which nest algebras admit a non-commutative Marshall theorem.

Let us establish some notation and terminology. A *nest* $\mathcal{N}$ is a totally ordered collection of projections, acting on a separable Hilbert space $\mathcal{H}$, which is closed under the operations "intersection" and "closure of the union" (symb. $\vee$). An *interval* for $\mathcal{N}$ is any projection of the form $E - F$, where $E, F \in \mathcal{N}$ and $F \subset E$. A minimal interval for $\mathcal{N}$ is called an atom; if $\mathcal{N}$ contains no atoms then it is *continuous*. If a nest $\mathcal{N}$ is ordered like the two point compactification of $\mathbb{Z}$, it is called *$\mathbb{Z}$-ordered*. If E is any element of $\mathcal{N}$ then E_- is the *immediate predecessor* of E in $\mathcal{N}$; if such predecessor does not exist then $E_- = E$. Similarly, we define E_+ as the *immediate successor* of E, for any $E \in \mathcal{N}$. It is plain that $\mathcal{N}$ is continuous iff $E_- = E = E_+$, for all $E \in \mathcal{N}$.

If $\mathcal{N}$ is a nest then the *nest algebra* $\operatorname{Alg}\mathcal{N}$ consists of all operators in $B(\mathcal{H})$ leaving invariant each element of $\mathcal{N}$. In general, an operator algebra $\mathcal{A}$ is of *infinite multiplicity* if it is (not necessarily isometrically) isomorphic to $\mathcal{A} \otimes B(\mathcal{H})$; it turns out that a nest algebra $\operatorname{Alg}\mathcal{N}$ is of infinite multiplicity if all the atoms for $\mathcal{N}$ are of infinite dimension (see [4]). In particular, the algebra of any continuous nest is of infinite multiplicity.

The nest algebras form a well behaved class of non-selfadjoint operator algebras which are under investigation since the late sixties. They were introduced by J. Ringrose [16] as the infinite-dimensional analog of the $n \times n$ upper triangular matrices; from this point of view, our results are rather surprising. The monograph of K. Davidson [3] contains most of the fundamental results in the field and is strongly recommended as a reference.

1 The main results

This is the main body of the paper. We start with a few definitions which isolate a useful class of projections.

Definition 1. *Let $\mathcal{N}$ be any nest. A projection E is said to be compatible with $\mathcal{N}$ if either $E \in \mathcal{N}$ or one of the following occurs: $E \subseteq 0_+$, $I_- \subseteq E$.*

It is clear that if $\mathcal{N}$ is continuous then all projections compatible with $\mathcal{N}$, belong to $\mathcal{N}$.

Definition 2. *Let $\mathcal{N}$ be any nest. A projection $R \in \operatorname{Alg}\mathcal{N}$ is called sliced if there exist sequences $\{E_n\}_{n=-\infty}^{+\infty}$, $\{F_n\}_{n=-\infty}^{+\infty}$ of projections compatible with $\mathcal{N}$, which satisfy,*

(i) $$F_n \subset E_n \subset F_{n+1}\,,$$

$$\dim E_n - F_n = \dim F_{n+1} - E_n = \infty, \quad \forall n \in \mathbb{Z}\,,$$

(ii) $$\lim_{n\to-\infty} E_n = 0, \quad \lim_{n\to\infty} E_n = I\,,$$

so that

$$R = \sum_{n=-\infty}^{+\infty} E_n - F_n\,.$$

The intervals $E_n - F_n$, $n \in \mathbb{Z}$, are said to be the slices for R while the holes for R are the intervals $F_n - E_{n-1}$, $n \in \mathbb{Z}$.

In general, an operator $A \in \operatorname{Alg}\mathcal{N}$ is said to be sliced if $A = RAR'$, where R, R' are sliced projections in $\operatorname{Alg}\mathcal{N}$ which have at least one common hole P; we then call P a hole for A.

There are two attractive features for a sliced projection R. First, its complement $I-R$ is also a sliced projection. The second is described in the following lemma.

Lemma 3. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms and let R be a sliced projection in* $\operatorname{Alg}\mathcal{N}$. *Then, there exists an isometry $V \in \operatorname{Alg}\mathcal{N}$ so that $VV^* = R$.*

Proof. Let $\{E_n\}_{n=-\infty}^{+\infty}$, $\{F_n\}_{n=-\infty}^{+\infty}$ be as in the above definition. Let V_n be any partial isometry with initial space $E_{n+1} - E_n$ and final space $E_n - F_n$, $n \in \mathbb{Z}$. Clearly $V_n \in \operatorname{Alg}\mathcal{N}$ and so $V = \sum_{n=-\infty}^{+\infty} V_n$ is the desired isometry. □

It is easy to see that the previous lemma admits the following generalization, which we will use frequently in the paper. If R is a sliced projection and E any non-zero element of $\mathcal{N}$ then there exists an isometry $V \in \operatorname{Alg}\mathcal{N}$ so that $VV^* = ER$.

Definition 4. *Let $\mathcal{N}$ be any nest. An operator $A \in \operatorname{Alg}\mathcal{N}$ is called separated if there exist projections E, F, compatible with $\mathcal{N}$, so that $F \subset E$ and $A = FAF + (I-E)A(I-E)$; the pair (F,E) is said to be a separating pair for A.*

A few remarks are in order. It is clear that every separated operator is block diagonal. In general, if A is a sliced operator with a hole $E-F$ then its star diagram, with respect to the decomposition $F(\mathcal{H}) \oplus (E-F)(\mathcal{H}) \oplus (I-E)(\mathcal{H})$, is as in Fig. 1; if A is separated then its star is as in Fig. 2.

The proof of the main theorem (Theorem 11) follows from a series of propositions and lemmas. As we shall see, the class of sliced operators plays an important role in the proof. Indeed, Proposition 6 shows that every operator in the open unit ball of $\operatorname{Alg}\mathcal{N}$ is the mean of sliced operators; this allows us to focus our attention on this particular class of operators. We mention that the most delicate part of our proof is Proposition 10; it is a factorization result which shows that certain sliced contractions factor as the product of sliced and separated ones. Needless to say, sliced and separated operators are (relatively) easily seen to be means of unitaries.

$$\begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}$$

Fig. 1

$$\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}$$

Fig. 2

Lemma 5. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms and let a be an integer greater than 2. Then there exist families $\{R_1, R_2, \ldots, R_a\}$ and $\{R'_1, R'_2, \ldots, R'_{a+1}\}$ of sliced projections such that:*

(i) $\sum_{i=1}^{a} R_i = (a-1)I$, $\sum_{j=1}^{a+1} R'_j = aI$

(ii) *for each pair (i,j) $1 \leqq i \leqq a, 1 \leqq j \leqq a+1$, the projections R_i and R'_j have (infinitely many) common holes.*

Proof. Let $\{E_n\}_{n=-\infty}^{+\infty}$ be any increasing sequence of projections compatible with $\mathcal{N}$ so that $\lim_{n\to-\infty} E_n = 0$ and $\lim_{n\to\infty} E_n = I$. We define

$$R_i = \vee\{E_{n+1} - E_n | n \in \mathbb{Z}, n \not\equiv i \bmod a\}, \quad i = 1,2,\ldots,a$$

$$R'_j = \vee\{E_{n+1} - E_n | n \in \mathbb{Z}, n \not\equiv j \bmod (a+1)\}, \quad j = 1,2,\ldots,a, a+1 .$$

It is plain that $\sum R_i = (a-1)I$ and $\sum R'_j = aI$. Assume that i,j are given; then the projections

$$E_{n+1} - E_n, \quad n = i(a+1) - ak, \ k \equiv j \bmod (a+1)$$

are common holes for R_i, R'_j and the conclusion follows. □

Proposition 6. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms and let A be an operator in* $\operatorname{Alg}\mathcal{N}$. *If A has the property that $\|A\| < 1$, then A is a mean of sliced operators in* $\operatorname{Alg}\mathcal{N}$, *whose norms are less than* 1.

Proof. Let a be an integer, greater than 2, such that $\|A\| < \dfrac{a-1}{a+1}$. Let $\{R_1, R_2, \ldots, R_a\}$ and $\{R'_1, R'_2, \ldots, R'_{a+1}\}$ be as in Lemma 5 and let $A_{ij} = \dfrac{a+1}{a-1} R_i A R'_j$. The properties of $\{R_i\}_{i=1}^{a}, \{R'_j\}_{j=1}^{a+1}$ guarantee that each A_{ij} is a sliced contraction in $\operatorname{Alg}\mathcal{N}$ of norm strictly less than 1. Moreover,

$$\begin{aligned} \frac{1}{a(a+1)} \sum_{i=1}^{a} \sum_{j=1}^{a+1} A_{ij} &= \frac{1}{a(a-1)} \sum_{i=1}^{a} \sum_{j=1}^{a+1} R_i A R'_j \\ &= \frac{1}{a(a-1)} \left(\sum_{i=1}^{a} R_i \right) A \left(\sum_{j=1}^{a+1} R'_j \right) \\ &= A . \end{aligned}$$

which completes the proof. □

The following proposition (which we consider as a kind of "non-selfadjoint polar decomposition") is our main tool for constructing partial isometries with prescribed properties.

Proposition 7. (Larson, [10]) *Let $\mathcal{N}$ be any nest and let Y be an invertible operator in $\mathcal{B}(\mathcal{H})$. Then there exist an operator $X \in$* $\operatorname{Alg}\mathcal{N}$ *such that $X^*X = Y^*Y$.*

Lemma 8. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms, let R, R' be sliced projections in* $\operatorname{Alg}\mathcal{N}$ *and let A be an operator in* $\operatorname{Alg}\mathcal{N}$ *such that $\|A\| < 1$. Then:*

(i) *If $A = RAR'$ is sliced then A is a mean of two sliced partial isometries whose initial spaces are equal to R'.*

(ii) *If* $A = RAE$ *or* $A = E^{\perp}AR'$, *for some* $E \in \mathcal{N}|\{0, I\}$, *then* A *is a mean of four unitary operators from* $\operatorname{Alg}\mathcal{N}$.
(iii) *If* $A = RAR'$ *is sliced and* $I_- \neq I$, *then* A *is a mean of four unitary operators from* $\operatorname{Alg}\mathcal{N}$.

Proof. Let E_n, F_n and E'_n, F'_n be as in Definition 2 so that $R = \sum_{n=-\infty}^{+\infty} E_n - F_n$ and $R' = \sum_{n=-\infty}^{+\infty} E'_n - F'_n$.

(i) With no loss of generality we may assume that $F_1 - E_0 = F'_1 - E'_0$ and so $F_1 - E_0$ is a hole for A.

For each $n \in \mathbb{Z}\backslash\{1\}$, let P_n be a proper subinterval of $F_n - E_{n-1}$ and let $P = \sum_{\substack{n=-\infty \\ n \neq 1}}^{+\infty} P_n$; clearly P is a sliced projection in $\operatorname{Alg}\mathcal{N}$. Since $\|A\| < 1$, Proposition 7 shows that there exists an operator $X \in \operatorname{Alg}\mathcal{N}$ so that $X^*X = I - A^*A$. Let

$$V_i = A + (-1)^i VXR', \quad i = 1, 2,$$

where V is any isometry in $\operatorname{Alg}\mathcal{N}$ with final space P (Lemma 3). We observe that

$$\begin{aligned} V_i^* V_i &= A^*A + R'X^*V^*VXR' \\ &= A^*A + R'(I - A^*A)R' \\ &= R'A^*AR' + R'(I - A^*A)R' = R'. \end{aligned}$$

Thus, both V_i are partial isometries with initial spaces equal to R'. Their final spaces are contained in $P + R$ and, in addition, $V_i(F_1 - E_0) = (F_1 - E_0)V_i = 0$, which shows that each V_i is sliced. Finally

$$A = \tfrac{1}{2}(V_1 + V_2)$$

and the conclusion follows.

(ii) Assume that $A = RAE$, for some $E \in \mathcal{N}\backslash\{I\}$. Let V be an isometry in $\operatorname{Alg}\mathcal{N}$ such that $VV^* = E(I - R)$, (see the remarks following Lemma 3), and let X be an operator in $\operatorname{Alg}\mathcal{N}$ such that $X^*X = I - A^*A$. Let $V_i = A + (-1)^i VXE$, $i = 1, 2$. It is easily seen that each V_i is a partial isometry in $\operatorname{Alg}\mathcal{N}$ such that $V_i^*V_i = E$ and $V_iV_i^* \subseteq E$. For each $i = 1, 2$, let W_i be a partial isometry in $\operatorname{Alg}\mathcal{N}$ whose initial and final spaces are $E^{\perp}$ and $(V_iV_i^*)^{\perp}$ respectively (the fact that the range of V_i is contained in E is crucial for the existence of such W_i). We define $U_1 = V_1 + W_1$, $U_2 = V_1 - W_1$, $U_3 = V_2 + W_2$ and $U_4 = V_2 - W_2$. Each U_i is then a unitary operator in $\operatorname{Alg}\mathcal{N}$ and

$$\tfrac{1}{4}(U_1 + U_2 + U_3 + U_4) = \tfrac{1}{4}(2V_1 + 2V_2) = \tfrac{1}{2}(V_1 + V_2) = A.$$

If $A = E^{\perp}AR'$, for some $E \in \mathcal{N}\backslash\{0\}$, then the previous considerations show that A^* is a mean of four unitaries in $\operatorname{Alg}\mathcal{N}^{\perp}$ and the conclusion follows.

(iii) The proof follows from arguments similar to that of part (ii). □

Since every sliced and separated operator is the direct sum of operators which satisfy the requirements of Lemma 8(ii), we have

Corollary 9. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms and let A be a sliced and separated operator in* $\operatorname{Alg}\mathcal{N}$ *such that* $\|A\| < 1$. *Then A is a mean of four unitary operators from* $\operatorname{Alg}\mathcal{N}$.

We would like to comment on some technical difficulties arising from the use of Proposition 7 in constructing partial isometries. One might hope that the partial isometries constructed in Lemma 8(i) are easily "completed" to unitaries (i.e., they are a mean of two unitaries) since their ranges seem to be "broken up" into slices. This may not be as easy; their ranges are indeed contained in the range of a sliced projection but there is a possibility that they do not commute even with a single element of $\mathcal{N}$! (Notice that Proposition 7 gives no information on the range of the operator X).

In order to overcome this difficulty we adopt a different approach.

Proposition 10. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms, let E, F be projections in $\mathcal{N}|\{0,I\}$ so that $F \subset E$ and let A be a partial isometry in* $\operatorname{Alg}\mathcal{N}$ *such that $(E-F)A = A(E-F) = 0$. If $A^*A \in \mathcal{N}'$ then there exist separated partial isometries $B^{(1)}, B^{(2)}$ in* $\operatorname{Alg}\mathcal{N}$ *so that $A = B^{(1)}B^{(2)}$. If A is sliced and $E-F$ a hole for A, then $B^{(1)}$ and $B^{(2)}$ can be chosen sliced.*

Proof. Let E', F' be projections in $\mathcal{N}'$ such that $F \subset F' \subset E' \subset E$ and let V be a partial isometry in $\operatorname{Alg}\mathcal{N}$ so that $V^*V = E' - F'$ and $VV^* = F$.

Since $A^*A \in \mathcal{N}'$, the operators $AF, AE^\perp$ are partial isometries in $\operatorname{Alg}\mathcal{N}$ whose final spaces P, Q are mutually orthogonal (Caution: we do not claim that P or Q belong to $\operatorname{Alg}\mathcal{N}$. Notice, however that $P \subseteq F$ and so $P^\perp$ and F commute). Let,

$$B^{(1)} = AF + P^\perp V + E^\perp$$

$$B^{(2)} = F + (V^* + E^\perp)AE^\perp .$$

Since $P^\perp V = F(P^\perp V)F^\perp$ and $V^*AE^\perp = E(V^*A)E^\perp$, both $B^{(1)}$ and $B^{(2)}$ belong to $\operatorname{Alg}\mathcal{N}$. In addition, $V^* + E^\perp$ is a partial isometry with initial space $F + E^\perp$ and so the operators $B^{(1)}$ and $B^{(2)}$ are partial isometries with separating pairs (E', E) and (F, F') respectively. Finally,

$$\begin{aligned} B^{(1)}B^{(2)} &= AF + PVV^*AE^\perp + E^\perp AE^\perp \\ &= AF + P^\perp FAE^\perp + E^\perp AE^\perp \\ &= FAF + FP^\perp AE^\perp + E^\perp A \\ &= FAF + FAE^\perp + E^\perp A \qquad (\text{since, } P^\perp AE^\perp = AE^\perp) \\ &= A . \end{aligned}$$

If A is sliced (and $E-F$ a hole for A) then minor modifications on the definition of $B^{(1)}, B^{(2)}$ show that these operators can be chosen sliced. □

We are in position now to state and prove the main result of the paper.

Theorem 11. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms and let A be an operator in* Alg $\mathcal{N}$. *If A has the property that $\|A\| < \frac{a-1}{a+1}$, for an integer a greater than 2, then there are $16a(a+1)$ unitary operators $U_1, U_2, \ldots, U_{16a(a+1)}$ in* Alg $\mathcal{N}$ *so that*

$$A = \frac{1}{16a(a+1)}(U_1 + U_2 + \cdots + U_{16a(a+1)}).$$

Proof. Let θ be a positive number, greater than 1, such that $\theta^2\|A\| < \frac{a-1}{a+1}$. Proposition 6 shows now that there exist $a(a+1)$ sliced operators A_j, $j = 1, 2, \ldots, a(a+1)$ whose average equals $\theta^2 A$. Each A_j is the average of two sliced partial isometries with initial space in $\mathcal{N}'$ (Lemma 8(i)); hence, there exist $2a(a+1)$ sliced partial isometries V_i, $i = 1, 2, \ldots, 2a(a+1)$, such that $V_i^* V_i \in \mathcal{N}'$ and,

$$\theta^2 A = \frac{1}{2a(a+1)} \sum_{i=1}^{2a(a+1)} V_i .$$

We now distinguish two cases.

Case 1. $\mathcal{N}$ has no maximal element different from I (i.e. $I_- = I$).

If this is the case, our construction (Lemma 5) guarantees that each V_i has at least one hole whose endpoints belong to $\mathcal{N}$ and so it factors as $V_i = B_i^{(1)} B_i^{(2)}$ where both $B_i^{(1)}$ and $B_i^{(2)}$ are sliced and separated contractions in Alg $\mathcal{N}$ (Proposition 10). Thus,

$$A = \frac{1}{2a(a+1)} \sum_{i=1}^{2a(a+1)} (\theta^{-1} B_i^{(1)})(\theta^{-1} B_i^{(2)}) .$$

By Corollary 9, each $\theta^{-1} B_i^{(2)}$ is a mean of four unitaries $U_{i1}^{(2)}, U_{i2}^{(2)}, U_{i3}^{(2)}, U_{i4}^{(2)}$ from Alg $\mathcal{N}$. Although a similar statement is also valid for each $\theta^{-1} B_i^{(1)}$, actually more is true! Indeed, the reader will not find it difficult to verify that each $\theta^{-1} B_i^{(1)}$ can be expressed as a mean of (only) two unitary operators $U_{i1}^{(1)}, U_{i2}^{(1)}$. Hence,

$$\begin{aligned} A &= \frac{1}{2a(a+1)} \sum_{i=1}^{2a(a+1)} \left(\frac{1}{2} \sum_{m=1}^{2} U_{im}^{(1)}\right) \left(\frac{1}{4} \sum_{n=1}^{4} U_{in}^{(2)}\right) \\ &= \frac{1}{16a(a+1)} \sum_{i=1}^{2a(a+1)} \sum_{m=1}^{2} \sum_{n=1}^{4} U_{im}^{(1)} U_{in}^{(2)} . \end{aligned}$$

Since each $U_{im}^{(1)} U_{in}^{(2)}$ is a unitary, the proof of the theorem, in the first case, is complete.

Case 2. $\mathcal{N}$ has a maximal element different from I.

In this case, it is possible that all holes for A have endpoints which do not belong to $\mathcal{N}$ (for instance, when $\mathcal{N}$ has only three elements!) and so Proposition 10 does not apply. Instead, we use Lemma 8(iii). Indeed, each $\theta^{-2}V_i$ is the mean of four unitaries and so A is a mean of $8a(a+1)$ unitaries. □

An immediate corollary of Theorem 11 is that the norm closure of the unitaries in $\operatorname{Alg}\mathcal{N}$ equals the closed unit ball of $\operatorname{Alg}\mathcal{N}$; indeed, every contraction A in $\operatorname{Alg}\mathcal{N}$ is a norm limit of $(n-1/n+1)A$ and $(n-1/n+1)A$ is a mean of $16n(n+1)$ unitaries in $\operatorname{Alg}\mathcal{N}$. At the same time, we can apply Theorem 11 to produce a special representation of an element of an infinite multiplicity nest algebra.

Corollary 12. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms. Then every operator in* $\operatorname{Alg}\mathcal{N}$ *is a sum of unitary operators which belong to* $\operatorname{Alg}\mathcal{N}$.

Proof. If $T \in \operatorname{Alg}\mathcal{N}$ and $\|T\| < 32$ then $\|\frac{1}{96}T\| < \frac{1}{3} = \frac{2-1}{2+1}$ and so there exist 96 unitaries $U_1, U_2, \ldots, U_{96}$ such that $\frac{1}{96}T = \frac{1}{96}(U_1 + U_2 + \cdots + U_{96})$. Thus, $T = U_1 + U_2 + \cdots + U_{96}$.

If $\|T\| \geqq 32$, then T can be expressed as the sum of operators of norm less than 32 and the conclusion follows. □

As we have already mentioned, the estimate appearing in the statement of Theorem 11 depends on the norm of the operator A; the number of unitaries required in expressing A as a mean of unitaries grows as A gets nearer to the surface of the unit ball. This growth is not artificial; Kadison and Pedersen have shown [8] that if A is any non-unitary isometry then λA, $1-2(n-1)^{-1} < \lambda < 1$, $n \in \mathbb{N}$, cannot be written as a mean of less than n unitaries. However, for the attractive class of compact operators we can offer the following.

Corollary 13. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms and let K be a compact operator in* $\operatorname{Alg}\mathcal{N}$, *so that $\|K\| < 1$. Then K is the convex combination of* 32 *unitary operators, which belong to* $\operatorname{Alg}\mathcal{N}$.

Proof. Let θ be a positive number, smaller than 1, such that $\|K\| < (1-\theta)^2$. Choose $E \in \mathcal{N}$ such that $\|E^{\perp}K\| < \theta(1-\theta)$ and let $K_1 = (1-\theta)^{-1}EK$ and $K_2 = \theta^{-1}E^{\perp}K$. Then, each K_i satisfies $\|K_i\| < 1-\theta$ and also $K = (1-\theta)K_1 + \theta K_2$. Let R be any sliced projection with the property that

$$\max_{i=1,2} \{\|RK_i\|, \|K_iR\|\} < \frac{\theta}{3}.$$

We now show that each K_i is the convex combination of 16 unitaries. Indeed, let $K_{i1} = (1-\theta)^{-1}R^{\perp}K_iR^{\perp}$, $K_{i2} = \frac{3}{\theta}RK_iR^{\perp}$, $K_{i3} = \frac{3}{\theta}R^{\perp}K_iR$ and $K_{i4} = RK_iR$. Then

$$K_i = (1-\theta)K_{i1} + \sum_{j=2}^{4} \frac{\theta}{3}K_{ij}.$$

Each K_{ij} satisfies the requirements of Lemma 8(ii) and the proof of the Corollary is complete. □

Our main result translates into a statement concerning the extreme points of the unit ball of an infinite multiplicity nest algebra $\operatorname{Alg}\mathcal{N}$: the (norm) closed convex hull of the extreme points of $\operatorname{Alg}\mathcal{N}$ equals the closed unit ball of $\operatorname{Alg}\mathcal{N}$. This is clearly an improvement of the Krein-Milman Theorem for infinite multiplicity nest algebras.

The next result is another improvement of the Krein-Milman Theorem for nest algebras and also a non-selfadjoint analogue of classical results due to Nagy [13] and Conway-Scucs [2]: the w^*-closure of the extreme points equals the closed unit ball of $\operatorname{Alg}\mathcal{N}$. Actually more is true; every contraction in $\operatorname{Alg}\mathcal{N}$ is the w^*-limits of unitaries from $\operatorname{Alg}\mathcal{N}$.

We start with a definition.

An operator $A \in \operatorname{Alg}\mathcal{N}$ is called *proper* if there exist projections E, F, compatible with $\mathcal{N}$, so that $F \subset E$ and $A = (E - F)A(E - F)$.

Lemma 14. *Let A be a proper contraction in the algebra of a nest $\mathcal{N}$ with no finite dimensional atoms. Then, A is the w^*-limit of proper partial isometries from* $\operatorname{Alg}\mathcal{N}$.

Proof. Let E, F be projections compatible with $\mathcal{N}$ so that $(E - F)T(E - F) = T$ and let $E = E_0 \subset E_1 \subset E_2 \dots$ be any increasing sequence of projections compatible with $\mathcal{N}$. For each $n \in \mathbb{N}$ choose partial isometry V_n with initial space $E_n - E_{n-1}$ and final space $E - F$. Let

$$A_n = A + (E - F - AA^*)^{1/2} V_n, \quad n \in \mathbb{N}\,.$$

Since $A_n A_n^* = E - F$, $n \in \mathbb{N}$ A_n is a partial isometry in $\operatorname{Alg}\mathcal{N}$. Also, the w^*-limit of $\{A_n\}_{n=1}^{\infty}$ equals A. □

Theorem 15. *If $\mathcal{N}$ is a nest with no finite dimensional atoms then every contraction in* $\operatorname{Alg}\mathcal{N}$ *is the w^*-limit of unitary operators from* $\operatorname{Alg}\mathcal{N}$.

Proof. Let $A \in \operatorname{Alg}\mathcal{N}$, $\|A\| \leqq 1$. Since any contraction in $\operatorname{Alg}\mathcal{N}$ is the w^*-limit of proper contractions there is no loss of generality assuming that A is a proper operator. However, Lemma 14 shows that A is the w^*-limit of proper partial isometries. Thus, we may assume that A is a proper partial isometry; let E, F so that $A = (E - F)A(E - F)$. Let $E \subset E_0 \subset E_1 \subset E_2 \subset \dots$ be a sequence of projections compatible with $\mathcal{N}$, which increases to I and let $\dots \subset F_2 \subset F_1 \subset F_0 \subset F$ be a sequence of projections compatible with $\mathcal{N}$, which decreases to 0. We define $P_i = E_i - E_{i-1}$, $Q_i = F_i - F_{i-1}$ and we let $P_0 = (E - F) - AA^*$ and $Q_0 = (E_0 - F_0) \cap \operatorname{Ker} A$; clearly both P_0, Q_0 are of infinite dimension.

For every $i = 1, 2, \dots$, let V_i be a partial isometry with initial space P_i and final space P_{i-1} and let W_i be a partial isometry with initial space Q_{i-1} and final space Q_i, $i = 1, 2, \dots$. For each $i \in \mathbb{N}$, let $\{X_i^{(n)}\}_{n=1}^{\infty}$ (resp. $\{Y_i^{(n)}\}_{n=1}^{\infty}$) be a sequence of partial isometries with initial and final spaces P_i (resp. Q_i) so that $\lim X_i^{(n)} = \lim Y_i^{(n)} = 0$. Let

$$U_n = A + \sum_{i=1}^{\infty} V_i X_i^{(n)} + \sum_{i=1}^{\infty} Y_i^{(n)} W_i, \quad n = 1, 2, \dots\,.$$

Then, each U_n is a unitary operator in $\operatorname{Alg}\mathcal{N}$ and $\lim U_n = A$, which proves the theorem. □

2 Beyond infinite multiplicity nest algebras

We say that an operator algebra $\mathcal{A}$ satisfies the *Russo-Dye property* iff the convex hull of the unitaries in $\mathcal{A}$ contains the open unit ball of $\mathcal{A}$. The reader should notice that Theorem 1.11 is valid for a class of nest algebras much larger than that of infinite multiplicity. Indeed, *let $\mathcal{N}$ be any nest which contains a $\mathbb{Z}$-ordered subnest with infinitely many atoms of infinite dimension. Then, minor modifications on our arguments show that* $\operatorname{Alg}\mathcal{N}$ *satisfies the Russo-Dye property*! In particular, if $\mathcal{N}$ is the "Cantor nest" $\mathcal{N}(\mathbb{Q})$ (see [10, pg 423]) then every operator in the open unit ball of $\operatorname{Alg}\mathcal{N}$ is a mean of unitaries.

However there is an exceptional case where our "infinite dimensional" techniques shed little light. Does $\operatorname{Alg}\mathcal{N}$ satisfy the Russo-Dye property, when $\mathcal{N}$ is a $\mathbb{Z}$-ordered nest whose atoms are finite dimensional? The techniques developed so far show that if $A \in \operatorname{Alg}\mathcal{N}$, so that $\|A\| < \frac{2}{3}$, then A is a mean of unitaries from $\operatorname{Alg}\mathcal{N}$. (This can be shown as follows: if $E, F \in \mathcal{N}\backslash\{0, I\}$ so that $F \subset E$ then $A = \frac{1}{3}(A_1 + A_2 + A_3)$, where $A_1 = \frac{3}{2}AF^\perp$, $A_2 = \frac{3}{2}EA$ and $A_3 = \frac{3}{2}(AF + E^\perp A)$. The operators A_i are easily seen to be means of unitary operators). We do not know if this is true when $1 > \|A\| > \frac{2}{3}$.

The following is an alternative not only for algebras of $\mathbb{Z}$-ordered nests but also for nest algebras for which a Russo-Dye Theorem does not hold.

Theorem 1. *Let $\mathcal{N}$ be a nest such that* $\dim E = \infty$ (resp. $\dim E^\perp = \infty$), *for all $E \in \mathcal{N}$. If A is an operator in* $\operatorname{Alg}\mathcal{N}$ *such that* $\|A\| < \frac{a-1}{a}$, *for an integer a greater than* 2, *then A is a mean of* $2a$ *isometries* (*resp. co-isometries*) *from* $\operatorname{Alg}\mathcal{N}$.

Proof. We start with a definition. A projection $S \in \operatorname{Alg}\mathcal{N}$ is called *semisliced* iff $S = \sum_{n=1}^{\infty} E_n - F_n$ where $\{E_n\}_{n=1}^{\infty}$, $\{F_n\}_{n=1}^{\infty}$ are sequences of projections, compatible with $\mathcal{N}$, which satisfy,

$$F_n \subset E_n \subset F_{n-1},\ \forall n \in \mathbb{N} \quad \text{and} \quad \lim E_n = 0\,.$$

It is easily seen that every semisliced projection is the range of some isometry in $\operatorname{Alg}\mathcal{N}$.

For the proof, let $S_1, S_2, \ldots, S_a$ be semisliced projections so that $S_1 + S_2 + \cdots + S_a = (a-1)I$ and let $A_i = \frac{a}{a-1}S_iA$, $i = 1, 2, \ldots, a$. Then each A_i is an operator of norm less than 1 and techniques similar to that of Lemma 1.8 show that there exist isometries $V_i^{(1)}$ and $V_i^{(2)}$ in $\operatorname{Alg}\mathcal{N}$ such that $A_i = \frac{1}{2}(V_i^{(1)} + V_i^{(2)})$. Thus,

$$A = \frac{1}{a}(A_1 + A_2 + \cdots + A_a) = \frac{1}{2a}\sum_{i=1}^{a} V_i^{(1)} + V_i^{(2)}$$

which proves the theorem. □

We would like to mention that our techniques are applicable to certain nest subalgebras of von Neumann algebras, thus leading to the following.

Theorem 2. *Let $\mathfrak{R}$ be any factor von Neumann algebra and let $\mathcal{N}$ be any nest with no finite dimensional atoms. Then, $\mathfrak{R} \otimes \operatorname{Alg} \mathcal{N}$ satisfies the Russo-Dye property. Moreover, the estimate of Theorem* 1.11 *is valid.*

The proof of the above theorem is identical to the proof of Theorem 1.11; the reader only needs to notice that Proposition 1.7 is valid for nest subalgebras of factor von Neumann algebras.

In the rest of this section we observe that the techniques already developed, combined with a geometric Hahn-Banach Theorem, are sufficient to prove that certain quasitriangular algebras also satisfy the Russo-Dye property.

If $\mathcal{N}$ is a nest then $\operatorname{Qtr} \mathcal{N}$ denotes the "quasitriangular algebra" $\operatorname{Alg} \mathcal{N} + C_\infty(\mathcal{H})$, where $C_\infty(\mathcal{H})$ denotes the set of compact operators acting on $\mathcal{H}$ (see [3] for more details on quasitriangular algebras). The unitary semigroup of $\operatorname{Qtr} \mathcal{N}$ is denoted by $\mathcal{U}(\operatorname{Qtr} \mathcal{N})$ and the convex hull of $\mathcal{U}(\operatorname{Qtr} \mathcal{N})$ by $\operatorname{co} \mathcal{U}(\operatorname{Qtr} \mathcal{N})$.

Theorem 3. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms. Then $\operatorname{Qtr} \mathcal{N}$ satisfies the Russo-Dye property.*

Proof. For the proof we need to show that
(i) $\operatorname{co} \mathcal{U}(\operatorname{Qtr} \mathcal{N})^- = (\operatorname{Qtr} \mathcal{N})_1$
(ii) $\operatorname{co} \mathcal{U}(\operatorname{Qtr} \mathcal{N})^0 \neq \phi$.
Assume for the moment that both (i) and (ii) have been established. Let A be any contraction in $\operatorname{Qtr} \mathcal{N}$ and assume that $A \notin \operatorname{co} \mathcal{U}(\operatorname{Qtr} \mathcal{N})$. Since $\operatorname{co} \mathcal{U}(\operatorname{Qtr} \mathcal{N})$ has non-empty interior, Theorem 7.20 in [20] shows that there exists a non-zero norm-continuous linear functional ϕ and real number λ such that

$$\operatorname{Re} \phi(\operatorname{co} \mathcal{U}(\operatorname{Qtr}(\mathcal{N})) \leqq \lambda \leqq \operatorname{Re} \phi(A) .$$

However, condition (i) implies that

$$\begin{aligned} \|\phi\| &= \sup \{|\phi(U)|,\ U \in \mathcal{U}(\operatorname{Qtr} \mathcal{N})\} \\ &\leqq \sup \{\operatorname{Re} \phi(U),\ U \in \mathcal{U}(\operatorname{Qtr} \mathcal{N})\} \\ &\leqq \lambda . \end{aligned}$$

Thus, $\lambda \neq 0$. Moreover,

$$\lambda \leqq \operatorname{Re} \phi(A) \leqq \|\phi\| \, \|A\| \leqq \lambda \|A\|$$

and so $\|A\| \geqq 1$, which proves the Theorem. □

It only remains to establish (i) and (ii). In order to do so we start with a few comments on the results already proven.

An operator $A \in \operatorname{Qtr} \mathcal{N}$ is called sliced iff there exist sliced projections R, R' in $\operatorname{Alg} \mathcal{N}$ which have at least one common hole and satisfy $A = RAR'$. Similarly, the operator A is said to be separated iff there exist projections

$E, F \in \mathcal{N}$ so that $E \subset F$ and $A = EAE + F^{\perp}AF^{\perp}$. With this definitions one can easily see that Lemma 1.5, Proposition 1.6, Lemma 1.8 and Corollary 1.9 in the present paper remain valid if one replaces $\operatorname{Alg}\mathcal{N}$ by $\operatorname{Q tr}\mathcal{N}$ in their statements. Caution is needed with Proposition 1.10. Its proof establishes the following "weaker" statement.

Proposition 4. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms, let E, F be projections in $\mathcal{N}$ such that $F \subset E$ and let A be a partial isometry in $\operatorname{Q tr}\mathcal{N}$ such that $E^{\perp}AE = F^{\perp}AF = 0$ and $(E - F)A = A(E - F) = 0$. If $A^*A \in \mathcal{N}'$ then there exist separated partial isometries $B^{(1)}, B^{(2)}$ in $\operatorname{Q tr}\mathcal{N}$ so that $A = B^{(1)}B^{(2)}$. If A is sliced then $B^{(1)}, B^{(2)}$ can be closed sliced.*

The proposition above leads to the following.

Corollary 5. *Let $\mathcal{N}$ be a nest with no finite dimensional atoms, let $E \in \mathcal{N}\backslash\{I\}$, let K be a compact operator and let A be an element of $\operatorname{Alg}\mathcal{N}$. If $\|A + EKE\| < 1$ then $A + EKE \in \operatorname{co}\mathcal{U}(\operatorname{Q tr}\mathcal{N})$.*

Proof. Follow the same steps as in the proof of Theorem 1.11. □

With these observations in hand we are in position now to show that (i) and (ii) do hold. We remark that we will not treat the case where $I_- \neq I$ since in this case, the validity of (i) and (ii) follows easily from arguments similar to the ones in the proof of Lemma 1.8.

Lemma 6. *The norm closure of $\operatorname{co}\mathcal{U}(\operatorname{Q tr}\mathcal{N})$ equals the closed unit ball of $\operatorname{Q tr}\mathcal{N}$.*

Proof. Let $A \in \operatorname{Alg}\mathcal{N}$ and let K be a compact operator such that $\|A + K\| < 1$. Let $\{E_n\}_{n=1}^{\infty}$ be any sequence in $\mathcal{N}$ which increases to I. Then, for large $n \in \mathbb{N}$, $\|A + E_nKE_n\| < 1$. Thus, Corollary 5 shows that $A + E_nKE_n \in \operatorname{co}\mathcal{U}(\operatorname{Q tr}\mathcal{N})$ and the conclusion follows. □

Lemma 7. *The set $\operatorname{co}\mathcal{U}(\operatorname{Q tr}\mathcal{N})$ has non-empty interior.*

Proof. We will show that $\operatorname{Q tr}(\mathcal{N})_{1/16} \subseteq (\operatorname{co}\mathcal{U}(\operatorname{Q tr}(\mathcal{N})))^0$.

Let $A \in \operatorname{Alg}\mathcal{N}$ and let K be a compact operator such that $\|A + K\| < \frac{1}{16}$. Let $E \in \mathcal{N}\backslash\{0, I\}$ and let

$$X_1 = 4E(A + K)E^{\perp}, \qquad X_2 = 4E^{\perp}(A + K)E^{\perp}$$
$$X_3 = 4E(A + K)E, \qquad X_4 = 4E^{\perp}(A + K)E.$$

Then $A + K = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$. Notice that X_1, X_2 and X_3 are easily seen to be means of unitaries. Indeed, let R be any sliced projection in $\operatorname{Alg}\mathcal{N}$ and let

$$X_{i1} = 4RX_i(I - R), \qquad X_{i2} = 4RX_iR$$
$$X_{i3} = 4(I - R)X_iR, \qquad X_{i4} = 4(I - R)X_i(I - R),$$

where $i = 1, 2, 3$. Then, each X_{ij} satisfies $\|X_{ij}\| < 1$ and also $X_i = \frac{1}{4}\sum_{j=1}^{4} X_{ij}$, $i = 1, 2, 3$. On the other hand, each X_{ij} satisfies the requirements of Lemma 1.8 (ii) (where $\operatorname{Alg}\mathcal{N}$ is replaced by $\operatorname{Q tr}\mathcal{N}$) and the conclusion follows.

Finally, a simple application of the selfadjoint Russo-Dye Theorem (for the c^*-algebra generated by I and $X_4 = E^{\perp}KE$) shows that X_4 is a mean of unitaries from the algebra $\mathrm{Qtr}\,\mathcal{N}$; this proves the proposition. □

3 Concluding remarks

The class of CSL algebras, whose invariant subspace lattices contain no atoms of infinite dimension, seems to be the most appropriate class of operator algebras for a continuation of the present investigation. However, there are serious obstacles, as the following example suggests.

Example. Let $\mathcal{H} = L^2([0,1],\lambda)$, where λ is the Lebesgue measure, and let L^∞ be the multiplication algebra with symbols in $L^\infty([0,1],\lambda)$. Let $\mathcal{B}$ be an appropriate weakly closed L^∞-bimodule. Then, the algebra

$$\mathcal{A} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \middle| \; A, C \in L^\infty, \; B \in \mathcal{B} \right\}$$

is a CSL algebra whose invariant subspace lattice contains no atoms of finite dimension. However, $\mathcal{A}$ does not satisfy the Russo-Dye property since all unitaries in $\mathcal{A}$ are diagonal.

In spite of the previous example, there is a variety of CSL algebras which satisfy the Russo-Dye property, such as $\mathrm{Alg}\,\mathcal{N} \otimes \mathcal{N}$, where $\mathcal{N}$ is any nest with $\dim 0_+ = \infty$. We plan to continue this investigation in a future paper.

Finally, a simple modification of the example above shows that there are infinite multiplicity operator algebras which do not satisfy a Russo-Dye theorem.

Acknowledgement. The authors express their gratitude to X. Dai for helpful discussions on the subject of the present paper. We also thank A. Arias, R. Kadison and A. Katavolos for several comments on the content of the paper. Many thanks are due to D. Larson, for suggesting the counterexamples appearing in the last section of the paper. Finally, we thank G. Pisier and V. Paulsen for drawing our attention to Marshall's Theorem.

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