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Operator algebras with contractive approximate identities

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Abstract

We give several applications of a recent theorem of the second author, which solved a conjecture of the first author with Hay and Neal, concerning contractive approximate identities; and another of Hay from the theory of noncommutative peak sets, thereby putting the latter theory on a much firmer foundation. From this theorem it emerges there is a surprising amount of positivity present in any operator algebras with contractive approximate identity. We exploit this to generalize several results previously available only for C^*-algebras, and we give many other applications.

Keywords: Operator algebras; One-sided ideals; Hereditary subalgebra; Approximate identity; Peak set; Pseudo-invertible elements; Completely positive operator

1. Introduction

An *operator algebra* is a closed subalgebra of $B(H)$, for a Hilbert space H. We recall that by a theorem due to Ralf Meyer, every operator algebra A has a unique unitization A^1 (see [30] or [10, Section 2.1]). Below 1 always refers to the identity of A^1 if A has no identity. We are mostly interested in operator algebras with contractive approximate identities (cai's). We

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also call these *approximately unital* operator algebras. In our paper we give several applications of the following recent result, which was prompted by, and solves, a question on p. 351 of [9]:

Theorem 1.1. *(See* [35].*) An operator algebra with a cai, has a cai (e_t) with $\|1 - e_t\| \leqslant 1$, and even with $\|1 - 2e_t\| \leqslant 1$, for all t.*

This result draws attention to the set of operators x in an operator algebra A satisfying $\|1 - x\| \leqslant 1$. We denote this set by $\mathfrak{F}_A$; it will play a role for us very much akin to the role of the positive cone in a C^*-algebra. This surprising claim will be justified at many points in our paper, but the reader can begin to see this by considering the following fact: a linear map $T : A \to B$ between C^*-algebras or operator systems is completely positive in the usual sense iff there is a constant $C > 0$ such that $T(\mathfrak{F}_A) \subset C\mathfrak{F}_B$, and similarly at the matrix levels (see Section 8). Indeed we use Theorem 1.1 to see that there is a remarkable amount of positivity present in any operator algebra with cai. We exploit this, and various properties of operators in $\mathfrak{F}_A$, to generalize several results previously available only for C^*-algebras. Many of the applications which we give are to the structure theory of operator algebras. Some of these advances are mentioned in more detail in the next paragraph. We recall that a classical principle is to study a ring or algebra A in terms of its ideals, both two-sided and one-sided. Unfortunately, not much is known about general closed ideals in A, even for common examples of function algebras, and so we focus on the *r-ideals* (right ideals with a left cai) and *ℓ-ideals* (left ideals with a right cai). As proved in [9] using a deep result from [26], these objects are in an inclusion-preserving, bijective correspondence with each other, and also with the *hereditary subalgebras* (or *HSA's*; defined below). HSA's are frequently more useful, for example in C^*-algebra theory, because they are more symmetrical objects, and because many important properties pass to HSA's [7].

The layout of our paper is as follows: At the end of Section 1 we give some quick consequences of Theorem 1.1. The long Section 2 contains a number of facts about $\mathfrak{F}_A$, and uses these together with Theorem 1.1 to give many applications to the structure of operator algebras. For example, one theme of our paper is how cai's may be built. We dissolve the remaining mysteries concerning r-ideals by showing how they all arise. The separable r-ideals in an operator algebra are precisely the subspaces $\overline{xA}$, for an element $x \in \mathfrak{F}_A$ which we may select to be as close as we like to a positive norm 1 operator. The nonseparable r-ideals are limits of increasing nets of such subspaces $\overline{xA}$. Similarly for the matching class of ℓ-ideals, or HSA's. Other sample results: we show that as in the C^*-algebra case, any separable operator algebra with cai has a countable cai consisting of mutually commuting elements; and we prove a noncommutative Urysohn lemma. In Section 3 we study the pseudo-invertible (sometimes called 'generalized invertible') elements in operator algebras, a topic connected to the notion of 'well-supported' elements. This topic is also very intimately connected to the question of when a 'principal ideal' xA is already closed. In Section 4 we study operator algebras possessing no r-ideals or HSA's. We also give several interesting examples of such algebras. In Section 5 we display a radical, approximately unital operator algebra which is an integral domain, and whose ideal structure can be completely determined. Hence this is an excellent example against which to test certain conjectures concerning the structure theory of operator algebras. In Section 6 we consider pre-images of r-ideals, HSA's, etc. In Section 7 we describe some other interesting constructions of r-ideals in operator algebras. In the final Section 8 we introduce and study a notion of *completely positive maps* between general operator algebras, or between unital operator spaces, and give an Arveson type extension theorem [6], and a Stinespring type characterization [38], for such maps.

We remark that most of our results apply immediately to function algebras, that is to uniformly closed subalgebras of $C(K)$ spaces, since these are special cases of operator algebras. We will not take the time to point these out, although some of these applications are new.

We now state our notation, and some facts. We refer the reader to [10] for additional background on operator algebras, and for some of the details and notation below. For us a *projection* is always an orthogonal projection, and an *idempotent* merely satisfies $x^2 = x$. If X, Y are sets, then XY denotes the closure of the span of products of the form xy for $x \in X$, $y \in Y$. We write X_+ for the positive operators that happen to belong to X. Returning to the unitization, if A is a nonunital operator algebra represented (completely) isometrically on a Hilbert space H then one may identify A^1 with $A + \mathbb{C}I_H$. The second dual A^{**} is also an operator algebra with its (unique) Arens product, this is also the product inherited from the von Neumann algebra B^{**} if A is a subalgebra of a C^*-algebra B. Meets and joins in B^{**} of projections in A^{**} remain in A^{**}, as can be readily seen for example by inspecting some of the classical formulae for meets and joins of Hilbert space projections, or by noting that these meets and joins may be computed in the biggest von Neumann algebra contained inside A^{**}. Note that A has a cai iff A^{**} has an identity $1_{A^{**}}$ of norm 1, and then A^1 is sometimes identified with $A + \mathbb{C}1_{A^{**}}$. In this case the multiplier algebra $M(A)$ is identified with the idealizer of A in A^{**} (that is, the set of elements $\alpha \in A^{**}$ such that $\alpha A \subset A$ and $A\alpha \subset A$). It can also be viewed as the idealizer of A in $B(H)$, if the above representation on H is nondegenerate. If A is unital then $M(A) = A$, and in this case we often assume that $1_A = I_H$.

Let A be an operator algebra. The set $\mathfrak{F}_A = \{x \in A\colon \|1 - x\| \leqslant 1\}$ equals $\{x \in A\colon \|1 - x\| = 1\}$ if A is nonunital, whereas if A is unital then $\mathfrak{F}_A = 1 + \mathrm{Ball}(A)$. If $x \in \mathfrak{F}_A$ then the numerical range of x is contained in the closed disk of center 1 and radius 1, and in particular is in the right half plane (that is, x is *accretive*). Clearly x is a sectorial operator. See [21] for more information on sectorial and accretive operators and their functional calculus. Note that $x \in \mathfrak{F}_A$ iff $xx^* \leqslant x + x^* = 2\,\mathrm{Re}(x)$, and iff $x^*x \leqslant x + x^* = 2\,\mathrm{Re}(x)$. If A is a closed subalgebra of an operator algebra B then it is easy to see, using the uniqueness of the unitization, that $\mathfrak{F}_A = A \cap \mathfrak{F}_B$. We write $\frac{1}{2}\mathfrak{F}_A$ for $\{x \in A\colon \|1 - 2x\| \leqslant 1\}$. We remark that the condition $\|1 - 2x\| \leqslant 1$ implies both $\|x\| \leqslant 1$ and $\|1 - x\| \leqslant 1$. In much of our paper, where we have $\mathfrak{F}_A$ it probably would be preferable to employ $\frac{1}{2}\mathfrak{F}_A$ instead. However since in these occurrences it will not matter technically, we use the simpler notation.

We recall that an *r-ideal* is a right ideal with a left cai, and an *ℓ-ideal* is a left ideal with a right cai. We say that an operator algebra D with cai, which is a subalgebra of another operator algebra A, is an HSA (hereditary subalgebra) of A, if $DAD \subset D$. For the theory of HSA's see [9]. These objects are in an order preserving, bijective correspondence with the r-ideals in A, and also with the *open projections* $p \in A^{**}$, by which we mean that there is a net $x_t \in A$ with $x_t = px_tp \to p$ weak*. These are also the open projections p in the sense of Akemann [1,2] in B^{**}, where B is a C^*-algebra containing A, such that $p \in A^{\perp\perp}$. The complement ('perp') of an open projection is called a *closed projection*. We spell out some of the correspondences above: if D is an HSA in A, then $J = DA$ is the matching r-ideal. The weak* limit of a cai for D, or of a left cai for J, is an open projection, and is called the *support projection* of J or D. Conversely, if p is an open projection in A^{**}, then $pA^{**} \cap A$ and $pA^{**}p \cap A$ is the matching r-ideal and HSA pair in A. We also mention that suprema (resp. infima) of open (resp. closed) projections in A^{**}, remain in A^{**}, by the fact mentioned two paragraphs earlier about meets and joins, together with the C^*-algebraic case of these facts [1,2].

The *peak* and *p-projections* studied in [25] and [9], are certain closed projections which generalize the important notions of *peak sets* and *p-sets* from the theory of function spaces. We recall that a peak set for a unital space A of continuous functions on a compact set K, is a set of form $E = f^{-1}(\{1\})$ for some $f \in A$, $\|f\| = 1$. Equivalently, E is a peak set iff there exists $g \in A$ with $|g|_{|E^c} < \|g\| = 1 = g_{|E}$. A p-set is an intersection of peak sets. Hay defined a peak projection for a unital subspace A of a C^*-algebra B to be a closed projection in B^{**}, such that there exists an $a \in \mathrm{Ball}(A)$ with $aq = q$ and satisfying any one of a long list of equivalent conditions; for example $\|ar\| < 1$ for every closed projection r in B^{**} with $r \leqslant q^\perp$. If A is a unital operator algebra, then peak projections are also the complements of support projections of r-ideals in A of the form $\overline{(1-z)A}$ for $z \in \mathrm{Ball}(A)$ (see Proposition 6.7 in [9]). By [9, Remark 6.10 (ii)], the latter support projections are the right support projections $r(1-z)$ for contractions $z \in A$ (this also follows from results in Section 2 below). A p-projection is defined to be the infimum of a family of peak projections, or equivalently a weak* limit of a decreasing net of peak projections.

If A has a cai, then a *state* of A is a functional $\varphi \in \mathrm{Ball}(A^*)$ with $\varphi(e_t) \to 1$, for some (or every) cai (e_t) for A. We write $S(A)$ for the space of states. We write $Q(A)$ for the quasistate space $\{t\varphi\colon\ t \in [0,1],\ \varphi \in S(A)\}$. States extend uniquely to states on the unitization A^1 (see [10, 2.1.19]). We will sometimes use C^*-algebras generated by an operator algebra A. If $C^*(A)$ is such a C^*-algebra, then it is known that any bounded approximate identity (bai) for A is a bai for $C^*(A)$, and hence states of A are precisely the restrictions to A of states on $C^*(A)$ (see [10, 2.1.19]). We will often use the *numerical range* of an operator (see e.g. [14]), as opposed to its spectrum. This distinction is important: for example, for an operator T, having spectrum $\{0\}$ or contained in $[0,1]$ tells one very little, whereas having numerical range in these sets gives $T = 0$ in the first case, and $0 \leqslant T \leqslant I$ in the second. Of course the (closed) numerical range of an operator contains its spectrum.

For an operator algebra A, and $x \in A$, we define $\mathrm{oa}(x)$ to be the closed subalgebra of A generated by x. We define the *left* (resp. *right*) *support projection* of $x \in A$ to be the smallest projection $p \in A^{**}$ such that $px = x$ (resp. $xp = x$), if such a projection exists (it always exists if A has a cai). If the left and right support projections exist, and are equal, then we call it the *support projection* written $s(x)$.

We end this section with some quick consequences of Theorem 1.1.

Theorem 1.1 answers several questions posed in [26,9,4]. For example, it solves the biggest open problem in Hay's thesis [25,26]. This problem concerns noncommutative peak sets, and the first part of the following result may be viewed as the noncommutative version of a fundamental theorem of Glicksberg on which the theory of peak sets rests (see e.g. Theorem II.12.7 and II.12.5 in [20] or [39]). Thus the result puts the theory of noncommutative peak sets on a much firmer foundation.

Theorem 1.2. *If A is a unital operator algebra and if q is a closed projection in A^{**}, then q is a p-projection, and indeed is a strong limit of a decreasing net of peak projections for A.*

The r-ideals in a unital operator algebra A are precisely the right ideals which are the closure of the union of an increasing net of right ideals of the form $\overline{(1-z)A}$ for $z \in \mathrm{Ball}(A)$.

Proof. The first statement was reduced, in the first two pages of [9, Section 6], to the existence in any operator algebra with a cai, of a bai (e_t) with $\|1 - e_t\| \leqslant 1$ for all t. The latter follows from Theorem 1.1.

For the second statement, we use the first statement, together with the fact mentioned earlier that peak projections are the complements of support projections of r-ideals in A of the given form $\overline{(1-z)A}$ for $z \in \mathrm{Ball}(A)$. It was shown in [9, Proposition 6.8] that such $\overline{(1-z)A}$ is an r-ideal (this also follows from Lemma 2.1 below). Moreover, the ordering of open projections in A^{**} corresponds to the inclusion of the matching r-ideals. Hence, by the correspondence between r-ideals and open/closed projections, closures of sums of r-ideals corresponds to infs of closed projections (or sups of the complementary open projections). More precisely, suppose that (e_i) is a family of open projections corresponding to r-ideals J_i in a (possibly nonunital) operator algebra A. Then J, the closure of the span of the J_i, is known (and is easily seen) to be an r-ideal, and its matching open projection r equals $e = \bigvee_i e_i$. Indeed $e \leqslant r$ clearly (since $J_i \subset J$). Conversely, if $a \in J_i$ then $e_i a = a$, so that $ea = a$. Hence $a = ea$ for any $a \in J$, so that $r \leqslant e$, and $r = e$.

Putting this all together, any r-ideal is the closure of the union of an increasing net of r-ideals of the given form. □

Remark. In particular, every nonzero r-ideal in a unital operator algebra A is what we called 1-*regular* in [4]: that is it contains $(1 - y)A$ for some $y \in \mathrm{Ball}(A) \setminus \{1\}$. This was stated as a question in that paper.

Corollary 1.3. *If A is a nonunital operator algebra with cai, and $x \in A^1 \setminus A$, then there are always more than one closest point in A to x. That is, A is* never *a Chebychev subspace of A^1.*

Proof. The existence of nonzero $x \in A$ with $\|1 - x\| = 1$ is saying that there are always more than one closest point in A to 1. If $a + \lambda 1 \in A^1$, for $a \in A$, $\lambda \neq 0$, and if $\|1 - x\| = 1$ for $x \in A \setminus \{0\}$, then $\|a + \lambda 1 - (a + \lambda x)\| = |\lambda|\|1 - x\| = |\lambda| = \|a + \lambda 1 - a\| \leqslant \|a + \lambda 1 - b\|$ for all $b \in A$, using [10, Lemma 2.1.12]. □

See e.g. [33] for more information on Chebychev subspaces of operator algebras.

Corollary 1.4. *Every r-ideal in an operator algebra, has a left cai (e_t) with $\|1 - 2e_t\| \leqslant 1$ for all t, and $e_s e_t \to e_s$ with t for any fixed s.*

Proof. If J is an r-ideal, and if D is the matching HSA, then by Theorem 1.1, D has a cai (e_t) with $\|1 - 2e_t\| \leqslant 1$. Since $J = DA$, as explained in the introduction, the result follows. □

Corollary 1.5. *If J is a closed two-sided ideal in an operator algebra A, and if J has a cai, then J has a cai (e_t) with $\|1 - 2e_t\| \leqslant 1$ for all t, which is also quasicentral* (*that is, $e_t a - a e_t \to 0$ for all $a \in A$*).

Proof. Let (e_t) be a cai for J with $\|1 - 2e_t\| \leqslant 1$ for all t (see Theorem 1.1). The weak* limit q of (e_t) is well known to be a central projection in A^{**}, and so $e_t a - a e_t \to 0$ weakly for all $a \in A$. A routine argument using Mazur's theorem shows that convex combinations of the e_t comprise the desired cai, and they will still have the property of being in the convex set $\frac{1}{2}\mathfrak{F}_A$ defined earlier. □

Corollary 1.6. *If A is an operator algebra with a countable cai (f_n), then A has a countable cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. By Theorem 1.1, A has a cai (e_t) in $\frac{1}{2}\mathfrak{F}_A$. Choosing t_n with $\|f_n e_{t_n} - f_n\| \vee \|e_{t_n} f_n - f_n\| < 2^{-n}$, it is easy to see that (e_{t_n}) is a countable cai in $\frac{1}{2}\mathfrak{F}_A$. □

2. Consequences involving $\mathfrak{F}_A$

Lemma 2.1. *If $x \in \mathfrak{F}_A$, with $x \neq 0$, then the operator algebra* $\mathrm{oa}(x)$ *has a cai. Indeed, the operator algebra* $\mathrm{oa}(x)$ *has a sequential cai belonging to $\frac{1}{2}\mathfrak{F}_A$, consisting of elements $u_n = (\frac{x}{2})^{1/n}$, the nth roots being suitably defined below.*

Proof. We will give two proofs of the fact that $\mathrm{oa}(x)$ has a cai, since both will be needed later. The operator algebra $\mathrm{oa}(x)$ is an ideal in C, its unitization, which is the closed algebra generated by 1 and x. Indeed the closure of xC is $\mathrm{oa}(x)$. Note too that $\mathrm{oa}(x)$ has a bai (e_n) where $e_n = 1 - \frac{1}{n}\sum_{k=1}^{n}(1-x)^k$, since

$$\frac{x}{n}\sum_{k=1}^{n}(1-x)^k = \frac{1}{n}\big(1-(1-x)\big)\sum_{k=1}^{n}(1-x)^k = \frac{1}{n}\big(1-(1-x)^{n+1}\big) \to 0$$

with n. Also, $\|\frac{1}{n}\sum_{k=1}^{n}(1-x)^k\| \leqslant 1$. By [9, Theorem 6.1], $\mathrm{oa}(x)$ has a cai (the argument is that any weak* limit point p of (e_n) in A^{**} has to be the identity for $\mathrm{oa}(x)^{**}$, hence is idempotent. Since $\|1-p\| \leqslant 1$, we see that $1-p$ and hence $p = 1_{\mathrm{oa}(x)^{**}}$ are projections. So $\mathrm{oa}(x)$ has a cai by a well-known principle stated in the introduction (see also [10, Proposition 2.5.8])).

The second proof will be presented after Proposition 2.3, and it will include the extra information about the sequential cai (u_n). $\square$

The following fact about the ‘disk algebra functional calculus’ arising from von Neumann’s inequality, is well known:

Lemma 2.2. *If $f, g \in A(\mathbb{D})$, with $\|g\|_{A(\mathbb{D})} \leqslant 1$, and if $T \in B(H)$ is a contractive operator, then $f(g(T)) = (f \circ g)(T)$.*

Proposition 2.3. *The sets $\mathfrak{F}_A$ and $\frac{1}{2}\mathfrak{F}_A$ are closed under taking roots. That is, for $0 < r \leqslant 1$ and $x \in \mathfrak{F}_A$ (resp. $x \in \frac{1}{2}\mathfrak{F}_A$), a suitably defined rth power x^r is in $\mathfrak{F}_A$ (resp. $x^r \in \frac{1}{2}\mathfrak{F}_A$), and $x^r \in \mathrm{oa}(x)$, and $(x^r)^{\frac{1}{r}} = x$.*

Proof. If $y \in A^1$ with $\|y\| \leqslant 1$, then the disk algebra functional calculus is a contractive algebra homomorphism θ from $A(\mathbb{D})$ to A^1 with $\theta(1) = 1$ and $\theta(z) = y$. If $r > 0$ then there is a unique analytic branch of $f(z) = (1-z)^r$ defined on $\mathbb{D}$ such that $f(0) = 1$. For $x \in \mathfrak{F}_A$ set $y = 1 - x$. Applying the functional calculus for this value of y, $\theta(f)$ will be our suitable rth power of x. The image $\theta(f)$ is a norm limit of polynomials $p_n(y)$, such that $p_n(z)$ converges uniformly to $(1-z)^r$ on the unit disk. In particular the values at $z = 1$ must tend to zero, and so we may assume that $p_n(1) = 0$. That is, $x^r = \theta(f)$ is a norm limit of polynomials $q_n(x)$ with $q_n(0) = 0$, and these are in $\mathrm{oa}(x)$. Hence $x^r \in \mathrm{oa}(x)$. Indeed for $0 < r \leqslant 1$, the binomial expansion of $1 - (1-z)^r$ is an absolutely convergent sum $\sum_{n=1}^{\infty} a_n z^n$ with $a_n \geqslant 0$ and $\sum_{n=1}^{\infty} a_n = 1$. Therefore $1 - x^r$ is in the closed convex hull of the powers $(1-x)^n$, so $\|1 - x^r\| \leqslant 1$ and $x^r \in \mathfrak{F}_A$. A routine application of Lemma 2.2, with $g(z) = 1 - (1-z)^r$ and the f there equal to $1 - (1-z)^{\frac{1}{r}}$, yields $(x^r)^{\frac{1}{r}} = x$.

Suppose that $x \in \frac{1}{2}\mathfrak{F}_A$. It is a pleasant exercise in complex numbers that $\frac{1}{2}\mathfrak{F}_{\mathbb{C}}$ is closed under taking roots. Equivalently, $|1 - 2(\frac{1-z}{2})^r| \leqslant 1$ for $0 < r \leqslant 1$ and $|z| \leqslant 1$. Replacing z by $1 - 2x$, that is by applying the functional calculus arising from von Neumann's inequality in a routine way, we have $\|1-2x^r\| \leqslant 1$. $\square$

Conclusion of proof of Lemma 2.1. Suppose that $x \in \mathfrak{F}_A$. It is not hard to see that $z^{\frac{1}{n}}z \to z$ uniformly on the closed disk of radius 1 center 1. Writing $y = 1-x$, and applying the functional calculus, we find that $\|x^{1/n}x - x\| \to 0$. The elements $u_n = x^{1/n}$ satisfy $u_n x = x u_n \to x$, and so they are a bai for $\mathrm{oa}(x)$. If $x \in \frac{1}{2}\mathfrak{F}_A$ then $u_n \in \frac{1}{2}\mathfrak{F}_A$, and so (u_n) is a cai. $\square$

Theorem 2.4. *For $0 < \rho < \frac{\pi}{2}$ let W_ρ be the wedge-shaped region containing the real interval $[0,1]$ consisting of numbers $re^{i\theta}$ with argument θ such that $|\theta| < \rho$, which are also inside the circle $|\frac{1}{2} - z| \leqslant \frac{1}{2}$.*

An operator algebra A with cai, has a cai (e_t) in $\frac{1}{2}\mathfrak{F}_A$, with the spectrum and numerical range of e_t contained in W_ρ. In fact this can be done with $\rho \to 0$ as t runs over its directed set.

Proof. If $x \in \frac{1}{2}\mathfrak{F}_A$ then $x^{\frac{1}{k}}$ is in $\mathrm{oa}(x)$ and in $\frac{1}{2}\mathfrak{F}_A$, by Proposition 2.3, and it clearly has spectrum contained inside a 'wedge-shaped region' of the type described; and the spectrum in A is smaller. The numerical range of x is also in this wedge, for example from a result of Macaev and Palant [29] (see also e.g. [21, Corollary 7.1.13]), stating that the numerical range of a kth root of an operator whose numerical range avoids the negative real axis, lies in the appropriate 'wedge' or sector centered on the positive real axis of angle $\frac{\pi}{k}$. It is also clearly inside the desired circle.

By Theorem 1.1, there is a cai (u_t) in $\frac{1}{2}\mathfrak{F}_A$. Let $v_{t,n} = u_t^{\frac{1}{n}}$ for $n \in \mathbb{N}$. If $b \in A$ then using (a_k) as we did in the proof of Proposition 2.3,

$$\|b - v_{t,n}b\| = \left\|\sum_{k=1}^{\infty} a_k(1-u_t)^k b\right\| \leqslant \left(\sum_{k=1}^{\infty} a_k\right)\big\|(1-u_t)b\big\| = \big\|(1-u_t)b\big\| \to 0,$$

with t, for fixed n. Similarly, $\|b - bv_{t,k}\| \to 0$. Thus $(v_{t,k})$ is also a cai in A. By the last paragraph we can ensure it has numerical range in the appropriate 'wedges', and that these wedges shrink to the interval $[0,1]$ with (t,k). $\square$

An operator with numerical range contained in $[0,1] \times [-\epsilon, \epsilon]$, in fact is near to a positive operator. Indeed $\mathrm{Re}(x) = \frac{x+x^*}{2} \geqslant 0$ (since $\varphi(\frac{x+x^*}{2}) = \mathrm{Re}(\varphi(x)) \in [0,1]$ for states φ), and $\|x - \mathrm{Re}(x)\| = \|\mathrm{Im}(x)\| \leqslant \epsilon$ (since $\mathrm{Im}(x)$ is Hermitian, so its norm is a supremum of quantities $|\varphi(\frac{x-x^*}{2})| = |\mathrm{Im}(\varphi(x))| \leqslant \epsilon$). It thus follows from Theorem 2.4 that any operator algebra with cai has a cai that gets arbitrarily close to being positive. In fact this is not the deep thing (the latter also follows by routine convexity methods of [5,17,36,37]). What seems deep here is the position of the numerical range (being accretive and sectorial, etc.).

Lemma 2.5. *For any operator algebra A, if $x \in \mathfrak{F}_A$, with $x \neq 0$, then the left support projection of x equals the right support projection. If $A \subset B(H)$ via a representation π, for a Hilbert space H, such that the unique weak* continuous extension $\tilde{\pi} : A^{**} \to B(H)$ is* (*completely*) *isometric, then $s(x)$ also may be identified with the smallest projection p on H such that $px = x$* (*and $xp = x$*)*. That is, $s(x)H = \overline{\mathrm{Ran}(x)} = \mathrm{Ker}(x)^{\perp}$. Also, $s(x)$ is an open projection in A^{**} in the sense of* [9]. *If A is a subalgebra of a C^*-algebra B then $s(x)$ is open in B^{**} in the sense of Akemann* [1,2].

Proof. Viewing $\mathrm{oa}(x) \subset A$, the identity of $\mathrm{oa}(x)^{**}$ corresponds to a projection $e \in A^{**}$ with $ex = xe = x$. If A is represented on H as described, suppose that $px = x$. Then $pe_n = e_n$, where (e_n) is the usual bai of $\mathrm{oa}(x)$ from Lemma 2.1, so that in the weak* limit we have $pe = e$ and $e \leqslant p$. Similarly, $e \leqslant p$ if $xp = x$. So $e = s(x)$. The equalities for $s(x)H$ are now routine.

This projection e, being the identity of $\mathrm{oa}(x)^{**}$, is open in the sense of [9]. The last statement of the proof follows from e.g. [9, Theorem 2.4]. □

Corollary 2.6. *For any operator algebra A, if $x \in \mathfrak{F}_A$, with $x \neq 0$, then the closure of xA is an r-ideal in A and $s(x)$ is the support projection of this r-ideal. We have $\overline{xA} = s(x)A^{**} \cap A$. Also, $\overline{xAx}$ is the HSA matching $\overline{xA}$, and $x \in \overline{xAx}$.*

Proof. The first assertion follows for example from Lemma 2.1: any cai for $\mathrm{oa}(x)$ serves as a left cai for the closure of xA. The second assertion follows from this, since the weak* limit of this left cai is $s(x)$. Clearly $\overline{xA} \subset s(x)A^{**} \cap A$, and since (e_n) in the proofs above converges weak* to $s(x)$, if $a \in s(x)A^{**} \cap A$ we have $e_n a \to a$ weakly. By Mazur's theorem, a convex combination converges in norm, so $a \in \overline{xA}$.

For the last assertion notice that by the argument in the first line of this proof, $\overline{xAx}$ has a cai, and so it is an HSA. It is the HSA matching $\overline{xA}$ by the correspondence described in the introduction, since $\overline{xAxA} = \overline{xA}$. The latter follows because $x \in \overline{xAx}$, which in turn follows easily from Lemma 2.1. □

Corollary 2.7. *If A is a closed subalgebra of an operator algebra B, and $x \in \mathfrak{F}_A$, then the support projection of x computed in A^{**} is the same, via the canonical embedding $A^{**} \cong A^{\perp\perp} \subset B^{**}$, as the support projection of x computed in B^{**}.*

Proof. This is obvious given the formula $s(x) = \mathrm{w}^* \lim_n e_n$ above. □

Corollary 2.8. *If A is a closed subalgebra of a C^*-algebra B, and $x \in \mathfrak{F}_A$, then $s(x)$ is the support projection of x^*x in B^{**}. Indeed $s(x) = s(x^*x) = s(xx^*) = s(x^*)$, where the latter three support projections are with respect to B.*

Proof. We have $x^*xs(x) = x^*x$, so $s(x) \geqslant s(x^*x)$. Conversely, if p is a projection in B with $x^*xp = x^*x$, then $(1-p)x^*x(1-p) = 0$, so that $x = xp$, and so $s(x) \leqslant p$ (using Corollary 2.7). Thus $s(x) \leqslant s(x^*x)$. So $s(x) = s(x^*x)$ and the other equalities are similar, or now obvious. □

Lemma 2.9. *Let A be an operator algebra with cai. If $x \in \mathfrak{F}_A$, then for any state φ of A, $\varphi(x) = 0$ iff $\varphi(s(x)) = 0$.*

Proof. Let $B = C^*(A)$, then as we said in the introduction, states φ on A are precisely the restrictions of states on B. Continuing to write φ for its canonical extension to A^{**}, if $\varphi(s(x)) = 0$ then by Cauchy–Schwarz,

$$\left|\varphi(x)\right| = \left|\varphi\big(s(x)x\big)\right| \leqslant \varphi\big(s(x)\big)^{\frac{1}{2}}\varphi\big(x^*x\big)^{\frac{1}{2}} = 0.$$

Conversely, if $\varphi(x) = 0$ then $\varphi(x^*x) \leqslant \varphi(x + x^*) = 0$, since $x \in \mathfrak{F}_A$. By Cauchy–Schwarz, $\varphi(ax) = 0$ for all $a \in A$. Since any bai for $\mathrm{oa}(x)$ converges to $s(x)$ weak* we have $\varphi(s(x)) = 0$. □

Lemma 2.10. *If $x, y \in \mathfrak{F}_A$, for any operator algebra A, then $\overline{xA} \subset \overline{yA}$ iff $s(x) \leqslant s(y)$. If A has a cai and $x \in \mathfrak{F}_A$, then the following are equivalent*:

(i) $\overline{xA} = A$.
(ii) $\overline{xAx} = A$.
(iii) $s(x) = 1_{A^{**}}$.
(iv) $\varphi(x) \neq 0$ *for every state φ of A.*
(v) $\mathrm{Re}(x)$ *is strictly positive (that is, $\varphi(\mathrm{Re}(x)) > 0$ for every state φ of $C^*(A)$).*

Proof. Since $\overline{xA} = s(x)A^{**} \cap A$, it is clear that if $s(x) = 1$ then $\overline{xA} = A$; and also that $\overline{xA} \subset \overline{yA}$ if $s(x) \leqslant s(y)$. Conversely, if $\overline{xA} \subset \overline{yA}$, then $x \in \overline{yA} = s(y)A^{**} \cap A$. We have $s(y)x = x$, so that $s(x) \leqslant s(y)$ by definition of $s(x)$.

(i) $\Leftrightarrow$ (iii) Corollary 2.6.

(i) $\Leftrightarrow$ (ii) Follows by the bijective correspondence between r-ideals and HSA's, and Corollary 2.6.

(iii) $\Rightarrow$ (iv) Obvious by the last lemma.

(iv) $\Rightarrow$ (iii) If $s(x) \neq 1$ choose a state φ on A (or equivalently on $C^*(A)$) with $\varphi(1 - s(x)) = 1$. Then $\varphi(s(x)) = 0$, and so $\varphi(x) = 0$ by Lemma 2.9.

(v) $\Rightarrow$ (iv) Follows since $\mathrm{Re}(\varphi(x)) = \varphi(\mathrm{Re}(x))$.

(iv) $\Rightarrow$ (v) If $\mathrm{Re}(\varphi(x)) = \varphi(\mathrm{Re}(x)) = 0$, then because $|1 - \varphi(x)| \leqslant 1$, we must have $\varphi(x) = 0$. $\square$

Remark. It is easy to see that in the last result we can replace A by any C^*-algebra $C^*(A)$ generated by A. Thus for example $\overline{xA} = A$ iff $\overline{xC^*(A)} = C^*(A)$.

An element in $\mathfrak{F}_A$ with $\mathrm{Re}(x)$ strictly positive, and hence satisfying the equivalent conditions in the last result, will be called *strictly real positive*. Note that roots $x^{\frac{1}{k}}$ of a strictly real positive x are strictly real positive, and they become as close as we like to a positive operator, as $k \to \infty$.

Proposition 2.11. *If $x \in \mathfrak{F}_A$ is strictly real positive, then pxp is invertible in pAp for every projection $p \in A$.*

Proof. The state space $S(pAp)$ is weak* compact since pAp has an identity, and the map $\varphi \mapsto \varphi(pxp)$ on $S(pAp)$ is continuous. It is also never zero, as can be seen using Lemma 2.10, since for any $\varphi \in S(pAp)$, $\varphi(p \cdot p)$ extends to a nonzero positive functional on $C^*(A)$, so is a nonzero multiple of a state on A, hence $\varphi(pxp) \neq 0$. Thus $|\varphi(pxp)|$ is bounded away from 0, so pxp has numerical range, hence spectrum with respect to pAp, excluding 0. $\square$

We will need the following 'Fredholm alternative' type result, a 'sharp form' of the Neumann lemma.

Theorem 2.12. *Let T be an operator in $B(H)$ with $\|I - T\| \leqslant 1$. Then T is not invertible if and only if $\|I - T\| = \|I - \frac{1}{2}T\| = 1$. Also, T is invertible iff T is invertible in the closed algebra generated by I and T, and iff $\mathrm{oa}(T)$ contains I. Here $I = I_H$ of course.*

Proof. Since $\|I - T\| \leqslant 1$ implies $\|I - \frac{1}{2}T\| \leqslant 1$ by convexity, the ($\Rightarrow$) direction of the first 'iff' is clear by the Neumann lemma. Conversely, if $\|I - T\| = \|I - \frac{1}{2}T\| = 1$, then by the parallelogram law

$$\left\|\frac{1}{2}T\zeta\right\|^2 + \left\|\left(I - \frac{1}{2}T\right)\zeta\right\|^2 = \left\|\frac{1}{2}\zeta\right\|^2 + \left\|\frac{1}{2}(I - T)\zeta\right\|^2 \leqslant 1, \quad \zeta \in \mathrm{Ball}(H).$$

Hence $I - \frac{1}{2}T$ approximately achieves its norm at some norm one vector ζ with $\|T\zeta\|$ as close as we wish to 0. Hence T is not invertible, or else $\|T\zeta\| \geqslant \|T^{-1}\|^{-1}$.

If $\mathrm{oa}(T)$ contains I_H then $\|I - RT\| < 1$ for some R in $\mathrm{oa}(T)$, which by commutativity of $\mathrm{oa}(T)$ and the Neumann lemma implies that T is invertible in $\mathrm{oa}(T)$, and hence in A. Conversely, if T is invertible in A then by the above, $\|I - T\| < 1$ or $\|I - \frac{1}{2}T\| < 1$. In the first case, the bai (e_n) for $\mathrm{oa}(T)$ in Lemma 2.1, converges in norm to I, so $I \in \mathrm{oa}(T)$. The second case follows from the first by replacing T with $T/2$. □

Lemma 2.13. *If (J_i) is a family of r-ideals in an operator algebra A, with matching family of HSA's (D_i), and if $J = \overline{\sum_i J_i}$ then the HSA matching J is the HSA D generated by the (D_i) (that is, the smallest HSA in A containing all the D_i). Here 'matching' means with respect to the correspondence between r-ideals and HSA's described in the introduction.*

Proof. Let D' be the HSA generated by the (D_i). Since $J_i \subset J$ we have $D_i \subset D$, and so $D' \subset D$. Conversely, since $D_i \subset D'$ we have $J_i \subset D'A$, so that $J \subset D'A$. Hence $D \subset D'$. □

An r-ideal (resp. HSA, ℓ-ideal) of the form $\overline{xA}$ (resp. $\overline{xAx}$, $\overline{Ax}$), for $x \in \mathfrak{F}_A$, will be called *peak-principal*. We note that the peak-principal r-ideals in a uniform algebra A are precisely the $J_E = \{f \in A\colon f_{|E} = 0\}$ for a peak set E. Thus peak-principal r-ideals (or peak-principal HSA's) may be thought of as a noncommutative variant of peak sets (see also [9, p. 354]).

If J is a peak-principal r-ideal, for example, then for any $\epsilon > 0$, we may write $J = \overline{xA}$ for some $x \in \frac{1}{2}\mathfrak{F}_A$, where the numerical range of x is contained in the thin wedge W_ϵ from Theorem 2.4. This is because $\overline{xA} = \overline{x^{\frac{1}{n}}A}$ for all $n \in \mathbb{N}$ (since $x^{\frac{1}{n}} \in \mathrm{oa}(x)$ by Proposition 2.3).

As pointed out in [9, Section 4], there is a bijective correspondence between r-ideals, and certain weak* closed faces in the quasistate space $Q(A)$, for an approximately unital operator algebra A. In fact, there are simple arguments for what we will need: If J is an r-ideal with support projection p, let $F_p = \{\varphi \in Q(A)\colon \varphi(p) = 0\} = Q(A) \cap J^\perp$. (The one direction of the last equality follows from Cauchy–Schwarz, the other from the fact that a left cai of J converges to p.) Note that if $F_{p_2} \subset F_{p_1}$ then by extending states to $C^*(A)$ we get a similar inclusion with respect to $C^*(A)$. Since p_1, p_2 are open with respect to $C^*(A)$, we obtain $p_1 \leqslant p_2$ by the C^*-algebra theory, and so $J_1 \subset J_2$. Thus $J \mapsto F_p$ is a one-to-one order injection. It also takes 'closures of sums of r-ideals' to 'intersections', since $Q(A) \cap (\sum_i J_i)^\perp = \bigcap_i (Q(A) \cap J_i^\perp)$.

Proposition 2.14. *Let A be any operator algebra (not necessarily with an identity or approximate identity). Suppose that (x_k) is a sequence in $\mathfrak{F}_A$, and that $\alpha_k \in (0, 1]$ add to 1. Then the closure of the sum of the r-ideals $\overline{x_kA}$, is the r-ideal $\overline{zA}$, where $z = \sum_{k=1}^\infty \alpha_k x_k \in \mathfrak{F}_A$. Similarly, the HSA generated by all the $\overline{x_kAx_k}$ equals $\overline{zAz}$.*

Proof. Since $x \in \mathrm{oa}(x)$, it is easy to see that $\overline{xA} = \overline{xA^1}$. Thus we may assume that A is unital if we want. The statement to be proved corresponds to the identity $\bigvee_k s(x_k) = s(z)$. We have

$$\left\{\varphi \in Q(A)\colon \varphi\left(\sum_k \alpha_k x_k\right) = 0\right\} = \bigcap_k \{\varphi \in Q(A)\colon \varphi(x_k) = 0\}$$

(because $\varphi(\sum_k \alpha_k x_k) = 0$ implies that $\sum_k \alpha_k \operatorname{Re}\varphi(x_k) = 0$; and the latter implies that $\operatorname{Re}\varphi(x_k) = 0$ since x_k is accretive, and so $\varphi(x_k) = 0$ because of the shape of the numerical range of elements in $\mathfrak{F}_A$). Hence, by Lemma 2.9, $F_{s(z)} = \bigcap_k F_{s(x_k)}$, which implies, in the light of the discussion above the proposition, that the closure of the sum of the r-ideals $\overline{x_k A}$, is the r-ideal $\overline{zA}$. So $\bigvee_k s(x_k) = s(z)$. The HSA assertion follows from the r-ideal assertion, by Lemma 2.13, and the last assertions of Corollary 2.6. □

Theorem 2.15. *Let A be any operator algebra* (*not necessarily with an identity or approximate identity*). *The r-ideals* (*resp. HSA's*) *in A, are precisely the closures of unions of an increasing net of ideals* (*resp. HSA's*) *of the form $\overline{xA}$* (*resp. $\overline{xAx}$*), *for $x \in \mathfrak{F}_A$.*

Proof. The r-ideal case is done in Theorem 1.2 if A is unital. If A is not unital, and if J is an r-ideal in A, then it is also an r-ideal in A^1. Theorem 1.2 gives that J is the closure of an increasing unions of ideals of the form $(1-z)A^1$, for $z \in \mathrm{Ball}(A^1)$. If $z = \lambda 1 - x$ for $x \in A$, $\lambda \in \mathbb{C}$, then $\lambda = 1$ (or else there is a nonzero scalar $t = 1 - \lambda$ with $t1 + x = 1 - z \in (1-z)A^1 \subset J \subset A$, which forces $1 = \frac{1}{t}((t1 + x) - x) \in A$). So $(1-z)A^1 = xA^1$. Since $\mathrm{oa}(x)$ has a cai by Lemma 2.1, $x \in \overline{x\,\mathrm{oa}(x)} \subset \overline{xA}$. It follows that xA^1 has the same closure as xA. Thus the closure of $(1-z)A^1$ is $\overline{xA}$. This completes the proof of the statements concerning r-ideals.

We saw in Corollary 2.6 that $\overline{xAx}$ is an HSA if $x \in \mathfrak{F}_A$, with matching r-ideal $\overline{xA}$. Also, the closure D of an increasing union of HSA's D_i is an HSA, and therefore it is the HSA generated by the (D_i). Indeed, it clearly satisfies $DAD \subset D$, and to see that it has a cai it is well known that it is enough to show that if $x_1, \dots, x_n \in D$ and $\epsilon > 0$ are given, then there exists $d \in \mathrm{Ball}(D)$ with $\|x_k d - x_k\| \vee \|d x_k - x_k\| < \epsilon$. Picking j and $y_1, \dots, y_n \in D_j$ with $\|x_k - y_k\| < \epsilon/3$ for all $k = 1, \dots, n$, we have for a cai (e_t) for D_j that

$$\|x_k e_t - x_k\| \leqslant \big\|(x_k - y_k)e_t\big\| + \|y_k e_t - y_k\| + \|x_k - y_k\| < \epsilon$$

for all k, and for some choice of t. Similarly, $\|d x_k - x_k\| < \epsilon$, as desired.

Finally, if D is an HSA in A, with matching r-ideal J, express J as the closure of an increasing unions of r-ideals J_i of the form $\overline{xA}$, by the first paragraph. Then by Lemma 2.13, D is the HSA generated by, and is also the closure of, the increasing net of HSA's (D_i) matching the (J_i). By Corollary 2.6, D_i is of the desired form $\overline{xAx}$. □

Peak sets were defined in the introduction. It is well known that a countable intersection of peak sets is a peak set, and that a p-set which is a G_δ set is a peak set, so that for uniform algebras on metrizable spaces the p-sets are exactly the peak sets (using the fact that $C(K)$ is separable if K is metrizable). Analogous results hold in C^*-algebras: e.g. separable closed right ideals are all of the form $\overline{xA}$ for some $x \in A_+$. Similar facts hold in our context:

Theorem 2.16. *Let A be any operator algebra* (*not necessarily with an identity or approximate identity*).

(1) *Every separable r-ideal (resp. HSA) in A, is peak-principal (that is, equal to $\overline{xA}$ (resp. $\overline{xAx}$), for some $x \in \mathfrak{F}_A$).*
(2) *The closure of a countable sum of peak-principal r-ideals (resp. the HSA generated by a countable number of peak-principal HSA's) is peak-principal.*

Proof. (2) By Proposition 2.14, the closure of $\sum_{n=1}^{\infty} \overline{x_n A}$ is $\overline{xA}$, where $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$. The HSA assertion then follows from this, as in the proofs of Proposition 2.14 and Theorem 2.15.

(1) Let D be a separable r-ideal (resp. HSA) in A. By Theorem 2.15, D is the closure of a union of r-ideals (resp. HSA's) of the form $\overline{wA}$ (resp. $\overline{wAw}$), for $w \in \mathfrak{F}_A$, and we can clearly assume that there are a countable number of these. Now apply (2) (in the HSA case recall that the HSA generated by an increasing net of HSA's is the closure of the union of these HSA's). □

Remark. The above considerations give an 'algorithm' for building useful cai in r-ideals, ℓ-ideals, or HSA's (and hence in any approximately unital operator algebra). In the separable case, we can just take $(y^{\frac{1}{k}})$ where $y = \frac{x}{2}$ for x as in Theorem 2.16. Indeed as we saw in the second proof of Lemma 2.1, $y^{\frac{1}{k}} y \to y$. Similarly in the nonseparable case, any r-ideal J, for example, in a unital operator algebra A may be written as the closure of the union of an increasing net of r-ideals $J_t = \overline{x_t A}$ for $x_t \in \frac{1}{2}\mathfrak{F}_A$, by Theorem 1.2. Then as before, $(x_t^{\frac{1}{k}})$ is a left cai for J.

Corollary 2.17. *If A is a separable operator algebra, generating a C^*-algebra B, then the open projections in $A^{\perp\perp}$ are precisely the $s(x)$ for $x \in \mathfrak{F}_A$.*

Proof. This follows from Theorem 2.16 (1), Lemma 2.5, and Corollary 2.6. □

Remark. Of course if in the last result A is also unital, then these projections are precisely the 'perps' of peak projections for A, by a fact mentioned in the introduction.

Corollary 2.18. *If A is a separable operator algebra with cai, then there exists an $x \in \mathfrak{F}_A$ with $A = \overline{xA} = \overline{Ax} = \overline{xAx}$.*

Any separable operator algebra with cai has a countable cai consisting of mutually commuting elements, indeed of form $(x^{\frac{1}{k}})$ for an $x \in \frac{1}{2}\mathfrak{F}_A$.

Proof. The first part is immediate from Theorem 2.16; if $A = \overline{xAx}$ then this agrees with $\overline{xA}$ and $\overline{Ax}$, since for example $\overline{xAx} \subset \overline{xA} \subset A = \overline{xAx}$. The second part is clear from the fact that $x^{\frac{1}{k}} x \to x$ (see the second proof of Lemma 2.1). □

Theorem 2.19. *Let A be any operator algebra with cai. The following are equivalent:*

(i) *A has a countable cai.*
(ii) *A has a strictly real positive element.*
(iii) *There is an element x in $\mathfrak{F}_A$ with $s(x) = 1_{A^{**}}$.*

Proof. If A has a strictly real positive element x then $A = \overline{xAx}$ by Lemma 2.10, and (a scaling of) $(x^{\frac{1}{k}})$ is a countable cai.

If A has a countable cai (f_n), then A has a countable cai (e_n) in $\mathfrak{F}_A$ by Corollary 1.6. By Lemma 2.10 and Theorem 2.16 (2), $A = \overline{\bigcup_n e_n A} = \overline{zA}$ for a strictly real positive element $z \in A$.

The equivalence of (ii) and (iii) comes from Lemma 2.10. □

Definition 2.20. If A is an approximately unital operator algebra, then we define a *peak projection* for A to be the complement of a support projection $s(x)$, for an element $x \in \mathfrak{F}_A$. A *p-projection* for A is the infimum of a collection of peak projections for A.

If A is unital, this definition coincides with the ones discussed in [26,9], as was pointed out in the introduction (following from [9, Remark 6.10]).

Remark. If A is a nonunital C^*-algebra, then our Definition 2.20 is connected, via Corollary 2.8, to the one in [28], but it is not the same. Indeed, the function in $B = C_0((0,1))$ which is 0 until $\frac{1}{2}$, and then makes an inverted 'vee' of height 1, is in $\mathfrak{F}_B$, and the corresponding peak projection is the characteristic function of $[0, \frac{1}{2}]$. However, the latter is not an essential support projection in their sense.

Corollary 2.21. *For any approximately unital operator algebra A, a projection $q \in A^{**}$ is the complement of the support projection of an r-ideal iff q is the infimum of a collection of peak projections. These can be chosen to be a decreasing net.*

Proof. $(\Rightarrow)$ This follows from Theorem 2.15, and from the fact in the proof of Theorem 1.2 that the open projection corresponding to the closure of a sum of r-ideals, is the supremum of the open projections p_t corresponding to each of these r-ideals. In our case here each $p_t = s(x_t)$ for some $x_t \in \mathfrak{F}_A$, so that $p^\perp = \bigwedge_t s(x_t)^\perp$.

$(\Leftarrow)$ This follows from the fact that $s(x)^\perp$ is closed, and that the infimum of closed projections in A^{**} remains a closed projection in A^{**}, as we said in the introduction. □

Just as in the unital case [26,9], one may write down several equivalent characterizations of peak projections matching some of the characterizations in these papers. We will not take the time to do this here since most of these become cumbersome to state in the nonunital case. We will mention a characterization in terms of the tripotent $u(z) = \mathrm{w}^* \lim_n z(z^*z)^n$ considered by Edwards and Rüttimann [19]:

Proposition 2.22. *If A is any operator algebra and if $x \in \mathfrak{F}_A$, set $z = 1 - \frac{x}{2}$, where 1 is the identity of a C^*-algebra B containing A. Then $u(z)$ (computed with respect to B) is a projection and $u(z) = s(x)^\perp = \mathrm{w}^* \lim_n (z^*z)^n$.*

Proof. We may assume that $A = B$, and view $A \subset A^{**} \subset B(H)$. Since $z = \frac{1+(1-x)}{2}$, it is a contraction. It is well known (and easy to see) that $(z^*z)^n \to P$ weak*, where P is the projection onto $\mathrm{Ker}(1 - z^*z)$. We claim that $\mathrm{Ker}(1 - z^*z) = \mathrm{Ker}(x)$. Indeed clearly $\mathrm{Ker}(x) \subset \mathrm{Ker}(1 - z^*z)$. If R is the contraction $1 - x$ and $\zeta \in H$ is a unit vector, then $z^*z\zeta = \zeta$ implies that $(2\,\mathrm{Re}(R) + R^*R)\zeta = 3\zeta$, which implies that $\langle R\zeta, \zeta\rangle = \langle R^*R\zeta, \zeta\rangle = 1$. Hence $\|R\zeta - \zeta\|^2 = 0$, so $\zeta \in \mathrm{Ker}(x)$. By Lemma 2.5, $s(x)^\perp$ is the projection onto $\mathrm{Ker}(x)$, namely P above. Of course $zP = (1 - x/2)P = P$. Thus $(z^*z)^n \to s(x)^\perp$ weak*, and $z(z^*z)^n \to zP = P$ weak*. □

Corollary 2.23. *If A is an approximately unital operator algebra then every projection in $M(A)$ is a strong limit of a decreasing net of peak projections, and is also a strong limit of an increasing* *net of support projections of elements of $\mathfrak{F}_A$. If A is separable then we do not need to take limits here.*

Proof. The first statements follow from Corollary 2.21, since every projection in $M(A)$ is open and closed (see [9, Proposition 5.1]). The last statement follows from this and Corollary 2.17. $\square$

During the writing of the papers [9,26], we had believed (on the basis of a proof that had a gap) the following fact about compact projections q, hence about closed projections in the second dual of a unital algebra: If $\{u_i\colon i \in I\}$ is a collection of open projections whose supremum $\bigvee_{i\in I} u_i$ dominates a compact projection q, then $q \leqslant \bigvee_{i\in F} u_i$ for a finite set $F \subset I$. This is true if $q = 1$, or under some strong commutativity hypotheses, but is false in general (as may be seen by considering $A = \mathbb{K}(\ell^2)$ (or its unitization if one prefers a unital algebra), q the projection onto $\mathbb{C}e_1$, and u_k the projection onto $\mathrm{Span}(\{e_1 + e_2, e_2 + e_3, \ldots, e_k + e_{k+1}\})$. Then $\bigvee_k u_k = I$, but we do not have $q \leqslant \bigvee_{k=1}^n u_k$ for any finite n).

This incorrect statement was used only twice in those papers, namely in [26, Proposition 5.6] and [9, Theorem 6.4]. Fortunately both of these proofs can be fixed. There is a very short direct proof for [26, Proposition 5.6]: note that the result is true for peak projections since these are weak* limits of terms in A. Every p-projection q is a limit of a decreasing net of peak projections q_i, so $\varphi(q) = \lim_i \varphi(q_i) = 0$. We can fix the gap in the first two lines of one direction of the proof of [9, Theorem 6.4], and at the same time improve the result as follows:

Theorem 2.24 *(Noncommutative Urysohn lemma for nonselfadjoint operator algebras). Let A be a subalgebra of a unital C^*-algebra B, with $1_B \in A$, and let $q \in B^{**}$ be a closed projection. Then $q \in A^{\perp\perp}$ iff for any open projection $u \geqslant q$, and any $\epsilon > 0$, there exists an $a \in \mathrm{Ball}(A)$ with $aq = q$ and $\|a(1-u)\| < \epsilon$ and $\|(1-u)a\| < \epsilon$. Indeed this can be done with, in addition, $\|1 - 2a\| \leqslant 1$.*

Proof. ($\Leftarrow$) As in [9, Theorem 6.4].

($\Rightarrow$) Let $q \in A^{\perp\perp}$, let u be an open projection with $u \geqslant q$, and let $\epsilon > 0$ be given. Using Theorem 1.1, let (e_t) be a cai with $\|1 - 2e_t\| \leqslant 1$, for the HSA $q^\perp A^{**} q^\perp \cap A$ associated with q as in the introduction. Then $1 - e_t \to q$ weak*, and $(1 - e_t)q = q$. We follow the idea in the last seven lines of the proof of [9, Theorem 6.4]. By the noncommutative Urysohn lemma [1], there is an $x \in B$ with $q \leqslant x \leqslant u$. Then $(1 - e_t)(1 - x) \to q(1 - x) = 0$ weak*, and hence weakly in B. Similarly, $(1 - x)(1 - e_t) \to 0$ weakly. By a routine convexity argument in $B \oplus B$, given $\epsilon > 0$ there is a convex combination a of the $1 - e_t$ such that $\|a(1 - x)\| < \epsilon$ and $\|(1 - x)a\| < \epsilon$. Clearly $\|1 - a\| \leqslant 1$ and $aq = q$. Therefore $\|a(1 - u)\| = \|a(1 - x)(1 - u)\| < \epsilon$. Similarly for $\|(1 - u)a\| < \epsilon$. The estimate $\|1 - 2a\| \leqslant 1$ follows since $\|1 - 2(1 - e_t)\| = \|1 - 2e_t\| \leqslant 1$, and this formula persists with convex combinations. $\square$

We now show that the elements a in Theorem 2.24 constitute a left cai for the r-ideal associated with q, with the net constituting the cai indexed by the directed set of open projections $u \geqslant q$.

Corollary 2.25. *Let A be a unital subalgebra of C^*-algebra B and let $q \in A^{\perp\perp}$ be a closed projection associated with an r-ideal J in A. Then an explicit left cai for J is given by $x_{(u,\epsilon)} = 1 - a$, where a is an element which satisfies the conclusions of the last theorem, and is associated to an open projection $u \geqslant q$, and a scalar $\epsilon > 0$. This left cai is indexed by such pairs (u, ϵ), that* *is, by the product of the directed set of open projections $u \geqslant q$, and the set of $\epsilon > 0$. This right cai is also in $\frac{1}{2}\mathfrak{F}_A$; that is, $\|1 - 2x_{(u,\epsilon)}\| \leqslant 1$.*

Proof. By [3, Theorem 2.9], if p_1, p_2 are two open projections dominating q, then there exists a third open projection $p \geqslant q$ with $\{x \in B_+ \colon q \leqslant x \leqslant p\} \subset \{x \in B_+ \colon q \leqslant x \leqslant p_k\}$. By Lemma 2.7 in [3], there is an increasing net in the first of these sets with strong limit q, hence $q \leqslant p_k$ for $k = 1, 2$. Thus the set of open projections dominating q is a directed set.

By the last theorem, for each open projection $u \geqslant q$, and any $\epsilon > 0$, there exists an $a \in \mathrm{Ball}(A)$ with $aq = q$ and $\|a(1-u)\| < \epsilon$, and $\|1 - 2a\| \leqslant 1$. Let $x_{(u,\epsilon)} = 1 - a$. We claim that $(x_{(u,\epsilon)})$ is a cai for J. Certainly $x_{(u,\epsilon)}q = (1-a)q = q - q = 0$, so that $x_{(u,\epsilon)} \in J$. Also, $\|x_{(u,\epsilon)}\| \leqslant 1$, indeed $\|1 - 2x_{(u,\epsilon)}\| \leqslant 1$. If $b \in \mathrm{Ball}(J)$, we have

$$\|x_{(u,\epsilon)}b - b\| = \|ab\| \leqslant \big\|(a - au)b\big\| + \|aub\| \leqslant \big\|a(1-u)\big\| + \|ub\| < \epsilon + \|ub\|.$$

Thus we need to show that $ub \to 0$ in norm with u, over the directed set of open projections $u \geqslant q$. This is easy, for example in Akemann's proof in [26, Proposition 2.3] one associates to an increasing left cai (a_t) for $q^\perp B^{**} \cap B$, an open projection r_t with $q \leqslant r_t \leqslant 2(1 - a_t)$. It follows that $\|r_t b\| \leqslant 2\|(1 - a_t)b\| \to 0$, since $b \in J \subset q^\perp B^{**} \cap B$. Thus if $\|r_t b\| < \epsilon$, then

$$\|x_{(u,\epsilon)}b - b\| < \epsilon + \|ub\| < \epsilon + \|r_t b\| < 2\epsilon,$$

if $q \leqslant u \leqslant r_t$. $\square$

3. When xA and Ax are closed

Proposition 3.1. *If A has a cai but no identity, and $x \in \mathfrak{F}_A$ with $\overline{Ax} = A$, then $xA \neq A$. Hence for no strictly real positive $x \in A$ is xA closed.*

Proof. If $x = xy$ for some $y \in A$, then $e_t y = e_t \to y$ for the cai (e_t), so that A has identity. For the last statement use Lemma 2.10. $\square$

We recall that 'well-supported' operators are those operators x that have a 'spectral gap' (for $|x|$) at 0, that is 0 is absent from, or is isolated in, the spectrum (of $|x|$). It is a well-known principle in operator theory and C^*-algebras, that x is a well-supported operator (resp. element if a C^*-algebra A) iff x has closed range (resp. xA is closed), and this is equivalent to the existence of an operator y (resp. element $y \in A$) with $xyx = x$. Such a y is called a *generalized inverse* or *pseudoinverse*. See e.g. [7, II.3.2.11], [23,24]. With this in mind, it is tempting to conjecture that for an operator algebra A, a noninvertible element $x \in \mathfrak{F}_A$ has 0 isolated in (or absent from) its spectrum, iff xA is closed, and iff there exists $y \in A$ with $xyx = x$. However there are two issues that we have to deal with, for $x \in \mathfrak{F}_A$. First, we do not know if it is true that xA is closed iff Ax is closed. What is true is that xA and Ax is closed iff xAx is closed. Second, in a nonsemisimple setting, 0 being an isolated point in $\mathrm{Sp}(x)$ need not imply that xA is closed. Indeed there can exist quasinilpotent operators without closed range. Thus for example suppose that A is the radical operator algebra in Example 4.3, with cai $(e_t) \subset \mathfrak{F}_A$. Then e_t is quasinilpotent (any character of A certainly annihilates e_t, so $\mathrm{Sp}(e_t) = (0)$). Hence 0 is an isolated point of its spectrum, but $\overline{e_t A} = A$ since this algebra has no proper r-ideals, and this differs from $e_t A$ by Proposition 3.1. So $e_t A$ is not closed.

With the above in mind, the following result may be the best possible:

Theorem 3.2. *For an operator algebra A, if $x \in \mathfrak{F}_A$, the following are equivalent*:

(i) $x\,\mathrm{oa}(x)$ *is closed.*
(ii) $\mathrm{oa}(x)$ *is unital* (*which implies that x is invertible in* $\mathrm{oa}(x)$).
(iii) *There exists $y \in \mathrm{oa}(x)$ with $xyx = x$.*
(iv) *xAx is closed.*
(v) *xA and Ax are closed.*
(vi) *There exists $y \in A$ with $xyx = x$.*

Also, the latter conditions imply

(vii) 0 *is isolated in, or absent from,* $\mathrm{Sp}_A(x)$.

Finally, if further $\mathrm{oa}(x)$ *is semisimple, then conditions* (i)–(vii) *are all equivalent.*

Proof. If A is unital, $x \in \mathfrak{F}_A$, and x is invertible in A, then by Theorem 2.12 we have (ii), and indeed in this case (i)–(vii) are all obvious. So we can assume that x is not invertible in A.

That (i) implies (ii) follows since in this case $x \in \overline{x\,\mathrm{oa}(x)} = x\,\mathrm{oa}(x)$, and if $x = xy$ for $y \in \mathrm{oa}(x)$ then $y = 1_{\mathrm{oa}(x)}$. That the first condition in (ii) implies the second is clear (as in Theorem 2.12). Now the equivalences (i)–(iii) are obvious (some also follow from (iv)–(vi)), as is the fact that these imply some of (iv)–(vi).

(iv) $\Rightarrow$ (vi) Suppose that xAx is closed. Now $x = (x^{\frac{1}{3}})^3$, and

$$x^{\frac{1}{3}} \in \mathrm{oa}(x) = \overline{x\,\mathrm{oa}(x)x} \subset \overline{xAx} = xAx,$$

and so $x = xyx$ for some $y \in A$.

(vi) $\Rightarrow$ (v) (vi) implies that $xA = xyA$ is closed since xy is idempotent. Similarly Ax is closed.

(vi) $\Rightarrow$ (iv) $xAx = xyAyx$ is closed as in the last line.

(v) $\Rightarrow$ (vi) If xA and Ax are closed, then by a similar argument, $x^{\frac{1}{3}} \in xA$, and similarly $x^{\frac{1}{3}} \in Ax$. Hence $x = x^{\frac{1}{3}}x^{\frac{1}{3}}x^{\frac{1}{3}} \in xAx$. Thus $x = xyx$ for some $y \in A$.

(vi) $\Rightarrow$ (vii) We may assume that $A \subset B(H)$, and that x is pseudo-invertible as an operator on H. Then $x(H) \subset xy(H) \subset x(H)$, so that $x(H)$ is closed. Let P be the projection onto $K = \mathrm{Ker}(x)^{\perp} = x(H)$ (see Lemma 2.5), an invariant subspace for x. Let S be the restriction of x to K, then S is bicontinuous and invertible. If $K = H$ then x is invertible, and we discussed this case at the start of the proof. If $K \neq H$ then since $x = PSP$, it follows that $\mathrm{Sp}_{B(H)}(x) = \{0\} \cup \mathrm{Sp}_{B(K)}(S)$, which has 0 as an isolated point since S is invertible. If 0 was not isolated in $\mathrm{Sp}_A(x)$ then, by the topology of compact sets in the plane, there is a sequence of boundary points in $\mathrm{Sp}_A(x)$ converging to 0. Since $\partial\,\mathrm{Sp}_A(x) \subset \mathrm{Sp}_{B(H)}(x)$, this is a contradiction.

(vi) $\Rightarrow$ (ii) We saw in the last paragraph or two that (vi) implies that $x = 0 \oplus S$ for an invertible S. Note that $\mathrm{oa}(x) \cong \mathrm{oa}(S)$. By Theorem 2.12, $\mathrm{oa}(S)$ is unital, and S is invertible there. Thus the same is true for x.

(vii) $\Rightarrow$ (ii) Suppose that $\mathrm{oa}(x)$ is semisimple but nonunital, and that 0 is an isolated point in $\mathrm{Sp}_A(x)$. The latter is equivalent (by the basic spectral result for singly generated subalgebras, and because the spectrum is contained in the disk $B(1,1)$), to 0 being isolated in $K = \mathrm{Sp}_{\mathrm{oa}(x)}(x)$. Consider E, the spectral projection of x corresponding to $\{0\}$. Namely, $E = f(x)$ where f is 1 on a neighborhood of 0, and is 0 on a neighborhood of $K \setminus \{0\}$. We have $\mathrm{Sp}(Ex) = \mathrm{Sp}((fz)(x)) = (fz)(\mathrm{Sp}(x)) = \{0\}$. By semisimplicity, $Ex = 0$, and $(1 - E)x = x$. Since $(1 - f)(0) = 0$ we have $1 - E \in \mathrm{oa}(x)$. Note that if g is 0 on a neighborhood of 0, and is $1/z$ on a neighborhood of $K \setminus \{0\}$, then $gz^2 - z$ is zero on $\mathrm{Sp}(x)$, and so by semisimplicity we have $xg(x)x = x$. Since $g(0) = 0$ we have $g(x) \in \mathrm{oa}(x)$, and we deduce from the above that $\mathrm{oa}(x)$ is unital, and (ii) holds. □

Remark. The conditions in the theorem are not necessarily equivalent under the assumption that A is semisimple. For example, if $A = B(L^2([0,1]))$, suppose that $T \in A$ is any quasinilpotent operator with $T - I$ contractive. Then T does not have closed range, for if it did have closed range then as in the proof that (vi) ⇒ (vii) above, T is of the form $0 \oplus S$ for an invertible S. This is impossible for a quasinilpotent (since if $0 \neq t \in \mathrm{Sp}(S)$ then $t \in \mathrm{Sp}(T) = (0)$, a contradiction). It is easy to see that then (vi) fails, and so (i)–(v) fail too. However 0 is isolated in the spectrum of this quasinilpotent operator.

Also in this connection we remark that all C^*-algebras are semisimple, yet C^*-algebras may have dense subalgebras consisting entirely of nilpotents (hence quasinilpotents) [34].

4. Operator algebras without HSA's

In this section we study operator algebras A without nontrivial HSA's or r-ideals. By a trivial HSA (or r-ideal) of A we mean of course (0) or A.

Theorem 4.1. *For a unital operator algebra A, the following are equivalent*:

(i) *A has no nontrivial r-ideals (or equivalently, HSA's).*
(ii) $a^n \to 0$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}1$.
(iii) *The spectral radius* $r(a) < \|a\|$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}1$.
(iv) *The numerical radius* $\nu(a) < \|a\|$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}1$.
(v) $\|1 + a\| < 2$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}1$.
(vi) $\mathrm{Ball}(A) \setminus \mathbb{C}1$ *consists entirely of quasi-invertibles.*

If A has a cai but no identity, then the following are equivalent:

(a) *A has no nontrivial r-ideals (or equivalently, HSA's).*
(b) *A^1 has one nontrivial r-ideal.*
(c) $\mathrm{Re}(x)$ *is strictly positive for every* $x \in \mathfrak{F}_A \setminus \{0\}$.

Proof. The equivalences (i)–(vi) are in [4, Section 3], if one uses the fact that every r-ideal contains what we called a 1-regular ideal (defined before Corollary 1.3), which is a consequence of Theorem 1.1. The one direction of the equivalence of (a) and (b) is obvious. For the other, suppose that $x \in \mathrm{Ball}(A^1)$. Then $\overline{(1-x)A}$ is an r-ideal in A by [9, Proposition 3.1] (or Theorem 7.1 below). So if A has no nontrivial r-ideals then either $\overline{(1-x)A} = (0)$ or $\overline{(1-x)A} = A$. In the first case, $xa = a$ for all $a \in A$, which forces (if A has a cai) $x = 1$ and $\overline{(1-x)A^1} = (0)$. In the second case: in the notation of the proof of Theorem 7.1 and the Remark after it, $\overline{(1-x)A}$ has bai $(e_n f_t)$, which has weak* limit pf if f is the left identity of $A^{\perp\perp}$ and p is the weak* limit of e_n. So $pf = f$, which forces $p = f$ or $p = 1$. If $p = f$ then $\overline{(1-x)A^1} = A$. If $p = 1$ then $\overline{(1-x)A^1} = A^1$. By the Remark after Theorem 1.2, for example, A^1 has A as its only nontrivial r-ideal.

That (a) is equivalent to (c) follows easily from Lemma 2.10, and the fact that r-ideals are 'sups' of peak-principal ones (see Theorem 2.15). □

Remarks. (1) Simple examples (the 2×2 matrices supported on the first row) show that the equivalence with (b) in the last result is not true without a cai.

(2) If there exist nontrivial r-ideals in a unital operator algebra A, then there exist proper maximal r-ideals in A. This follows from [4, Proposition 3.6] and the Remark after Theorem 1.2 above.

Theorem 4.2. *An approximately unital operator algebra with no countable cai, has nontrivial r-ideals.*

Proof. If A has no countable cai then by Theorem 2.19 there is no element in $\mathfrak{F}_A$ with $s(x) = 1$. Thus for any nonzero $x \in \mathfrak{F}_A$, we have $\overline{xA} \neq A$ by Lemma 2.10, and this is a nontrivial r-ideal. □

Proposition 4.3. *If a nonunital operator algebra A contains a nonzero $x \in \mathfrak{F}_A$ with xAx closed, or with 0 isolated in $\mathrm{Sp}_A(x)$ and $\mathrm{oa}(x)$ semisimple, then A has a nontrivial r-ideal.*

Proof. By Theorem 3.2, under these conditions xA is closed, and so $\overline{xA} \neq A$ by Proposition 3.1. This is a nontrivial r-ideal. □

Proposition 4.4. *A nontrivial r-ideal in the unitization of an approximately unital radical operator algebra A, is an r-ideal in A.*

Proof. If $\lambda \neq 1$ then we claim that $z = \lambda 1 + a$ is quasi-invertible in A^1, so that $(1-z)A^1 = A^1$, for all $a \in A$. Indeed we know that $\frac{a}{1-\lambda}$ is quasi-invertible in A, and easy algebra shows that its quasi-inverse gives rise to a quasi-inverse of $\lambda 1 + a$. □

We now give several examples of operator algebras with cai, with only trivial r-ideals.

4.1. Example. A unital two-dimensional algebra

Consider the upper triangular 2×2 matrices whose 1–2 entry is the difference of the diagonal entries.

4.2. Example. A nonunital commutative semisimple algebra

Set $A = RDR^{-1}$, where D is the diagonal copy of c_0 in $B(\ell^2)$, and R is an invertible operator in $B(\ell^2)$, such that the commutant of R^*R contains no nontrivial projections in the diagonal copy of ℓ^∞ in $B(\ell^2)$. For example, $R = I + S/2$ where S is the backwards shift. Since A is a subalgebra of the compact operators, its second dual may be identified with its σ-weak closure in $B(\ell^2)$. Thus A^{**} is unital, so that A has cai. In this case, there are no nontrivial projections in $A^{**} = A^{\perp\perp} = R\overline{D}^{w*}R^{-1}$. Indeed any projection q in $R\overline{D}^{w*}R^{-1}$ corresponds to an idempotent, hence projection, p in $\overline{D}^{w*}$. Any projection in $\overline{D}^{w*}$ is a sequence of 0's and 1's. That $q = q^*$ forces p to commute with R^*R, so that $p = 0$ or $p = 1$. Thus A has no nontrivial r-ideals.

4.3. Example. A commutative radical algebra

Let A be the norm closed algebra generated by the Volterra operator V. This is commutative, and so since V is quasinilpotent we have that A is radical. By [16, Corollary 5.11] we have $\bar{A}^{w*} = V'$. As in the last example this coincides with A^{**}, and since this is unital we see that A has a cai. By [16, Lemma 5.1], V' contains no nontrivial projections, hence the same is true for A^{**}. Thus A has no HSA's or r-ideals.

In the next section we will continue looking at examples.

5. An approximately unital radical operator algebra which is an integral domain

In this section we present an interesting commutative approximately unital operator algebra, which happens to be radical and semiprime, and in fact is an integral domain (so $ab = 0$ exactly when $a = 0$ or $b = 0$). It is also an operator algebra whose ideal structure we can completely describe. This is achieved by adapting the work of Domar [18] on convolution algebras $L^1(\mathbb{R}^+, \omega)$; he showed that certain conditions on ω imply that $L^1(\mathbb{R}^+, \omega)$ is a radical Banach algebra and that all of its closed ideals are of a certain 'standard' type which we will describe below. A simplified exposition of Domar's result can be found on p. 554 of Dales [15]. The operator algebras we produce will clearly have no nontrivial ideals having approximate identities. They will have some features in common with Example 4.3, but in other ways they are very different.

For our purposes, a *radical weight* $\omega : [0, \infty) \to (0, \infty)$ is a continuous function such that $\omega(0) = 1$, $\omega(s + t) \leqslant \omega(s)\omega(t)$ for all $s, t \geqslant 0$, and $\omega(t)^{1/t} \to 0$ as $t \to \infty$. The Banach spaces $L^p(\mathbb{R}^+, \omega)$ $(1 \leqslant p < \infty)$ consist of equivalence classes of measurable functions $f : \mathbb{R}^+ \to \mathbb{C}$ such that $\|f\|_p = (\int_0^\infty |f(t)|^p \omega(t)^p \, dt)^{1/p} < \infty$. The space $L^1(\mathbb{R}^+, \omega)$ is a Banach algebra when given the convolution multiplication (see [15, Section 4.7]). For each $\alpha \geqslant 0$, there is the "standard ideal" $J_\alpha \subset L^1(\mathbb{R}^+, \omega)$ consisting of functions supported on $[\alpha, \infty)$, and this ideal is always norm closed.

We say that the radical weight ω satisfies Domar's criterion if the function $\eta(t) = -\log \omega(t)$ is a convex function on $(0, \infty)$, and for some $\varepsilon > 0$ we have $\eta(t)/t^{1+\varepsilon} \to \infty$ as $t \to \infty$. An obvious example of such a weight is $\omega(t) = e^{-t^2}$. Domar's theorem asserts that if the radical weight ω satisfies Domar's criterion, then the standard ideals are the only nonzero closed ideals in $L^1(\mathbb{R}^+, \omega)$. We will use this result to obtain radical operator algebras with interesting properties. Let ω denote any radical weight. The algebra $L^1(\mathbb{R}^+, \omega)$ acts on $L^2(\mathbb{R}^+, \omega)$ by convolution, for one may readily check that the familiar inequality $\|f * g\|_2 \leqslant \|f\|_1 \cdot \|g\|_2$ still holds when the L^p spaces are given their radical weighting according to ω. If we write H for the Hilbert space $L^2(\mathbb{R}^+, \omega)$ and M_f for the operator on H with $M_f(g) = f * g$, then the norm closure of the operators M_f $(f \in L^1(\mathbb{R}^+, \omega))$, is an operator algebra $\mathcal{A} = \mathcal{A}(\omega)$. Indeed, the set of operators M_f is already a subalgebra of $B(H)$ since $M_f \cdot M_g = M_{f*g}$.

Clearly $\mathcal{A}$ is a commutative operator algebra, and we claim that it has a dense set of quasinilpotent elements consisting of operators M_f, so that $\mathcal{A}$ is also radical. Let $|\cdot|_1$ denote the *unweighted* L^1 norm, that is, the usual norm on $L^1(\mathbb{R}^+)$, and let $\|\cdot\|_1$ be the weighted norm on $L^1(\mathbb{R}^+, \omega)$. If

$f \in L^1(\mathbb{R}^+, \omega)$ is supported on $[a, b]$, where $0 < a < b$, then $\|f\|_1 \geqslant |f|_1 \cdot \min\{\omega(t)\colon t \in [a, b]\}$. The convolution power $*^n f$ is supported on $[na, nb]$, and so $\|*^n f\|_1 \leqslant |*^n f|_1 \cdot \max\{\omega(t)\colon t \in [na, nb]\}$. We deduce that

$$\left\|*^n f\right\|_1 \leqslant \|f\|_1^n \cdot \frac{\max\{\omega(t)\colon t \in [na, nb]\}}{(\min\{\omega(t)\colon t \in [a, b]\})^n}.$$

Taking nth roots and using the spectral radius formula and the fact that $\omega(t)^{1/t} \to 0$, we see that f is a quasinilpotent element of $L^1(\mathbb{R}^+, \omega)$. Since the operator norm in $\mathcal{A}$ is bounded by the L^1 norm $\|\cdot\|_1$, it follows that M_f is quasinilpotent in $\mathcal{A}$ too.

The fact that $\omega(t) \to 1$ as $t \to 0$ ensures that for small $\varepsilon > 0$, the L^1 norm of any nonnegative function f whose integral is 1 and which is supported on $[0, \varepsilon]$, is close to 1. The corresponding operators M_f form a contractive approximate identity for $\mathcal{A}$. Also, $\mathcal{A}$ has, for each $\alpha \geqslant 0$, a "standard ideal" J_α consisting of the norm closure of operators M_f with $f \in L^1$ supported on $[\alpha, \infty)$. We shall show:

Theorem 5.1. *For any radical weight ω, the algebra $\mathcal{A}(\omega)$ is an integral domain with cai. If the radical weight ω satisfies Domar's criterion, then the standard ideals are the only nonzero closed ideals of $\mathcal{A}$.*

Note that $J_\alpha \cdot J_\beta \subset J_{\alpha+\beta}$, so $J_\alpha \neq \overline{J_\alpha^2}$ for $\alpha > 0$. In particular, the nontrivial standard ideals do not have any approximate identity and are not r-ideals. So when the above theorem is proved, we will have shown that the algebra $\mathcal{A}$ has all the properties claimed at the start of the section, provided the weight ω satisfies Domar's criterion.

Recall that $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ denotes the Fréchet space of locally integrable measurable functions on $\mathbb{R}^+$. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, we define $\alpha(f)$ to be the minimum of the support of f (or $+\infty$ if $f = 0$). We will use the Titchmarsh convolution theorem (see e.g. [15, Theorem 4.7.22]), which states for example that $\alpha(f * g) = \alpha(f) + \alpha(g)$ for $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$. In particular this is the case when $f \in L^p(\mathbb{R}^+, \omega)$ and $g \in L^q(\mathbb{R}^+, \omega)$ for some $p, q \geqslant 1$ (for on compact intervals ω is bounded away from 0 and so by the Hölder inequality it is clear that $L^p(\mathbb{R}^+, \omega) \subset L^1_{\mathrm{loc}}$ for every $p \geqslant 1$). For an operator $T \in \mathcal{A}$ we define $\alpha(T) = \inf\{\alpha(Tf)\colon f \in L^2(\mathbb{R}^+, \omega)\}$.

Lemma 5.2. *For each radical weight ω and each $S, T \in \mathcal{A}(\omega)$, we have $\alpha(ST) = \alpha(S) + \alpha(T)$.*

Proof. Let $f \in L^2(\mathbb{R}^+, \omega)$ and let $g = T(f)$, so that $\alpha(g) \geqslant \alpha(T)$. Write $g_1(x) = g(x) \cdot \mathbf{1}_{x \leqslant \alpha(S)+\alpha(T)}$. Then g_1 is compactly supported with $|g_1|^2 \in L^1$, and so $g_1 = \delta_{\alpha(T)} * g_0$ for some $g_0 \in L^2$. Then $S(g_1) = \delta_{\alpha(T)} * S(g_0)$, because S is a norm limit of convolution operators. Hence $\alpha(S(g_1)) = \alpha(T) + \alpha(S(g_0)) \geqslant \alpha(S) + \alpha(T)$. The functions $Sg_1(x)$ and $Sg(x)$ agree for $x \leqslant \alpha(T) + \alpha(S)$, because $g(x)$ and $g_1(x)$ agree for such values, and the subspace of functions f with $\alpha(f) \geqslant \alpha$ is invariant for all operators under consideration. Thus $\alpha(STf) = \alpha(Sg) \geqslant \alpha(T) + \alpha(S)$, so for all S and T we have $\alpha(ST) \geqslant \alpha(S) + \alpha(T)$.

To prove the converse, we will use the fact that the function $f \mapsto \alpha(f)$ is upper semicontinuous on $L^p(\mathbb{R}^+, \omega)$. This is because if the minimum of the support of $f \in L^p$ is α, then for $\varepsilon > 0$ the integral $\int_\alpha^{\alpha+\varepsilon} |f(t)|^p\, dt$ is strictly positive. Hence for functions g sufficiently close to f in p norm, we will have $\int_\alpha^{\alpha+\varepsilon} |g(t)|^p\, dt > 0$ also, and in particular $\alpha(g) < \alpha + \varepsilon$.

Since the set $C_{00}(\mathbb{R}^+)$ of continuous functions of compact support is dense in L^2, given $\varepsilon > 0$ we may pick $f, g \in C_{00}(\mathbb{R}^+)$ such that $\alpha(Sf) \leqslant \alpha(S) + \varepsilon/2$ and $\alpha(Tg) \leqslant \alpha(T) + \varepsilon/2$. Then $ST(f * g) = S(f * (Tg))$, because T is a norm limit of convolution operators. This is equal to $S(f) * T(g)$ because S is a norm limit of convolution operators. By the Titchmarsh convolution theorem, $\alpha(S(f) * T(g)) = \alpha(Sf) + \alpha(Tg)$, and so

$$\alpha(ST) \leqslant \alpha\big(ST(f * g)\big) = \alpha(Sf) + \alpha(Tg) \leqslant \alpha(S) + \alpha(T) + \varepsilon.$$

Hence $\alpha(ST) \leqslant \alpha(S) + \alpha(T)$, and the lemma is proved. □

Corollary 5.3. *The algebra $\mathcal{A}(\omega)$ is an integral domain.*

We have defined the standard ideal J_α to be the closure in $\mathcal{A}$ of the operators M_f with $f \in L^1(\mathbb{R}^+, \omega)$ and $\alpha(f) \geqslant \alpha$. There is another obvious closed ideal, namely $I_\alpha = \{T \in \mathcal{A}\colon \alpha(T) \geqslant \alpha\}$. We now show that these two ideals coincide.

Lemma 5.4. *Let ω be any radical weight. Then for each $\alpha \geqslant 0$, the ideals I_α and $J_\alpha \subset \mathcal{A}(\omega)$ are the same.*

Proof. Every operator M_f with $f \in L^1(\mathbb{R}^+, \omega)$ and $\alpha(f) \geqslant \alpha$ is plainly in I_α, which is closed because it is the set of operators in $\mathcal{A}$ which map H into the closed subspace of functions supported on $[\alpha, \infty)$. Therefore $J_\alpha \subset I_\alpha$. Conversely, let $T \in I_\alpha$ with $T = \lim_i M_{f_i}$, $f_i \in L^1(\mathbb{R}^+, \omega)$. We claim that $T \in J_\alpha$, which will imply that the two ideals are the same. To prove this, let f be smooth and compactly supported in $[0, \infty)$. The operator $T \cdot M_f = \lim_i M_{f_i * f}$, and the function $\gamma = T(f) = \lim_i f_i * f$, are supported on $[\alpha, \infty)$. Thus $(f_i * f) \cdot \mathbf{1}_{[0,\alpha]} \to 0$ in $L^2([0, \alpha], \omega)$, and even in $L^1([0, \alpha], \omega)$, because the L^1 norm on the compact set $[0, \alpha]$ is bounded by a constant times the L^2 norm. It follows that if $\gamma_i = (f_i * f) \cdot \mathbf{1}_{[0,\alpha]} \in \mathcal{A}$ then the operator norm $\|\gamma_i\| \to 0$ (for the operator norm is at most the L^1 norm, which is known to tend to zero). So $TM_f = \lim_i M_{(f_i * f) \cdot \mathbf{1}_{(\alpha,\infty)}} \in J_\alpha$. However, the algebra $\mathcal{A}$ is known to have a sequential cai consisting of (convolution operators by) functions u_i which are smooth and compactly supported in $[0, \infty)$. So $T = \lim_i TM_{u_i} \in J_\alpha$ also, and $I_\alpha = J_\alpha$. □

Lemma 5.5. *If the radical weight ω satisfies Domar's criterion, then for each $t > 0$ the integral $\int_0^\infty (\omega(x + t)/\omega(x))^2\, dx$ is finite.*

Proof. The function $\eta(x) = -\log \omega(x)$ is a convex function by Domar's criterion. Thus the ratio $\omega(t + x)/\omega(x) = \exp(\eta(x) - \eta(t + x))$ is a decreasing function of x. We know that $\eta(x) \geqslant x^{1+\varepsilon}$ for large x, so that $\eta(nt) \geqslant t^{1+\varepsilon} n^{1+\varepsilon}$ for large enough n. Since $\eta(0) = 0$, we have $\sum_{r=1}^n \eta(rt) - \eta((r - 1)t) \geqslant t^{1+\varepsilon} n^{1+\varepsilon}$, and so the largest term $\eta(nt) - \eta((n - 1)t)$ must dominate $n^\varepsilon t^{1+\varepsilon}$. This implies that for all large enough n we have $\omega(nt)/\omega((n - 1)t) \leqslant e^{-cn^\varepsilon}$, where $c = t^{1+\varepsilon}$. Since $\omega(x + t)/\omega(x)$ is a decreasing function, this implies that $\int_0^\infty (\omega(x + t)/\omega(x))^2\, dx < \infty$. □

Corollary 5.6. *If ω satisfies Domar's criterion, $f \in L^2(\mathbb{R}^+, \omega)$ and $t > 0$ then $f * \delta_t \in L^1(\mathbb{R}^+, \omega)$.*

Proof. By the Cauchy–Schwarz inequality we have

$$\int_0^\infty |f(x)|\omega(x+t)\,dx \leqslant \left(\int_0^\infty |f(x)|^2\omega(x)^2\,dx\right)^{1/2}\left(\int_0^\infty \bigl(\omega(x+t)/\omega(x)\bigr)^2\,dx\right)^{1/2} < \infty. \qquad \square$$

Lemma 5.7. *Suppose a radical weight ω satisfies Domar's criterion, $T \in \mathcal{A}(\omega)$ is nonzero, and $\alpha(T) = \alpha$. Then the closed principal ideal $\overline{T \cdot \mathcal{A}}$ is equal to I_α.*

Proof. By Lemma 5.2 we have $\overline{T \cdot \mathcal{A}} \subset I_\alpha$. Conversely, since $I_\alpha = J_\alpha$ it is enough to show that for each $f \in L^1(\mathbb{R}^+, \omega)$ with $\alpha(f) \geqslant \alpha$ we have $M_f \in \overline{T \cdot \mathcal{A}}$. Any such f is a norm limit of functions $f_n \in L^1(\omega)$ with $\alpha(f_n) > \alpha$; and so it is enough to show that $M_f \in \overline{T \cdot \mathcal{A}}$ when $\alpha(f) > \alpha$. Given such a function f, pick $g \in L^2(\mathbb{R}^+, \omega)$ with $\alpha(Tg) < \alpha(f)$ (this is possible because the infimum of values $\alpha(Tg)$ is by hypothesis equal to α), and pick $t > 0$ such that we have $t + \alpha(Tg) = \alpha(\delta_t * Tg) < \alpha(f)$. By Corollary 5.6 the function $h_0 = \delta_t * g$ is in $L^1(\mathbb{R}^+, \omega)$, as also is $h = \delta_t * Tg$.

By Domar's theorem the closed ideal generated by h in $L^1(\mathbb{R}^+, \omega)$ is standard. It therefore contains every function k in $L^1(\mathbb{R}^+, \omega)$ with $\alpha(k) \geqslant \alpha(h)$, so it contains the function f. There is a sequence of functions $u_i \in L^1(\mathbb{R}^+, \omega)$ with $u_i * h \to f$. The operator norm on $\mathcal{A}$ is bounded by the L^1 norm so $M_{u_i*h} \to M_f$ in $\mathcal{A}$. Now T is a norm limit of convolution operators: $T = \lim_j M_{\tau_j}$ with $\tau_j \in L^1(\mathbb{R}^+, \omega)$. So for any $\gamma \in C_{00}(\mathbb{R}^+)$, we have

$$\begin{aligned} M_{u_i*h}(\gamma) &= u_i * (\delta_t * Tg) * \gamma = u_i * \delta_t * \Bigl(\lim_j \tau_j * g\Bigr) * \gamma \\ &= \lim_j \tau_j * u_i * \delta_t * g * \gamma = T \cdot M_{u_i*h_0}(\gamma), \end{aligned}$$

because convergence of $\tau_j * g$ occurs in $L^2(\mathbb{R}^+, \omega)$, and both $u_i * \delta_t$ and γ are in $L^1(\mathbb{R}^+, \omega)$, and $L^1(\mathbb{R}^+, \omega)$ acts continuously on $L^2(\mathbb{R}^+, \omega)$ by convolution.

Since C_{00} is dense in L^2, the operators $T \cdot M_{u_i*h_0}$ and M_{u_i*h} are equal. Hence the operator $T \cdot M_{u_i*h}$ is in the principal ideal $T\mathcal{A}$. Therefore the closure $\overline{T\mathcal{A}}$ contains M_f for every $f \in L^1(\mathbb{R}^+, \omega)$ with $\alpha(f) > \alpha$. Thus $\overline{T\mathcal{A}} = I_\alpha$ as claimed. $\square$

Proof of Theorem 5.1. By Corollary 5.3, $\mathcal{A}(\omega)$ is an integral domain for any radical weight ω. Let $J \subset \mathcal{A}$ be any nonzero closed ideal and let $\alpha = \alpha(J) = \inf\{\alpha(T)\colon T \in J\}$. We claim that $J = I_\alpha$. From the definition it is plain that $J \subset I_\alpha$. For the converse, choose $T_n \in J$ with $\alpha(T_n) \to \alpha$. By Lemma 5.7, J contains I_β for a sequence of values β tending to α. In particular J contains the operator M_f for every $f \in L^1(\mathbb{R}^+, \omega)$ with $\alpha(f) > \alpha$. The closure of this set includes every M_g with $\alpha(g) \geqslant \alpha$. Hence it contains $J_\alpha = I_\alpha$. Thus $J = I_\alpha$ as claimed. $\square$

It is easy to see that the algebras $A(\omega)$ above contain no idempotents, and are 'modular annihilator algebras'. As in [15], the function $u = 1$ in $L^1(\mathbb{R}^+, \omega)$ corresponds to a single generator for the algebra A.

6. Pre-images of HSA's

If J is a closed ideal in an approximately unital operator algebra, we examine the relation between $\mathfrak{F}_A$ and $\mathfrak{F}_{A/J}$. From Meyer's theorem ([30], [10, Theorem 2.1.13]) one can see that the 'image' of $\mathfrak{F}_A$ in A/J is a subset of $\mathfrak{F}_{A/J}$.

Proposition 6.1. *If J is a closed ideal in an operator algebra A, and if J has a cai, then $q(\mathfrak{F}_A) = \mathfrak{F}_{A/J}$, where $q : A \to A/J$ is the canonical map.*

Proof. Indeed suppose that $x \in A/J$ with $\|1 - x\| \leqslant 1$ in $A^1/J \cong (A/J)^1$. Since J is an M-ideal in A^1 (see e.g. [10, Theorem 4.8.5]), it is proximinal [22]. Hence there is an element $z = \lambda 1 + a$ in $\mathrm{Ball}(A^1)$, with $\lambda \in \mathbb{C}$, $a \in A$, such that $\lambda 1 + a + J = 1 - x$. It is easy to see now that $\lambda = 1$, and $a + J = -x$. Let $y = -a$. Then $\|1 - y\| = \|1 + a\| = \|z\| \leqslant 1$, so $y \in \mathfrak{F}_A$, and $q(y) = x$. $\square$

Proposition 6.2. *If J is a closed ideal in an operator algebra A, and if J has a cai, then any closed approximately unital subalgebra D in A/J is the image of a closed approximately unital subalgebra in A, under the quotient map q_J from A onto A/J. In fact $q_J^{-1}(D)$ will serve here.*

Proof. The idea for this proof was found independently by M. Almus. Note that J is an approximately unital ideal in $q_J^{-1}(D)$. Moreover, $q_J^{-1}(D)/J \cong D$, which is approximately unital. So $q_J^{-1}(D)$ is approximately unital by [12, Proposition 3.1]. Another proof follows immediately from 3.4 in [12], since, in the language there, $B \oplus_C C' = \beta^{-1}(C')$ clearly. $\square$

Corollary 6.3. *Let J be a closed approximately unital two-sided ideal in an operator algebra A, and let $q_J : A \to A/J$ be the quotient map.*

(i) *The open projections in $(A/J)^{**}$ are exactly the $q_J^{**}(p)$, for open projections p in A^{**}.*
(ii) *The HSA's in A/J are precisely the images of the HSA's in A, under q_J.*
(iii) *The r-ideals in A/J are precisely the images of the r-ideals in A, under q_J.*
(iv) *An r-ideal (resp. HSA) in A/J of the form $\overline{x(A/J)}$ (resp. $\overline{x(A/J)x}$) for some $x \in \mathfrak{F}_{A/J}$, is the image of an r-ideal (resp. HSA) in A of the form $\overline{yA}$ (resp. $\overline{yAy}$) for some $y \in \mathfrak{F}_A$.*

Proof. It is easy to see that the images of HSA's (resp. r-ideals) in A, are HSA's (resp. r-ideals) in A/J. If p is open in A^{**} then p is the weak* limit of a net (a_t) in A with $a_t = pa_tp$. Then $q_J^{**}(p)$ is the weak* limit of a similar net in A/J, so is open there. Items (i)–(iii) follow easily from these observations, and Proposition 6.2. For (i), if p is the support projection of $D' = q_J^{-1}(D)$, where D is the HSA associated with p, then $q_J^{**}(p)$ is the support projection of D. So the open projections in $(A/J)^{**}$ are precisely the $q_J^{**}(p)$, for open projections $p \in A^{**}$.

Item (iv) follows easily from Proposition 6.1; and that result also leads to another proof of (i)–(iii), which seems to give a possibly different pre-image. We give the argument in the r-ideal case: Let K be an r-ideal in A/J. By Corollary 1.4, there is a lcai (e_t) in K with $\|1 - 2e_t\| \leqslant 1$. As above we obtain $x_t \in \mathfrak{F}_A$, with $q_J(x_t) = e_t$. The closure of the sum of the right ideals x_tA, is an r-ideal K' in A by Theorem 2.15. Moreover $q_J(K')$ is contained in the closure of the union of the $q_J(x_t)(A/J) = e_t(A/J) \subset K$. Conversely, for any $a \in A$, we have $e_t(a + J) = q_J(x_ta)$; and since (e_t) is a lcai for K it follows that $K \subset q_J(K')$. So $q_J(K') = K$. $\square$

This technique seems applicable to other 'constructions' besides quotients, such as direct limits, ultrapowers, interpolated operator algebras, etc. See [10, Sections 2.2 and 2.4] for some of these constructions. Indeed the results apply directly to ultraproducts because they are quotients of the type described in this section.

7. Other constructions of r-ideals

The following is an improvement of [4, Proposition 3.1], and also answers the question in the Remark following it.

Theorem 7.1. *If A is an operator algebra with left cai, which is a left ideal in an operator algebra B, then $\overline{(1-x)A}$ is an r-ideal in A for all $x \in \mathrm{Ball}(B)$.*

Proof. We may assume that B is unital. Certainly $J = \overline{(1-x)A}$ is a right ideal, the question is whether it has a left cai. Note that $e_n = 1 - \frac{1}{n}\sum_{k=1}^{n} x^k$ defines a bounded net, $\|1 - e_n\| \leqslant 1$, and $e_n(1-x) = 1 - x - \frac{1}{n}(1 - x^{n+1}) \to 1 - x$. Suppose that (f_t) is a left cai for A, with weak* limit $f \in A^{\perp\perp}$, which is a projection and a left identity for $A^{\perp\perp}$. We may view $(e_n f_t)$ as a net in $(1-x)A \subset J$, with the product indexing, and it is easy to see using the above that $e_n f_t(1-x)a \to (1-x)a$ for all $a \in A$. Suppose that a subnet $((1 - e_{n_\mu}) f_{t_\mu})$ converges weak* to an element r. Then it is easy to see that r is a contraction in $A^{\perp\perp}$, so that $fr = r$. Also, $e_{n_\mu} f_{t_\mu} \to f - r$, so that $f - r \in J^{\perp\perp}$. Since $e_{n_\mu} f_{t_\mu}(1-x)a \to (1-x)a$, we have $(f-r)z = z$ for all $z \in J$, hence for all $z \in J^{\perp\perp}$. So $f - r$ is a left identity for $J^{\perp\perp}$, hence it is idempotent. That is, $f - fr - rf + r^2 = f - r$, which by a fact above implies that $r^2 = rf$. If we choose (f_t) so that $f_s f_t \to f_t$ with t (as in Corollary 1.4 above, or [9, Corollary 2.6]), we may assume that $f_t f = f_t$, which forces $rf = r$ by definition of r. Thus r is idempotent, hence is a projection, and so $f - r$ is a projection too. By e.g. [10, Proposition 2.5.8], J has a left cai. □

Remarks. (1) We do not have a clean formula for the left cai in the last result, although there is one for a left bai: the net $(e_n f_t)$ in the proof is a left bai. This illustrates the fact that although we may know that left cai exist of a nice form (as in Theorem 1.1 or in the Remark after Theorem 2.16), we may not be able to write a simple expression for them.

(2) Considering the example of the 2×2 matrices supported on the first column, shows that the last result is best possible. That is, the hypothesis of a left cai is not removable.

The following result is actually equivalent to the last theorem:

Corollary 7.2. *If A is an operator algebra with left cai, and if $\eta : A \to A$ is a completely contractive left A-module map, or if $\eta \in \mathrm{Ball}(A^{**})$ satisfies $\eta A \subset A$, then $\overline{(1-\eta)A}$ is an r-ideal of A.*

Proof. Consider $B = \{\eta \in A^{**}\colon\ \eta A \subset A\}$, an operator algebra containing A as a left ideal. Thus the second case of our result follows from Theorem 7.1.

The set of completely bounded left A-module maps 'equals' B by [8, Theorem 6.1] (note that a hypothesis in the latter theorem was removed in [9, Corollary 2.6]), and hence this case follows by the last paragraph. □

Remarks. (1) By another equivalence in [8, Theorem 6.1], the last result is correct with η a contraction in the operator space left multiplier algebra $\mathcal{M}_\ell(A)$ (see [10, Chapter 4] for the definition of the latter). If A is approximately unital then $\mathcal{M}_\ell(A) = LM(A)$, the ordinary left multiplier algebra.

(2) The first result of this type that we are aware of dates to 2005 (see [9, Lemma 6.8], but this is much less general). See also [27], for some recent Banach algebra variants.

According to [9, Corollary 2.7], there is a bijective correspondence between the classes of r-ideals, ℓ-ideals, and HSA's, of A. One may ask what is the ℓ-ideal and HSA matching the r-ideal in Theorem 7.1, in terms of x? In general we do not have a simple answer. However we have:

Proposition 7.3. *If A is an operator algebra with cai, which is an ideal in an operator algebra B, and $x \in \mathrm{Ball}(B)$, then the ℓ-ideal and HSA matching the r-ideal $\overline{(1-x)A}$, are $\overline{A(1-x)}$ and $\overline{(1-x)A(1-x)}$.*

Proof. We may assume that B is unital. Then A corresponds to a central projection p in B^{**}, whereas the r-ideal $J = \overline{(1-x)B}$ in B has a support projection $e \in B^{**}$, say. Then $J^{\perp\perp} = eB^{**}$, $\overline{B(1-x)}^{\perp\perp} = B^{**}e$, and $\overline{(1-x)B(1-x)}^{\perp\perp} = eB^{**}e$ by facts in [9]. Since e and p commute, we have that $eB^{**} \cap B^{**}p = epB^{**}$. By [13, 5.2.7], $A^\perp + J^\perp$ is closed, and by a formula in the proof of [13, 5.2.9], we have that

$$(A \cap J)^{\perp\perp} = \left(A^\perp + J^\perp\right)^\perp = A^{\perp\perp} \cap J^{\perp\perp} = pB^{**} \cap eB^{**} = epB^{**}.$$

Similarly, $(A \cap \overline{B(1-x)})^{\perp\perp} = B^{**}ep$. Now $\overline{A(1-x)} \subset A \cap \overline{B(1-x)}$. Conversely if $z \in A \cap \overline{B(1-x)}$ then since A has a cai (e_t) and A is an ideal in B, we have $z = \lim_t e_t z \in \overline{A(1-x)}$. Thus $A \cap \overline{B(1-x)} = \overline{A(1-x)}$, and, similarly, $A \cap \overline{(1-x)B} = \overline{(1-x)A}$. It follows that $\overline{A(1-x)}$ is the ℓ-ideal matching $\overline{(1-x)A}$. By [9, Corollary 2.8], the corresponding HSA will be the intersection of these, which also equals their product, which can be seen to be $\overline{(1-x)A(1-x)}$, using the cai for A. We do not need this here, but this HSA also equals $A \cap \overline{(1-x)B(1-x)}$, since the latter equals $A \cap \overline{(1-x)B} \cap \overline{B(1-x)} = \overline{(1-x)A} \cap \overline{A(1-x)}$. $\square$

Remark. The result is not true if A only has a one-sided cai. For example if $x = E_{11}$, and $A = C_2$ as in the example after Theorem 7.1.

8. Positive maps between operator algebras

The *size* of $\mathfrak{F}_A$, and what all it contains, seems to be an important and possibly quite difficult question for nonunital operator algebras A. Of course for unital A the answer is trivial. Note too that for a C^*-algebra A with positive cai (e_t), then one obtains a probably quite good idea of what is contained in $\mathfrak{F}_A$, by meditating on the simple fact that $a^2 + axa \in \mathfrak{F}_A$ for all $x \in \mathrm{Ball}(A)$ and $a \in \mathrm{Ball}(A)_+$ (this follows since the product $y\,\mathrm{diag}(1,x)y^*$ is a contraction where $y = [\sqrt{1-a^2} \quad a]$). In particular, $e_t^2 + e_t x e_t \in \mathfrak{F}_A$ in this case. The method in the next proof shows that for general approximately unital operator algebras, if $v \in \frac{1}{2}\mathfrak{F}_A$ then $v^{\frac{1}{2}} = a^2 + iaxa$ for some selfadjoint contractions $a, x \in C^*(A^1)$ with $a \geqslant 0$. Conversely, any such $a^2 + iaxa$ is in $\mathfrak{F}_A$.

Remark. We remark in passing that the only idempotents that could be contained in $\mathbb{R}^+\mathfrak{F}_A$, are orthogonal projections. Also, note that $\mathfrak{F}_A$ can contain selected unitaries (e.g. certain functions valued in the unit circle on certain subsets of [0, 1]), but not nonunitary isometries (by e.g. Corollary 2.8).

Lemma 8.1. *If A is an approximately unital operator algebra, then $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$.*

Proof. (We are indebted to the referee for supplying this proof.) Assume that $A \subset B(H)$. Let (v_t) be a cai as in Theorem 2.4, with numerical range in the wedge shape region of angle 2ρ described there, where $\rho \to 0$ as t increases through the directed set. Write $v_t = a_t + ib_t$, for selfadjoint $a = a_t$ and $b = b_t$. Because of the position of the numerical range of v_t, a is a positive contraction. Also, for all states φ on a C^*-algebra generated by A, we have $|\varphi(b)| \leqslant (\tan\rho)\varphi(a)$. So $a\tan\rho \pm b \geqslant 0$. By a well-known fact about selfadjoint operators (which is a pleasant exercise to prove), there exists a selfadjoint $c \in B(H)$ with $b = a^{\frac{1}{2}}ca^{\frac{1}{2}}$ and $\|c\| \leqslant \tan\rho$. Then $v_t = a^{\frac{1}{2}}(1+ic)a^{\frac{1}{2}}$. Setting $c_t = c$ we have $c_t \to 0$ with t.

Let $z \in \mathrm{Ball}(A^{**})$; by Goldstine's lemma we may choose $z_i \in \mathrm{Ball}(A)$ with $z_i \to z$ weak*. Fix $\delta > 0$ and set $x_{i,t,\delta} = (1-\delta)v_t + (1-2\delta)v_t z_i v_t$. It is easy to check that $1 - x_{i,t,\delta} = 1 - a + a^{\frac{1}{2}}wa^{\frac{1}{2}}$, where

$$w = \delta 1 - (1-\delta)ic - (1-2\delta)(1+ic)a^{\frac{1}{2}}z_i a^{\frac{1}{2}}(1+ic).$$

Since $c = c_t \to 0$ with t, for t 'large' we have

$$\|w\| \leqslant \delta + (1-\delta)\|c\| + (1-2\delta)\|1+ic\|^2 \leqslant 1.$$

Hence $\|1 - x_{i,t,\delta}\| \leqslant 1$, since the product $y\,\mathrm{diag}(1,w)y^*$ is a contraction where $y = [\sqrt{1-a} \quad a^{\frac{1}{2}}]$. Thus $x_{i,t,\delta} \in \mathfrak{F}_A$. Since $v_t z_i v_t \to z_i$ in norm with t, it follows that $(1-\delta)1 + (1-2\delta)z_i$ is in the weak* closure of $\mathfrak{F}_A$ for every i. Hence $1 + z$ is in this weak* closure too. $\square$

Below we will also consider *unital operator spaces*: subspaces A of $B(H)$ containing I_H (see e.g. [11] for a matrix norm characterization of these). Here $\mathfrak{F}_A = \{x \in A\colon \|1_A - x\| \leqslant 1\}$. One may define a cone in any operator algebra (or unital operator space) A by considering $\mathfrak{c} = \mathfrak{c}_A = \mathbb{R}^+\mathfrak{F}_A$. Probably $\frac{1}{2}\mathfrak{F}_A$ should be considered to be the analogue of the positive part of the *unit ball* of a C^*-algebra. Similarly, one obtains cones $\mathfrak{c}_n$ in $M_n(A)$ for every $n \in \mathbb{N}$.

The following shows that $\mathfrak{c}_A$ is large enough to determine A:

Corollary 8.2. *Let A and B be approximately unital closed subalgebras of $B(H)$. Or, let A and B be unital subspaces of $B(H)$ with identities 1_A and 1_B corresponding to projections on H. If $\mathfrak{c}_A \subset \mathfrak{c}_B$ then $A \subset B$. Hence $A = B$ iff $\mathfrak{c}_A = \mathfrak{c}_B$.*

Proof. First assume that A and B are unital. If $x \in \mathrm{Ball}(A)$ then 1_A and $1_A + x$ are in $\mathfrak{F}_A \subset \mathfrak{c}_B$, and so $1_A, x \in B$. Hence $A \subset B$.

In the general case, taking weak* closures in $B(H)^{**}$, we have by Lemma 8.1 that $\mathfrak{F}_{A^{\perp\perp}} = \overline{\mathfrak{F}_A}^{w*} \subset \overline{\mathfrak{F}_B}^{w*} = \mathfrak{F}_{B^{\perp\perp}}$. By the last paragraph, $A^{\perp\perp} \subset B^{\perp\perp}$, and hence $A = A^{\perp\perp} \cap B(H) \subset B = B^{\perp\perp} \cap B(H)$. $\square$

Definition 8.3. We say that a map $T : A \to B$ between operator algebras, or between unital operator spaces, is *operator completely positive*, or *OCP*, if there is a constant $C > 0$ such that $T_n(\mathfrak{F}_{M_n(A)}) \subset C\mathfrak{F}_{M_n(B)}$ for every $n \in \mathbb{N}$. We study these maps below. If A and B are operator algebras, but not unital, then we will also require T to be completely bounded (this is automatic if A is unital).

Some remarks on Definition 8.3: First, the definition is 'positive homogeneous' in C. That is, $T : A \to B$ satisfies $T_n(\mathfrak{F}_{M_n(A)}) \subset C\mathfrak{F}_{M_n(B)}$, iff $R_n(\mathfrak{F}_{M_n(A)}) \subset \mathfrak{F}_{M_n(B)}$ where $R = \frac{T}{C}$. Thus we may usually assume that $C = 1$. Second, we will also use the fact that $x \in \mathfrak{c}$ iff there is a constant $C > 0$ with $x + x^* \geqslant Cx^*x$. Third, it is obvious that a completely contractive unital linear map between unital operator spaces is OCP. Finally, we remark that if $\varphi : A \to B$ is a completely contractive homomorphism between operator algebras, then φ is OCP. Indeed by Meyer's theorem ([30], [10, Theorem 2.1.13]), we can extend φ to a completely contractive unital homomorphism between unitizations, and then the result is obvious by the third remark.

We write $C^*(A)$ for a C^*-algebra that contains A completely isometrically as a subalgebra if A is an operator algebra, or as a unital subspace if A is a unital operator space (with $1_A = 1_{C^*(A)}$ in this case), and which is generated by A.

Lemma 8.4. *If $\varphi : A \to B(H)$ is a map from an operator algebra, or from a unital operator space, that extends to a completely positive map from $C^*(A)$ into $B(H)$, then φ is OCP.*

Proof. We may assume without loss of generality that A is a C^*-algebra. Suppose that $x \in \mathfrak{F}_A$. Then

$$\varphi(x) + \varphi(x)^* = \varphi\big(x + x^*\big) \geqslant \varphi\big(x^*x\big) \geqslant C\varphi(x)^*\varphi(x),$$

for a constant $C = \|\varphi\|_{cb}^{-1} > 0$, by the Kadison–Schwarz inequality (see e.g. [6,31]). Thus $\varphi(\mathfrak{F}_A) \subset \|\varphi\|_{cb}\mathfrak{F}_B$. Similarly for matrices. So φ is OCP. $\square$

Lemma 8.5. *If A is a C^*-algebra or operator system, then $x \in \mathrm{Ball}(A)_+$ iff $zx \in \mathfrak{F}_A$ for all $z \in \mathfrak{F}_{\mathbb{C}}$.*

Proof. ($\Rightarrow$) Left to the reader.

($\Leftarrow$) If x satisfies this property then for any $z \in \mathfrak{F}_{\mathbb{C}}$, we have $|z|^2 xx^* \leqslant 2\,\mathrm{Re}(zx)$, so that $\mathrm{Re}(z\langle x\zeta, \zeta\rangle) \geqslant 0$ for any unit vector $\zeta \in H$. It is a pleasant exercise in calculus that if the latter holds for all $z \in \mathfrak{F}_{\mathbb{C}}$ then $\langle x\zeta, \zeta\rangle \geqslant 0$. So x is positive, and it is easy to see that it has to be a contraction. $\square$

Theorem 8.6. *If $T : A \to B$ is a map between C^*-algebras or operator systems then T is completely positive iff T is OCP.*

Proof. By virtue of Lemma 8.4 we need only prove one direction. Suppose that T is OCP. By one of the observations below Definition 8.3, we may assume that $C = 1$ in the definition of OCP. If $x \in \mathrm{Ball}(A)_+$ and $z \in \mathfrak{F}_{\mathbb{C}}$ then $zx \in \mathfrak{F}_A$ by Lemma 8.5. Thus $zT(x) = T(zx) \in \mathfrak{F}_B$, and so $T(x) \geqslant 0$ by Lemma 8.5. A similar argument applies to matrices. $\square$

Theorem 8.7. *If $T : A \to B(H)$ is an OCP map on a unital operator space A, then the canonical extension $\tilde{T} : A + A^* \to B(H) : x + y^* \mapsto T(x) + T(y)^*$ is well defined and completely positive.*

Proof. As in the last proof, we may assume that $C = 1$ in the definition of OCP. In this case notice that by the last result applied to the restriction of T to $\mathbb{C}1$, we have $0 \leqslant T(1) \leqslant I$. Assume first that $\varphi : A \to \mathbb{C}$ is OCP. Since $|1 - \varphi(1) - \varphi(x)| \leqslant 1$ for all $x \in \mathrm{Ball}(A)$, we have $1 - \varphi(1) + |\varphi(x)| \leqslant 1$, so that $\|\varphi\| \leqslant \varphi(1)$. Hence $\|\varphi\| = \varphi(1)$. Thus φ extends by the Hahn–Banach theorem to a functional $\psi : A + A^* \to \mathbb{C}$ satisfying $\|\psi\| = \psi(1)$. The latter implies that ψ is positive [6,31].

To see that $\tilde{T}$ is well defined, notice that if $x + y^* = 0$, and if φ is any state on $B = B(H)$, then by the last paragraph, $\varphi \circ T$ extends to a positive map ψ on $A + A^*$, so that $\varphi(T(x) + T(y)^*) = \psi(x + y^*) = 0$. Since this holds for every state on B we have $T(x) + T(y)^* = 0$.

Similarly, if $x + y^* \geqslant 0$ then $\varphi(T(x) + T(y)^*) = \psi(x + y^*) \geqslant 0$. Since this holds for every state on B we have $T(x) + T(y)^* \geqslant 0$. Thus $\tilde{T}$ is positive on $A + A^*$. Applying this at every matrix level to T_n, we see that $\tilde{T}$ is completely positive on $A + A^*$. $\square$

Lemma 8.8. *If $T : A \to B(H)$ is an OCP map on an approximately unital operator algebra, and if $\mathfrak{F}_{M_n(A)}$ is weak* dense in $\mathfrak{F}_{M_n(A^{**})}$ for all $n \in \mathbb{N}$, then the canonical weak* continuous extension $\tilde{T} : A^{**} \to B(H)$ on the unital operator algebra A^{**} is OCP.*

Proof. As in the last proofs, we may assume that $C = 1$ in the definition of OCP. Suppose that $\eta \in \mathrm{Ball}(A^{**})$. By hypothesis, there exists $(y_\lambda) \subset \mathfrak{F}_A$, with $y_t \to 1 + \eta$ weak*. Then $\|1 - T(y_t)\| \leqslant 1$, and in the weak* limit, $\|1 - \tilde{T}(1 + \eta)\| \leqslant 1$. A similar argument prevails at the matrix level, so that $\tilde{T}$ is OCP. $\square$

Theorem 8.9 *(Extension and Stinespring dilation for OCP maps). Suppose that A is an approximately unital operator algebra (resp. a unital operator space), and that B is a C^*-algebra containing A as a subalgebra (resp. as a subspace, with 1_A a projection in B). If $T : A \to B(H)$ is a linear map, then T is OCP iff T has a completely positive extension $\tilde{T} : B \to B(H)$. This is equivalent to being able to write T as the restriction to A of $V^*\pi(\cdot)V$ for a $*$-representation $\pi : B \to B(K)$, and an operator $V : H \to K$. Moreover this can be done with $\|T\| = \|T\|_{\mathrm{cb}} = \|V\|^2$, and this equals $\|T(1)\|$ if A is unital.*

Proof. As before, we only need prove one direction of the first 'iff'. If T is OCP and A is a unital operator space, then by the last theorem we can extend T to a completely positive map on $A + A^*$. By Arveson's extension theorem [6,31], we may extend further to a completely positive map $\tilde{T} : pBp \to B(H)$, where $p = 1_A$, and this has a canonical completely positive extension to B.

If A is an approximately unital operator algebra, then by Lemmas 8.1 and 8.8, the canonical weak* continuous extension of T to a map from the unital operator algebra A^{**} into $B(H)$, is OCP. By the last paragraph, the latter map has a completely positive extension $S : B^{**} \to B(H)$, and $S = V^*\pi(\cdot)V$ for a $*$-representation $\pi : B^{**} \to B(K)$ as above. Restricting S and π to B we obtain the desired extension $\tilde{T} = V^*\pi_{|B}(\cdot)V$.

The last assertion, about the norm, follows immediately in the unital space case, since it is well known for completely positive maps on C^*-algebras, and indeed all of our extensions preserve norms. If A is an algebra with cai (e_t), and $B = C^*(A)$, then $T(e_t) \to S(1)$ weak*. Thus

$\|S(1)\| \leqslant \sup_t \|T(e_t)\|$ by Alaoglu's theorem. Consequently, by the unital space case, $\|T\|_{cb} \leqslant \|S\|_{cb} = \|S(1)\| = \|V\|^2 \leqslant \|T\|$, and so $\|T\| = \|T\|_{cb} = \sup_t \|T(e_t)\|$. □

The following complement gives a positive extension into a general C^*-algebra, under a hypothesis that is often satisfied.

Proposition 8.10. *If $T : A \to B$ is OCP, from a unital operator space A into a C^*-algebra B, and if there is a (resp. weak* continuous) affine map $L : Q(A) \to Q(C^*(A))$ taking 0 to 0, which is a retract of the restriction map $Q(C^*(A)) \to Q(A)$. Then there exists a positive map $\tilde{T} : C^*(A) \to B^{**}$ (resp. $\tilde{T} : C^*(A) \to B$) extending T.*

Proof. As before, we may assume that $C = 1$ in the definition of OCP. If $\varphi \in S(B)$ then $\varphi \circ T \in Q(A)$. Hence $T^\sharp : Q(B) \to Q(A) : \varphi \to \varphi \circ T$ is a weak* continuous affine map. Then $L \circ T^\sharp : Q(B) \to Q(C^*(A))$ is a (resp. weak* continuous) affine map taking 0 to 0. For any $c \in C^*(A)_{\mathrm{sa}}$, the map $\epsilon_c : Q(C^*(A)) \to \mathbb{C}$ of evaluation at c, is a weak* continuous bounded affine map taking 0 to 0. Hence $\epsilon_c \circ L \circ T^\sharp : Q(B) \to \mathbb{C}$ equals ϵ_b for a unique $b \in B^{**}_{\mathrm{sa}}$ (resp. B_{sa}), by [32, 3.10.3]. Define $\tilde{T}(c) = b$. Then $\tilde{T} : C^*(A)_{\mathrm{sa}} \to B^{**}_{\mathrm{sa}}$ (resp. B_{sa}) is real linear. Extend $\tilde{T}$ to $C^*(A)$ by linearity. If $c \in C^*(A)_+$ then it is clear that $\psi(\tilde{T}(c)) \geqslant 0$ for all $\psi \in S(B)$, so $\tilde{T}$ is positive. □

Remark. In the light of the last result, it is worth pointing out that there need not exist a weak* continuous retract $L : S(A) \to S(C^*(A))$. For example, suppose that such a retract existed when A is the sum of the compact operators on ℓ^2 and the upper triangular operators with constant entries on the leading diagonal (that is, $t_{ij} = 0$ unless $j \geqslant i$, and $t_{ii} = t_{jj}$ for all i, j). The states φ_n on $B(\ell^2)$ picking out the nth entry on the leading diagonal, when restricted to A, converge weak* on A. However $(L(\varphi_n|_A))$ has no weak* limit. Indeed, the restriction of φ_n to the compact operators is well known to have a unique state extension, so $L(\varphi_n|_A) = \varphi_n$. If $t = (t_{ij})$ with $t_{ii} = 1$ and $t_{2k-1,2k} = 1$ ($k \in \mathbb{N}$), and all other $t_{ij} = 0$, then the diagonal entries of tt^* are $2, 1, 2, 1, \ldots$, so $(\phi_n(tt^*))$ does not converge to any limit.

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