THE DUAL OF THE HAAGERUP TENSOR PRODUCT

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ABSTRACT

The weak*-Haagerup tensor product $\mathscr{M} \otimes_{w*h} \mathscr{N}$ of two von Neumann algebras is related to the Haagerup tensor product $\mathscr{M} \otimes_{h} \mathscr{N}$ in the same way that the von Neumann algebra tensor product is related to the spatial tensor product. Many of the fundamental theorems about completely bounded multilinear maps may be deduced from elementary properties of the weak*-Haagerup tensor product. We show that $X^* \otimes_{w*h} Y^* = (X \otimes_h Y)^*$ for all operator spaces X and Y. The weak*-Haagerup tensor product has simple characterizations and behaviour with reference to slice map properties. The tensor product of two (not necessarily self-adjoint) operator algebras is proven to have many strong commutant properties. All operator spaces possess a certain approximation property which is related to this tensor product. The connection between bimodule maps and commutants is explored.

1. *Introduction*

Completely bounded maps have played a significant role in recent years—see [**20**, **9**, **10**] for an overview. The completely bounded multilinear maps are naturally associated with the *Haagerup tensor product* of the underlying spaces. In this paper we define and study the *weak*-Haagerup tensor product* $X^* \otimes_{w*h} Y^*$ of two dual operator spaces X^* and Y^*. This is the completion of the ordinary Haagerup tensor product $X^* \otimes_h Y^*$ in a certain weak*-topology. Its relationship with the Haagerup tensor product is analogous to the relationship between the von Neumann algebra tensor product and the spatial tensor product. It will be seen that the weak*-Haagerup tensor product has many properties in common with the Haagerup norm. The title of this paper is due to the relation $(X \otimes_h Y)^* = X^* \otimes_{w*h} Y^*$, for all operator spaces X and Y, which we prove in Section 3. This is related to the fact that the dual tensor norm of h is again h. This last result appears in [**16**, **5**]; however, it also follows naturally and immediately from our methods. In fact many of the fundamental theorems about completely bounded multilinear maps may be deduced from elementary properties of our tensor product.

The weak*-Haagerup tensor product $\mathscr{M} \otimes_{w*h} \mathscr{N}$ of two von Neumann algebras is completely isometrically isomorphic to a direct summand of the normal Haagerup tensor product $\mathscr{M} \otimes_{\sigma h} \mathscr{N}$ defined by Effros and Kishimoto [**12**], with a weak*-continuous projection implementing the summand. The normal Haagerup tensor product seems more analogous to the projective tensor product.

In Section 2 we set up some machinery we shall need. In Section 3 we exhibit some characterizations and behaviour of the weak*-Haagerup tensor product with reference to slice map properties. This leads to the relation $(X \otimes_h Y)^* = X^* \otimes_{w*h} Y^*$ mentioned above, and several corollaries. For instance $\mathscr{H}_c \otimes_{w*h} \mathscr{H}_r = B(\mathscr{H})$ for any

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Hilbert space $\mathscr{H}$ (see also [16, 5]). We define a related approximation property and prove that all operator spaces possess this property. In Section 4 we prove some commutant theorems. For instance we show that if $\mathscr{A}$ and $\mathscr{B}$ are uniformly closed unital subalgebras of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively, then $(\mathscr{A} \otimes_{\mathrm{h}} \mathscr{B})' = \mathscr{A}' \otimes_{\mathrm{w*h}} \mathscr{B}'$, where the commutant is in $B(\mathscr{H}) \otimes_{\mathrm{w*h}} B(\mathscr{K})$. We also explore the connection between commutants and bimodule maps, and give a characterization of operator algebras which have a virtual diagonal in our context.

As with the normal Haagerup tensor product, the weak*-Haagerup tensor product $X \otimes_{\mathrm{w*h}} Y$ may be viewed as the dual of a certain space of completely bounded bilinear functionals on $X \times Y$ which are separately weak*-continuous. In our case this is the space of bilinear functionals with representations $\langle (x \otimes I_\infty) k (y \otimes I_\infty) \zeta, \eta \rangle$, where k is a bounded $\infty \times \infty$ matrix whose entries are compact operators. We postpone this approach until Section 5.

Our methods give elementary proofs of the fundamental theorems about completely bounded multilinear maps, for example, the representation theorem for completely bounded bilinear forms [12, 26], the Christensen–Sinclair representation theorem [8, 21, 5], the injectivity of the Haagerup norm [21, 6, 5], and the self-duality of the Haagerup norm [16, 5]. In fact we shall quote these results freely, but only for motivational purposes. Similarly, in the interest of a self-contained and elementary presentation we shall not use Ruan's characterization of operator spaces; we remark that the results from [6] quoted here do not essentially use this result.

We begin with some notation. We assume that the reader is familiar with the definition of completely bounded maps and operator spaces, as may be found in [20, 9]. If X is an operator space then the dual X^* may be regarded as an operator space in a natural way [6, 4, 14], by identifying $M_n(X^*)$, the $n \times n$ matrices with entries in X^*, with the space $\mathrm{CB}(X, M_n)$ of completely bounded maps from X into M_n. We call this operator space the standard dual of X, and denote it by X^*. A dual operator space is an operator space Y of the form X^*, where X is a closed operator space. We often write $X = Y_*$ in this case. In [4] an elementary proof is given of the fact that if Y is a dual operator space then there is a Hilbert space $\mathscr{H}$ and a weak*-homeomorphic complete isometry from Y into $B(\mathscr{H})$.

If Y is another operator space then it is also convenient to regard $\mathrm{CB}(X, Y)$, the space of completely bounded maps from X to Y, as an operator space by identifying $M_n(\mathrm{CB}(X, Y))$ with $\mathrm{CB}(X, M_n(Y))$ [13]. A direct-sum argument as in [4, Proposition 2.1] shows that this is an operator space.

If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces then we write $B(\mathscr{H}, \mathscr{K})$, $T(\mathscr{H}, \mathscr{K})$ and $K(\mathscr{H}, \mathscr{K})$ respectively for the spaces of bounded, trace class and compact linear operators from $\mathscr{H}$ to $\mathscr{K}$. Of course $B(\mathscr{H}, \mathscr{K})$ and $K(\mathscr{H}, \mathscr{K})$ are operator spaces, but we may also define an operator space structure on $T(\mathscr{H}, \mathscr{K})$ by identifying $T(\mathscr{H}, \mathscr{K})$ with the standard dual of $K(\mathscr{K}, \mathscr{H})$. If $\mathscr{H} = \mathscr{K}$ then we write $B(\mathscr{H})$, $T(\mathscr{H})$ and $K(\mathscr{H})$ for these spaces. We set $\mathscr{H}_c = B(\mathbb{C}, \mathscr{H})$, and $\mathscr{H}_r = B(\mathscr{H}, \mathbb{C})$, as operator spaces. We call these Hilbert operator spaces; the first is also called *Hilbert column space*, the second *Hilbert row space*.

If X and Y are linear spaces we write $X \otimes Y$ for the algebraic tensor product of X and Y. A subscript after the $\otimes$ symbol (for example $\otimes_\alpha$) indicates that the tensor product has been completed in an appropriate topology which is specified by the subscript. A pair of linear maps $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ between linear spaces extends to a linear map $S \otimes T: X_1 \otimes Y_1 \to X_2 \otimes Y_2$ between the algebraic tensor products. If this map further extends to a map $X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2$ then we denote this new map by $S \otimes_\alpha T$. We usually require that $S \otimes_\alpha T$ be continuous with respect to the topology α; if this is the case then of course $S \otimes_\alpha T$ is the unique extension of $S \otimes T$.

There are at least three interesting operator space structures on the tensor product of two operator spaces. The first is the spatial norm **[20, 6]**, which we denote by min. The second is the Haagerup norm, the first version of which was introduced in **[12, 26]**, and in full generality in **[10, 21]**. This tensor product seems to have become increasingly significant, and there are now many papers in which this norm plays a vital role. The third is the operator space projective norm, introduced independently in **[6]** and **[14]**, which we denote by max. We shall use very often the canonical completely isometric identifications $(X \otimes_{\max} Y)^* = \mathrm{CB}(X, Y^*) = \mathrm{CB}(Y, X^*)$. In **[6]** the reader will find a treatment of the elementary duality theory of tensor norms.

We now define the Haagerup norm on the algebraic tensor product $X \otimes Y$ of operator spaces X and Y. In fact it is convenient to define the norm more generally on matrices with entries in $X \otimes Y$. If X and Y are operator spaces then the Haagerup tensor norm is given for $U \in M_n(X \otimes Y)$ by $\|U\|_h = \inf\{\|A\| \, \|B\|\}$, where the infimum is taken over all integers p, and all matrices $A \in M_{n,p}(X)$ and $B \in M_{p,n}(Y)$, such that $U = [\sum_k a_{ik} \otimes b_{kj}]$. This last matrix is often denoted by $A \odot B$.

If X and Y are operator spaces then in analogy to the Banach space theory **[22]** we define $\Gamma_c(X, Y)$ (respectively $\Gamma_r(X, Y)$) to be the space of linear operators $X \to Y$ with completely bounded factorization through Hilbert column (respectively row) space (for more details see **[16, 5, 6]**). For instance $T \in \Gamma_r(X, Y)$ if and only if there is a Hilbert row space $\mathscr{H}_r$ and operators R and S in $\mathrm{CB}(X, \mathscr{H}_r)$ and $\mathrm{CB}(\mathscr{H}_r, Y)$ respectively, with $T = SR$; and in this case the norm of T is given by $\inf\{\|S\|_{\mathrm{cb}} \|R\|_{\mathrm{cb}}$: all factorizations $T = SR\}$. Effros and Ruan have observed that $(X \otimes_{\mathrm{h}} Y)^* = \Gamma_c(Y, X^*) = \Gamma_r(X, Y^*)$ **[16, 5]**; we shall only use this fact for motivational purposes.

If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces and if $\mathscr{H} \otimes \mathscr{K}$ is the Hilbert space tensor product, then the algebraic tensor product $B(\mathscr{H}) \otimes B(\mathscr{K})$ may be regarded as a subalgebra of $B(\mathscr{H} \otimes \mathscr{K})$ in a natural way. Of course $T(\mathscr{H} \otimes \mathscr{K})^* = B(\mathscr{H} \otimes \mathscr{K})$, and the spatial tensor product $B(\mathscr{H}) \otimes_{\min} B(\mathscr{K})$ is completely isometrically contained in $T(\mathscr{H} \otimes \mathscr{K})^* = B(\mathscr{H} \otimes \mathscr{K})$. In fact we have the identity $T(\mathscr{H} \otimes \mathscr{K}) = T(\mathscr{H}) \otimes_{\max} T(\mathscr{K})$ **[15, 5]**. We may define the von Neumann algebra tensor product $B(\mathscr{H}) \otimes_{\mathrm{w^*min}} B(\mathscr{K})$ of $B(\mathscr{H})$ and $B(\mathscr{K})$ (this is traditionally denoted by $B(\mathscr{H}) \bar{\otimes} B(\mathscr{K})$) to be the weak*-closure of $B(\mathscr{H}) \otimes B(\mathscr{K})$ in $(T(\mathscr{H}) \otimes_{\max} T(\mathscr{K}))^* = T(\mathscr{H} \otimes \mathscr{K})^* = B(\mathscr{H} \otimes \mathscr{K})$. Now it is well known that $B(\mathscr{H}) \otimes B(\mathscr{K})$ is dense, that is $B(\mathscr{H}) \otimes_{\mathrm{w^*min}} B(\mathscr{K}) = B(\mathscr{H} \otimes \mathscr{K})$. The fact that we have w*min on the left of the equation and max on the right is related to the fact that max* = min **[6]**. Now in view of the fact that $h = h^*$ **[16, 5]** we may expect by analogy that $B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}) = (T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}))^*$.

Since h = h* we have that $B(\mathscr{H}) \otimes_{\mathrm{h}} B(\mathscr{K})$ is completely isometrically contained in $(T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}))^*$. We define $B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ to be the weak*-closure of $B(\mathscr{H}) \otimes B(\mathscr{K})$ in $(T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}))^*$. We shall see shortly that this is in fact the whole space.

In analogy to **[12]** we may define the normal Haagerup tensor product $X \otimes_{\sigma\mathrm{h}} Y$ of two dual operator spaces X and Y to be the dual space of the space $\mathrm{Bil}_{\mathrm{h}}^{\sigma}(X, Y)$ of completely bounded bilinear functionals on $X \times Y$, which are weak*-continuous in each variable. One may regard this as an operator space by identifying $\mathrm{Bil}_{\mathrm{h}}^{\sigma}(X, Y)$ with a subspace of $(X \otimes_{\mathrm{h}} Y)^*$. Effros and Kishimoto showed in **[12]** that if $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ then $\mathscr{M} \otimes_{\sigma\mathrm{h}} \mathscr{N} =$

$CB_{\mathcal{M}',\mathcal{N}'}(B(\mathcal{K},\mathcal{H}),B(\mathcal{K},\mathcal{H}))$ isometrically, where $CB_{\mathcal{M}',\mathcal{N}'}(B(\mathcal{K},\mathcal{H}),B(\mathcal{K},\mathcal{H}))$ is the set of completely bounded $(\mathcal{M}',\mathcal{N}')$-module maps from $B(\mathcal{K},\mathcal{H})$ to $B(\mathcal{K},\mathcal{H})$. These authors do not point this out, but $\mathcal{M}\otimes_{\mathrm{h}}\mathcal{N}$ is contained in $\mathcal{M}\otimes_{\sigma\mathrm{h}}\mathcal{N}$ isometrically. This follows from the well-known isometric inclusion of $\mathcal{M}\otimes_{\mathrm{h}}\mathcal{N}$ in $\mathrm{CB}(B(\mathcal{K},\mathcal{H}), B(\mathcal{K},\mathcal{H}))$ (see [**23**, **5**] for two recent proofs of this fact).

2. *The space* $B(\mathcal{H})\otimes_{\mathrm{w}^*\mathrm{h}}B(\mathcal{K})$*, slice maps and duality*

First we give some alternative characterizations of the space $(T(\mathcal{H})\otimes_{\mathrm{h}}T(\mathcal{K}))^*$. We shall subsequently refer to all of these identifications, so it will be helpful to number them.

PROPOSITION 2.1. *The following operator spaces are all completely isometrically isomorphic:*

(1) $(T(\mathcal{H})\otimes_{\mathrm{h}}T(\mathcal{K}))^*$,
(2) $(\mathcal{H}_r\otimes_{\mathrm{h}}K(\mathcal{K},\mathcal{H})\otimes_{\mathrm{h}}\mathcal{K}_c)^*$,
(3) $(K(\mathcal{K},\mathcal{H})\otimes_{\max}T(\mathcal{H},\mathcal{K}))^*$,
(4) $\mathrm{w}^*\mathrm{CB}(B(\mathcal{K},\mathcal{H}),B(\mathcal{K},\mathcal{H}))$, *the weak*-continuous completely bounded operators on* $B(\mathcal{K},\mathcal{H})$,
(5) $\mathrm{CB}(K(\mathcal{K},\mathcal{H}),B(\mathcal{K},\mathcal{H}))$,
(6) $\mathrm{CB}(T(\mathcal{H},\mathcal{K}),T(\mathcal{H},\mathcal{K}))$,
(7) $\mathrm{CB}(\mathcal{H}_r\otimes_{\max}\mathcal{K}_c,\mathcal{H}_r\otimes_{\max}\mathcal{K}_c)$,
(8) $\Gamma_c(T(\mathcal{K}),B(\mathcal{H}))$, *the operators* $T(\mathcal{K})\to B(\mathcal{H})$ *with completely bounded factorization through Hilbert column space,*
(9) $\Gamma_r(T(\mathcal{H}),B(\mathcal{K}))$, *the operators* $T(\mathcal{H})\to B(\mathcal{K})$ *with completely bounded factorization through Hilbert row space.*

Proof. The equality (4) = (5) is well known; for completeness we sketch the proof. Any weak*-continuous completely bounded operator T on $B(\mathcal{K},\mathcal{H})$ gives an operator from $K(\mathcal{K},\mathcal{H})$ to $B(\mathcal{K},\mathcal{H})$ by restriction. Conversely, if we are given a completely bounded operator S from $K(\mathcal{K},\mathcal{H})$ to $B(\mathcal{K},\mathcal{H})$ then S^{**} is a weak*-continuous completely bounded operator T from $B(\mathcal{K},\mathcal{H})$ to $B(\mathcal{K},\mathcal{H})^{**}$. Composing S^{**} with the weak*-continuous projection from $B(\mathcal{K},\mathcal{H})^{**}$ to $B(\mathcal{K},\mathcal{H})$ gives a weak*-continuous completely bounded operator from $B(\mathcal{K},\mathcal{H})$ to $B(\mathcal{K},\mathcal{H})$. It is easy to check that these operations are inverses of each other.

We know by [**6**, Section 5] that (3), (5) and (6) are all completely isometrically isomorphic. Using elementary Hilbert operator space identifications [**16**, **5**] we see that (6) = (7). The equalities (1) = (8) = (9) follow from [**16**, identities (5.7) and (5.8)] (see also [**5**]). To complete the proof we show that $K(\mathcal{K},\mathcal{H})\otimes_{\max}T(\mathcal{H},\mathcal{K})$, $\mathcal{H}_r\otimes_{\mathrm{h}}K(\mathcal{K},\mathcal{H})\otimes_{\mathrm{h}}\mathcal{K}_c$ and $T(\mathcal{H})\otimes_{\mathrm{h}}T(\mathcal{K})$ are completely isometrically isomorphic. To see this note that

$$\begin{aligned}K(\mathcal{K},\mathcal{H})\otimes_{\max}T(\mathcal{H},\mathcal{K}) &= K(\mathcal{K},\mathcal{H})\otimes_{\max}(\mathcal{H}_r\otimes_{\max}\mathcal{K}_c)\\ &= \mathcal{H}_r\otimes_{\max}K(\mathcal{K},\mathcal{H})\otimes_{\max}\mathcal{K}_c\\ &= \mathcal{H}_r\otimes_{\mathrm{h}}K(\mathcal{K},\mathcal{H})\otimes_{\mathrm{h}}\mathcal{K}_c\\ &= \mathcal{H}_r\otimes_{\mathrm{h}}(\mathcal{H}_c\otimes_{\mathrm{h}}\mathcal{K}_r)\otimes_{\mathrm{h}}\mathcal{K}_c\\ &= (\mathcal{H}_r\otimes_{\mathrm{h}}\mathcal{H}_c)\otimes_{\mathrm{h}}(\mathcal{K}_r\otimes_{\mathrm{h}}\mathcal{K}_c)\\ &= T(\mathcal{H})\otimes_{\mathrm{h}}T(\mathcal{K}),\end{aligned}$$

using the Hilbert operator space identifications [**16**, **5**].

REMARK 1. In a while we shall see that $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$ is also equal to these nine spaces. The identification $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K}) = \mathrm{CB}(\mathscr{H}_r \otimes_{\max} \mathscr{K}_c, \mathscr{H}_r \otimes_{\max} \mathscr{K}_c)$ is analogous to the equality

$$B(\mathscr{H}) \otimes_{w^*\min} B(\mathscr{K}) = B(\mathscr{H} \otimes \mathscr{K}, \mathscr{H} \otimes \mathscr{K}) = \mathrm{CB}(\mathscr{H}_c \otimes_{\max} \mathscr{K}_c, \mathscr{H}_c \otimes_{\max} \mathscr{K}_c)$$
$$= \mathrm{CB}(\mathscr{H}_r \otimes_{\max} \mathscr{K}_r, \mathscr{H}_r \otimes_{\max} \mathscr{K}_r).$$

The equation $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K}) = \mathrm{w^*CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$ is reminiscent of the equality $B(\mathscr{H}) \otimes_{\sigma h} B(\mathscr{K}) = \mathrm{CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$ **[12]**, while the relation $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K}) = (T(\mathscr{H}) \otimes_h T(\mathscr{K}))^*$ parallels the identification $B(\mathscr{H}) \otimes_{w^*\min} = B(\mathscr{K})$ $(T(\mathscr{H}) \otimes_{\max} T(\mathscr{K}))^*$.

REMARK 2. We may also consider the space $B(\mathscr{F}, \mathscr{H}) \otimes_{w^*h} B(\mathscr{G}, \mathscr{K})$ for Hilbert spaces $\mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{K}$. There is a corresponding 'rectangular version' of Proposition 2.1 which is also useful. We remark that one may always deduce the rectangular versions from the usual version by considering the direct-sum Hilbert space $\mathscr{F} \oplus \mathscr{G} \oplus \mathscr{H} \oplus \mathscr{K}$. The same comments apply to rectangular versions of results in **[23]**.

In view of the final calculation in the proof, the weak*-topologies from (1), (2) and (3) are the same. We shall refer to this topology as *the weak*-topology* on any of the nine spaces above.

We shall use the following notation in the sequel. Define P to be the weak*-continuous projection from $B(\mathscr{K}, \mathscr{H})^{**}$ to $B(\mathscr{K}, \mathscr{H})$. Define

$$J: \mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H})) \longrightarrow \mathrm{w^*CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$$

to be the canonical inclusion. For $u \in B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$ we shall write Φ_u for the corresponding element of $\mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$. The embedding of $B(\mathscr{H}) \otimes B(\mathscr{K})$ into $\mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$ is well known: an elementary tensor $s \otimes t$ corresponds to the map $\Phi_{(s \otimes t)}: k \to skt$.

We shall also write L for the inclusion of $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$ in $\Gamma_c(T(\mathscr{K}), B(\mathscr{H}))$ guaranteed by the equivalence of (1) and (8) above; and R for the inclusion of $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$ in $\Gamma_r(T(\mathscr{H}), B(\mathscr{K}))$ guaranteed by the equivalence of (1) and (9) above. Finally, we write Π for the inclusion of $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$ in $\mathrm{CB}(T(\mathscr{H}, \mathscr{K}), T(\mathscr{H}, \mathscr{K}))$ guaranteed by the equivalence of (1) and (6).

From now on we shall assume that $\mathscr{H}$ and $\mathscr{K}$ are separable, and then all sums $\sum_i$ which follow are countable. The general case is identical, but with arbitrary index sets replacing the integers.

THEOREM 2.2. *Let $\Phi: K(\mathscr{K}, \mathscr{H}) \to B(\mathscr{K}, \mathscr{H})$ be a complete contraction. Then Φ may be written as $\Phi(k) = \sum_i s_i k t_i$ for all $k \in K(\mathscr{K}, \mathscr{H})$, where $s_i \in B(\mathscr{H})$, $t_i \in B(\mathscr{K})$, $\| \sum_i s_i s_i^* \| \leqslant 1$, $\| \sum_i t_i^* t_i \| \leqslant 1$, and the convergence of the partial sums of $\sum_i s_i \cdot t_i$ in* $\mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$ *is in the weak*-topology on* $\mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$.

Proof. We may assume without loss of generality that $\mathscr{H} = \mathscr{K}$, by taking the direct sum of $\mathscr{H}$ and $\mathscr{K}$ if necessary. Except for the final statement this result is in **[17]** (the result may also be found in **[23**, Theorem 3.1**]**).

Note that for any $k \in K(\mathscr{H})$ the sum $\sum_i s_i k t_i$ certainly converges in the weak operator topology. Thus if $k \in K(\mathscr{H})$ and if $\zeta, \eta \in \mathscr{H}$ then

$$\langle (\sum_i s_i k t_i)\zeta, \eta\rangle = \sum_i \langle s_i k t_i \zeta, \eta\rangle,$$

and so the partial sums of $\sum_i s_i \cdot t_i$ converge when applied to elementary tensors $k \otimes (\eta \otimes \zeta)$ from the predual $K(\mathscr{H}) \otimes_{\max} T(\mathscr{K})$. However, these elementary tensors span a norm dense set, and the partial sums of $\sum_i s_i \cdot t_i$ are uniformly bounded. Thus it follows that $\sum_i s_i \cdot t_i$ converges in the weak*-topology.

COROLLARY 2.3. *$B(\mathscr{H}) \otimes_{\mathrm{w*h}} B(\mathscr{K})$ coincides with the nine spaces above.*

Proof. We need to show that the image of $B(\mathscr{H}) \otimes B(\mathscr{K})$ under the map $u \to \Phi_u$ is weak*-dense in $\mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$. However, this is evident from Theorem 2.2.

Thus we can identify with each $u \in B(\mathscr{H}) \otimes_{\mathrm{w*h}} B(\mathscr{K})$ a representation $u = \sum_i s_i \otimes t_i$, which converges in the weak*-topology to u and which satisfies $\| \sum_i s_i s_i^* \| \, \| \sum_i t_i^* t_i \| < \infty$. We shall call this a *w*-representation of u*, and we remark that it is not in general unique. In fact Theorem 2.2 shows that for $u \in B(\mathscr{H}) \otimes_{\mathrm{w*h}} B(\mathscr{K})$ there exists a w*-representation $u = \sum_i s_i \otimes t_i$ with $\| \sum_i s_i s_i^* \| \, \| \sum_i t_i^* t_i \| = \|\Phi_u\|_{\mathrm{cb}}^2$.

In view of Proposition 2.1 we see that $B(\mathscr{H}) \otimes_{\mathrm{w*h}} B(\mathscr{K})$ is indeed isometrically isomorphic to a direct summand of $B(\mathscr{H}) \otimes_{\sigma \mathrm{h}} B(\mathscr{K})$, since Effros and Kishimoto showed in [12] that $B(\mathscr{H}) \otimes_{\sigma \mathrm{h}} B(\mathscr{K}) = \mathrm{CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$. In fact it is not hard to see that these are completely isometric isomorphisms (see for instance Corollary 3.5).

LEMMA 2.4. (i) *If u has w*-representation $u = \sum_i s_i \otimes t_i$, and if $x \in B(\mathscr{K}, \mathscr{H})$ and $\psi \in T(\mathscr{H}, \mathscr{K})$, then $\langle x, \Pi(u)(\psi)\rangle = \sum_i \langle s_i x t_i, \psi\rangle$, and this sum converges absolutely.*

(ii) *For $x \in B(\mathscr{K}, \mathscr{H})$ we have*

$$J(S)(x) = PS^{**}(x) = \textstyle\sum_i s_i x t_i,$$

if $S = \Phi_u$, where u has w-representation $u = \sum_i s_i \otimes t_i$.*

Proof. (i) We may suppose [24] that $\psi(\cdot) = \sum_j \langle \cdot \xi_j, \eta_j\rangle$, where $\sum_j \|\xi_j\|^2 < \infty$ and $\sum_j \|\eta_j\|^2 < \infty$. Then we have

$$\sum_i \psi(s_i x t_i) = \sum_i \sum_j \langle s_i x t_i \xi_j, \eta_j\rangle = \sum_i \sum_j \langle x t_i \xi_j, s_i^* \eta_j\rangle.$$

Now $\sum_{i,j} \|t_i \xi_j\|^2 < \infty$ and $\sum_{i,j} \|s_i^* \eta_j\|^2 < \infty$, and so $\sum_i \psi(s_i \cdot t_i)$ is a normal functional on $B(\mathscr{K}, \mathscr{H})$ [24]. Now $\Pi(u)(\psi)(k) = \langle \Phi_u(k), \psi\rangle = \sum_i \psi(s_i k t_i)$, for $k \in K(\mathscr{K}, \mathscr{H})$, and the last sum converges absolutely. Since $K(\mathscr{K}, \mathscr{H})$ is weak*-dense in $B(\mathscr{K}, \mathscr{H})$ we see that $\langle x, \Pi(u)(\psi)\rangle = \sum_i \langle s_i x t_i, \psi\rangle$ for all $x \in B(\mathscr{K}, \mathscr{H})$.

(ii) The first equality is contained in the proof of Proposition 2.1. Now if u has w*-representation $u = \sum_i s_i \otimes t_i$, then for $x \in B(\mathscr{K}, \mathscr{H})$ the sum $\sum_i s_i x t_i$ converges in the weak operator topology, and the map $\Psi_u : x \to \sum_i s_i x t_i$ is an element of $\mathrm{CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$. Part (i) shows that Ψ_u is weak*-continuous (we can interchange the order of summation in the identity in (i) and so $\sum_i \psi(s_i x t_i) = \psi(\sum_i s_i x t_i)$), and Ψ_u is equal to Φ_u on $K(\mathscr{K}, \mathscr{H})$. Since $K(\mathscr{K}, \mathscr{H})$ is weak*-dense in $B(\mathscr{K}, \mathscr{H})$ we have $\Psi_u = J(\Phi_u)$.

Now let φ and ψ be normal functionals on $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. One may define the right and left slice maps R_φ and L_ψ on $u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ by $R_\varphi(u) = R(u)(\varphi)$ and $L_\psi(u) = L(u)(\psi)$. The most elementary approach, however, is as follows. Let φ be a fixed normal functional on $B(\mathscr{H})$. Define a complete contraction $r(\varphi): T(\mathscr{K}) \to T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K})$ by $r(\varphi)(\psi) = \varphi \otimes \psi$. Now let

$$R_\varphi = r(\varphi)^*: B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}) \longrightarrow B(\mathscr{K})$$

(using Proposition 2.1). We define L_ψ similarly, and immediately obtain the following.

LEMMA 2.5. *Let φ and ψ be normal functionals on $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then R_φ and L_ψ are norm continuous, and also weak*-continuous. If $u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ has* w*-*representation $u = \sum_i s_i \otimes t_i$ then $R_\varphi(u) = \sum_i \varphi(s_i)\, t_i$ and $L_\psi(u) = \sum_i s_i\, \psi(t_i)$, and these two sums converge uniformly.*

Proof. Only the final statement needs proof. Since ϕ is completely bounded and $\| \sum_i s_i s_i^* \| < \infty$, we see that $\sum_i |\varphi(s_i)|^2 < \infty$. Thus the sum $\sum_i \varphi(s_i)\, t_i$ converges uniformly.

3. *Dual operator spaces*

Let X and Y be weak*-closed subspaces of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Define $X \otimes_{\mathrm{w^*h}} Y$ to be the weak*-closure in $B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ of $X \otimes Y$. It follows immediately from [**23**, Theorem 4.5] that $X \otimes_{\mathrm{h}} Y$ is isometrically embedded in $X \otimes_{\mathrm{w^*h}} Y$. We remark that this gives another proof of the injectivity of the Haagerup norm [**21**, **6**, **5**], and the fact that the Haagerup tensor product of operator spaces is again an operator space [**21**]. In fact this is a completely isometric embedding; this follows from [**6**, Proposition 3.5] and the rectangular version of [**23**, Theorem 4.5] (see Remark 2 after Proposition 2.1). We have

$$\begin{aligned} M_n(X \otimes_{\mathrm{h}} Y) &= C_n(X) \otimes_{\mathrm{h}} R_n(Y) \subset \mathrm{CB}\,(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}^{(n)}, \mathscr{H}^{(n)})) \\ &= \mathrm{CB}\,(K(\mathscr{K}, \mathscr{H}), M_n(B(\mathscr{K}, \mathscr{H})) \\ &= M_n(B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})). \end{aligned}$$

We now offer a proof of the self-duality of the Haagerup norm [**16**, **5**] which does not use the Christensen–Sinclair representation theorem. That is, we shall show that if X and Y are operator spaces then $X^* \otimes_{\mathrm{h}} Y^*$ is completely isometrically contained in $(X \otimes_{\mathrm{h}} Y)^*$. Suppose that X^* and Y^* are represented weak*-homeomorphically and completely isometrically in $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then the complete quotient maps $T(\mathscr{H}) \to X$ and $T(\mathscr{K}) \to Y$ induce a complete quotient map $T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}) \to X \otimes_{\mathrm{h}} Y$ [**16**], which gives a complete isometry $(X \otimes_{\mathrm{h}} Y)^* \to (T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}))^*$. We have the following commutative diagram.

$$\begin{array}{ccc} X^* \otimes_{\mathrm{h}} Y^* & \longrightarrow & (X \otimes_{\mathrm{h}} Y)^* \\ \downarrow & & \downarrow \\ B(\mathscr{H}) \otimes_{\mathrm{h}} B(\mathscr{K}) & \longrightarrow & (T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}))^* = B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}) \end{array}$$

We observed earlier that the bottom arrow is a complete isometry. It follows that the arrow at the top is a complete isometry, which completes the proof.

We now give two interesting characterizations of $X \otimes_{\mathrm{w^*h}} Y$.

If X and Y are weak*-closed subspaces of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively then we define the Fubini product $\mathscr{F}(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}))$ to be the set

$$\{u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}): R_\varphi(u) \in X, L_\psi(u) \in Y, \text{ all } \varphi \in B(\mathscr{H})_*,\ \psi \in B(\mathscr{K})\}_*.$$

If in addition E and F are weak*-closed subspaces of X and Y respectively, then we define the relative Fubini product $\mathscr{F}(E, F, X \otimes_{\mathrm{w^*h}} Y)$ to be the set $\{u \in X \otimes_{\mathrm{w^*h}} Y$: $R_\varphi(u) \in E$, $L_\psi(u) \in F$, all φ, ψ as above$\}$.

THEOREM 3.1. *Let X and Y be weak*-closed subspaces of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then the following hold.*

(i) *An element $u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ lies in $X \otimes_{\mathrm{w^*h}} Y$ if and only if u has a* w*-*representation $u = \sum_i s_i \otimes t_i$ with $s_i \in X$ and $t_i \in Y$. Moreover, $\|u\|_{\mathrm{w^*h}} = \inf\{\| \sum_i s_i s_i^* \|^{\frac{1}{2}} \| \sum_i t_i^* t_i \|^{\frac{1}{2}}\}$, where the infimum (which is actually achieved) is taken over all such* w*-*representations of u. In this case $u \in X \otimes_{\mathrm{h}} Y$ if and only if u has a* w*-*representation which converges uniformly, and then the infimum above may be taken over all such representations.*

(ii) $F(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})) = X \otimes_{\mathrm{w^*h}} Y$.

Proof. If $u \in X \otimes_{\mathrm{w^*h}} Y$ then there is a net in $X \otimes Y$ which converges to u in the weak*-topology. Since the slice maps are weak*-continuous (Lemma 2.5) it follows that $u \in F(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}))$.

Now suppose that $u \in F(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}))$. By Theorem 4.5 in **[23]** (the implication (ii) implies (iii) goes through in our case), in conjunction with the argument of Theorem 2.2 above, there is a w*-representation $u = \sum_i s_i \otimes t_i$ with $s_i \in X$ and $t_i \in Y$. Thus $u \in X \otimes_{\mathrm{w^*h}} Y$. This proves (ii). Except for the last statement in part (i) the remaining assertions are now immediate from the preceding paragraph.

Clearly if u has a uniformly convergent w*-representation then $u \in X \otimes_{\mathrm{h}} Y$. Conversely if $u \in X \otimes_{\mathrm{h}} Y$, $\|u\|_{\mathrm{h}} < 1$, then given $\varepsilon > 0$ we may choose $u_1 = \sum_{i \leqslant n_1} s_i \otimes t_i$, with $s_i \in X$ and $t_i \in Y$, $\|u - u_1\|_h < \varepsilon$, and $\| \sum_{i \leqslant n_1} s_i s_i^* \| = \| \sum_{i \leqslant n_1} t_i^* t_i \| < 1$. Now choose $u_2 = \sum_{n_1 < i \leqslant n_2} s_i \otimes t_i$, with $s_i \in X$ and $t_i \in Y$, $\|u - u_1 - u_2\|_h < \varepsilon/2$, and $\| \sum_{n_1 < i \leqslant n_2} s_i s_i^* \| = \| \sum_{n_1 < i \leqslant n_2} t_i^* t_i \| < \varepsilon$. Choose $u_3 = \sum_{n_2 < i \leqslant n_3} s_i \otimes t_i$, with $s_i \in X$ and $t_i \in Y$, $\|u - u_1 - u_2 - u_3\|_h < \varepsilon/4$, and so on. From the inequalities above it follows that $\| \sum_i s_i s_i^* \|$ and $\| \sum_i t_i^* t_i \|$ are each less than $1 + 2\varepsilon$, and that $\sum_i s_i \otimes t_i$ converges in the weak*-topology as in Theorem 2.2. Since $\sum_i u_i$ converges uniformly to u, it follows that u has w*-representation $\sum_i s_i \otimes t_i$. The remaining assertion is evident from the construction above.

REMARK 1. There is a matricial version of the formula in (i) above which we shall state presently.

REMARK 2. Define $F^f(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}))$ to be the Fubini product considering only normal functionals which are finite combinations of vector states. The proof above shows that this coincides with the usual Fubini product $F(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}))$.

THEOREM 3.2. *If X and Y are weak*-closed subspaces of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively then $F(X, Y, B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}))$, $(X_* \otimes_{\mathrm{h}} Y_*)^*$ and $X \otimes_{\mathrm{w^*h}} Y$ are weak*-homeomorphic and completely isometrically isomorphic.*

Proof. Let q_X and q_Y be the quotient maps $T(\mathscr{H}) \to X_*$ and $T(\mathscr{K}) \to Y_*$ respectively. Then since the Haagerup norm is completely projective **[16]** we see that $(q_X \otimes q_Y)^*$ is a complete isometry

$$(X_* \otimes_{\mathrm{h}} Y_*)^* \longrightarrow (T(\mathscr{H}) \otimes_{\mathrm{h}} T(\mathscr{K}))^* = B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}).$$

We shall show that the range of this complete isometry is $F(X, Y, B(\mathscr{H}) \otimes_{w*h} B(\mathscr{K}))$ which will complete the proof, using Theorem 3.1(ii).

Suppose that $f \in (X_* \otimes_h Y_*)^*$, and set $u = (q_X \otimes q_Y)^*(f)$. Let φ and ψ be normal functionals on $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. We have

$$\begin{aligned}\langle R_\varphi(u), \psi\rangle &= \langle u, \varphi \otimes \psi\rangle = \langle (q_X \otimes q_Y)^*(f), \varphi \otimes \psi\rangle = \langle f, (q_X \otimes q_Y)(\varphi \otimes \psi)\rangle \\ &= \langle f, q_X(\varphi) \otimes q_Y(\psi)\rangle = \langle f_r(q_X(\varphi)), q_Y(\psi)\rangle = \langle (q_Y)^* f_r(q_X(\varphi)), \psi\rangle,\end{aligned}$$

where f_r is the canonical map $X_* \to Y$ induced by f. Thus $R_\varphi(u) = (q_Y)^* f_r(q_X(\varphi)) \in Y$. Similarly $L_\psi(u) = (q_X)^* f_l(q_Y(\psi)) \in X$. Thus $u \in F(X, Y, B(\mathscr{H}) \otimes_{w*h} B(\mathscr{K}))$.

Conversely, let $u \in F(X, Y, B(\mathscr{H}) \otimes_{w*h} B(\mathscr{K})) = X \otimes_{w*h} Y$. Then u has a w*-representation $u = \sum_i s_i \otimes t_i$, with s_i in X and t_i in Y. For normal functionals φ and ψ on $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively we have $\langle u, \varphi \otimes \psi\rangle = \sum_i \varphi(s_i)\psi(t_i)$, and the map $\varphi \otimes \psi \to \langle u, \varphi \otimes \psi\rangle$ is an element of $(T(\mathscr{H}) \otimes_h T(\mathscr{K}))^*$ by Proposition 2.1. Now an element of X_* (respectively Y_*) may be regarded as the restriction of a normal functional on $B(\mathscr{H})$ (respectively $B(\mathscr{K})$). Define a functional f_u on $X_* \otimes Y_*$ by $f_u(q_X(\varphi) \otimes q_X(\psi)) = \sum_i \varphi(s_i)\psi(t_i)$. This is well defined and we have $f_u((q_X \otimes q_Y)(\varphi \otimes \psi)) = \langle u, \varphi \otimes \psi\rangle$. Clearly f extends to $X_* \otimes_h Y_*$, and $(q_X \otimes q_Y)^*(f_u) = u$. This completes the proof.

One may also prove the above theorem by using the equivalence of (1), (8), (9) in Proposition 2.1 and an argument similar to [**5**, Lemma 2.8], or by the argument following Corollary 3.4 below.

By the theorem above we see that $X \otimes_{w*h} Y$ does not depend on the particular containing $B(\mathscr{H})$ and $B(\mathscr{K})$. It also follows that the weak*-Haagerup norm is associative, completely weak*-injective and completely weak*-projective (with the obvious definition of these terms [**5**]).

COROLLARY 3.3. *If X and Y are operator spaces then $X \otimes_h Y$ is completely isometrically isomorphic to the space of weak*-continuous linear functionals on $X^* \otimes_{w*h} Y^*$.*

We also obtain yet another proof of the self-duality of the Haagerup tensor norm [**16, 5**].

COROLLARY 3.4 [**16, 5**]. *If X and Y are operator spaces then*

$$X^* \otimes_h Y^* \subset (X \otimes_h Y)^*$$

completely isometrically.

Proof. We have $X^* \otimes_h Y^* \subset X^* \otimes_{w*h} Y^* = (X \otimes_h Y)^*$.

The representation theorem for completely bounded bilinear functionals [**12, 26**] follows immediately from the above. The idea is that a contractive functional $f \in (X \otimes_h Y)^* = X^* \otimes_{w*h} Y^*$ has a w*-representation $f = \sum_i \varphi_i \otimes \psi_i$. Let $\mathscr{H}$ be the separable Hilbert space and define $\Phi: X \to \mathscr{H}_r = B(\mathscr{H}, \mathbb{C})$ by $\Phi(x) = [\varphi_1(x)\,\varphi_2(x)\ldots]$ for $x \in X$. Similarly define $\Psi: Y \to \mathscr{H}_c = B(\mathbb{C}, \mathscr{H})$ by $\Psi(x) = [\psi_1(x)\,\psi_2(x)\ldots]^t$. It is easy to see that Φ and Ψ are complete contractions and that $f(x \otimes y) = \Phi(x)\Psi(y)$. There is a similar proof for the full Christensen–Sinclair theorem [**8, 21, 5**]. However, since this result was shown in [**5**] to be a consequence of the representation theorem for bilinear functionals we omit this proof.

The following extends results in [**5, 16**].

COROLLARY 3.5. *If X is an operator space and if $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces then we have (completely isometrically):*

(i) $\mathrm{CB}(\mathscr{H}_t, X^*) = \mathrm{CB}(X, \mathscr{H}_c) = \mathscr{H}_c \otimes_{\mathrm{w^*h}} X^* = \mathscr{H}_c \otimes_{\mathrm{w^*min}} X^*$,
and $\mathrm{CB}(\mathscr{H}_c, X^*) = \mathrm{CB}(X, \mathscr{H}_t) = X^* \otimes_{\mathrm{w^*h}} \mathscr{H}_t = X^* \otimes_{\mathrm{w^*min}} \mathscr{H}_t$,

(ii) $(\mathscr{H}_c \otimes_{\mathrm{h}} X)^* = \mathscr{H}_t \otimes_{\mathrm{h}} X^* = \mathscr{H}_t \otimes_{\mathrm{w^*h}} X^*$,
and $(X \otimes_{\mathrm{h}} \mathscr{H}_t)^* = X^* \otimes_{\mathrm{h}} \mathscr{H}_t = X^* \otimes_{\mathrm{h}} \mathscr{H}_t$,

(iii) $(\mathscr{H}_t \otimes_{\mathrm{h}} X)^* = \mathscr{H}_c \otimes_{\mathrm{w^*h}} X^*$, *and* $(X \otimes_{\mathrm{h}} \mathscr{H}_c)^* = X^* \otimes_{\mathrm{w^*h}} \mathscr{H}_t$,

(iv) $\mathscr{H}_c \otimes_{\mathrm{w^*h}} \mathscr{K}_t = \mathscr{H}_c \otimes_{\mathrm{w^*min}} \mathscr{K}_t = B(\mathscr{K}, \mathscr{H})$,

(v) $\mathrm{CB}(X^*, B(\mathscr{K}, \mathscr{H})) = (\mathscr{H}_c \otimes_{\mathrm{h}} X \otimes_{\mathrm{h}} \mathscr{K}_t)^{**}$,

(vi) $(K(\mathscr{H}) \otimes_{\mathrm{h}} K(\mathscr{K}))^{**} = (T(\mathscr{H}) \otimes_{\mathrm{w^*h}} T(\mathscr{K}))^* = (\mathscr{H}_t \otimes_{\mathrm{h}} B(\mathscr{K}, \mathscr{H}) \otimes_{\mathrm{h}} \mathscr{K}_c)^*$
$= \mathrm{CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H})) = B(\mathscr{H}) \otimes_{\sigma\mathrm{h}} B(\mathscr{K})$.

Proof. We first establish (i). Notice that

$$\mathrm{CB}(\mathscr{H}_t, X^*) = (\mathscr{H}_t \otimes_{\max} X)^* = (\mathscr{H}_t \otimes_{\mathrm{h}} X)^* = \mathscr{H}_c \otimes_{\mathrm{w^*h}} X^*.$$

Since $\mathscr{H}_c$ has the slice-map property [19] we have by [5, Theorem 2.5] that $(\mathscr{H}_t \otimes_{\max} X)^* = \mathscr{H}_c \otimes_{\mathrm{w^*min}} X^*$. Similarly, $\mathrm{CB}(X, \mathscr{H}_c) = (X \otimes_{\max} \mathscr{H}_t)^*$, which is equal to the above. The identities in the second line of (i) are similar. To obtain (ii) we need to show that $\mathscr{H}_t \otimes_{\mathrm{w^*h}} X^* = \mathscr{H}_t \otimes_{\mathrm{h}} X^*$. Certainly $\mathscr{H}_t \otimes_{\mathrm{h}} X^* \subset \mathscr{H}_t \otimes_{\mathrm{w^*h}} X^*$; however, if $u \in \mathscr{H}_t \otimes_{\mathrm{w^*h}} X^*$ then it is evident that the partial sums in a w*-representation of u actually form a Cauchy sequence, and so are uniformly convergent to an element of $\mathscr{H}_t \otimes_{\mathrm{h}} X^*$.

Statements (iii) and (iv) are evident from Theorem 3.2. Identity (v) follows from (ii) and [5, 2.3 (v); 16]. The first equality in (vi) is evident, the second is proven using (iv) and (ii) analogously to the proof of the comparable assertion in Proposition 2.1. The third equality is well known [16, 5]. The isometric version of the fourth equality in (vi) is in [12]. To prove the complete isometry we need to show (see also Section 5) that the canonical inclusion of $\mathscr{H}_t \otimes_{\mathrm{h}} B(\mathscr{K}, \mathscr{H}) \otimes_{\mathrm{h}} \mathscr{K}_c$ in $(B(\mathscr{H}) \otimes_{\mathrm{h}} B(\mathscr{K}))^*$ is a complete isometry. To see this observe that

$$\begin{aligned}\mathscr{H}_t \otimes_{\mathrm{h}} B(\mathscr{K}, \mathscr{H}) \otimes_{\mathrm{h}} \mathscr{K}_c &= (\mathscr{H}_t \otimes_{\mathrm{h}} \mathscr{H}_c) \otimes_{\mathrm{w^*h}} (\mathscr{K}_t \otimes_{\mathrm{h}} \mathscr{K}_c) = T(\mathscr{H}) \otimes_{\mathrm{w^*h}} T(\mathscr{K}) \\ &\subset B(\mathscr{H})^* \otimes_{\mathrm{w^*h}} B(\mathscr{K})^* = (B(\mathscr{H}) \otimes_{\mathrm{h}} B(\mathscr{K}))^*,\end{aligned}$$

completely isometrically.

There is a natural notion of *matricial* w*-*representations*. The idea is that if X and Y are dual operator spaces then there are weak*-homeomorphic completely isometric identifications [6]

$$\begin{aligned}M_n(X \otimes_{\mathrm{w^*h}} Y) &= C_n \otimes_{\mathrm{w^*h}} (X \otimes_{\mathrm{w^*h}} Y) \otimes_{\mathrm{w^*h}} R_n \\ &= (C_n \otimes_{\mathrm{w^*h}} X) \otimes_{\mathrm{w^*h}} (Y \otimes_{\mathrm{w^*h}} R_n) \\ &= C_n(X) \otimes_{\mathrm{w^*h}} R_n(Y).\end{aligned}$$

We may thus associate a w*-representation $\sum_i s_i \otimes t_i$ in $C_n(X) \otimes_{\mathrm{w^*h}} R_n(Y)$ with each element U of $M_n(X \otimes_{\mathrm{w^*h}} Y)$. Set $A = [s_1\, s_2 \ldots]$ and $B = [t_1\, t_2 \ldots]^t$. If X is a subspace of $B(\mathscr{H})$ then $C_n(X)$ is a subspace of $B(\mathscr{H}, \mathscr{H}^n)$; and A may be interpreted as an element of $B(\mathscr{H}^\infty, \mathscr{H}^n)$, with norm $\|A\| = \|\sum_i s_i s_i^*\|^{\frac{1}{2}}$. Similarly, if X is a subspace of $B(\mathscr{K})$ then B may be interpreted as an element of $B(\mathscr{K}^n, \mathscr{K}^\infty)$, with norm $\|B\| = \|\sum_i t_i^* t_i\|^{\frac{1}{2}}$. Write $A \odot B$ for $\sum_i s_i \otimes t_i$; we shall call this a w*-representation of U. The uniformly bounded partial sums of this w*-representation may be viewed as elements of $M_n(X \otimes_{\mathrm{h}} Y)$ using [6, Proposition 3.5]. Writing A and B as matrices $[a_{ij}]$ and $[b_{ij}]$ with entries in $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively, then we claim that the (i,j) entry of U has w*-

representation $\sum_k a_{ik} \otimes b_{kj}$. To see this notice that the weak*-continuous (i,j) entry function on $M_n(X \otimes_{\mathrm{w^*h}} Y)$ agrees on elementary tensors with the weak*-continuous map

$$\pi_i \otimes_{\mathrm{w^*h}} \rho_j : C_n(X) \otimes_{\mathrm{w^*h}} R_n(Y) \longrightarrow X \otimes_{\mathrm{w^*h}} Y;$$

here $\pi_i : C_n(X) \to X$ and $\rho_i : R_n(Y) \to Y$ are the weak*-continuous coordinate maps.

One obtains immediately a matricial version of the formula in Theorem 3.1(i) above.

PROPOSITION 3.6. *If X and Y are dual operator spaces and if $U \in M_n(X \otimes_{\mathrm{w^*h}} Y)$ then $\|U\|_n = \inf\{\|A\|\,\|B\| : U = A \odot B\}$, where the infimum (which is achieved) is taken over all* w*-*representations $U = A \odot B$ of U. Moreover, $U \in M_n(X \otimes_{\mathrm{h}} Y)$ if and only if U has a uniformly convergent* w*-*representation, and in this case $\|U\|_n$ is the infimum over such representations.*

Suppose that X_1, X_2, Y_1 and Y_2 are dual operator spaces, and that $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ are completely bounded maps. If these maps are weak*-continuous then the map

$$(T_1)_* \otimes_{\mathrm{h}} (T_2)_* : (Y_1)_* \otimes_{\mathrm{h}} (Y_2)_* \longrightarrow (X_1)_* \otimes_{\mathrm{h}} (X_2)_*$$

dualizes to give a weak*-continuous completely bounded map

$$T_1 \otimes_{\mathrm{w^*h}} T_2 : X_1 \otimes_{\mathrm{w^*h}} X_2 \longrightarrow Y_1 \otimes_{\mathrm{w^*h}} Y_2$$

which clearly agrees with $T_1 \otimes T_2$ on elementary tensors. Since weak*-representations converge in the weak*-topology it follows that $T_1 \otimes_{\mathrm{w^*h}} T_2$ is the unique weak*-continuous extension of $T_1 \otimes T_2$.

Even if T_1 and T_2 are not weak*-continuous it is still possible to define a (non-weak*-continuous) extension

$$T_1 \otimes_{\mathrm{w^*h}} T_2 : X_1 \otimes_{\mathrm{w^*h}} X_2 \longrightarrow Y_1 \otimes_{\mathrm{w^*h}} Y_2$$

of $T_1 \otimes T_2$. If $\sum_\iota s_\iota \otimes t_\iota$ is a w*-representation of $u \in X_1 \otimes_{\mathrm{w^*h}} X_2$ then we define $(T_1 \otimes_{\mathrm{w^*h}} T_2)(\sum_\iota s_\iota \otimes t_\iota)$ to be $\sum_\iota T_1(s_\iota) \otimes T_2(t_\iota)$. The second sum is an element of $Y_1 \otimes_{\mathrm{w^*h}} Y_2$.

PROPOSITION 3.7. *If X_1, X_2, Y_1 and Y_2 are dual operator spaces, and if $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ are completely bounded maps, then*

(i) *$T_1 \otimes_{\mathrm{w^*h}} T_2$ is a well-defined completely bounded map from $X_1 \otimes_{\mathrm{w^*h}} X_2$ to $Y_1 \otimes_{\mathrm{w^*h}} Y_2$, and, moreover,*

$$\|T_1 \otimes_{\mathrm{w^*h}} T_2\|_{\mathrm{cb}} \leqslant \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}}.$$

(ii) *For dual operator spaces X and Y and Hilbert space $\mathscr{H}$ we have that* $\mathrm{CB}(X \otimes_{\mathrm{h}} Y, B(\mathscr{H}))$ *is completely isometrically contained in* $\mathrm{CB}(X \otimes_{\mathrm{w^*h}} Y, B(\mathscr{H}))$.

Proof. To establish (i) we need to show that $T_1 \otimes_{\mathrm{w^*h}} T_2$ is well-defined as a map $X_1 \otimes_{\mathrm{w^*h}} X_2 \to Y_1 \otimes_{\mathrm{w^*h}} Y_2$. For if this is the case then the formula given in Proposition 3.6 shows that $T_1 \otimes_{\mathrm{w^*h}} T_2$ is completely bounded, and also that $\|T_1 \otimes_{\mathrm{w^*h}} T_2\|_{\mathrm{cb}} \leqslant \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}}$.

Without loss of generality we may assume that the four dual spaces are represented on the same Hilbert space $\mathscr{H}$. Suppose that $\sum_\iota s_\iota k t_\iota = 0$ for all

$k \in K(\mathscr{H})$. Then for all $\zeta, \eta, \xi, \omega \in \mathscr{H}$ we have $\langle \sum_i s_i(\zeta \otimes \eta^*) t_i \xi, \omega \rangle = 0$, and so $\sum_i \langle t_i \xi, \eta \rangle \langle s_i \zeta, \omega \rangle = 0$. Actually for all ξ, η we know that $\sum_i \langle t_i \xi, \eta \rangle s_i$ converges in norm, and from the above we see that it converges to 0. Thus $\sum_i \langle t_i \xi, \eta \rangle T_1(s_i) = 0$, and consequently $\sum_i \langle t_i \xi, \eta \rangle \langle T_1(s_i) \zeta, \omega \rangle = 0$ for all ζ, ω. Hence $\sum_i \langle T_1(s_i) \zeta, \omega \rangle t_i$ converges in norm to 0 for all ζ, ω. Thus $\sum_i \langle T_1(s_i) \zeta, \omega \rangle T_2(t_i) = 0$, and so

$$\sum_i \langle T_1(s_i) \zeta, \omega \rangle \langle T_2(t_i) \xi, \eta \rangle = \langle \sum_i T_1(s_i)(\zeta \otimes \eta^*) T_2(t_i) \xi, \omega \rangle = 0$$

for all $\zeta, \eta, \xi, \omega \in \mathscr{H}$. By continuity we see that $\sum_i T_1(s_i) k T_2(t_i) = 0$ for all $k \in K(\mathscr{H})$, so that $\sum_i T_1(s_i) \otimes T_2(t_i) = 0$.

To establish (ii) we note that a map $\varphi \in \mathrm{CB}(X \otimes_{\mathrm{h}} Y, B(\mathscr{H}))$ has a Christensen–Sinclair factorization $\varphi(x, y) = \Phi(x) \Psi(y)$, where Φ is a completely bounded map $X \to B(\mathscr{K}, \mathscr{H})$ and Ψ is a completely bounded map $Y \to B(\mathscr{H}, \mathscr{K})$, for some Hilbert space $\mathscr{K}$. Then

$$\Phi \otimes_{\mathrm{w^*h}} \Psi : X \otimes_{\mathrm{w^*h}} Y \longrightarrow B(\mathscr{K}, \mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{H}, \mathscr{K}).$$

Composing $\Phi \otimes_{\mathrm{w^*h}} \Psi$ with the multiplication map $B(\mathscr{K}, \mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{H}, \mathscr{K}) \to B(\mathscr{H})$ (this is essentially the map $u \to J(\Phi_u)(1))$ gives the required extension of $\varphi^{\cdot}$ of φ to $X \otimes_{\mathrm{w^*h}} Y$. By construction $\|\varphi^{\cdot}\|_{\mathrm{cb}} \leqslant \|\varphi\|_{\mathrm{cb}}$, with equality since $X \otimes_{\mathrm{h}} Y$ is contained in $X \otimes_{\mathrm{w^*h}} Y$. The complete isometry follows from the isometry and the relation $M_n(\mathrm{CB}(Z, B(\mathscr{H}))) = \mathrm{CB}(Z, B(\mathscr{H}^{(n)}))$ for all operator spaces Z.

We remark that Proposition 3.7 shows that slice maps on $X \otimes_{\mathrm{w^*h}} Y$ make sense even for non-weak*-continuous functionals.

We note that the canonical complete contraction

$$\rho : X_* \otimes_{\max} Y_* \longrightarrow X_* \otimes_{\mathrm{h}} Y_*$$

induces a canonical weak*-continuous complete contraction

$$\rho^* : X \otimes_{\mathrm{w^*h}} Y \longrightarrow X \otimes_{\mathrm{w^*min}} Y$$

which extends the identity map on elementary tensors. Since ρ has dense range it follows that ρ^* is injective. We have shown the following.

Corollary 3.8. *If X and Y are weak*-closed subspaces of $B(\mathscr{H})$ and $B(\mathscr{K})$ then the canonical map $X \otimes_{\mathrm{w^*h}} Y \to X \otimes_{\mathrm{w^*min}} Y \subset B(\mathscr{H} \otimes \mathscr{K})$ is injective.*

Remark. Using these ideas, particularly the notion of strong independence, we have been able to show the following result. Let X_1, X_2, Y_1 and Y_2 be operator spaces, and suppose that $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ are completely bounded maps. If E and F are norm-closed subspaces of Y_1 and Y_2 respectively, then we have

$$(T_1 \otimes_{\mathrm{h}} T_2)^{-1}(E \otimes_{\mathrm{h}} F) = X_1 \otimes_{\mathrm{h}} T_2^{-1}(0) + T_1^{-1}(E) \otimes_{\mathrm{h}} T_2^{-1}(F) + T_1^{-1}(0) \otimes_{\mathrm{h}} X_2.$$

It follows that $(T_1 \otimes_{\mathrm{h}} T_2)^{-1}(0) = X_1 \otimes_{\mathrm{h}} T_2^{-1}(0) + T_1^{-1}(0) \otimes_{\mathrm{h}} X_2$. Also if T_1 and T_2 are injective then $(T_1 \otimes T_2)^{-1}(E \otimes_{\mathrm{h}} F) = T_1^{-1}(E) \otimes_{\mathrm{h}} T_2^{-1}(F)$; in particular $T_1 \otimes_{\mathrm{h}} T_2$ is injective.

Effros and Ruan have shown in [16] that, given $E \subset X$ and $F \subset Y$, there is a complete quotient map $X \otimes_{\mathrm{h}} Y \to (X/E) \otimes_{\mathrm{h}} (Y/F)$. As another corollary of the result of the previous paragraph we see that the kernel of this quotient map is $E \otimes_{\mathrm{h}} Y + X \otimes_{\mathrm{h}} F$. Thus we see that there is an isomorphism between $X \otimes_{\mathrm{h}} Y / (E \otimes_{\mathrm{h}} Y + X \otimes_{\mathrm{h}} F)$ and $(X/E) \otimes_{\mathrm{h}} (Y/F)$.

The results contained in this remark are also valid for the weak*-Haagerup tensor product.

We now turn to approximation properties of operator spaces. Using the analogy between w*h and w*min we may essentially follow Kraus [19].

Let X be a norm-closed subspace of $B(\mathscr{H})$, write $\mathrm{Fin}(X)$ for the set of finite rank operators $X \to X$. We say that X has the HAP (Haagerup approximation property) if for every (infinite-dimensional) Hilbert space $\mathscr{H}$ there is a net $\phi_\alpha \in \mathrm{Fin}(X)$ such that for all $u \in X \otimes_{\mathrm{h}} B(\mathscr{H})$ we have $\phi_\alpha^{\sim}(u) \to u$ in norm; here $\phi_\alpha^{\sim} = \phi_\alpha \otimes_{\mathrm{h}} I$.

THEOREM 3.9. (i) *Every operator space has the HAP.*

(ii) *If X and Y are two operator spaces then there exist nets $S_\alpha \in \mathrm{Fin}(X)$ and $T_\alpha \in \mathrm{Fin}(Y)$ such that $(S_\alpha \otimes_{\mathrm{h}} T_\alpha)(u) \to u$ in norm for any $u \in X \otimes_{\mathrm{h}} Y$.*

Proof. (i) Fix $u \in X \otimes_{\mathrm{h}} B(\mathscr{H})$, and let $\mathscr{R}_u = \{R_\varphi(u) : \varphi \in X^*\} \subset B(\mathscr{H})$, as in [23]. Then $u \in \mathscr{R}_u$ from [23]. Given $\varepsilon > 0$ there exists a finite sum $\sum x_i \otimes r_i$ with $x_i \in X$ and $r_i \in \mathscr{R}_u$ such that $\|u - \sum x_i \otimes r_i\| < \varepsilon$. By approximating each r_i if necessary we may assume that each r_i has the form $R_{\varphi_i}(u)$ for $\varphi_i \in X^*$. Thus $\|u - \sum x_i \otimes R_{\varphi_i}(u)\| < \varepsilon$.

Now if $a \otimes b \in X \otimes_{\mathrm{h}} B(\mathscr{H})$ then

$$\sum x_i \otimes R_{\varphi_i}(a \otimes b) = \sum x_i \otimes \varphi_i(a)\, b = \left(\sum \varphi_i(a)\, x_i\right) \otimes b = \phi^{\sim}(a \otimes b),$$

where $\phi \in \mathrm{Fin}(X)$ is defined by $\varphi(a) = \sum \varphi_i(a)\, x_i$. By continuity we have $\sum x_i \otimes R_{\varphi_i}(u) = \phi^{\sim}(u)$, and so $\|u - \phi^{\sim}(u)\| < \varepsilon$.

Now let Λ be the net $\{U, \varepsilon\}$, where U is a finite subset of $X \otimes_{\mathrm{h}} B(\mathscr{H})$, and $\varepsilon > 0$; ordered by $\{U_1, \varepsilon_1\} \leqslant \{U_2, \varepsilon_2\}$ if and only if $U_1 \subseteq U_2$ and $\varepsilon_2 \leqslant \varepsilon_1$. If $\{U, \varepsilon\} \in \Lambda$ let $u_1, \ldots, u_n$ be the elements of U. Since $\mathscr{H}$ is infinite dimensional it is unitarily equivalent to $\mathscr{H}^n$. Consider $u = u_1 \oplus \ldots \oplus u_n \in X \otimes_{\mathrm{h}} B(\mathscr{H}^n)$ (we are using the notation of [19] here). From the first part there is a $\phi \in \mathrm{Fin}(X)$ such that $\|(\phi \otimes I_{\mathscr{H}^n})(u) - u\| < \varepsilon$, from which it is clear that $\|(\phi \otimes I_{\mathscr{H}})(u_i) - u_i\| < \varepsilon$, for each i. Put $\phi_{U,\varepsilon} = \phi$. Then for all $u \in X \otimes_{\mathrm{w^*h}} B(\mathscr{H})$ we have $\lim_{(U,\varepsilon)} \|\phi_{U,\varepsilon}(u) - u\| = 0$. Hence X has the HAP.

(ii) The first part gives a net $S_\beta \in \mathrm{Fin}(X)$ such that $S_\beta \otimes_{\mathrm{h}} I_Y \to I_{X \otimes_{\mathrm{h}} Y}$ in the point-norm topology. Similarly there is a net $T_\gamma \in \mathrm{Fin}(Y)$ such that $I_X \otimes_{\mathrm{h}} T_\gamma \to I_{X \otimes_{\mathrm{h}} Y}$ in the point-norm topology.

Consider the net of pairs $\{U, \varepsilon\}$, where U is a finite subset of $X \otimes_{\mathrm{h}} B(\mathscr{H})$, and $\varepsilon > 0$; ordered by $\{U_1, \varepsilon_1\} \leqslant \{U_2, \varepsilon_2\}$ if and only if $U_1 \subseteq U_2$ and $\varepsilon_2 \leqslant \varepsilon_1$. Consider $\{U, \varepsilon\} = \alpha$, and let $u_1, \ldots, u_n$ be the elements of U. Choose S_{β_0} such that $\|(S_{\beta_0} \otimes_{\mathrm{h}} I)(u_i) - u_i\| < \frac{1}{2}\varepsilon$ for each i, and then choose T_{γ_0} such that $\|(I \otimes_{\mathrm{h}} T_{\gamma_0})(S_{\beta_0} \otimes_{\mathrm{h}} I)(u_i) - (S_{\beta_0} \otimes_{\mathrm{h}} I)\, u_i\| < \frac{1}{2}\varepsilon$ for each i. By the triangle inequality $\|(S_{\beta_0} \otimes_{\mathrm{h}} T_{\gamma_0})(u_i) - u_i\| < \varepsilon$ for each i. Put $S_\alpha = S_{\beta_0}$ and $T_\alpha = T_{\gamma_0}$. Then $(S_\alpha \otimes_{\mathrm{h}} T_\alpha)(u) \to u$ in norm for any $u \in X \otimes_{\mathrm{h}} Y$.

Let X be a dual operator space, and write $\mathrm{w^*Fin}(X)$ for the set of finite rank weak*-continuous operators $X \to X$. We say that X has the w*HAP if for every (infinite-dimensional) Hilbert space $\mathscr{H}$ there is a net $\phi_\alpha \in \mathrm{w^*Fin}(X)$ such that for all $u \in X \otimes_{\mathrm{w^*h}} B(\mathscr{H})$ we have $\phi_\alpha^{\sim}(u) \to u$ in the weak*-topology; here $\phi_\alpha^{\sim} = \phi_\alpha \otimes_{\mathrm{w^*h}} I$.

COROLLARY 3.10. (i) *Every dual operator space has the* w*HAP.

(ii) *If X and Y are two dual operator spaces then there exist nets $S_\alpha \in \mathrm{w^*Fin}(X)$ and $T_\alpha \in \mathrm{w^*Fin}(Y)$ such that $(S_\alpha \otimes_{\mathrm{w^*h}} T_\alpha)(u) \to u$ in the weak*-topology for any $u \in X \otimes_{\mathrm{w^*h}} Y$.*

Proof. These assertions follow from applying Theorem 3.9 to the predual, and then dualizing the nets of operators provided by Theorem 3.9. For instance in (i), considering $T(\mathscr{H})$ as a subspace of some $B(\mathscr{K})$, we obtain a net $\psi_\alpha \in \mathrm{Fin}(X_*)$ such that $\psi_\alpha \otimes_{\mathrm{h}} I_{T(\mathscr{H})} \to I_{X_* \otimes_{\mathrm{h}} T(\mathscr{H})}$ in norm. Let $\phi_\alpha = \psi_\alpha^*$, then $\phi_\alpha \otimes_{\mathrm{w^*h}} I_{B(\mathscr{H})} \to I_{X \otimes_{\mathrm{w^*h}} B(\mathscr{H})}$ in the point weak* topology.

We remark that as in [19] these approximation properties are equivalent to a certain Fubini property, but of course we know from [23] and the first part of this section that every space has this Fubini property.

We now establish some other slice map results for the weak*-Haagerup norm analogous to those in [23] for the Haagerup norm. As in [23] define for $u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ the space $\mathscr{R}_u$ (respectively $\mathscr{L}_u$) to be the weak*-closure of the set of images of u under all right (respectively left) slice maps.

COROLLARY 3.11. (i) *If X_1 and X_2 are weak*-closed subspaces of $B(\mathscr{H})$, and if Y_1 and Y_2 are weak*-closed subspaces of $B(\mathscr{K})$, then*

$$(X_1 \otimes_{\mathrm{w^*h}} Y_1) \cap (X_2 \otimes_{\mathrm{w^*h}} Y_2) = (X_1 \cap X_2) \otimes_{\mathrm{w^*h}} (Y_1 \cap X_2).$$

(ii) *If $X_1 \subset X_2$ are weak*-closed subspaces of $B(\mathscr{H})$, and if $Y_1 \subset Y_2$ are weak*-closed subspaces of $B(\mathscr{K})$, then*

$$X_1 \otimes_{\mathrm{w^*h}} Y_1 = F(X_1, Y_1, X_2 \otimes_{\mathrm{w^*h}} Y_2).$$

(iii) *If $u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ then $u \in \mathscr{L}_u \otimes_{\mathrm{w^*h}} \mathscr{R}_u$.*

Proof. These are all immediate from Theorem 3.1.

4. *Module maps and commutants*

Let $\mathscr{A}$ and $\mathscr{B}$ be unital subalgebras of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then $K(\mathscr{K}, \mathscr{H})$ and $B(\mathscr{K}, \mathscr{H})$ are $(\mathscr{A}, \mathscr{B})$-modules. Define $\mathrm{CB}_{\mathscr{A},\mathscr{B}}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$ and $\mathrm{w^*CB}_{\mathscr{A},\mathscr{B}}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$ to be the subsets of $\mathrm{CB}(K(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H})$ and $\mathrm{w^*CB}(B(\mathscr{K}, \mathscr{H}), B(\mathscr{K}, \mathscr{H}))$, respectively, consisting of the $(\mathscr{A}, \mathscr{B})$-module maps.

If $u, v \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ have w*-representations $u = \sum_i s_i \otimes t_i$ and $v = \sum_j q_j \otimes r_j$, then define $u \cdot v = \sum_{i,j} s_i q_j \otimes r_j t_i$. As in Theorem 2.2 we see that this sum converges in the weak*-topology. Also

$$\begin{aligned}\langle \Phi_{u \cdot v}(k)\zeta, \eta\rangle &= \sum_i \sum_j \langle s_i q_j k r_j t_i \zeta, \eta\rangle = \sum_i \langle s_i \Phi_v(k) t_i \zeta, \eta\rangle \\ &= \langle J(\Phi_u)(\Phi_v(k))\zeta, \eta\rangle,\end{aligned}$$

using Lemma 2.4. Hence $J(\Phi_u)\Phi_v = \Phi_{u \cdot v}$. In particular this shows that $u \cdot v$ is well defined. Now using Theorem 3.1(ii) it is easy to see the following assertion.

PROPOSITION 4.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be weak*-closed subalgebras of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then $\mathscr{A} \otimes_{\mathrm{w^*h}} \mathscr{B}$ is a dual Banach algebra with the multiplication $u \cdot v$ defined above.*

We remark that the multiplication on $\mathscr{A} \otimes_{\mathrm{w^*h}} \mathscr{B}$ is not weak*-continuous in the second variable, although it is weak*-continuous in the first variable. We omit the proof since we shall not use this fact.

THEOREM 4.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital subalgebras of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively.*

(i) *Let $u \in B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$. Then $\Phi_u \in \mathrm{CB}_{\mathscr{A},\mathscr{B}}(K(\mathscr{K},\mathscr{H}), B(\mathscr{K},\mathscr{H}))$ if and only if u has a w*-representation $u = \sum_i s_i \otimes t_i$ with each $s_i \in \mathscr{A}'$ and $t_i \in \mathscr{B}'$.*

(ii) *$\mathscr{A}' \otimes_{\mathrm{w^*h}} \mathscr{B}' = \mathrm{CB}_{\mathscr{A},\mathscr{B}}(K(\mathscr{K},\mathscr{H}), B(\mathscr{K},\mathscr{H})) = \mathrm{w^*CB}_{\mathscr{A},\mathscr{B}}(B(\mathscr{K},\mathscr{H}), B(\mathscr{K},\mathscr{H}))$ completely isometrically. The first identification is also a weak*-homeomorphism.*

(iii) *$\mathrm{CB}_{\mathscr{A},\mathscr{B}}(K(\mathscr{K},\mathscr{H}), B(\mathscr{K},\mathscr{H}))$ is a Banach algebra with the multiplication $S \cdot T = J(S)\,T$; and the identification in* (ii) *is also a Banach algebra isomorphism.*

(iv) *If $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras on $\mathscr{H}$ and $\mathscr{K}$ respectively, then*

$$\mathscr{M} \otimes_{\mathrm{w^*h}} \mathscr{N} = \mathrm{CB}_{\mathscr{M}',\mathscr{N}'}(K(\mathscr{K},\mathscr{H}), B(\mathscr{K},\mathscr{H}))$$

completely isometrically and weak-homeomorphically.*

Proof. The necessity in (i) is [**23**, Theorem 3.1] modified as in Theorem 2.2 above. The sufficiency is evident. The first identity in (ii) follows immediately from (i) and Theorem 3.1(i). It is a weak*-homeomorphic identification since it is the restriction of a weak*-homeomorphism. The second identity in (ii) follows from (i) and Lemma 2.4(ii). Part (iii) follows from (i), Proposition 4.1 and the identity $J(\Phi_u)\Phi_v = \Phi_{u \cdot v}$. Part (iv) follows from (ii) and the double commutant theorem.

REMARK. Theorem 4.2(iv) is the version of [**12**, Theorem 2.5] appropriate to our setting. Notice that we need the self-adjoint condition here, since the double commutant theorem is not true in the non-self-adjoint case.

In particular $B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K}) = \mathrm{CB}(K(\mathscr{K},\mathscr{H}), B(\mathscr{K},\mathscr{H}))$ is a Banach algebra. We remark that Lemma 2.4(i) shows that the multiplication here corresponds to the reversed natural multiplication on the isomorphic space (6) of Proposition 2.1. For a subset X of $B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$ we shall write X' for the commutant of X in $B(\mathscr{H}) \otimes_{\mathrm{w^*h}} B(\mathscr{K})$.

The following theorem is similar to [**23**, Corollary 4.7] and Tomita's commutant theorem.

THEOREM 4.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital subalgebras of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then $(\mathscr{A} \otimes_{\mathrm{h}} \mathscr{B})' = \mathscr{A}' \otimes_{\mathrm{w^*h}} \mathscr{B}'$. If in addition $\mathscr{A}$ and $\mathscr{B}$ are weak*-closed then $(\mathscr{A} \otimes_{\mathrm{w^*h}} \mathscr{B})' = \mathscr{A}' \otimes_{\mathrm{w^*h}} \mathscr{B}'$.*

Proof. If $u \in \mathscr{A}' \otimes_{\mathrm{w^*h}} \mathscr{B}'$ then u clearly commutes with elementary tensors in $\mathscr{A} \otimes \mathscr{B}$, and so by continuity $u \in (\mathscr{A} \otimes_{\mathrm{h}} \mathscr{B})'$. Now let $u \in (\mathscr{A} \otimes_{\mathrm{h}} \mathscr{B})'$. We claim that Φ_u is an $(\mathscr{A},\mathscr{B})$-bimodule map. For if this is true then Theorem 4.2(ii) shows that $u \in \mathscr{A}' \otimes_{\mathrm{w^*h}} \mathscr{B}'$.

Suppose that u has w*-representation $u = \sum_i s_i \otimes t_i$, then

$$\begin{aligned}\langle a\Phi_u(k)\zeta,\eta\rangle &= \sum_i \langle as_i k t_i \zeta,\eta\rangle = \langle \Phi_{(a\otimes 1)\cdot u}(k)\zeta,\eta\rangle \\ &= \langle \Phi_{u\cdot(a\otimes 1)}(k)\zeta,\eta\rangle = \sum_i \langle s_i a k t_i \zeta,\eta\rangle = \langle \Phi_u(ak)\zeta,\eta\rangle,\end{aligned}$$

which proves the claim.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are weak*-closed. If $u \in \mathcal{A}' \otimes_{w^*h} \mathcal{B}'$ then $u \in (\mathcal{A} \otimes_{w^*h} \mathcal{B})'$ as is evident from considering the expression for a product $u \cdot v$ obtained before Proposition 4.1. The final inclusion follows from the first part of the proof.

COROLLARY 4.4. *Let $\mathcal{A}_1 \subset \mathcal{A}_2$ and $\mathcal{B}_1 \subset \mathcal{B}_2$ be unital weak*-closed subalgebras of $\mathcal{B}(\mathcal{H})$ and $B(\mathcal{K})$ respectively. Then the relative commutant of $\mathcal{A}_1 \otimes_{w^*h} \mathcal{B}_1$ in $\mathcal{A}_2 \otimes_{w^*h} \mathcal{B}_2$ is $(\mathcal{A}_1' \cap \mathcal{A}_2) \otimes_{w^*h} (\mathcal{B}_1' \cap \mathcal{B}_2)$.*

Proof. The relative commutant of $\mathcal{A}_1 \otimes_{w^*h} \mathcal{B}_1$ in $\mathcal{A}_2 \otimes_{w^*h} \mathcal{B}_2$ is by definition $(\mathcal{A}_1 \otimes_{w^*h} \mathcal{B}_1)' \cap (\mathcal{A}_2 \otimes_{w^*h} \mathcal{B}_2)$, which by Theorem 4.3 and Corollary 3.4(i), is $(\mathcal{A}_1' \cap \mathcal{A}_2) \otimes_{w^*h} (\mathcal{B}_1' \cap \mathcal{B}_2)$.

We now define the appropriate virtual diagonal for our tensor product, following **[12]**. If $\mathcal{A}$ is a unital subalgebra of $B(\mathcal{H})$ then an element M in $\mathcal{A}'' \otimes_{w^*h} \mathcal{A}''$ is a virtual w*-diagonal for $\mathcal{A}$ if $aM = Ma$ for all $a \in \mathcal{A}$, and if $J(\Phi_M)(1) = 1$. Here aM may be viewed as $(a \otimes 1)M$, and Ma as $M(1 \otimes a)$.

THEOREM 4.5. *If $\mathcal{A}$ is a unital subalgebra of $B(\mathcal{H})$ then the following are equivalent:*

(i) *$\mathcal{A}$ has a virtual* w*-*diagonal,*

(ii) *there is a weak*-continuous completely bounded $\mathcal{A}'$-bimodule projection from $B(\mathcal{H})$ onto $\mathcal{A}'$,*

Proof. If Q is a weak*-continuous completely bounded $\mathcal{A}'$-bimodule projection from $B(\mathcal{H})$ onto $\mathcal{A}'$, then using Theorem 4.2(i) we may view Q as $J(\Phi_u)$ for some u in $\mathcal{A}'' \otimes_{w^*h} \mathcal{A}''$. It is easy to see that u is a virtual w*-diagonal. Conversely, if M in $\mathcal{A}'' \otimes_{w^*h} \mathcal{A}''$ is a virtual w*-diagonal, then M has a w*-representation $M = \sum_i s_i \otimes t_i$, with s_i, t_i in $\mathcal{A}''$. Now $J(\Phi_M)(1) = \sum_i s_i t_i = 1$, using Lemma 2.4(ii), and so $J(\Phi_M)(a') = \sum_i s_i a' t_i = a'$ for all $a' \in \mathcal{A}'$. The condition $aM = Ma$ implies that $J(\Phi_M)$ maps into $\mathcal{A}'$, and so $J(\Phi_M)$ is the required projection.

We remark that (ii) is equivalent to $\mathcal{A}'$ being injective in some strong sense. There must be some characterization of this property in terms of some sort of amenability or nuclearity.

5. *Another approach to the predual*

Let $\mathcal{K}$ and $\mathcal{H}$ be Hilbert spaces. Define $\mathrm{Bil}_{w^*h}(B(\mathcal{H}), B(\mathcal{K}))$ to be the space of completely bounded bilinear functionals on $B(\mathcal{H}) \times B(\mathcal{K})$ with representation

$$\langle (x \otimes I_\infty) k (y \otimes I_\infty) \zeta, \eta \rangle,$$

where k is a bounded linear operator which has an $\infty \times \infty$ matrix representation whose entries are compact operators from $\mathcal{K}$ to $\mathcal{H}$, and ζ and η are in $\mathcal{H}^\infty$ and $\mathcal{K}^\infty$ respectively (cf. **[12, 26]**). We view this as an operator space by regarding it as a subspace of $(B(\mathcal{H}) \otimes_h B(\mathcal{K}))^*$.

More generally if X and Y are weak*-closed subspaces of $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively, then we can define $\mathrm{Bil}_{w^*h}(X, Y)$ to be the space of completely bounded bilinear functionals on $X \times Y$ consisting of the restrictions to $X \times Y$ of elements of $\mathrm{Bil}_{w^*h}(B(\mathcal{H}), B(\mathcal{K}))$. This space is regarded as a subspace of $(X \otimes_h Y)^*$.

We begin by showing that $\mathrm{Bil}_{w^*h}(B(\mathscr{H}), B(\mathscr{K}))$ is completely isometrically isomorphic to the predual $K(\mathscr{K}, \mathscr{H}) \otimes_{\max} T(\mathscr{H}, \mathscr{K})$ of $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$. It will follow that as with the normal Haagerup tensor product, the weak*-Haagerup tensor product $X \otimes_{w^*h} Y$ may be viewed as the dual of a certain space of completely bounded bilinear functionals on $X \times Y$ which are separately weak*-continuous.

Let X be an operator space, acting on a Hilbert space $\mathscr{F}$. Recall that $M_\infty(X)$ is the space of $\infty \times \infty$ matrices with entries in X, whose finite submatrices are uniformly bounded [13, 15]. Each element in $M_\infty(X)$ may be regarded as a bounded operator on $\mathscr{F}^\infty$.

The following lemma is essentially in [11]. If $\mathscr{K}$ and $\mathscr{H}$ are Hilbert spaces then $(\mathscr{H}_r \otimes_h X \otimes_h \mathscr{K}_c)^* = \mathrm{CB}(X, B(\mathscr{K}, \mathscr{H}))$ [16, 5]. Suppose that $x = [x_{ij}] \in M_\infty(X)$, $\zeta = \oplus \zeta_i \in \mathscr{K}^\infty$, and $\eta = \oplus \eta_i \in \mathscr{H}^\infty$. Consider the formal sum $\sum_{i,j} \eta_i^* \otimes x_{ij} \otimes \zeta_j$. The partial sums of this form a Cauchy sequence in $\mathscr{H}_r \otimes_h X \otimes_h \mathscr{K}_c$, and consequently converge to an element of $\mathscr{H}_r \otimes_h X \otimes_h \mathscr{K}_c$. We write this element as $\eta^* \odot x \odot \zeta$ or $\sum_{i,j} \eta_i^* \otimes x_{ij} \otimes \zeta_j$. We have by continuity that

$$\langle \sum_{i,j} \eta_i^* \otimes x_{ij} \otimes \zeta_j, \Phi \rangle = \sum_{i,j} \langle \Phi(x_{ij}) \zeta_j, \eta_i \rangle,$$

for $\Phi \in \mathrm{CB}(X, B(\mathscr{K}, \mathscr{H}))$.

LEMMA 5.1 [11]. *Let X be an operator space, and let $\mathscr{K}$ and $\mathscr{H}$ be Hilbert spaces. If $u \in \mathscr{H}_r \otimes_h X \otimes_h \mathscr{K}_c$ then $u = \sum_{i,j} \eta_i^* \otimes x_{ij} \otimes \zeta_j$, where $[x_{ij}] \in M_\infty(X)$, and where ζ_j and η_i are elements of $\mathscr{K}$ and $\mathscr{H}$ respectively, with $\sum_j \|\zeta_j\|^2 < \infty$, and $\sum_i \|\eta_i\|^2 < \infty$.*

THEOREM 5.2. *We have $\mathscr{H}_r \otimes_h K(\mathscr{K}, \mathscr{H}) \otimes_h \mathscr{K}_c = \mathrm{Bil}_{w^*h}(B(\mathscr{H}), B(\mathscr{K}))$ completely isometrically. Thus $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K}) = (\mathrm{Bil}_{w^*h}(B(\mathscr{H}), B(\mathscr{K})))^*$ weak*-homeomorphically and completely isometrically.*

Proof. Write κ for the completely isometric inclusion of $\mathscr{H}_r \otimes_h K(\mathscr{K}, \mathscr{H}) \otimes_h \mathscr{K}_c$ in $(B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K}))^*$ (see Proposition 2.1), and write λ for the completely isometric inclusion of $B(\mathscr{H}) \otimes_h B(\mathscr{K})$ in $B(\mathscr{H}) \otimes_{w^*h} B(\mathscr{K})$. Then the composition $\rho = \lambda^* \kappa$ is a complete contraction from $\mathscr{H}_r \otimes_h K(\mathscr{K}, \mathscr{H}) \otimes_h \mathscr{K}_c$ to $(B(\mathscr{H}) \otimes_h B(\mathscr{K}))^*$. We have

$$\langle \rho(\eta^* \otimes k \otimes \zeta), x \otimes y \rangle = \langle \kappa(\eta^* \otimes k \otimes \zeta), x \otimes y \rangle = \langle xky\zeta, \eta \rangle,$$

and so by continuity

$$\langle \rho(\sum_{i,j} \eta_i^* \otimes k_{ij} \otimes \zeta_j), x \otimes y \rangle = \sum_{i,j} \langle xk_{ij} y \zeta_j, \eta_i \rangle,$$

if $[k_{ij}] \in M_\infty(K(\mathscr{K}, \mathscr{H}))$, and if ζ_j and η_i are elements of $\mathscr{K}$ and $\mathscr{H}$ respectively, with $\sum_j \|\zeta_j\|^2 < \infty$, and $\sum_i \|\eta_i\|^2 < \infty$. This shows that $\rho(B(\mathscr{H}) \otimes_h B(\mathscr{K})) = \mathrm{Bil}_{w^*h}(B(\mathscr{H}), B(\mathscr{K}))$, using Lemma 5.1. Thus we need only show that ρ is a complete isometry. To see this consider the complete isometry

$$\theta : \mathscr{H}_r \otimes_h K(\mathscr{K}, \mathscr{H}) \otimes_h \mathscr{K}_c = T(\mathscr{H}) \otimes_h T(\mathscr{K}) \longrightarrow (B(\mathscr{H}) \otimes_h B(\mathscr{K}))^*$$

given by the self-duality of h. This agrees with ρ on rank 1 tensors of the form $\eta^* \otimes (\xi \otimes \omega^*) \otimes \zeta$, and by continuity agrees everywhere.

COROLLARY 5.3. *Suppose that X and Y are weak*-closed subspaces of $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively. Then $X \otimes_{w^*h} Y = (\mathrm{Bil}_{w^*h}(X, Y))^*$ weak*-homeomorphically and completely isometrically. Thus $X_* \otimes_h Y_* = \mathrm{Bil}_{w^*h}(X, Y)$ completely isometrically.*

Proof. This is similar to the previous proof. Since $X \otimes_{w*h} Y$ is weak*-closed in $B(\mathscr{H}) \otimes_{w*h} B(\mathscr{K})$, there is an induced complete quotient map

$$r: \mathrm{Bil}_{w*h}(B(\mathscr{H}), B(\mathscr{K})) \longrightarrow (X \otimes_{w*h} Y)_*.$$

Let $\lambda: X \otimes_h Y \to X \otimes_{w*h} Y$ and $\kappa: (X \otimes_{w*h} Y)_* \to (X \otimes_{w*h} Y)^*$ be the canonical completely isometric inclusions. Put $\rho = \lambda^* \kappa$. If $v = r(\varphi)$ is an element of $(X \otimes_{w*h} Y)_*$, then

$$\langle \rho(v), x \otimes y \rangle = \langle \kappa(v), x \otimes y \rangle = \langle x \otimes y, v \rangle = \varphi(x \otimes y).$$

Thus $\rho(v) = \varphi_{|X \otimes_h Y}$. Hence the range of ρ is $\mathrm{Bil}_{w*h}(X, Y)$. We need to show that ρ is a complete isometry, but again the canonical complete isometry $\theta: (X \otimes_{w*h} Y)_* = X_* \otimes_h Y_* \to (X \otimes_h Y)^*$ (given by the self-duality of h) agrees with ρ, as may be seen by checking the relation on elementary tensors.

One may also prove these two results using matricial w*-representations.

In view of the above there is a complete isometry $\mathrm{Bil}_{w*h}(X, Y) \to \mathrm{Bil}_h^\sigma(X, Y)$. Dualizing this map gives a weak*-continuous complete quotient map $X \otimes_{\sigma h} Y \to X \otimes_{w*h} Y$. Thus $X \otimes_{w*h} Y$ is completely isometrically isomorphic to a direct summand of $X \otimes_h^\sigma Y$, with a weak*-continuous projection implementing the retraction.

We remark that the result above shows that $\mathrm{Bil}_{w*h}(X, Y)$ does not depend on the particular containing $B(\mathscr{H})$ and $B(\mathscr{K})$, but only on the dual operator space structure of X and Y. Hence we may as well define $\mathrm{Bil}_{w*h}(X, Y)$, for dual operator spaces X and Y, to be the space of completely bounded bilinear functionals on $X \times Y$ of the form

$$\langle (\pi(x) \otimes I_\infty) k (\sigma(y) \otimes I_\infty) \zeta, \eta \rangle,$$

where π and σ are weak*-homeomorphic completely isometric maps from X and Y into $B(\mathscr{H})$ and $B(\mathscr{K})$ respectively, and where k, ζ and η are as usual; with $\mathscr{H}$ and $\mathscr{K}$ varying. With this definition we have the following.

COROLLARY 5.4. *If X and Y are operator spaces then* $X \otimes_h Y = \mathrm{Bil}_{w*h}(X^*, Y^*)$.

At this point it is natural to ask for a characterization of all weak*-continuous completely bounded maps $X \otimes_{w*h} Y \to B(\mathscr{F})$; here X and Y are dual operator spaces. More particularly one might ask for a characterization of all weak*-continuous completely bounded maps $B(\mathscr{H}) \otimes_{w*h} B(\mathscr{H}) \to B(\mathscr{H})$. The space of such maps can be shown to contain completely isometrically (cf. Corollary 3.7(ii)) the completely bounded maps of the form $k_1(x \otimes I_\infty) k_2 (y \otimes I_\infty) k_3$, where k_1, k_2 and k_3 are bounded linear operators which have $\infty \times \infty$ matrix representations whose entries are compact operators (between appropriate Hilbert spaces). One interesting such map is the Schur product on $B(\mathscr{H})$. The difficulty with a general representation theorem for such maps appear to be related to the fact that $B(\mathscr{H})$ is not weak*-injective in the sense of [4] (see also [13]). However, if $\mathscr{M}$ is a weak*-injective W*-algebra (these algebras were completely described in [4]) then the weak*-continuous completely bounded maps $\phi: X \otimes_{w*h} Y \to \mathscr{M}$ may be characterized. Indeed if $\mathscr{M} = M_n$ then Theorems 5.2 and 5.3 easily generalize to a matricial version which is the required characterization. For a general weak*-injective W*-algebra we now obtain a characterization from the identity $\oplus_i \mathrm{w^*CB}(Z, M_{n_i}) = \mathrm{w^*CB}(Z, \otimes_i M_{n_i})$ for all dual operator spaces Z.

Note added in proof. We thank Professor E. G. Effros for bringing to our attention the fact that the embedding $\theta: X \otimes_{w*h} Y \to X \otimes_h^\sigma Y$ is isometric is not clear. This may be seen as follows: if $\varphi \in \mathrm{Bil}_{\mathrm{h}}^{\sigma}(X, Y)$ then we may describe $\theta(u)(\varphi)$ as $\varphi^{\sim}(u)$ (with notation as in Proposition 3.7(ii)). Combining this observation with the comment after Corollary 5.3 (existence of a completely contractive retraction) yields the desired result.

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