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THE OPERATOR HILBERT SPACE OH AND TYPE III VON NEUMANN ALGEBRAS

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Abstract

A proof is given to show that the operator Hilbert space OH does not embed completely isomorphically into the predual of a semi-finite von Neumann algebra. This complements Junge's recent result, which admits such an embedding in the non-semi-finite case.

In remarkable recent work [**5**], Marius Junge proves that the operator Hilbert space OH (from [**8**]; see also [**10**]) embeds completely isomorphically into the predual M_* of a von Neumann algebra M which is of type III; thus this algebra M is not semi-finite. In this paper, we show that no such embedding can exist when M is semi-finite.

The results that we have just stated all belong to the currently very active field of 'operator spaces', for which we refer the reader to the monographs [**2**, **11**]. We merely recall a few basic facts, relevant to the present paper. An *operator space* is a Banach space, given together with an isometric embedding $E \subset B(H)$ into the algebra $B(H)$ of all bounded operators on a Hilbert space H. Using this embedding, we equip the space $M_n(E)$ (consisting of the $n \times n$ matrices with entries in E) with the norm induced by the space $M_n(B(H))$, naturally identified isometrically with $B(H \oplus \cdots \oplus H)$.

Let $F \subset B(K)$ be another operator space. In operator space theory, the morphisms are the completely bounded linear maps: a linear map $u\colon E \longrightarrow F$ is called *completely bounded* if the mappings $u_n\colon M_n(E) \longrightarrow M_n(F)$ defined by $[x_{ij}] \longrightarrow [u(x_{ij})]$ are uniformly bounded when n ranges over all integers greater than or equal to 1, and the *cb-norm* is defined as $\|u\|_{\mathrm{cb}} = \sup_n \|u_n\|$. The resulting *normed space* of all completely bounded maps $u\colon E \longrightarrow F$ equipped with the cb-norm is denoted by $\mathrm{CB}(E, F)$. If u is invertible with completely bounded inverse, then u is called a *complete isomorphism*. For any operator space $E \subset B(H)$, the Banach dual E^* can be equipped with a specific operator space structure, say $E^* \subset B(\mathcal{H})$, for which the natural identification $M_n(E^*) \simeq \mathrm{CB}(E, M_n)$ is isometric. On the other hand, the complex conjugate $\bar{E}$ can obviously be viewed as an operator space using the canonical embeddings $\bar{E} \subset \overline{B(H)} \simeq B(\bar{H})$. Let I be any set. In [**8**], we exhibited an operator space E that is isometric to $\ell_2(I)$ as a Banach space, and such that the canonical isometry (associated to the inner product) $E \simeq \overline{E^*}$ is a complete isometry. The latter operator space, which is uniquely characterized by the preceding self-dual property, is denoted by $\mathrm{OH}(I)$, or simply by OH when $I = \mathbb{N}$. We call it the *operator Hilbert space*.

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In Banach space theory (and in commutative harmonic analysis), the existence of an isomorphic (actually, isometric) embedding of ℓ_2 (or $\ell_2(I)$) in an L_1-space plays a very important role in connection with the Khintchine and Grothendieck inequalities. However, the non-commutative version of the Khintchine inequality due to Lust-Piquard and the author (see [**10**, Theorem 8.4.1]), when properly interpreted, leads to the embedding of a different (Hilbertian but not OH) operator space into L_1, namely the space called $R+C$ in [**10**]. This essentially implies that OH does not embed (completely isomorphically) into a commutative L_1 space (see [**6**] for details and more general results). Thus the question (raised in [**8**]) of whether OH itself embeds (completely isomorphically) in a *non-commutative* L_1 space remained open, and was only recently solved affirmatively by Junge [**5**]. By a *non-commutative* L_1*-space*, we mean the predual M_* of a von Neumann algebra M, equipped with the operator space structure induced by the dual M^*. Recall that M is called *semi-finite* if it admits a normal and faithful trace that is also semi-finite; that is, although it is not necessarily finite, it admits sufficiently many elements on which it is finite (see [**16**] for more).

Our main theorem below shows that if OH embeds in M_*, then M cannot be semi-finite. This proves the need for a type III algebra in Junge's work, and perhaps explains the delay in resolving this embedding problem.

We use the following easy consequence of [**13**, Corollary 3.4]. (A more direct proof, with a better value of the constant K, appears in [**12**].)

Lemma 1. *Let I be any set, let $E = \mathrm{OH}(I)$, and let M be a von Neumann algebra. Consider a linear map $v\colon E \longrightarrow M_*$.*

(i) *If v is completely bounded, then there is a normal state φ on M such that*

$$\|v^*x\|^2 \leqslant K^2(\varphi(xx^*)\varphi(x^*x))^{1/2}, \quad \text{for all } x \in M, \tag{1}$$

where $K = 2^{9/4}\|v\|_{\mathrm{cb}}$.

(ii) *Conversely, if there are f_1, f_2 in M_+^* such that*

$$\|v^*x\|^2 \leqslant (f_1(xx^*)f_2(x^*x))^{1/2}, \quad \text{for all } x \in M,$$

then necessarily

$$\|v\|_{\mathrm{cb}} \leqslant (f_1(1)f_2(1))^{1/4}. \tag{2}$$

Proof. Let $K = 2^{9/4}\|v\|_{\mathrm{cb}}$. By [**13**, Corollary 3.4], there is a state f on M such that

$$\|v^*x\|^2 \leqslant K^2(f(xx^*)f(x^*x))^{1/2}, \quad \text{for all } x \in M. \tag{3}$$

We use an argument that can be traced back to [**14**, p. 352] and, in the non-commutative case, to [**3**, Proposition 2.3]. Let $f = f_n + f_s$ be the decomposition of f into its normal and singular parts. We set $\varphi = f_n$. As explained in the proof of [**3**, Proposition 2.3], there is an increasing net (p_α) of projections in M such that $p_\alpha \longrightarrow 1$, say in the strong operator topology (SOT for short), and $f_s(p_\alpha) = 0$ for all α. Note that (by the SOT-continuity of the product of M on bounded sets) we have $p_\alpha x p_\alpha \longrightarrow x$, $p_\alpha x p_\alpha x^* p_\alpha \longrightarrow xx^*$ and $p_\alpha x^* p_\alpha x p_\alpha \longrightarrow x^*x$ for the strong operator topology. For any ξ in the unit ball of E^*, by assumption $x \longrightarrow \langle \xi, v^*x\rangle$ is in M_* (that is, it is 'normal'), and hence

$$\langle \xi, v^*x\rangle = \lim \langle \xi, v^*(p_\alpha x p_\alpha)\rangle.$$

Hence, by (3),

$$\langle \xi, v^*x\rangle \leqslant K^2 \lim(f(p_\alpha x p_\alpha x^* p_\alpha) f(p_\alpha x^* p_\alpha x p_\alpha))^{1/2},$$

but since $f_s(p_\alpha) = 0$, *a fortiori* $f_s(p_\alpha x p_\alpha x^* p_\alpha) = 0 = f_s(p_\alpha x^* p_\alpha x p_\alpha)$, and hence we obtain

$$|\langle \xi, v^*x\rangle| \leqslant K^2 \lim(\varphi(p_\alpha x p_\alpha x^* p_\alpha)\varphi(p_\alpha x^* p_\alpha x p_\alpha))^{1/2}.$$

Thus we conclude that

$$|\langle \xi, v^*x\rangle| \leqslant K^2(\varphi(xx^*)\varphi(x^*x))^{1/2},$$

which immediately yields statement (i). The proof of statement (ii) is identical to that of the last assertion in [**13**, Corollary 3.4]. □

Our main result is the following theorem.

THEOREM. *Let $E = \mathrm{OH}(I)$, with I an infinite set. Let $F \subset M_*$ be a subspace of the predual of a semi-finite von Neumann algebra M. Then for any completely bounded maps*

$$u\colon E \longrightarrow F \quad \text{and} \quad w\colon F \longrightarrow E,$$

the composition wv is compact.

Proof. Clearly, this reduces to $I = \mathbb{N}$ if we wish. Let τ be a normal faithful semi-finite trace on M.

We will argue by contradiction. Assume that wv is not compact. Then, by the homogeneity of OH (see [**8**, p. 18]), we may assume that wv is diagonal or even furthermore that wv is the identity on E. In other words, we may as well assume that u invertible, and that $w = u^{-1}$.

Let $v\colon E \longrightarrow M_*$ be the same map as u, but viewed as acting into M_*. By Lemma 1, there is a normal state φ such that (1) holds. Let e be the support projection of φ (that is, we have $\varphi(1-e) = 0$ and $\varphi(q) > 0$ for any non-zero projection q in M with $q \leqslant e$). Then (1) implies that for any x in M, we have $v^*(x(1-e)) = 0 = v^*((1-e)x)$; hence

$$v^*(x) = v^*(exe).$$

Thus if we replace M by eMe and φ by $e\varphi$, we may assume in addition that φ is faithful.

Since $M_* \simeq L_1(\tau)$, we may assume that $\varphi = \psi \cdot \tau$. Fix $0 < \delta < 1$. Let p be the spectral projection associated to ψ with respect to the set $[\delta, \delta^{-1}]$ (for more details, see, for example, [**15**, p. 338], or also [**7**]). Note that $\delta\tau(p) \leqslant \tau(\psi) = \varphi(1) \leqslant 1$, so that in particular $\tau(p) < \infty$; moreover,

$$p \cdot \psi = \psi \cdot p = p \cdot \psi \cdot p \leqslant \delta^{-1} p. \tag{4}$$

On the other hand, let $\varepsilon(\delta) = \tau((1-p)\psi) = \varphi(1-p)$. Clearly, (since φ is faithful) we have $\varepsilon(\delta) \to 0$ when $\delta \to 0$. Thus if we set, for all y in E,

$$v_\delta(y) = p \cdot v(y) \cdot p, \quad T_1(y) = v(y)(1-p) \quad \text{and} \quad T_2(y) = (1-p)v(y)p,$$

we have $v = v_\delta + T_1 + T_2$. We will show that $T_1 + T_2$ is small when $\delta \to 0$, so that v_δ can be viewed as a perturbation of v. Indeed, for x in M, we have $T_1^*(x) = v^*((1-p)x)$ and $x^*(1-p)x \leqslant x^*x$.

Hence by (1) we have

$$\|T_1^*(x)\|^2 \leqslant K^2(\varphi((1-p)xx^*(1-p))\varphi(x^*x))^{1/2},$$

and hence by (2)

$$\begin{aligned}\|T_1\|_{\mathrm{cb}} &\leqslant K(\varphi(1-p))^{1/4}\\ &\leqslant K(\varepsilon(\delta))^{1/4}.\end{aligned}$$

Similarly, $\|T_2\|_{\mathrm{cb}} \leqslant K(\varepsilon(\delta))^{1/4}$; hence $\|v-v_\delta\|_{\mathrm{cb}} \leqslant \|T_1\|_{\mathrm{cb}} + \|T_2\|_{\mathrm{cb}} \leqslant 2K(\varepsilon(\delta))^{1/4}$. Let $\varepsilon_0 = \|u^{-1}\|_{\mathrm{cb}}^{-1}$. Clearly, if we choose δ small enough so that $2K(\varepsilon(\delta))^{1/4} < \varepsilon_0$, we have $\|v-v_\delta\|_{\mathrm{cb}} < \varepsilon_0$. Hence, by elementary reasoning (based solely on the triangle inequality in $M_n(M_*)$), the map $v_\delta\colon E \longrightarrow pM_*p \subset M_*$ is a completely isomorphic embedding. But now by (4) we have

$$\|v_\delta^*(x)\|^2 = \|v^*(pxp)\|^2 \leqslant K^2\delta^{-1}\|pxp\|_{L^2(\tau)}\|px^*p\|_{L^2(\tau)}.$$

Hence, since τ is *tracial* (this is where we make crucial use of the semi-finiteness assumption),

$$\begin{aligned}\|v_\delta^*(x)\|^2 &\leqslant K^2\delta^{-1}\|pxp\|_{L^2(\tau)}^2\\ &\leqslant K^2\delta^{-1}\min\left\{\|px\|_{L^2(\tau)}^2, \|xp\|_{L^2(\tau)}^2\right\}\\ &\leqslant K^2\delta^{-1}\|px\|_{L^2(\tau)}^2.\end{aligned}$$

By Lemma 2 below, since $\tau(p) < \infty$, this is impossible. □

LEMMA 2. *With the above notation, let $V\colon \mathrm{OH} \longrightarrow L_1(\tau)$ be a linear map for which there is a in the unit ball of $L_2(\tau)$ and a constant B such that, for any x in M, we have*

$$\|V^*(x)\| \leqslant B\|ax\|_{L^2(\tau)}. \tag{5}$$

Then, for any isometry $J\colon C \longrightarrow \mathrm{OH}$, VJ is completely bounded from C to $L_1(\tau)$. In particular, V cannot be a completely isomorphic embedding.

Proof. By (5), for any finite sequence (x_i) in M,

$$\sum \|(VJ)^*(x_i)\|^2 = \sum \|V^*(x_i)\|^2 \leqslant B^2 \left\|\sum x_i x_i^*\right\|. \tag{6}$$

By a well-known argument from [**1**], it follows that $\|(VJ)^*\colon M \longrightarrow C^*\|_{\mathrm{cb}} \leqslant B$, and hence VJ is completely bounded. Finally, if V were a completely isomorphic embedding, then VJ (when viewed as acting into the range of V) would be a completely bounded map from C to OH. Hence VJ would (by [**8**, Remark 2.11]) be in the Schatten class S_4, and *a fortiori* would be compact. But then J itself would have to be compact, which is absurd. □

REMARKS.

(i) Junge [**5**] proves that OH_n embeds completely isomorphically (with uniform constants) into the predual of a finite-dimensional (and hence semi-finite!) von Neumann algebra. More precisely, he proves that there is $C > 0$ such that, for any n, there are an integer N, a subspace $F_n \subset M_N^*$ and a (complete) isomorphism $u_n\colon \mathrm{OH}_n \longrightarrow F_n$ such that $\sup_n \|u_n\|_{\mathrm{cb}}\|u_n^{-1}\|_{\mathrm{cb}} \leqslant C$. It would be interesting to estimate N as a function of n.

(ii) The non-existence of embeddings of OH into M_* when M is commutative is rather easy to show. In that case, even the finite-dimensional case (as in the preceding point) is ruled out (see [**9**] for related facts). The paper [**6**] contains stronger results in the same direction.

(iii) The above theorem remains valid with essentially the same proof for $E=(R,C)_\theta$ (with $0<\theta<1$) in the sense of [**10**], but this requires the generalized version of Lemma 1 that is proved in [**12**]. This implies that, for any $1<p<2$, the Schatten classes S_p (and hence most non-commutative L_p-spaces) do not embed (completely isomorphically) into the predual of any semi-finite von Neumann algebra.

(iv) Let N_* be the predual of the injective factor of type III_λ with $0<\lambda\leqslant 1$, and let M be a (semi-finite) von Neumann algebra of type II_∞. Junge proved that OH embeds completely isomorphically into N_*. Hence Theorem 1 implies that N_* does not embed completely isomorphically into M_*. This gives a somewhat partial answer to the (still-open) question raised in [**4**] of the existence of an isomorphic (in the Banach space sense) embedding of N_* into M_*.

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