Tensor algebras of $C^*$-correspondences and their $C^*$-envelopes

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Received 3 February 2005; accepted 21 December 2005
Available online 30 January 2006
Communicated by D. Voiculescu

Abstract


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Keywords: $C^*$-envelope; Cuntz–Pimsner $C^*$-algebra; Tensor algebra; Fock space

1. Introduction

Fowler, Muhly and Raeburn have recently characterized [5, Theorem 5.3] the $C^*$-envelope of the tensor algebra $T^+_\mathcal{X}$ of a faithful and strict $C^*$-correspondence $\mathcal{X}$, as the associated universal Cuntz–Pimsner algebra. Their proof is based on a gauge invariant uniqueness theorem and earlier elaborate results of Muhly and Solel [11]. Beyond faithful strict $C^*$-correspondences, little is
known: if $\mathcal{X}$ is strict, but not necessary faithful, then the $C^*$-envelope of $\mathcal{T}_\mathcal{X}^+$ is known to be a quotient of the associated Toeplitz–Cuntz–Pimsner algebra, without any further information [11, Theorem 6.4]. In [5, Remark 5.4], the authors ask whether the above mentioned conditions on $\mathcal{X}$ are necessary for the validity of their [5, Theorem 5.3].

In this note we answer the question of Fowler, Muhly and Raeburn [5] (and Muhly and Solel [11]) by showing that the $C^*$-envelope of the tensor algebra of an arbitrary $C^*$-correspondence $\mathcal{X}$ coincides with the Cuntz–Pimsner algebra $\mathcal{O}_\mathcal{X}$, as defined by Katsura in [8]. Our proof does not require any of the results from [11] and is modelled upon the proof of our recent result [7] that identifies the $C^*$-envelope of the tensor algebra of a directed graph. We also make use of the result of Muhly and Tomforde [12] that generalizes the process of adding tails to a graph to the context of $C^*$-correspondences.

2. Preliminaries

Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{X}$ be a (right) Hilbert $\mathcal{A}$-module, whose inner product is denoted as $\langle \cdot | \cdot \rangle$. Let $\mathcal{L}(\mathcal{X})$ be the adjointable operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the norm-closed subalgebra of $\mathcal{L}(\mathcal{X})$ generated by the operators $\theta_{\xi,\eta} \in \mathcal{X}$, where $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta | \zeta \rangle$, $\zeta \in \mathcal{X}$.

A Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be a $C^*$-correspondence over $\mathcal{A}$ provided that there exists a $\star$-homomorphism $\phi_\mathcal{X} : \mathcal{A} \to \mathcal{L}(\mathcal{X})$. We refer to $\phi_\mathcal{X}$ as the left action of $\mathcal{A}$ on $\mathcal{X}$. A $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$ is said to be faithful if and only if the map $\phi_\mathcal{X}$ is faithful. A $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$ is called strict if $[\phi_\mathcal{X}(\mathcal{A})] \subseteq \mathcal{X}$ is complemented, as a submodule of the Hilbert $\mathcal{A}$-module $\mathcal{X}$. In particular, if $[\phi_\mathcal{X}(\mathcal{A})] = \mathcal{X}$, i.e., the map $\phi_\mathcal{X}$ is non-degenerate, then $\mathcal{X}$ is said to be essential.

From a given $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$, one can form new $C^*$-correspondences over $\mathcal{A}$, such as the $n$-fold ampliation or direct sum $\mathcal{X}^{(n)}$ [9, p. 5] and the $n$-fold interior tensor product $\mathcal{X}^{\otimes n} \equiv \mathcal{X} \otimes_{\phi_\mathcal{X}} \mathcal{X} \otimes_{\phi_\mathcal{X}} \cdots \otimes_{\phi_\mathcal{X}} \mathcal{X}$ [9, p. 39], $n \in \mathbb{N}$ ($\mathcal{X}^{\otimes 0} \equiv \mathcal{A}$). These operation are defined within the category of $C^*$-correspondences over $\mathcal{A}$. (See [9] for more details.)

A representation $(\pi, t)$ of a $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$ on a $C^*$-algebra $\mathcal{B}$ consists of a $\star$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ and a linear map $t : \mathcal{X} \to \mathcal{B}$ so that

\begin{align*}
(i) \quad & t(\xi)^{\ast} t(\eta) = \pi (\langle \xi | \eta \rangle), \quad \text{for } \xi, \eta \in \mathcal{X}, \\
(ii) \quad & \pi (a) t(\xi) = t(\phi_\mathcal{X}(a) \xi), \quad \text{for } a \in \mathcal{A}, \xi \in \mathcal{X}.
\end{align*}

For a representation $(\pi, t)$ of a $C^*$-correspondence $\mathcal{X}$ there exists a $\star$-homomorphism $\psi_t : \mathcal{K}(\mathcal{X}) \to \mathcal{B}$ so that $\psi_t (\theta_{\xi,\eta}) = t(\xi)^{\ast} t(\eta)^{\ast}$, for $\xi, \eta \in \mathcal{X}$. Following Katsura [8], we say that the representation $(\pi, t)$ is covariant iff $\psi_t (\phi_\mathcal{X}(a)) = \pi (a)$, for all $a \in \mathcal{J}_\mathcal{X}$, where

$$
\mathcal{J}_\mathcal{X} \equiv \phi_\mathcal{X}^{-1}\mathcal{K}(\mathcal{X}) \cap (\ker \phi_\mathcal{X})^{\perp}.
$$

If $(\pi, t)$ is a representation of $\mathcal{X}$ then the $C^*$-algebra (respectively norm-closed algebra) generated by the images of $\pi$ and $t$ is denoted as $C^*(\pi, t)$ (respectively alg$(\pi, t)$). There is a universal representation $(\tilde{\pi}_\mathcal{X}, \tilde{t}_\mathcal{X})$ for $\mathcal{X}$ and the $C^*$-algebra $C^*(\tilde{\pi}_\mathcal{X}, \tilde{t}_\mathcal{X})$ is the Toeplitz–Cuntz–Pimsner algebra $\mathcal{T}_\mathcal{X}$. Similarly, the Cuntz–Pimsner algebra $\mathcal{O}_\mathcal{X}$ is the $C^*$-algebra generated by the image of the universal covariant representation $(\pi_\mathcal{A}, t_\mathcal{X})$ for $\mathcal{X}$. 

A concrete presentation of both $T_X$ and $O_X$ can be given in terms of the generalized Fock space $F_X$ which we now describe. The Fock space $F_X$ over the correspondence $X$ is defined to be the direct sum of the $X^\otimes n$ with the structure of a direct sum of $C^*$-correspondences over $A$,$$F_X = A \oplus X \oplus X^\otimes 2 \oplus \cdots.$$Given $\xi \in X$, the (left) creation operator $t_\infty(\xi) \in L(F_X)$ is defined by the formula$$t_\infty(\xi)(a, \zeta_1, \zeta_2, \ldots) = (0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, \ldots),$$where $\zeta_n \in X^\otimes n$, $n \in \mathbb{N}$. Also, for $a \in A$, we define $\pi_\infty(a) \in L(F_X)$ to be the diagonal operator with $\phi_X(a) \otimes id_{n-1}$ at its $X^\otimes n$th entry. It is easy to verify that $(\pi_\infty, t_\infty)$ is a representation of $X$ which is called the Fock representation of $X$. Fowler and Raeburn [4] (respectively Katsura [8]) have shown that the $C^*$-algebra $C^*(\pi_\infty, t_\infty)$ (respectively $C^*(\pi_\infty, t_\infty)/K(F_XJ_X)$) is isomorphic to $T_X$ (respectively $O_X$).

**Definition 2.1.** The tensor algebra of a $C^*$-correspondence $X$ over $A$ is the norm-closed algebra $\text{alg}(\bar{\pi}_A, \bar{t}_X)$ and is denoted as $T^+_X$.

According to [4,8], the algebras $T^+_X \equiv \text{alg}(\bar{\pi}_A, \bar{t}_X)$ and $\text{alg}(\pi_\infty, t_\infty)$ are completely isometrically isomorphic and we will therefore identify them. The main result of this paper implies that $T^+_X$ is also completely isometrically isomorphic to $\text{alg}(\pi_A, t_X)$.

**3. Main result**

We begin with a useful description of the norm in $X^{(n)}$.

**Lemma 3.1.** Let $X$, $Y$ be Hilbert $A$-modules and let $\phi: A \to L(Y)$ be an injective $*$-homomorphism. If $(\xi_i)_{i=1}^n \in X^{(n)}$, then$$\|(\xi_i)_{i=1}^n\| = \sup \{ \| (\xi_1 \otimes \phi u)_{i=1}^n \| \mid u \in Y, \|u\| = 1 \}.$$**Proof.** Let us denote by $M$ the supremum in (1). Then, using the fact that $\phi$ is injective and therefore isometric,

$$M^2 = \sup \left\{ \sum_{i=1}^n \| u \phi((\xi_i \otimes \xi_i)) u \| \mid u \in Y, \|u\| = 1 \right\} = \sup \{ \| (\phi((\xi_i \otimes \xi_i)^{1/2}) u)_{i} \|^2 \mid u \in Y, \|u\| = 1 \} = \phi \left( \sum_{i=1}^n \langle \xi_i \mid \xi_i \rangle \right) = \| (\xi_i) \|^2$$

and the conclusion follows. $\square$
In the proof of our next lemma we make use of the right creation operators. If $\mathcal{Y}$ is a $C^*$-correspondence over $A$ and $\xi \in \mathcal{Y}^\otimes k$, then define the right creation operator $R_\xi$ by the formula

$$R_\xi(a, \zeta_1, \zeta_2, \ldots) = (0, 0, \ldots, 0, (\phi_X(a) \otimes id_{k-1})(\xi), \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, \ldots),$$

$\zeta_n \in \mathcal{Y}^\otimes n$, $n \in \mathbb{N}$. The operator, $R_\xi$ may not be adjointable but it is nevertheless bounded by $\|\xi\|$ and commutes with alg$(\pi_{\infty}, t_{\infty})$.

**Lemma 3.2.** If $\mathcal{X}$ be a faithful $C^*$-correspondence over $A$, then

$$\|A\| = \inf \{\|A + K\| \mid K \in M_n(\mathcal{K}(\mathcal{F}\mathcal{X}))\}$$

for all $A \in M_n(T_{\mathcal{X}^+})$, $n \in \mathbb{N}$.

**Proof.** Let $K \in M_n(\mathcal{K}(\mathcal{F}\mathcal{X}))$ be an $n \times n$ matrix with entries in $\mathcal{K}(\mathcal{F}\mathcal{X})$ and let $\epsilon > 0$. We choose unit vector $\xi \in \mathcal{F}_n$ so that $\|A\xi\| \|A\| - \epsilon$. Since $K \in M_n(\mathcal{K}(\mathcal{F}\mathcal{X}))$, there exists $k \in \mathbb{N}$ so that $\|KR_\xi\| \leq \epsilon$, for all unit vectors $u \in \mathcal{X}^\otimes k$. (Here $R_\xi^{(n)}$ denotes the $n$th ampliation of the right creation operator $R_\xi$.) Note that for any vector $u \in \mathcal{X}^\otimes k$ we have

$$\|R_\xi^{(n)}A\xi\| = \|A\xi \otimes u\|.$$

Therefore, using Lemma 3.1, we choose unit vector $u \in \mathcal{X}^\otimes k$ so that

$$\|R_\xi^{(n)}A\xi\| \geq \|A\xi\| - \epsilon \geq \|A\| - 2\epsilon.$$

We compute,

$$\|A + K\| \|A + K\| R_\xi^{(n)}\xi\| \geq \|AR_\xi^{(n)}\xi\| - \epsilon = \|R_\xi^{(n)}A\xi\| - \epsilon \geq \|A\| - 3\epsilon.$$

Since $\epsilon$ and $K$ are arbitrary, the proof is complete. □

**Corollary 3.3.** Let $\mathcal{X}$ be a faithful $C^*$-correspondence over $A$, and let $(\pi_A, t_{\mathcal{X}})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry

$$\tau_{\mathcal{X}}: T_{\mathcal{X}^+} \rightarrow \text{alg}(\pi_A, t_{\mathcal{X}})$$

so that $\tau_{\mathcal{X}}(\pi_{\infty}(a)) = \pi_A(a)$, for all $a \in A$, and $\tau_{\mathcal{X}}(t_{\infty}(\xi)) = t_{\mathcal{X}}(\xi)$, for all $\xi \in \mathcal{X}$.

In particular, the algebra $\text{alg}(\pi_A, t_{\mathcal{X}})$ is completely isometrically isomorphic to the tensor algebra $T_{\mathcal{X}^+}$.

**Proof.** Let $\tau_{\mathcal{X}}$ be the restriction of the natural quotient map

$$C^*(\pi_{\infty}, t_{\infty}) \rightarrow C^*(\pi_{\infty}, t_{\infty})/\mathcal{K}(\mathcal{F}\mathcal{X}, \mathcal{F}\mathcal{X}).$$
on the non-selfadjoint subalgebra $\text{alg}(\pi_\infty, t_\infty)$. By Lemma 3.2, this map is a complete isometry. □

Remark 3.4. Note that the above lemma already implies the result of Fowler, Muhly and Raeburn [5, Theorem 5.3] without their requirement of $\mathcal{X}$ being strict.

We now remove the requirement of $\mathcal{X}$ being faithful from the statement of the above lemma. In the special case of a graph correspondence, this was done in [7] with the help of a well-known process called “adding tails to a graph.” This process has been generalized to arbitrary correspondences by Muhly and Tomforde [12]. Indeed, let $\mathcal{X}$ be an arbitrary $C^*$-correspondence over $\mathcal{A}$ and let $\Sigma = c_0(\ker \phi_{\mathcal{X}})$ consist of all null sequences in $\ker \phi_{\mathcal{X}}$. Muhly and Tomforde show that there exists a well-defined left action of $\mathcal{B} = \mathcal{A} \oplus \Sigma$ on $\mathcal{Y} = \mathcal{X} \oplus \Sigma$ so that $\mathcal{Y}$ becomes a faithful $C^*$-correspondence over $\mathcal{B}$. One can view $\mathcal{A}$ and the $C^*$-correspondence $\mathcal{X}$ as subsets of $\mathcal{B}$ and $\mathcal{Y}$ respectively, via the identifications

$$A \ni a \to (a, 0) \in \mathcal{A} \oplus 0,$$

$$\mathcal{X} \ni \xi \to (\xi, 0) \in \mathcal{X} \oplus 0$$

and by noting that the action of $\phi_{\mathcal{Y}}$ on $\mathcal{A} \oplus 0$ coincides with that of $\phi_{\mathcal{X}}$ on $\mathcal{A}$. (The restriction of a representation $(\pi, t)$ of $\mathcal{Y}$ on that subset of $\mathcal{Y}$ will be denoted as $(\pi|_{\mathcal{A}}, t|_{\mathcal{X}})$ and is indeed a representation of $\mathcal{X}$.) In [12, Theorem 4.3(b)] it is shown that if $(\pi, t)$ is a covariant representation of $\mathcal{Y}$, then $(\pi|_{\mathcal{A}}, t|_{\mathcal{X}})$ is a covariant representation of $\mathcal{X}$.

Lemma 3.5. Let $\mathcal{X}$ be a $C^*$-correspondence over $\mathcal{A}$, and let $(\pi_{\mathcal{A}}, t|_{\mathcal{X}})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry

$$\tau_{\mathcal{X}} : T_{\mathcal{X}}^+ \to \text{alg}(\pi_{\mathcal{A}}, t|_{\mathcal{X}})$$

so that $\tau_{\mathcal{X}}(\pi_\infty(a)) = \pi_{\mathcal{A}}(a)$, for all $a \in \mathcal{A}$, and $\tau_{\mathcal{X}}(t_\infty(\xi)) = t_{\mathcal{X}}(\xi)$, for all $\xi \in \mathcal{X}$.

Proof. Let $(\pi_\infty, t_\infty)$ be the Fock representation of $\mathcal{Y}$ and note that [8, Corollary 4.5] shows that

$$\pi_\infty(B) \cap \psi_\infty(\mathcal{K}(\mathcal{Y})) = \{0\}.$$ 

Therefore, the restriction $(\pi_\infty|_{\mathcal{A}}, t_\infty|_{\mathcal{X}})$ satisfies the same property and so [8, Theorem 6.2] implies that the integrated representation $\pi_{\mathcal{A}} \times t_{\mathcal{X}}$ is a $C^*$-isomorphism from the universal Toeplitz algebra $T_{\mathcal{X}}$ onto $C^*_{\mathcal{A}}(\pi_\infty|_{\mathcal{X}}, t_\infty|_{\mathcal{X}})$. We therefore view $T_{\mathcal{X}}^+$ as a subalgebra of $T_{\mathcal{Y}}^+$. Corollary 3.3 shows now that there exists a complete isometry

$$\tau_{\mathcal{Y}} : T_{\mathcal{Y}}^+ \to \text{alg}(\pi_B, t|_{\mathcal{Y}})$$

so that $\tau_{\mathcal{Y}}(\pi_\infty(b)) = \pi_B(b)$, for all $b \in B$, and $\tau_{\mathcal{Y}}(\pi_\infty(\xi)) = t_{\mathcal{Y}}(\xi)$, for all $\xi \in \mathcal{Y}$. As we discussed earlier, [12, Theorem 4.3(b)] shows that the restriction $(\pi_B|_{\mathcal{A}}, t|_{\mathcal{X}})$ is covariant for $\mathcal{X}$. Since it is also injective, the gauge invariant uniqueness theorem [8, Theorem 6.4] shows that the restriction $\tau_{\mathcal{X}} \equiv \tau_{\mathcal{Y}}|_{T_{\mathcal{X}}}$ has range isomorphic to $\text{alg}(\pi_{\mathcal{A}}, t|_{\mathcal{X}})$ and satisfies the desired properties. □
Let $B$ be a $C^*$-algebra and let $B^+$ be a (nonselfadjoint) subalgebra of $B$ which generates $B$ as a $C^*$-algebra and contains a two-sided contractive approximate unit for $B$, i.e., $B^+$ is an essential subalgebra for $B$. A two-sided ideal $J$ of $B^+$ is said to be a boundary ideal for $B^+$ if and only if the quotient map $\pi : B \to B/J$ is a complete isometry when restricted to $B^+$. It is a result of Hamana [6], following the seminal work of Arveson [1], that there exists a boundary ideal $J_S(B^+)$, the Shilov boundary ideal, that contains all other boundary ideals. In that case, the quotient $B/J_S(B^+)$ is called the C*-envelope of $B^+$ and it is denoted as $C^*_e(B^+)$. The C*-envelope is unique in the following sense. Assume that $\phi' : B^+ \to B'$ is a completely isometric isomorphism of $B^+$ onto an essential subalgebra of a $C^*$-algebra $B'$ and suppose that the Shilov boundary for $\phi'(B^+) \subseteq B'$ is zero. Then $B$ and $B'$ are *-isomorphic, via an isomorphism $\phi$ so that $\phi(\pi(x)) = \phi'(x)$, for all $x \in B$.

In the case where an operator algebra $B^+$ has no contractive approximate identity, the $C^*_e(B^+)$ is defined by utilizing the unitization [10] $(B^+)_{1}$ of $B^+$: the C*-envelope of $B^+$ is the $C^*$-subalgebra of $C^*_e((B^+)_{1})$ generated by $B^+$. (See [2,3] for a comprehensive discussion regarding the implications of [10] on the theory of C*-envelopes.)

**Lemma 3.6.** Let $B$ be a non-unital $C^*$-algebra and let $J \subseteq B_1$ be a closed two-sided ideal in its unitization. If $J \cap B = \{0\}$ then $J = \{0\}$.

**Proof.** Assume that $J \neq \{0\}$. Since $B_1 \subseteq B$ has codimension 1, $J$ is of the form $J = \{\lambda I\}$, for some $\lambda \in \mathbb{C}$. Then, easy manipulations show that there is no loss of generality assuming that $\lambda \in \mathbb{R}$ (because $J \not\subseteq \{0\}$, $A$ is selfadjoint (because $J \cap J^* \neq 0$) and

\[(A + \lambda I)^2 = A + \lambda,
\]  

(2) after perhaps scaling (since $J^2 \neq 0$). It is easy to see now that (2) implies that $A = -P$, for some projection $P \in B$. But then, $(I - P)B = 0$ and so $P$ is a unit for $B$, a contradiction. □

We have arrived to the main result of the paper.

**Theorem 3.7.** If $\chi$ is a $C^*$-correspondence over $A$, then the C*-envelope of $\mathcal{T}_\chi^+$ coincides with the universal Cuntz–Pimsner algebra $O_\chi$.

**Proof.** According to Lemma 3.5, it suffices to show that the C*-envelope of $\text{alg}(\pi_A, t_\chi)$ equals $O_\chi$. 

Assume first that $\text{alg}(\pi_A, t_\chi)$ is unital. In light of the above discussion, we need to verify that the Shilov boundary ideal $J_S(\text{alg}(\pi_A, t_\chi))$ is zero. However, the maximality of $J_S(\text{alg}(\pi_A, t_\chi))$ and the invariance of $\text{alg}(\pi_A, t_\chi)$ under the gauge action of $\mathbb{T}$ on $O_\chi$ imply that $J_S(\text{alg}(\pi_A, t_\chi))$ is a gauge-invariant ideal. By the gauge invariant uniqueness theorem [8, Theorem 6.4], any non-zero gauge-invariant ideal has non-zero intersection with $\pi_A(A)$. Hence $J_S(\text{alg}(\pi_A, t_\chi)) = \{0\}$, or otherwise the quotient map would not be faithful on $\text{alg}(\pi_A, t_\chi)$.

Assume now that $\text{alg}(\pi_A, t_\chi)$ is not unital. We distinguish two cases.

If $O_\chi$ has a unit $I \in O_\chi$ then let

\[\text{alg}(\pi_A, t_\chi)_1 \equiv \text{alg}(\pi_A, t_\chi) + \mathbb{C}I \subseteq O_\chi.\]
Clearly, \( \text{alg}(\pi_A, t\mathcal{X})_1 \) is gauge invariant and so a repetition of the arguments in the second paragraph of the proof shows that

\[
\mathbb{C}^*_{\text{env}}(\text{alg}(\pi_A, t\mathcal{X})_1) = \mathcal{O}_\mathcal{X}.
\]

The \( C^* \)-subalgebra of \( \mathcal{O}_\mathcal{X} \) generated by \( \text{alg}(\pi_A, t\mathcal{X}) \) equals \( \mathcal{O}_\mathcal{X} \), which by convention will be its \( C^* \)-envelope.

Finally, if \( \mathcal{O}_\mathcal{X} \) does not have a unit then unitize \( \mathcal{O}_\mathcal{X} \) by joining a unit \( I \) and let

\[
\text{alg}(\pi_A, t\mathcal{X})_1 \equiv \text{alg}(\pi_A, t\mathcal{X}) + CI \subseteq \mathcal{O}_\mathcal{X} + CI.
\]

Since the Shilov ideal \( J_S(\text{alg}(\pi_A, t\mathcal{X})_1) \) is gauge invariant,

\[
J_S(\text{alg}(\pi_A, t\mathcal{X})_1) \cap \mathcal{O}_\mathcal{X} \subseteq \mathcal{O}_\mathcal{X}
\]

is gauge invariant. Therefore,

\[
J_S(\text{alg}(\pi_A, t\mathcal{X})_1) \cap \mathcal{O}_\mathcal{X} = \{0\},
\]

or else it meets \( \pi_A(A) \). By Lemma 3.6, \( J_S(\text{alg}(\pi_A, t\mathcal{X})_1) = \{0\} \) and so \( \mathbb{C}^*_{\text{env}}(\text{alg}(\pi_A, t\mathcal{X})_1) = \mathcal{O}_\mathcal{X} + CI \). The \( C^* \)-subalgebra of \( \mathcal{O}_\mathcal{X} + CI \) generated by \( \text{alg}(\pi_A, t\mathcal{X}) \) is \( \mathcal{O}_\mathcal{X} \), and the conclusion follows. \( \square \)

**Remark 3.8.** In [5, p. 596], it is claimed that if a \( \mathcal{X} \) is a \( C^* \)-correspondence over \( A \), with universal Toeplitz representation \((\pi_A, t\mathcal{X})\), then \( \pi_A \) maps an approximate unit of \( A \) to an approximate unit for both \( T\mathcal{X} \) and \( T\mathcal{X}^+ \). It is not hard to see that this claim is valid if and only if \( \phi_\mathcal{X} \) is non-degenerate. Therefore, there is a gap in the proof of [5, Theorem 5.3] in the case where \( \mathcal{X} \) is strict but not essential. Nevertheless, our Theorem 3.7 incorporates all possible cases and hence completes the proof of [5, Theorem 5.3].

We now obtain one of the main results of [7] as a corollary.

**Corollary 3.9** ([7, Theorem 2.5]). If \( G \) is a countable directed graph then the \( C^* \)-envelope of \( T_+(G) \) coincides with the universal Cuntz–Krieger algebra associated with \( G \).

Note that in [7], the proof of the above corollary is essentially self-contained and avoids the heavy machinery used in this paper. The reader would actually benefit from reading that proof and then making comparisons with the proof of Theorem 3.7 here.

**Acknowledgment**

We are grateful to David Blecher for directing us to the work of Meyer [10] and its impact on the theory of \( C^* \)-envelopes [2].
References