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Translation and dilation invariant subspaces of $L^2(\mathbb{R})$

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Abstract. The closed subspaces of the Hilbert space $L^2(\mathbb{R})$ which are invariant under multiplication by $H^\infty(\mathbb{R})$ functions and the dilation operators $f(x) \to f(sx)$, $1 < s < \infty$, are determined as the two parameter family of subspaces $L^2[-a, b]$, $0 \leqq a$, $b \leqq \infty$, which are reducing for multiplication operators, together with a four parameter family of nonreducing subspaces. The lattice and topological structure are determined and using operator algebra methods the corresponding family of orthogonal projections, with the weak operator topology, is identified as a compact connected 4-manifold.

1. Introduction

An important result in operator function theory, usually referred to as Beurling's theorem, asserts that the closed subspaces of $L^2(\mathbb{R})$ which are invariant for multiplication by functions in $H^\infty(\mathbb{R})$ are either of the form $L^2(E)$ for some Lebesgue measurable subset E or are of the form $uH^2(\mathbb{R})$ for some unimodular function u in $L^\infty(\mathbb{R})$. See Lax [10], Helson [7] or Nikolskii [14] for example.

Using Beurling's theorem and additional arguments, involving cocycles of unimodular functions and the structure of singular inner functions, we have recently determined the $H^\infty(\mathbb{R})$-invariant closed subspaces which are also invariant under right translations. In addition to the obvious subspaces $L^2([t, \infty])$ and $e^{i\lambda x}H^2(\mathbb{R})$, for t, λ real, there is the unexpected family of jointly invariant subspaces,

$$\{e^{-isx^2/2}e^{i\lambda x}H^2(\mathbb{R}) : s > 0, \lambda \in \mathbb{R}\}.$$

Moreover, by analysing the Hilbert-Schmidt operators in the w*-closed operator algebra $\mathscr{A}_{FB}$ generated by $H^\infty(\mathbb{R})$ and the right shifts—this is one formulation of the Fourier binest algebra of [9]—it was found that the lattice $\mathscr{L}_{FB}$ of (projections onto) these jointly invariant subspaces is compact in the strong operator topology and is homeomorphic to the unit disc. The interior of this disc corresponds to those subspaces with parameter $s > 0$.

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In what follows we analyse the jointly invariant subspaces arising when the right translation semigroup is replaced by the unitary dilation semigroup $\{V_t \colon t \geqq 0\}$ where $V_t f(x) = e^{t/2} f(e^t x)$. From the perspective of composition operators determined by biholomorphic automorphisms of the upper half plane this amounts to replacing a parabolic 1-parameter semigroup by a hyperbolic 1-parameter semigroup.

Once again we find that there is a surprisingly rich topological and lattice theoretic structure as well as interesting connections with operator algebras and function theory. We show that in addition to the trivial subspaces and the obvious subspaces $L^2[-a, b]$, for $a, b \geqq 0$, there is the four parameter family

$$\{u_{0,\theta}(x)|x|^{is} e^{i\lambda x} e^{i\mu x^{-1}} H^2(\mathbb{R}) : \lambda, \mu \geqq 0, s \in \mathbb{R}, \theta \in S^1\}$$

where $u_{0,\theta}(x)$ is constant and unimodular on $\mathbb{R}_+$ and $\mathbb{R}_-$ taking values 1 and θ respectively. Moreover this collection accounts for all the hyperbolically jointly invariant subspaces and the lattice $\mathscr{L}$ of orthogonal projections onto these subspaces forms a compact connected 4-manifold in the strong operator topology. To obtain the connectedness of $\mathscr{L}$ we determine various strong operator topology limits of orthogonal projections and in particular we find that

$$[e^{i\lambda x} e^{i\lambda x^{-1}} H^2(\mathbb{R})] \to [L^2[-1, 1]],$$

$$[|x|^{is_n} e^{i\lambda_n x} H^2(\mathbb{R})] \to [L^2[-a, 0]],$$

where $[K]$ denotes the orthogonal projection onto K, where $\lambda \to \infty$, and where (s_n, λ_n) is an appropriate sequence, for $a > 0$, tending to (∞, ∞). The arguments to determine these seemingly classical facts are rather deep; they require the main structure theorem for hyperbolically invariant subspaces, the compactness of the projection lattice, indicated below, and additional arguments building on the standard model of Halmos [6] for a pair of projections in generic position.

To determine that the projection lattice is compact we consider the w*-closed nonselfadjoint algebra, denoted $\mathscr{A}_h$, generated by the dilation semigroup and $H^\infty(\mathbb{R})$. This algebra is analogous to the Fourier binest algebra and has similar properties; it contains no nontrivial finite rank operators and is antisymmetric. Identifying Hilbert-Schmidt operators in $\mathscr{A}_h$ which are related to pseudodifferential operators with bianalytic symbols, we obtain (as in [9]) a sequence of such operators which tends strongly to the identity. From this and a result of Wagner [21] the compactness of the projection lattice follows.

We have indicated above the results of Sections 2, 3 and 4. In Section 5 we determine the lattice structure of the hyperbolically invariant subspaces and consider associated operator algebras and in Section 6 we determine the isometric automorphism group of $\mathscr{A}_h$ as $\mathbb{R}^2 \times S^1$. In fact the hyperbolic algebra $\mathscr{A}_h$ is doubly generated and the classical lifting theorem of Sz-Nagy and Foias [19] can be applied to show that a contractive w*-continuous representation of $\mathscr{A}_h$ is completely contractive. This is used to show that the isometric automorphisms of $\mathscr{A}_h$ are precisely the unitary automorphisms and these are identified explicitly.

In the final section we complete the identification of $\mathscr{L}$ as a topological manifold and determine the inclusions

$$\mathscr{L} \subset \tilde{\mathscr{L}} \subset \hat{\mathscr{L}}$$

where $\tilde{\mathscr{L}}$ is the closed set arising from the extended parameter range $\lambda, \mu \in \mathbb{R}$ and $\hat{\mathscr{L}}$ is the manifold $\tilde{\mathscr{L}} \cup \tilde{\mathscr{L}}^{\perp}$. In particular $\tilde{\mathscr{L}}$ is homeomorphic to a topological identification space $(B^3 \times S^1)/\sim$ where B^3 is the closed ball of $\mathbb{R}^3$ and where circles on the boundary of B^3 are collapsed. The lattice $\mathscr{L}$ may be viewed as a submanifold corresponding to a deformed hemisphere of B^3. The $\theta = 1$ sections of these inclusions bear an analogy with the corresponding inclusions

$$\mathscr{L}_{FB} \subset \tilde{\mathscr{L}}_{FB} \subset \hat{\mathscr{L}}_{FB}$$

for the Fourier binest algebra. We show that $\hat{\mathscr{L}}_{FB}$ is a foliated 2-sphere, which we refer to as the Fourier-Plancherel 2-sphere, whilst the section $\hat{\mathscr{L}}_{\theta=1}$ is a topological 3-sphere.

The results obtained here, and in [9], suggest some interesting directions in the theory of nonself-adjoint operator algebras. Firstly they suggest the development of a general theory of what one might refer to as *Euclidean lattice algebras*, meaning those operator algebras whose invariant projection lattices, with the strong operator topology, are Euclidean manifolds. This context contrasts with the fact that extant weakly closed operator algebras generally have exotic invariant subspace lattices when these lattices are not trivial, as the following table suggests.

algebra $\mathscr{A}$	lattice Lat $\mathscr{A}$
von Neumann algebra	continuous geometry [12], [13]
commutative subspace lattice algebra	increasing sets [2]
free semigroup algebras	generalised inner functions [5]
Bergman shift algebra	contains copy of Proj$(\mathscr{H})$ [3]

The simplest algebras for which there is a topological injection Lat $\mathscr{A} \to \mathbb{R}^n$ come from the Volterra nest algebra $\mathscr{A}_v$; the n-fold direct sum $\mathscr{A}_v \oplus \cdots \oplus \mathscr{A}_v$ has lattice homeomorphic to the product $[0,1] \times \cdots \times [0,1]$. It would be interesting to determine which manifolds are attainable by direct-sum-indecomposable algebras and how the order topological structure relates to the operator algebra structure.

There is, furthermore, an additional Lie group perspective. The analysis of the hyperbolic algebra and the Fourier binest algebra can be viewed as part of a theory of *Lie semigroup algebras*. By this term we mean a weak operator topology closed operator algebra generated by the image of a Lie semigroup [8] in a unitary representation of the ambient Lie group. It is the specific setting of a non compact locally compact group and an indecomposable representation which is of relevance here. In contrast the free semigroup algebras of Davidson and Pitts [5] derive from discrete groups. It is the $ax + b$ group, with the Lie semigroup for $a \geqq 0$ and $b \geqq 0$ that provides the hyperbolic algebra studied here.

The Fourier binest algebra on the other hand is obtained from the Lie semigroup of the 3-dimensional Heisenberg group given by

$$\begin{bmatrix} 1 & \lambda & t \\ & 1 & \mu \\ & & 1 \end{bmatrix}, \quad \lambda \geqq 0, \mu \geqq 0.$$

For these new algebras and perspectives many natural and fundamental structural problems are ready-to-hand.

2. Hyperbolically invariant subspaces

Let $\{V_t : t \geqq 0\}$ and $\{M_\lambda : \lambda \geqq 0\}$ be the continuous semigroups given by the unitary dilation operators $(V_t f)(x) = e^{t/2} f(e^t x)$ and the multiplication operators $(M_\lambda f)(x) = e^{i\lambda x} f(x)$ for f in $L^2(\mathbb{R})$. In this section we determine the lattice

$$\mathscr{L} = \mathrm{Lat}\{M_\lambda, V_t : \lambda \geqq 0, t \geqq 0\}$$

of all (closed) subspaces invariant under both semigroups.

Since $\{M_\lambda : \lambda \geqq 0\}$ generates $\{M_h : h \in H^\infty(\mathbb{R})\}$, the algebra of multiplication operators, for which $M_h f = hf$, we see that an invariant subspace K in $\mathscr{L}$ is necessarily invariant for $\{M_h : h \in H^\infty(\mathbb{R})\}$ and so either K is reducing for the semigroup $\{M_\lambda : \lambda \geqq 0\}$ and hence has the form $L^2(E)$ for some measurable subset of $\mathbb{R}$, or K is simply invariant. In the latter case by Beurling's theorem, indicated in the introduction, we have

$$K = uH^2(\mathbb{R})$$

for some unimodular function u in $L^\infty(\mathbb{R})$, whilst in the former case, since K must also be invariant for the dilation semigroup, it follows that $K = L^2[-a, b]$ for some $0 \leqq a, b \leqq \infty$. In the latter case we shall derive the nature of the unimodular function u.

First we simply present a four parameter family of unimodular functions and confirm that they provide jointly invariant subspaces for the two semigroups. Then we set about the more technical argument to establish that these are the only such functions.

Define $g_{s,\theta}$, for $s \in \mathbb{R}$ and θ a unimodular scalar to be the function $u_\theta(x)|x|^{is}$, where u_θ is the 2-valued unimodular function given in the introduction. Explicitly,

$$g_{s,\theta}(x) = \begin{cases} \exp(is \log|x|), & x > 0, \\ \theta \exp(is \log|x|), & x < 0. \end{cases}$$

Then $g_{s,\theta}$ is a unimodular function in $L^\infty(\mathbb{R})$ for which

$$V_t(g_{s,\theta} f) = e^{ist} g_{s,\theta} V_t f.$$

Since $V_t H^2(\mathbb{R}) = H^2(\mathbb{R})$ it follows that $V_t g_{s,\theta} H^2(\mathbb{R}) = g_{s,\theta} H^2(\mathbb{R})$, for all $s \in \mathbb{R}$ and $|\theta| = 1$.

Let $u_{s,\theta}$ be the two-valued function on $\mathbb{R}$ given by

$$u_{s,\theta}(x) = \begin{cases} 1, & x > 0, \\ \theta e^{s\pi}, & x \leqq 0. \end{cases}$$

Then $g_{s,\theta}H^2(\mathbb{R})$ may be more simply described as $u_{s,\theta}H^2(\mathbb{R})$. To see this note that

$$g_{s,\theta}(x) = u_{s,\theta}(x)g_s(x)$$

where g_s is the invertible function in H^∞ given by

$$g_s(z) = \exp(is\log z) = z^{is}$$

where $\log z$ is defined on $\mathbb{C}\backslash\{\mathrm{Im}\, z \leqq 0\}$ with principal value of the argument.

Consider now the unimodular functions

$$e_{\lambda,\mu}(x) = e^{i(\lambda x + \mu x^{-1})}$$

where λ, μ are real. Note that for $\lambda \geqq 0$ and $\mu \leqq 0$ this function is an inner function. In view of the commutation relation

$$V_t M_\lambda = M_{e^t\lambda} V_t,$$

and the fact that $H^2(\mathbb{R})$ is a reducing subspace for the unitary V_t, we have

$$V_t e_{\lambda,\mu} H^2(\mathbb{R}) = e^{i\lambda e^t x} e^{i\mu e^{-t} x^{-1}} H^2(\mathbb{R}) = e_{\lambda,\mu}(x)(e^{i\lambda(e^t-1)x} e^{i\mu(e^{-t}-1)x^{-1}}) H^2(\mathbb{R}).$$

For $t, \lambda, \mu \geqq 0$ we have $\lambda(e^t - 1) \geqq 0$ and $\mu(e^{-t} - 1) \leqq 0$ and so it follows that $e_{\lambda,\mu}H^2(\mathbb{R})$ is an invariant subspace for $\{V_t : t \geqq 0\}$.

It has been shown that the four parameter family of subspaces

$$K_{s,\theta,\lambda,\mu} = u_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R}), \quad s \in \mathbb{R}, |\theta| = 1, \lambda \geqq 0, \mu \geqq 0,$$

is a family of closed subspaces which are invariant for $\{V_t : t \geqq 0\}$ and simply invariant for $\{M_\lambda : \lambda \geqq 0\}$.

Theorem 2.1. *Let K be a nonzero closed subspace of $L^2(\mathbb{R})$ which is invariant for the dilation semigroup $\{V_t : t \geqq 0\}$ and simply invariant for the Fourier shift semigroup $\{M_\lambda : \lambda \geqq 0\}$. Then $K = K_{s,\theta,\lambda,\mu}$ for some $\lambda, \mu \geqq 0$, real s and unimodular complex number θ.*

Proof. By Beurling's characterisation of simply invariant subspaces we have $K = uH^2(\mathbb{R})$ for some unimodular function u in $L^\infty(\mathbb{R})$. For real t the subspace V_tK is also simply invariant for $\{M_\lambda : \lambda \geqq 0\}$ since

$$M_\lambda V_t K = V_t M_{e^{-t}\lambda} K \subseteqq V_t K$$

and $e^{-t}\lambda \to \infty$ as $\lambda \to \infty$. Therefore $V_tK = \phi_t H^2(\mathbb{R})$ for some unimodular function ϕ_t in $L^\infty(\mathbb{R})$. By assumption, for $t \geqq 0$, $V_tK \subseteqq K$ and so $\phi_t H^2(\mathbb{R}) \subseteqq uH^2(\mathbb{R})$ so that $\phi_t\bar{u}$ is an inner function; in other words $V_tK = w_t uH^2(\mathbb{R})$ for some inner function w_t.

Since we also have

$$V_tK = V_t uH^2(\mathbb{R}) = u(e^t x)V_t H^2(\mathbb{R}) = u(e^t x)H^2(\mathbb{R})$$

it follows that

$$w_t(x)u(x) = c_t u(e^t x)$$

for some unimodular constant c_t. Replacing $w_t(x)$ by $\bar{c}_t w_t(x)$ we may assume that there is a chain $\{w_t : t \geqq 0\}$ of inner functions for which $w_t(x)u(x) = u(e^t x)$ for almost every x (depending on t). For positive s, t we have

$$w_{s+t}(x) = \frac{u(e^{s+t}x)}{u(x)} = \frac{u(e^{s+t}x)}{u(e^s x)} \cdot \frac{u(e^s x)}{u(x)} = w_t(e^s x)w_s(x)$$

for almost every x, and so we obtain a cocycle equation for the inner function chain. This implies that the inner function $w_t(e^s x)$ divides $w_{s+t}(x)$. Since $V_rK \subseteqq V_{s+t}K$ when $r > s+t$ the function w_{s+t} divides w_r and so, in fact $w_t(e^s x)$ divides $w_r(x)$ if $r > s+t$. Thus for fixed $r > 0$ there is an inner function h_s such that $w_t(e^s z)h_s(z) = w_r(z)$ for $s < r-t$. It follows that if $w_t(z)$ has a zero in the upper half plane then $w_r(z)$ has a radial line segment of zeros, which is impossible.

Thus w_t is a singular inner function and we can write, for some unimodular α, some nonnegative β and some singular measure μ (depending on t),

$$w_t(z) = \alpha e^{i\beta z} \exp\left(i \int_{\mathbb{R}} \frac{sz+1}{s-z} \frac{1}{s^2+1} d\mu(s) \right) \quad (\operatorname{Im} z > 0).$$

Moreover, the support of the singular measure for w_t must be concentrated at 0. To see this recall (see for example [9]) that if w_t divides w_r and ν is the singular measure corresponding to w_r, then μ is dominated by ν. So here we deduce that $\mu(e^s.)$ is dominated by ν for all $0 < s < r-t$. But now the measure μ_0 on $\mathbb{R}\backslash\{0\}$ defined by

$$\mu_0(A) = \int_0^{r-t} \mu(e^s A)\, ds$$

is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ and at the same time is dominated by the singular measure $(r-t)\nu$. Thus $\mu_0 = 0$ and $\mu(e^s A) = 0$ for almost all $s \in (0, r-t)$. From this it follows that the support of μ is contained in $\{0\}$.

We have shown then that w_t has the form

$$w_t(x) = \alpha(t)e^{i\beta(t)x}e^{-i\gamma(t)x^{-1}}$$

where $\alpha(t)$ is unimodular and $\beta(t), \gamma(t)$ are nonnegative. From the cocycle identity

$$w_{s+t}(x) = w_t(e^s x) w_s(x)$$

we obtain

$$\alpha(s+t) e^{i\beta(s+t)x} e^{-i\gamma(s+t)x^{-1}} = \alpha(t) e^{i\beta(t)e^s x} e^{-i\gamma(t)(e^s x)^{-1}} \alpha(s) e^{i\beta(s)x} e^{-\gamma(s)x^{-1}}$$

and so

$$\alpha(s+t) = \alpha(s)\alpha(t),$$
$$\beta(s+t) = \beta(t)e^s + \beta(s),$$
$$\gamma(s+t) = \gamma(t)e^{-s} + \gamma(s).$$

Since w_t divides w_r whenever $t < r$ it follows that the functions $\beta(t), \gamma(t)$ are increasing, and so it follows from the equations above that $\beta(t)$ and $\gamma(t)$ are differentiable.

To see this note that since β is increasing on $\mathbb{R}_+$ it is differentiable there, almost everywhere, by Lebesgue's Theorem. The functional equation shows that if s is a point of differentiability then so is $s + t$. Hence β is differentiable everywhere.

Setting $s = 0$ in the functional equation shows that $\beta(0) = 0$. Differentiate the functional equation with respect to s and set $s = 0$ to obtain $\beta'(t) = \beta(t)e^0 + \lambda$ where $\lambda = \beta'(0)$. This equation has unique solution $\beta(t) = \lambda(e^t - 1)$ (since $\beta(0) = 0$) and $\lambda \geqq 0$ since $\beta(t) \geqq \beta(0) = 0$ for $t \geqq 0$. The argument for γ is similar. Thus we have

$$\beta(t) = (e^t - 1)\lambda, \quad \gamma(t) = (1 - e^{-t})\mu$$

for some $\lambda, \mu \geqq 0$.

Furthermore, since

$$\frac{u(e^t x)}{u(x)} = w_t(x) = \alpha(t) e^{i(e^t - 1)\lambda x} e^{-i(1-e^{-t})\mu x^{-1}}$$

and the quotient is measurable in (x, t) and continuous in x for each fixed t, it follows that the quotient, and hence α, is measurable in t. Since $\alpha(s+t) = \alpha(s)\alpha(t)$ it follows that $\alpha(t) = e^{i\sigma t}$ for some real number σ.

For each $t \geqq 0$ we now have the equality

$$\frac{u(e^t x)}{u(x)} = e^{i\sigma t} e^{i(e^t - 1)\lambda x} e^{i(e^{-t} - 1)\mu x^{-1}}$$

for almost every x. Set

$$u_1(x) = \exp(i\sigma \log|x|) e_{\lambda,\mu}(x)$$

and observe that this equation implies that

$$\frac{u(e^t x)}{u(x)} = \frac{u_1(e^t x)}{u_1(x)}$$

for almost every x, for each $t \geqq 0$. We conclude that the unimodular function $v = u\bar{u}_1$ satisfies the equation $v(e^t x) = v(x)$ for almost every x for each $t > 0$. This in fact means that the equation holds for almost every x for each t in $\mathbb{R}$. By a standard argument (using Fubini's theorem)[1] for almost every x we have the equality $v(e^t x) = v(x)$ for almost all t. It follows now that $v(t)$ is a two-valued function, constant on $\mathbb{R}_+$ and $\mathbb{R}_-$. Thus, up to a multiplicative unimodular constant, the function $u = v\bar{u}_1$ has the form $g_{\sigma,\theta} e_{\lambda,\mu}$ and hence $K = K_{\sigma,\theta,\lambda,\mu}$ as required. □

It follows from the theorem above that the lattice $\mathscr{L}$ of all (closed) subspaces of $L^2(\mathbb{R})$ which are invariant for both semigroups is the disjoint union

$$\mathscr{L} = \{K_{s,\theta,\lambda,\mu} : s \in \mathbb{R}, |\theta| = 1, \lambda \geqq 0, \mu \geqq 0\} \cup \{L^2([-a,b]) : a, b \in [0,\infty]\}$$

and so $\mathscr{L}$ is parametrised by the set $(\mathbb{R}_+^2 \times \mathbb{R} \times S^1) \cup ([0,1]^2)$, the set $\{(a,b) : 0 \leqq a, b \leqq \infty\}$ being replaced by the square $[0,1]^2$. In Section 4 we consider $\mathscr{L}$ as a topological space of projections and show the way in which this square gives a compactification of $\mathbb{R}_+^2 \times \mathbb{R} \times S^1$.

3. Approximate identity and compactness

Let $\mathscr{A}_h$ be the w*-closed algebra generated by the set $\{M_\lambda, V_t : \lambda \geqq 0, t \geqq 0\}$. Note that the dilation and multiplication groups provide a unitary representation of the $ax + b$ group by means of the correspondence

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \to M_b V_{\log a}$$

for $a \geqq 0$ and b real. Thus $\mathscr{A}_h$ is a Lie semigroup algebra for the Lie semigroup with $a \geqq 1, b \geqq 0$.

Plainly, the lattice $\operatorname{Lat}\mathscr{A}_h$ of invariant subspaces for $\mathscr{A}_h$ is equal to $\mathscr{L}$. We shall show that $\mathscr{L}$, with the strong operator topology, is compact. To see this we first show that the algebra $\mathscr{A}_h$ contains a contractive approximate identity consisting of compact operators in the sense of Proposition 3.2. This we do by exploiting a connection with the Fourier binest algebra given in the next lemma.

[1] Let $f(x,y) = |v(xy) - v(x)|$. By hypothesis, for each $y > 0$ there is a co-null set $A_y \subseteqq \mathbb{R}$ such that $f(x,y) = 0$ whenever $x \in A_y$. Thus $\int_{\mathbb{R}} f(x,y)\,dx = 0$ for all $y > 0$ and so $\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(x,y)\,dx\,dy = 0$. By Fubini's theorem, $\int_{\mathbb{R}}\int_{\mathbb{R}_+} f(x,y)\,dy\,dx = 0$ so that $\int_{\mathbb{R}_+} f(x,y)\,dy = 0$ for all x in a co-null set. Pick $x_1 > 0$ and $x_2 < 0$ such that $\int_{\mathbb{R}_+} f(x_i,y)\,dy = 0$ for $i = 1,2$ to conclude that $f(x_i,y) = 0$ for almost $y > 0$ and hence $v(x) = v(x_1)$ for almost all $x > 0$ and $v(x) = v(x_2)$ for almost all $x < 0$.

We use the following notation. Let F be the Fourier transform on $L^2(\mathbb{R})$ such that for suitable functions f

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(s)\, ds,$$

and for ψ in $L^\infty(\mathbb{R})$ define the Fourier multiplication operator $D_\psi = FM_\psi F^*$. Also for real λ write D_λ for the right translation operator $D_\lambda = FM_\lambda F^*$.

Lemma 3.1. *The algebra* $\mathscr{A}_h$ *contains a subalgebra unitarily equivalent to the* w*-*closed algebra* $\mathscr{C}$ *on* $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *generated by the set of operators*

$$\{M_s \oplus e^{-s\pi} M_s, D_{-t} \oplus D_{-t} : t \geqq 0, s \in \mathbb{R}\}.$$

Proof. Recall that $g_s(x) = x^{is}$ is in $H^\infty(\mathbb{R})$ and so belongs to the w*-closed linear span of $\{e^{i\lambda x} : \lambda \geqq 0\}$. Let $\mathscr{B}$ be the w*-closed algebra generated by $\{M_{g_s}, V_t : s \in \mathbb{R}, t \geqq 0\}$. It follows that $\mathscr{B}$ is contained in $\mathscr{A}_h$. We now exhibit a unitary equivalence between the generators of $\mathscr{B}$ and those of $\mathscr{C}$.

Define the unitary operators $C_+ \colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ and $C_- \colon L^2(\mathbb{R}_-) \to L^2(\mathbb{R})$ by $(C_+ f)(x) = e^{x/2} f(e^x)$ and $(C_- g)(x) = e^{x/2} g(-e^x)$, and define $C \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ by $Cf = (C_+ f_+) \oplus (C_- f_-)$ where $f_\pm$ are the restrictions of f to the half-lines $\mathbb{R}_\pm$. A calculation shows that

$$CV_t = (D_{-t} \oplus D_{-t})C.$$

Also, on $L^2(\mathbb{R}_+)$ we have $C_+ M_{g_s} = M_s C_+$ while on $L^2(\mathbb{R}_-)$, $C_- M_{g_s} = e^{-s\pi} M_s C_-$. Therefore

$$CM_{g_s} = (M_s \oplus e^{-s\pi} M_s)C.$$

Thus we obtain the desired unitary equivalence $C\mathscr{B}C^* = \mathscr{C}$. □

Recall that the Fourier binest algebra is generated by right translations and $H^\infty(\mathbb{R})$ and so contains the pseudodifferential operators $M_\phi D_\psi$ for ϕ, ψ in $H^\infty(\mathbb{R})$. The unitary equivalence above allows us to make a connection between these operators and operators in $\mathscr{A}_h$ and this enables the construction of a contractive approximate identity of Hilbert Schmidt operators.

Proposition 3.2. *The algebra* $\mathscr{A}_h$ *contains a sequence of Hilbert-Schmidt contractions that converges to the identity operator in the* w*-*topology.*

Proof. By the lemma, it suffices to produce such a sequence in the algebra $\mathscr{C}$. Note that if $f \in H^\infty(\mathbb{R})$ then there is a sequence $\{p_m\}$ of analytic trigonometric polynomials that converge to f in the w*-topology of $H^\infty(\mathbb{R})$. Furthermore the functions q_m given by $q_m(z) = p_m(z + i\pi)$ converge weak star to the function h in $H^\infty(\mathbb{R})$ given by $h(z) = f(z + i\pi)$. Since $e^{-s\pi} M_s$ is the operator of multiplication by the function $\exp is(z + i\pi)$, the operators $M_{p_m} \oplus M_{q_m}$ belong to the w*-closed linear span of $\{M_s \oplus e^{-s\pi} M_s, s \in \mathbb{R}\}$ and hence so does the operator $M_f \oplus M_h$. Also if $g = \bar{f}$, then $g = \text{w}^*\text{-lim}\, \bar{p}_m$ and since p_m is an analytic trigonometric polynomial the operator D_g belongs to the w*-closed linear span of $\{D_{-t} \colon t \geqq 0\}$. It follows that the operator $M_f D_g \oplus M_h D_g$ belongs to $\mathscr{C}$.

Now consider the $H^\infty(\mathbb{R})$ functions

$$f_n(z) = \frac{in}{z + in}.$$

Note that the sequence f_n is uniformly bounded and tends to 1 uniformly on compact subsets of the upper half-plane. It follows that the corresponding sequence of operators $\{M_{f_n} D_{g_n} \oplus M_{h_n} D_{g_n}\}$, where $h_n(z) = f_n(z + i\pi)$ and $g_n = \bar{f}_n$, is a norm bounded sequence in $\mathscr{C}$ converging weakly, and hence ultraweakly, to the identity operator. Finally recall (for example from [9]) that if $f, g \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ then the operator $M_f D_g$ is a Hilbert-Schmidt operator. □

We can now use an argument of Wagner [21] to obtain the compactness of the lattice $\mathscr{L}$ of projections with respect to the strong operator topology. This fact will be needed in the next section and for completeness we give a proof.

Proposition 3.3. *The lattice $\mathscr{L}$ is compact in the strong operator topology.*

Proof. Suppose that $\mathscr{L}$ is not strongly compact. Then, since it is contained in the unit ball of $\mathscr{B}(L^2(\mathbb{R}))$ which is compact and metrisable in the weak operator topology, there is a sequence $\{P_n\} \subseteqq \mathscr{L}$ which converges in the weak operator topology to a non-projection Q which is a positive contraction.

For each compact operator $K \in \mathscr{A}_h$ we claim that $KQ = QKQ$. Indeed, for each $f, g \in L^2(\mathbb{R})$, since $K(P_n - Q)f \to 0$ in norm, we have

$$|\langle P_n K(P_n - Q)f, g\rangle| = |\langle K(P_n - Q)f, P_n g\rangle| \leqq \|K(P_n - Q)f\|.\|P_n g\| \to 0.$$

But $P_n K P_n = K P_n$ since K leaves P_n invariant and so

$$P_n K(P_n - Q) = KP_n - P_n KQ \to KQ - QKQ$$

in the weak operator topology. It follows that $KQ = QKQ$ as claimed.

Now if $\{K_n\} \subseteqq \mathscr{A}_h$ is a sequence of compact operators tending to the identity operator in the strong operator topology then we obtain that $Q^2 = \lim QK_n Q = \lim K_n Q = Q$, so that Q is a projection contrary to assumption. In view of Proposition 3.2, $\mathscr{L}$ must be strongly compact. □

4. The connectedness of $\mathscr{L}$

We now examine the topology of $\mathscr{L}$ and establish the boundary limits mentioned in the introduction. It will follow that the closure of the family of projections for the subspaces

$$K_{s,\theta,\lambda,\mu} = u_{0,\theta}(x)|x|^{is} e^{i\lambda x} e^{i\mu x^{-1}} H^2(\mathbb{R})$$

parametrised by $\mathbb{R} \times S^1 \times \mathbb{R}_+^2$, is a connected compact 4-manifold.

Continuity at a point $(s, \theta, \lambda, \mu)$ is elementary as the next proposition shows.

Proposition 4.1. (i) *The map* $(a,b) \to [L^2([-a,b])]$ *is strongly continuous at each point* (a,b) *in* $[0,+\infty] \times [0,+\infty]$.

(ii) *The map* $(s,\theta,\lambda,\mu) \to [K_{s,\theta,\lambda,\mu}]$ *is strongly continuous at each point* (s,θ,λ,μ) *in* $\mathbb{R} \times S^1 \times \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. The first assertion is obvious. For the second, if $P = [H^2(\mathbb{R})]$, then $[K_{s,\theta,\lambda,\mu}] = U_{s,\theta}T_{\lambda,\mu}PT^*_{\lambda,\mu}U^*_{s,\theta}$, where $U_{s,\theta}$ and $T_{\lambda,\mu}$ are the commuting unitary operators of multiplication by $g_{s,\theta}$ and $e_{\lambda,\mu}$ respectively. As these are unimodular functions and

$$\lim_{(s,\theta)\to(\sigma,\phi)} g_{s,\theta}(x) = g_{\sigma,\phi}(x), \qquad \lim_{(\lambda,\mu)\to(l,m)} e_{\lambda,\mu}(x) = e_{l,m}(x)$$

for (almost) all $x \in \mathbb{R}$, it follows readily that

$$\lim_{(s,\theta)\to(\sigma,\phi)} U_{s,\theta} = U_{\sigma,\phi}, \qquad \lim_{(\lambda,\mu)\to(l,m)} T_{\lambda,\mu} = T_{l,m}$$

strongly. Since these are unitary operators, we conclude that if $(s_n,\theta_n,\lambda_n,\mu_n) \to (s,\theta,\lambda,\mu)$ and $W_n = U_{s_n,\theta_n}T_{\lambda_n,\mu_n}$, $W = U_{s,\theta}T_{\lambda,\mu}$, then $W_n \to W$ strongly, and hence $W_nPW_n^* \to WPW^*$ strongly, and (ii) follows. □

We now turn to the limiting behaviour where s,θ are fixed and λ,μ tend to infinity.

Proposition 4.2. *If* $P_{\lambda,\mu} = [K_{0,1,\lambda,\mu}] = [e_{\lambda,\mu}H^2(\mathbb{R})]$, *then* $\lim_{\lambda\to\infty} P_{\lambda,\lambda} = E_1$ *strongly, where* $E_1 = [L^2([-1,1])]$.

Proof. Let d be any metric inducing the strong operator topology on $\mathscr{L}$. If the conclusion fails, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $\lambda_n > n$ with

$$d(P_{\lambda_n,\lambda_n}, E_1) \geqq \varepsilon.$$

By the compactness of $\mathscr{L}$, the sequence P_{λ_n,λ_n} has a subsequence, $\{Q_n\}$ say, that converges strongly to some projection $Q \in \mathscr{L}$. It follows that

$$d(Q, E_1) \geqq \varepsilon.$$

We claim that if U is the unitary operator defined by $(Uf)(x) = x^{-1}f(x^{-1})$, then $UQ^\perp U^* = Q$. Indeed since $e_{\lambda,\lambda}$ is unimodular we see that $P^\perp_{\lambda,\lambda}$ is the projection onto $e_{\lambda,\lambda}\overline{H^2}$. But U commutes with multiplication by $e_{\lambda,\lambda}$, hence transforms $e_{\lambda,\lambda}\overline{H^2}$ to $e_{\lambda,\lambda}H^2$, and so $UP^\perp_{\lambda,\lambda}U^* = P_{\lambda,\lambda}$. Thus each Q_n satisfies $Q_n = UQ_n^\perp U^*$, hence so does Q, proving the claim.

This shows that Q cannot be of the form $Q = [L^2([-a,b])]$ for some $a,b \in \mathbb{R}_+$. Indeed, since $U\big(L^2([-a,b])^\perp\big) = L^2\left(\left[\frac{-1}{a},\frac{1}{b}\right]\right)$ the claim gives $a = b = 1$ and so $Q = E_1$, contrary to hypothesis.

Thus by Theorem 2.1 the range K of Q must be of the form $K = g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})$ for appropriate s,θ,λ,μ. Now $K^\perp = g_{s,\theta}e_{\lambda,\mu}\overline{H^2(\mathbb{R})}$ and U transforms multiplication by $g_{s,\theta}e_{\lambda,\mu}$ to multiplication by $g_{-s,\theta}e_{\mu,\lambda}$. Since these multiplication operators are unitary and $U\overline{H^2} = H^2$, we obtain $UK^{\perp} = g_{-s,\theta}e_{\mu,\lambda}H^2(\mathbb{R})$. By the claim, we must have $UK^{\perp} = K$ and so

$$g_{-s,\theta}e_{\mu,\lambda}H^2(\mathbb{R}) = g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R}).$$

This implies that $g_{s,\theta}e_{\lambda,\mu}\bar{g}_{-s,\theta}\bar{e}_{\mu,\lambda}$ is a constant and hence gives $\mu = \lambda$ and $s = 0$.

We conclude that $Q_n \to [g_{0,\theta}e_{\mu,\mu}H^2(\mathbb{R})]$ for some $\mu \geqq 0$. Applying to this the unitary operator of multiplication by $e_{-\mu,-\mu}$ we see that there exists a sequence $\mu_n \to \infty$ such that

$$P_{\mu_n,\mu_n} \to Q'$$

where Q' is the projection onto $g_{0,\theta}H^2(\mathbb{R})$. It follows that $V_tP_{\mu_n,\mu_n}V_{-t} \to V_tQ'V_{-t}$, or

$$P_{e^t\mu_n,e^{-t}\mu_n} \to Q'$$

(since $V_t(g_{0,\theta}H^2) = g_{0,\theta}H^2$). Now recall that the projection $P_{\lambda,\mu}$ decreases as λ increases and increases as μ increases. It follows that if $t \geqq 0$ then

$$P_{\mu_n,\mu_n} \geqq P_{e^t\mu_n,\mu_n} \geqq P_{e^t\mu_n,e^{-t}\mu_n}.$$

But given any $\lambda \geqq 0$ we can find $t \geqq 0$ such that $e^t\mu_n \geqq \mu_n + \lambda \geqq \mu_n$ for all (large enough) n, and therefore

$$P_{\mu_n,\mu_n} \geqq P_{\mu_n+\lambda,\mu_n} \geqq P_{e^t\mu_n,\mu_n} \geqq P_{e^t\mu_n,e^{-t}\mu_n}.$$

Since $P_{e^t\mu_n,e^{-t}\mu_n} \to Q'$ and $P_{\mu_n,\mu_n} \to Q'$ it now follows that $P_{\mu_n+\lambda,\mu_n} \to Q'$. But $P_{\mu_n+\lambda,\mu_n} = M_\lambda P_{\mu_n,\mu_n}M_\lambda^*$ and thus

$$M_\lambda Q'M_\lambda^* = \lim_{n\to\infty} M_\lambda P_{\mu_n,\mu_n}M_\lambda^* = Q'$$

so that $e_\lambda g_{0,\theta}H^2(\mathbb{R}) = g_{0,\theta}H^2(\mathbb{R})$ for all $\lambda \geqq 0$. This is a contradiction and completes the proof. □

Proposition 4.3. *The strong operator topology closure of the set* $\{P_{\lambda,\mu} : \lambda \geqq 0, \mu \geqq 0\}$ *is* $\{P_{\lambda,\mu} : \lambda \geqq 0, \mu \geqq 0\} \cup \{E_a : 0 \leqq a \leqq +\infty\}$, *where* E_a *denotes the projection onto* $L^2([-a,a])$.

Proof. By the previous proposition we have $\lim_{\lambda\to\infty} P_{\lambda,\lambda} = E_1$ strongly, and hence

$$\lim_{\lambda\to\infty} V_tP_{\lambda,\lambda}V_{-t} = V_tE_1V_{-t}.$$

But $V_tP_{\lambda,\lambda}V_{-t} = P_{e^t\lambda,e^{-t}\lambda}$ as observed earlier, while $V_tE_1V_{-t} = E_a$ where $a = e^{-t}$. Thus $\lim_{\lambda\to\infty} P_{a^{-1}\lambda,a\lambda} = E_a$ and $\lim_{\lambda\to\infty} P_{\lambda,c^2\lambda} = E_c$ when $c > 0$.

Now suppose that a sequence P_{λ_n,μ_n} converges to some projection Q. Passing to a subsequence, we may assume that the sequence (λ_n,μ_n) converges to some $(\lambda,\mu) \in [0,+\infty]^2$. Suppose first that $\lim \frac{\mu_n}{\lambda_n} = 0$. Then for all $c > 0$ there exists $n_c \in \mathbb{N}$ such that $\frac{\mu_n}{\lambda_n} < c^2$ for $n \geqq n_c$. Then we will have $\mu_n < c^2\lambda_n$ so

$$P_{\lambda_n,\mu_n} \leqq P_{\lambda_n,c^2\lambda_n}$$

eventually. If $\lambda \in \mathbb{R}$ then $Q = P_{\lambda,0}$ by Proposition 4.1. If not,

$$Q = \lim P_{\lambda_n,\mu_n} \leqq \lim P_{\lambda_n,c^2\lambda_n} = E_c.$$

Since $c > 0$ is arbitrary and $\inf_{c>0} E_c = 0$, we obtain $Q = 0$. Similarly if $\lim \frac{\mu_n}{\lambda_n} = +\infty$ we conclude that $Q = P_{0,\mu}$ or $Q = I$.

It remains to consider the case when $\lambda \to +\infty$ and the sequence $\left(\frac{\mu_n}{\lambda_n}\right)$ has a subsequence converging to some $a \in (0, +\infty)$. Then for all $\varepsilon > 0$ we have

$$(a - \varepsilon)\lambda_n < \mu_n < (a + \varepsilon)\lambda_n$$

for infinitely many $n \in \mathbb{N}$ so that

$$P_{\lambda_n,(a-\varepsilon)\lambda_n} \leqq P_{\lambda_n,\mu_n} \leqq P_{\lambda_n,(a+\varepsilon)\lambda_n}$$

for infinitely many n and therefore $E_{\sqrt{a-\varepsilon}} \leqq Q \leqq E_{\sqrt{a+\varepsilon}}$. Since ε is arbitrary, this yields $Q = E_{\sqrt{a}}$ and completes the proof. □

Noting that the unitary operator of multiplication by $\bar{g}_{s,\theta}$ transforms the subspace $g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})$ to $e_{\lambda,\mu}H^2(\mathbb{R})$ and leaves $L^2[-a,a]$ invariant, it follows immediately from the proposition that for each pair (s,θ) the set of projections

$$\mathscr{P}_{s,\theta} = \{[g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})] : \lambda, \mu \geqq 0\} \cup \{[L^2[-a,a]] : a \geqq 0\} \cup \{[L^2(\mathbb{R})]\}$$

with the strong operator topology, is homeomorphic to a closed disc or, more intuitively, a closed triangle as indicated in Figure 1.

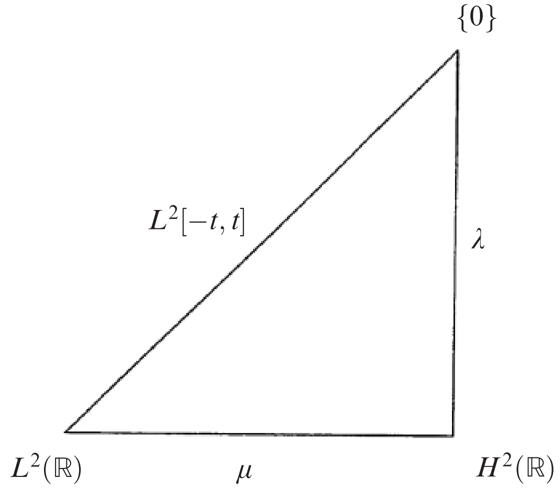

Figure 1

We now turn to the limiting behaviour as λ, s tend to infinity with θ, μ fixed.

Proposition 4.4. *The strong operator limit* $\lim_{s\to\infty}[u_{s,1}H^2(\mathbb{R})]$ *is* $[L^2(\mathbb{R}_-)]$.

Proof. Recall that

$$u_{s,1}(x) = \begin{cases} 1, & x > 0, \\ e^{s\pi}, & x \leqq 0. \end{cases}$$

If we denote $[L^2(\mathbb{R}_-)]$ by E_- and $[H^2(\mathbb{R})]$ by P, then the operator of multiplication by $u_{s,1}$ is $E_-^\perp + e^{s\pi}E_-$. Since multiplication by a constant leaves any subspace invariant, if $s > 0$ we have

$$\begin{aligned}[u_{s,1}H^2(\mathbb{R})] &= [(E_-^\perp + e^{s\pi}E_-)H^2(\mathbb{R})] = \big[\big((1+\delta)e^{-s\pi}E_-^\perp + (1+\delta)E_-\big)H^2(\mathbb{R})\big] \\ &= \big[\big(\delta E_-^\perp + (1+\delta)E_-\big)H^2(\mathbb{R})\big] = [(\delta I + E_-)H^2(\mathbb{R})]\end{aligned}$$

where we have chosen $\delta > 0$ so that $(1+\delta)e^{-s\pi} = \delta$. Therefore, noting that $E_- \wedge P = E_-^\perp \wedge P = E_- \wedge P^\perp = E_-^\perp \wedge P^\perp = 0$, it suffices to prove the following general fact:

Proposition 4.5. *Let E, P be two projections on an infinite dimensional Hilbert space $\mathcal{H}$ such that $E \wedge P = E^\perp \wedge P = E \wedge P^\perp = E^\perp \wedge P^\perp = 0$ (that is, the pair (E, P) is in 'generic position'). For $\delta > 0$, denote by Q_δ the projection onto the range of the operator $(E + \delta I)P$. Then $\lim_{\delta\to 0} Q_\delta = E$ in the strong operator topology.*

Proof. After a unitary equivalence we may assume that the pair (E, P) has the form

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}$$

where C, S are positive injective contractions and $S^2 = I - C^2$. This well-known model for a projection pair in generic position is discussed in Halmos [6].

Note that $P = XX^*$ where

$$X = \begin{pmatrix} C & 0 \\ S & 0 \end{pmatrix}.$$

Hence $[P\mathcal{H}] = [X\mathcal{H}]$ and therefore $[(\delta I + E)P\mathcal{H}] = [(\delta I + E)X\mathcal{H}]$ for $\delta > 0$, since $(\delta I + E)$ is invertible. Thus $Q_\delta = [(\delta I + E)P\mathcal{H}] = [Z_\delta\mathcal{H}]$ where

$$Z_\delta = (\delta I + E)X = \begin{pmatrix} (1+\delta)C & 0 \\ \delta S & 0 \end{pmatrix}.$$

If $Z_\delta = V_\delta|Z_\delta|$ is the polar decomposition, then

$$|Z_\delta|^2 = Z_\delta^* Z_\delta = \begin{pmatrix} (1+\delta)^2C^2 + \delta^2S^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (1+2\delta)C^2 + \delta^2 I & 0 \\ 0 & 0 \end{pmatrix}.$$

Noting that $W_\delta \equiv (1+2\delta)C^2 + \delta^2 I$ is positive and invertible, we obtain

$$V_\delta = \begin{pmatrix} (1+\delta)CW_\delta^{-1/2} & 0 \\ \delta S W_\delta^{-1/2} & 0 \end{pmatrix}.$$

Now the projection Q_δ onto the range of Z_δ equals $V_\delta V_\delta^*$ and so

$$Q_\delta = \begin{pmatrix} (1+\delta)^2 C^2 W_\delta^{-1} & (1+\delta)\delta C W_\delta^{-1} S \\ (1+\delta)\delta S W_\delta^{-1} C & \delta^2 S W_\delta^{-1} S \end{pmatrix}.$$

The $(1,2)$ entry of this matrix equals

$$F_\delta = \delta(1+\delta)CSW_\delta^{-1}.$$

If we represent C (resp. S) as (multiplication by) the non-negative function $c(x)$ (resp. $s(x) = \sqrt{1-c^2(x)}$) on a suitable $L^2(\mu)$ space, then F_δ is represented by the function

$$f_\delta(x) = \frac{\delta s(x)(1+\delta)c(x)}{\delta^2 s^2(x) + (1+\delta)^2 c^2(x)}.$$

Since $\lim_{\delta\to 0} f_\delta(x) = 0$ pointwise and $0 \leqq f_\delta(x) \leqq 1$ for all x and δ, it follows that $\lim F_\delta = 0$ strongly. Indeed, for each vector ξ, we have

$$\|F_\delta \xi\|^2 = \int |f_\delta(x)\xi(x)|^2 \, d\mu(x)$$

which tends to 0 as $\delta \to 0$ by dominated convergence.

Similarly the $(2,2)$ entry of Q_δ equals

$$G_\delta = \delta^2 S^2 \big((1+2\delta)C^2 + \delta^2 I\big)^{-1}$$

which is represented by the nonnegative function

$$g_\delta(x) = \frac{\delta^2 s^2(x)}{\delta^2 + (1+2\delta)c^2(x)} \leqq \frac{\delta^2 s^2(x)}{(1+2\delta)c^2(x)}.$$

Since C is injective, $c(x) \neq 0$ for almost all x, and so $\lim_{\delta\to 0} g_\delta(x) = 0$ for almost all x. But also $g_\delta(x) \leqq 1$ for (almost) all x and δ, and hence $\lim_{\delta\to 0} G_\delta = 0$ strongly.

Finally, the $(1,1)$ entry of Q_δ equals

$$(1+\delta)^2 C^2 \big((1+\delta)^2 C^2 + \delta^2 S^2\big)^{-1} = \left(C^2 + \frac{\delta^2}{(1+\delta)^2} S^2\right)^{-1} C^2$$

which tends to I strongly. □

Proposition 4.6. *The closure of the set $\{[u_{s,1}e_{\lambda,0}H^2(\mathbb{R})] : \lambda \geqq 0, s \geqq 0\}$ in the strong operator topology is $\{[u_{s,1}e_{\lambda,0}H^2(\mathbb{R})] : \lambda \geqq 0, s \geqq 0\} \cup \{[L^2([-a,0])] : 0 \leqq a \leqq +\infty\}$.*

Proof. Let $(s_n, \lambda_n) \to (+\infty, +\infty)$ and suppose that $\lim_n [g_{s_n,1}e_{\lambda_n,0}H^2(\mathbb{R})] = [K]$. Note that since $e_{\lambda_n,0}H^2(\mathbb{R}) \subseteqq H^2(\mathbb{R})$ we have

$$g_{s_n,1}e_{\lambda_n,0}H^2(\mathbb{R}) \subseteqq g_{s_n,1}H^2(\mathbb{R}).$$

But by Proposition 4.4 we know that $\lim_n [g_{s_n,1}H^2(\mathbb{R})] = [L^2(\mathbb{R}_-)]$. It follows that $K \subseteqq L^2(\mathbb{R}_-)$ and, so (by the F. and M. Riesz Theorem) K cannot be of the form $g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})$. Since $K \in \mathscr{L}$, it must be of the form $K = L^2([-a,0])$ for some $a \geqq 0$.

We now prove that all possible values of the parameter a arise as limits of such sequences. For brevity, we write $Q_{s,\lambda}$ for the projection onto $g_{s,1}e_{\lambda,0}H^2(\mathbb{R})$.

Let ζ be a unit vector in $L^2(\mathbb{R}_-)$ which is separating for the family of projections $[L^2[-a,0]]$ and let $\mathscr{C}_n$ be the set of projections $Q_{s,\lambda}$ with $|s| + |\lambda| \geqq n$. Since $\mathscr{C}_n$ is path connected (see Proposition 4.1) it follows that if we define $F: \mathscr{L} \to [0,1]$ by $F(L) = \langle L\zeta, \zeta\rangle$, then $F(\mathscr{C}_n)$ is connected. By Proposition 4.1 the real number 1 belongs to the closure of $F(\mathscr{C}_n)$. Since 0 also belongs to the closure it follows from connectedness that $F(\mathscr{C}_n) = (0,1)$ and hence that $F(\overline{\mathscr{C}_n}) = [0,1]$ (for the closure of $\mathscr{C}_n$ in the strong operator topology). By the compactness of $\mathscr{L}$ we have

$$F\left(\bigcap_{n=1}^{\infty} \overline{\mathscr{C}_n}\right) = \bigcap_{n=1}^{\infty} F(\overline{\mathscr{C}_n}) = [0,1]$$

and this identity completes the proof. □

Corollary 4.7. *Let $\theta \in S^1$.*

(i) *The strong operator limit $\lim_{s\to\infty}[u_{s,\theta}H^2(\mathbb{R})]$ is $[L^2(\mathbb{R}_-)]$, while $\lim_{s\to-\infty}[u_{s,\theta}H^2(\mathbb{R})]$ is $[L^2(\mathbb{R}_+)]$.*

(ii) *For all $\mu \geqq 0$, the set of limit points of sequences $\big([u_{s_n,\theta}e_{\lambda_n,\mu}H^2(\mathbb{R})]\big)$ as $(s_n, \lambda_n) \to (+\infty, +\infty)$ is $\{[L^2([-a,0])], a \geqq 0\} \cup \{[L^2(\mathbb{R}_-)]\}$, while if $(s_n, \lambda_n) \to (-\infty, +\infty)$ the set of limit points is $\{[L^2([0,b])], b \geqq 0\} \cup \{[L^2(\mathbb{R}_+)]\}$.*

(iii) *For all $\lambda \geqq 0$, the set of limit points of sequences $\big([u_{s_n,\theta}e_{\lambda,\mu_n}H^2(\mathbb{R})]\big)$ as $(s_n, \mu_n) \to (+\infty, +\infty)$ is $\big\{\big[L^2\big((-\infty,b]\big)\big], b \geqq 0\big\} \cup \{[L^2(\mathbb{R}_-)]\}$, while if $(s_n, \mu_n) \to (-\infty, +\infty)$ the set of limit points is $\big\{\big[L^2\big([-a,+\infty)\big)\big], a \geqq 0\big\} \cup \{[L^2(\mathbb{R}_+)]\}$.*

Proof. We may write $u_{s,\theta} = u_{s,1}u_{0,\theta}$. Noting that $u_{0,\theta}$ is unimodular, we see that the corresponding multiplication operator is unitary. Since it transforms $[u_{s,1}H^2(\mathbb{R})]$ to $[u_{s,\theta}H^2(\mathbb{R})]$ and leaves $L^2(\mathbb{R}_-)$ invariant, the first claim of (i) follows from Proposition 4.4. Since $[u_{s_n,\theta}e_{\lambda_n,\mu}H^2(\mathbb{R})]$ is unitarily equivalent to $[u_{s_n,1}e_{\lambda_n,0}H^2(\mathbb{R})]$, the first claim of (ii) follows in the same way from Proposition 4.6. Exactly the same arguments give the limits when $s \to -\infty$.

To obtain (iii), use (ii) and apply the unitary transformation U induced by the symmetry $x \to x^{-1}$. $\square$

It remains to consider the limits when all three parameters (s, λ, μ) blow up:

Proposition 4.8. *The set of limit points of sequences* $\left([g_{s_n,\theta} e_{\lambda_n,\mu_n} H^2(\mathbb{R})]\right)_n$ *as* $s_n \to +\infty$, $\lambda_n \to +\infty$ *and* $\mu_n \to \infty$ *is* $\{[L^2([-a, b])] : 0 \leqq a, b \leqq \infty\}$.

Proof. Suppose that $\lim_n [g_{s_n,\theta} e_{\lambda_n,\mu_n} H^2(\mathbb{R})] = [K]$. Note that

$$g_{s_n,\theta} e_{\lambda_n,0} H^2(\mathbb{R}) \subseteqq g_{s_n,\theta} e_{\lambda_n,\mu_n} H^2(\mathbb{R}).$$

Passing to a subsequence, if necessary, we may assume that both sequences of projections onto these subspaces converge. But by the preceding corollary we know that there exists $a_0 \geqq 0$ such that $\lim_n [g_{s_n,\theta} e_{\lambda_n,0} H^2(\mathbb{R})] = [L^2[-a_0, 0]]$. It follows that $L^2[-a_o, 0] \subseteqq K$ and, since $K \in \mathscr{L}$, it must be that $K = L^2([-a, b])$ for some $a, b \geqq 0$.

It remains to prove that all projections $[L^2[-a, b]]$ arise as limits of such sequences.

We use an argument similar to the proof of 4.6 with simple connectedness in place of connectedness.

Let ζ be a unit vector in $L^2(\mathbb{R})$ which is strictly positive on $\mathbb{R}_-$ and vanishes on $\mathbb{R}_+$. Let η be a unit vector which is strictly positive on $\mathbb{R}_+$ and zero on $\mathbb{R}_-$. Define the function $F: \mathscr{L} \to [0, 1] \times [0, 1]$ by

$$F(L) = (\langle L\zeta, \zeta \rangle, \langle L\eta, \eta \rangle)$$

and note that F separates the family $\mathscr{L}_M = \{[L^2[-a, b]], 0 \leqq a, b \leqq \infty\}$. (Indeed, if $L_i = [L^2[-a_i, b_i]]$, for $i = 1, 2$, the equality $\langle L_1\eta, \eta \rangle = \langle L_2\eta, \eta \rangle$ implies $b_1 = b_2$, and the equality $\langle L_1\zeta, \zeta \rangle = \langle L_2\zeta, \zeta \rangle$ gives $a_1 = a_2$.)

Furthermore, if ∂ denotes the boundary of $\mathscr{L}_M$, namely the union of the four families of projections $\{[L^2[-a, 0]] : a \geqq 0\}, \{[L^2[-a, \infty)] : a \geqq 0\}, \{[L^2[0, b]] : b \geqq 0\}$ and $\{[L^2(-\infty, b]] : b \geqq 0\}$ then $F(\partial)$ is the boundary of the unit square, whereas all other values of $F(L)$ lie in the interior of the square. Now fix such a value $F(L_*)$ for some projection $L_* \in \mathscr{L}_M \backslash \partial$.

Let $\mathscr{L}_n$ be the set of projections $[g_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R})]$ with $|s| + |\lambda| + |\mu| \geqq n$. By our previous results, all projections in ∂ belong to the closure of each $\mathscr{L}_n$. By compactness we may choose a path π_1 in $\mathscr{L}_1$ so that the closed curve $F(\pi_1)$ lies close to the boundary of I^2 and is such that the value $F(L_*)$ lies inside the curve.

Indeed, since $F(L_*) \in I^2 \backslash F(\partial)$, there exists a smaller open square $J^2 = (\delta, 1 - \delta) \times (\delta, 1 - \delta)$ with $F(L_*) \in J^2$ and $F(\partial) \subseteqq I^2 \backslash J^2$. Now cover the compact set $F(\partial)$ by a finite number of open sets U_i, $i = 1, \ldots, k$, all contained in $I^2 \backslash J^2$. For $i = 1, \ldots, k$ choose $L_i \in \mathscr{L}_1 \cap F^{-1}(U_i)$ and join the points L_i by a continuous simple path in $\mathscr{L}_1$ lying in the union of the $F^{-1}(U_i)$ to form the required closed curve π_1. (For instance if $L_i = [g_{s_i,\theta} e_{\lambda_i,\mu_i} H^2(\mathbb{R})]$, one may define a path $L_t = [g_{s_t,\theta} e_{\lambda_t,\mu_t} H^2(\mathbb{R})]$ $(t \in [0,1])$ joining L_1 to L_2 by choosing s_t, λ_t, μ_t to be linear paths joining s_1, λ_1, μ_1 to s_2, λ_2, μ_2, so that $|s_t| + |\lambda_t| + |\mu_t| \geqq 1$.) Plainly we can arrange that $F(\pi_1)$ is a continuous closed curve in $I^2 \backslash J^2$ with winding number 1 with respect to the point $F(L_*)$. By the contractibility of $\mathscr{L}_1$, there is a homotopy $\{\pi(t) : t \in [0,1]\}$ in $\mathscr{L}_1$ with $\pi(1) = \pi_1$ and $\pi(0)$ a single point. If $F(L_*)$ were not in $F(\mathscr{L}_1)$, then the image $\{F(\pi(t)) : t \in [0,1]\}$ would be a homotopy in the space $F(\mathscr{L}_1) \subseteqq I^2 \backslash F(L_*)$ deforming the curve $F(\pi_1)$ to a point. Since each curve $F(\pi(t))$ has winding number 1 with respect to $F(L_*)$, this is impossible.

Choose n_2 so that $L_1 \notin \mathscr{L}_{n_2}$ and similarly locate a projection L_2 in $\mathscr{L}_{n_2}$ with $F(L_2) = F(L_*)$. Continuing, construct a sequence L_n which, by compactness, we may assume converges, in the strong operator topology, to a projection L_∞. Since $L_n = [g_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R})]$ where $|s| + |\lambda| + |\mu| \geqq n$, the projection L_∞ cannot be of the form $[g_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R})]$ and it follows from our earlier remarks that $L_\infty = L_*$. $\square$

5. Lattice structure

We now consider the lattice structure of $\mathscr{L}$. Recall that $\mathrm{Alg}(\mathscr{M})$ denotes the algebra of all operators that leave invariant all the projections in a set $\mathscr{M}$. These are the *reflexive* operator algebras $\mathscr{A}$, which are reflexive in the sense that $\mathrm{Alg}\,\mathrm{Lat}\,\mathscr{A} = \mathscr{A}$. We begin by examining natural distinguished sublattices of $\mathscr{L}$ and their associated operator algebras.

Define

$$\mathscr{L}_M = \{[L^2([-a,b])] : 0 \leqq a, b \leqq \infty\}.$$

This is a commutative lattice consisting of those projections in $\mathscr{L}$ which are reducing for the multiplication semigroup $\{M_\lambda : \lambda \geqq 0\}$. In other words

$$\mathscr{L}_M = \mathscr{L} \cap \{M_\lambda : \lambda \in \mathbb{R}\}'.$$

Note that $\mathscr{L}_M$ is generated by the two projection nests $\mathscr{N}_1 = \{[L^2([-a,\infty])] : a \geqq 0\} \cup \{0\}$ and $\mathscr{N}_2 = \{[L^2([-\infty,b])] : b \geqq 0\} \cup \{0\}$.

The reflexive algebra $\mathrm{Alg}\,\mathscr{L}_M$ can be defined intrinsically in terms of generators as the algebra

$$\mathscr{A}_M = \mathrm{w}^*\text{-alg}\{V_t, M_\lambda : t \geqq 0, \lambda \in \mathbb{R}\}.$$

To see this let $\mathscr{N}$ be the multiplicity two projection nest consisting of the projections $[L^2[-t,t]]$ for $t \in [0,\infty]$ and let $\mathscr{M} = \{0, E_+, E_-, I\}$ where $E_+ = [L^2[0,\infty]]$, $E_- = [L^2[-\infty,0]]$. Plainly $\mathrm{Alg}\,\mathscr{L}_M = (\mathrm{Alg}\,\mathscr{N}) \cap (\mathrm{Alg}\,\mathscr{M})$ and on identifying $\mathbb{R}_-$ and $\mathbb{R}_+$ we see that

$$\mathrm{Alg}\,\mathscr{L}_M = (\mathrm{Alg}\,\mathscr{N}_+) \oplus (\mathrm{Alg}\,\mathscr{N}_+)$$

where $\mathrm{Alg}\,\mathscr{N}_+$ is the nest algebra on $L^2(\mathbb{R}_+)$ for the restriction $\mathscr{N}_+$ of the projection nest $\mathscr{N}_2$. With this identification the subalgebra $\mathscr{A}_M$ is generated by the operators

$$W_t \oplus W_t, \quad M_{\phi_+} \oplus M_{\phi_-}$$

where W_t is the restriction operator $V_t \mid L^2(\mathbb{R}_+)$, for $t \geqq 0$, and where ϕ_+, ϕ_- are the right and left parts of the function $\phi \in L^\infty(\mathbb{R})$. It follows from elementary nest algebra theory that $\mathscr{A}_M = \operatorname{Alg} \mathscr{L}_M$.

One could also deduce the equality from a result of Arveson [2] which asserts that a width two commutative subspace lattice is synthetic. Here "synthetic" means that every w*-closed algebra $\mathscr{A}$ that contains a masa and has invariant lattice equal to $\mathscr{L}_M$ is a reflexive operator algebra.

Consider now the sublattice of $\mathscr{L}$ consisting of the projections that reduce $\{V_t : t \geqq 0\}$, namely

$$\mathscr{L}_V = \mathscr{L} \cap \{V_t : t \in \mathbb{R}\}'.$$

Then we can obtain the explicit description

$$\mathscr{L}_V = \{[K_{s,\theta,0,0}] : s \in \mathbb{R}, \theta \in S^1\} \cup \{0, E_+, E_-, I\}.$$

Indeed it is clear that the only V_t-reducing subspaces in $\mathscr{L}_M$ are $\{0\}, L^2(\mathbb{R}_+), L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R})$. On the other hand, if a subspace $K \in \mathscr{L}$ is not one of these then by Theorem 2.1 $K = u_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R})$ for some choice of parameters. If this is V_t-reducing, then, since

$$V_t(K) = u_{s,\theta} e_{e^t\lambda, e^{-t}\mu} H^2(\mathbb{R}),$$

the equality $V_t(K) = K$ implies that the quotient

$$\frac{e_{e^t\lambda, e^{-t}\mu}}{e_{\lambda,\mu}}$$

is constant valued almost everywhere and so $\lambda = \mu = 0$.

It follows from the continuity obtained in Propositions 4.1 and 4.4 that the sublattice $\mathscr{L}_V$, with the strong operator topology, can be viewed as a topological sphere together with two isolated points, namely the zero projection 0 and the identity I. We see in Proposition 5.2 that the lattice structure is trivial in that the supremum of any two distinct points is the identity and the infimum is zero.

We now show that the sublattice structure for $\{e_{\lambda,\mu} H^2(\mathbb{R}) : \lambda, \mu \geqq 0\}$ is of product type. Recall that $e_{\lambda,\mu}$ is the unimodular function $e^{i\lambda x} e^{i\mu x^{-1}}$ and that for $\lambda, \mu \geqq 0$, $e_{\lambda,0}$ is an inner function and $e_{0,\mu}$ is a co-inner function.

Lemma 5.1. *If $\alpha, \beta, \lambda, \mu$ are (nonnegative) real numbers then*

$$e_{\lambda,\mu} H^2(\mathbb{R}) \wedge e_{\alpha,\beta} H^2(\mathbb{R}) = e_{\max(\lambda,\alpha),\min(\mu,\beta)} H^2(\mathbb{R})$$

and

$$e_{\lambda,\mu} H^2(\mathbb{R}) \vee e_{\alpha,\beta} H^2(\mathbb{R}) = e_{\min(\lambda,\alpha),\max(\mu,\beta)} H^2(\mathbb{R}).$$

Proof. Recall that $P_{\lambda,\mu} = [e_{\lambda,\mu} H^2(\mathbb{R})]$. Suppose that $\lambda \geqq \lambda'$ and $\mu \leqq \mu'$. We have $P_{\lambda,\mu} \leqq P_{\lambda',\mu}$ and $P_{\lambda,\mu} \leqq P_{\lambda,\mu'}$, thus $P_{\lambda,\mu} \leqq P_{\lambda',\mu} \wedge P_{\lambda,\mu'}$. Conversely if a subspace K with projection $P_K \in \mathscr{L}$ satisfies $K \subseteqq e_{\lambda',\mu} H^2(\mathbb{R})$ and $K \subseteqq e_{\lambda,\mu'} H^2(\mathbb{R})$ then we claim that $P_K \leqq P_{\lambda,\mu}$.

To see this note that $P_K \notin \mathscr{L}_M \backslash \{0\}$, since no proper subspace $L^2([-a,b])$ can be a subspace of $e_{\lambda,\mu'}H^2(\mathbb{R})$. Thus, by Theorem 2.1, $K = u_{s,\theta}e_{\alpha,\beta}H^2$ for appropriate indices s, θ, α, β. Consider first the inclusion $K \subseteqq e_{\lambda,\mu'}H^2(\mathbb{R})$. This gives $u_{s,\theta}e_{\alpha,\beta}\bar{e}_{\lambda,\mu'} \in H^\infty(\mathbb{R})$, which implies $u_{s,\theta} = 1$. Indeed if $u_{s,\theta}e_{\alpha,\beta}\bar{e}_{\lambda,\mu'} = h$ with $h \in H^\infty(\mathbb{R})$ then

$$u_{s,\theta}(x)e^{i\alpha x}e^{i\beta x^{-1}} = e^{i\lambda x}e^{i\mu' x^{-1}}h(x)$$

and so

$$e^{i\alpha x}e^{-i\mu' x^{-1}} = e^{i\lambda x}e^{-i\beta x^{-1}}h(x)$$

for almost all $x > 0$. Since this is an equality of H^∞ functions it holds for almost every $x \in \mathbb{R}$ and so $u_{s,\theta} = 1$. Now we have $K = e_{\alpha,\beta}H^2(\mathbb{R}) \subseteqq e_{\lambda,\mu'}H^2(\mathbb{R})$, and the analyticity of $\exp\big(i(\alpha-\lambda)x + i(\beta-\mu')x^{-1}\big)$ gives $\alpha \geqq \lambda$ and $\beta \leqq \mu'$. Similarly, the inclusion $K \subseteqq e_{\lambda',\mu}H^2(\mathbb{R})$ gives $\alpha \geqq \lambda'$ and $\beta \leqq \mu$. Thus $\alpha \geqq \lambda$ and $\beta \leqq \mu$, so that $P_K \leqq P_{\lambda,\mu}$ as claimed.

In a similar way one obtains $P_{\lambda',\mu} \vee P_{\lambda,\mu'} = P_{\lambda',\mu'}$. $\square$

Consider now the families of subspaces

$$\mathscr{L}_{s,\theta} = \{u_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R}) : \lambda, \mu \geqq 0\} \cup \{\{0\}, L^2(\mathbb{R})\}.$$

The next proposition summarises the lattice structure of $\mathscr{L}$, as a subspace lattice, in terms of these sublattices. The arguments for the proof are entirely similar to the methods of Lemma 5.1.

Proposition 5.2. *The lattice $\mathscr{L}$ can be written as the union*

$$\mathscr{L} = \left(\bigcup_{s\in\mathbb{R},\theta\in S^1} \mathscr{L}_{s,\theta}\right) \cup \mathscr{L}_M.$$

The lattice $\mathscr{L}_M$ is a width 2 commutative subspace lattice (parametrised by the square of points $(a,b) \in [0,\infty]^2$) and each $\mathscr{L}_{s,\theta}$ is a noncommutative projection lattice (with non-trivial projections parametrised by $(\lambda,\mu) \in [0,\infty)^2$). If $K_1, K_2 \in \mathscr{L}$ then $K_1 \cap K_2 = \{0\}$ and $K_1 \vee K_2 = L^2(\mathbb{R})$ unless both belong to the same sublattice $\mathscr{L}_{s,\theta}$ or $\mathscr{L}_M$. Finally, if $K_{s,\theta,\lambda,\mu}, K_{s,\theta,\lambda',\mu'} \in \mathscr{L}_{s,\theta}$ then

$$K_{s,\theta,\lambda,\mu} \wedge K_{s,\theta,\lambda',\mu'} = K_{s,\theta,\max(\lambda,\lambda'),\min(\mu,\mu')}$$

and

$$K_{s,\theta,\lambda,\mu} \vee K_{s,\theta,\lambda',\mu'} = K_{s,\theta,\min(\lambda,\lambda'),\max(\mu,\mu')}.$$

Proof. If $K_1 \in \mathscr{L}_M$ and $K_2 \in \mathscr{L}\backslash\mathscr{L}_M$ then $K_1 \cap K_2 = \{0\}$ and $K_1 \vee K_2 = L^2(\mathbb{R})$. Indeed writing $K_1 = L^2([-a,b])$ and $K_2 = g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})$, if $f \in K_1 \cap K_2$, then

$$(g_{s,\theta}e_{\lambda,\mu})^{-1}f \in H^2(\mathbb{R}) \cap L^2([-a,b]),$$

so $(g_{s,\theta}e_{\lambda,\mu})^{-1}f = 0$ by the F. and M. Riesz Theorem. Similarly $K_1^\perp \cap K_2^\perp = \{0\}$, so $K_1 \vee K_2 = L^2(\mathbb{R})$.

Suppose $K_1, K_2 \in \mathscr{L}\backslash\mathscr{L}_M$. Now $K_1 = K_{s,\theta,\lambda\mu}$ and $K_2 = K_{s',\theta',\lambda',\mu'}$. We claim that $K_1 \cap K_2 = \{0\}$ unless $s = s'$ and $\theta = \theta'$. Indeed, if $f \in K_1 \cap K_2$ is nonzero then there are nonzero H^2 functions h and k such that $f = u_{s,\theta}e_{\lambda,\mu}h = u_{s',\theta'}e_{\lambda',\mu'}k$. Using the F. and M. Riesz Theorem, as in the proof of Lemma 5.1, we obtain that $\frac{u_{s',\theta'}}{u_{s,\theta}}$ must be a constant, so that $s = s'$ and $\theta = \theta'$. A similar argument shows that $K_1 \vee K_2 = L^2(\mathbb{R})$ unless $s = s'$ and $\theta = \theta'$.

Suppose now that $K_1 = K_{s,\theta,\lambda\mu}$ and $K_2 = K_{s,\theta,\lambda',\mu'}$. Since $K_{s,\theta,\lambda\mu} = g_{s,\theta}\big(e_{\lambda,\mu}H^2(\mathbb{R})\big)$ and multiplication by $g_{s,\theta}$ is a unitary operator, the fact that

$$K_{s,\theta,\lambda,\mu} \wedge K_{s,\theta,\lambda',\mu'} = K_{s,\theta,\max(\lambda,\lambda'),\min(\mu,\mu')},$$
$$K_{s,\theta,\lambda,\mu} \vee K_{s,\theta,\lambda',\mu'} = K_{s,\theta,\min(\lambda,\lambda'),\max(\mu,\mu')}$$

follows from Lemma 5.1. In particular for each $s \in \mathbb{R}$ and $\theta \in S^1$ the set

$$\mathscr{L}_{s,\theta} = \{g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R}) : \lambda, \mu \geqq 0\} \cup \{L^2(\mathbb{R}), \{0\}\}$$

is a lattice, but is not commutative. The remaining assertions are obvious. □

Remark 5.3. Note that, unlike $\mathscr{L}_M$, the lattices $\mathscr{L}_{s,\theta}$ are not strongly closed. In fact it follows from Proposition 4.3 and the remarks following it that the strong operator closure of $\mathscr{L}_{s,\theta}$ is

$$\mathscr{P}_{s,\theta} = \{[g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})] : \lambda, \mu \geqq 0\} \cup \{[L^2[-a,a]] : a \geqq 0\} \cup \{L^2(\mathbb{R})\}.$$

Observe that $\mathscr{P}_{s,\theta}$ is also a sublattice of $\mathscr{L}$, but it is not commutative: the projections $[L^2[-1,1]]$ and $[g_{s,\theta}e_{\lambda,\mu}H^2(\mathbb{R})]$ do not commute (they are disjoint and not orthogonal). Hence $\mathscr{L}_{s,\theta}$ cannot be commutative either.

Remark 5.4. The algebra $\operatorname{Alg}\mathscr{L}_M$ contains no nonzero finite rank operators and so it follows that the (smaller) algebra $\operatorname{Alg}\mathscr{L}$ can contain no nonzero finite rank operators. To see this recall first that a reflexive algebra $\operatorname{Alg}\mathscr{M}$ contains a rank one operator if and only if there exists a nonzero $L \in \mathscr{M}$ such that $\bigvee\{K \in \mathscr{M} : K \ngeqq L\}$ is proper [11]. If $L = [L^2([-a,b])] \in \mathscr{L}_M$, then for large enough $n \in \mathbb{N}$, all subspaces of the form $[L^2([0,n])]$ or $[L^2([-n,0])]$ are not larger than L. Thus $\operatorname{Alg}\mathscr{L}_M$ contains no rank one operators. In a CSL algebra, a finite rank operator is approximable by linear combinations of rank one operators in the algebra [4], Theorem 23.16, and so it follows that $\operatorname{Alg}\mathscr{L}_M$ contains no nonzero finite rank operators.

Remark 5.5. The algebra $\mathscr{A} = \operatorname{Alg}\mathscr{L}$ is antisymmetric, that is, $\mathscr{A} \cap \mathscr{A}^* = \mathbb{C}I$. To see this let A be a selfadjoint operator in $\mathscr{A}$. Since A leaves each $L \in \mathscr{L}_M$ invariant, it must commute with $\mathscr{L}_M$. But the commutant of $\mathscr{L}_M$ is the multiplication algebra of $L^\infty(\mathbb{R})$. Thus A is the operator of multiplication by some function $f \in L^\infty(\mathbb{R})$. Since $A(H^2) \subseteqq H^2$, the function f must be in H^∞. Since $A = A^*$, f is real valued and so f must be constant.

The last two remarks show that $\operatorname{Alg}\mathscr{L}$ shares two of the basic properties of the Fourier binest algebra. In further analogy with this algebra define $\mathscr{L}_B = \mathscr{L}_V \cup \mathscr{L}_M$, the 'boundary' lattice. This is the analogue of the Fourier binest (in that each of its 'components' is reducing for one of the semigroups) and so it is natural to ask whether $\mathscr{L}_B$ has reflexive hull equal to $\mathscr{L}$. That is, is $\operatorname{Lat}\operatorname{Alg}\mathscr{L}_B = \mathscr{L}$?

Remark 5.6. The operator algebra $\mathrm{Alg}\,\mathcal{L}$ is a reflexive operator algebra with invariant subspace lattices $\mathcal{L}$. It would be interesting to determine whether $\mathrm{Alg}\,\mathcal{L} = \mathcal{A}_h$. This can be viewed as a noncommuting two variable variant of the classical result of Sarason [18] on the reflexivity of $H^\infty(\mathbb{R})$. This could possibly be established, as in the case of the Fourier binest algebra, by determining an explicit form for the Hilbert-Schmidt operators of each algebra.

There are many other basic structural questions that arise naturally for the algebras $\mathcal{A}_h$ and $\mathcal{A}_{FB}$. With the well-developed theory of nest algebras to hand one is naturally lead to the following problems.

Are the weakly closed ideals in correspondence with certain lower continuous endomorphisms of the invariant projection manifold with its partial order?

Is it possible to characterise the Jacobson radical in an explicit manner?

6. Automorphism groups

We now determine the group $\mathcal{U}(\mathcal{L})$, consisting of those unitaries U for which the map $K \to UK$ is a bijection of $\mathcal{L}$. With the relative strong operator topology this is a Lie group which is isomorphic to $\mathbb{W} \times S^1$ where $\mathbb{W}$ is the three dimensional Lie group determined by the Weyl commutation relations. The unitaries that induce the identity map are unimodular multiples of the identity and so it follows that the unitary automorphism group of $\mathcal{L}$ (that is, the group of automorphisms of $\mathcal{L}$ that are unitarily implemented) is isomorphic to the abelian quotient

$$\mathbb{R}^2 \times S^1 = (\mathbb{W}/S^1) \times S^1.$$

This identification also provides an identification of the unitary automorphism group of $\mathcal{A}_h$.

By making use of dilation theory for semicrossed products of the disc algebra (which in turn relies on the commutant lifting theorem of Sz-Nagy and Foias) we shall obtain the complete contractivity of certain representations of $\mathcal{A}_h$. We make use of this to show that in fact the unitary automorphism group of $\mathcal{A}_h$ coincides with the isometric automorphism group.

Recall the unimodular function $g_{s,\theta}$ and define the unitary operators $U_{s,\theta,t} = M_{g_{s,\theta}} V_t$, for $s, t \in \mathbb{R}$ and $\theta \in S^1$. Then from the description of $\mathcal{L}$ given in Section 2, it follows that the map $\beta_{s,\theta,t}$: $K \to U_{s,\theta,t}K$ is a lattice automorphism of $\mathcal{L}$ for each triple s, θ, t. In view of the commutation relations

$$V_t M_{g_{s,\theta}} = e^{ist} M_{g_{s,\theta}} V_t$$

the set of automorphisms $\beta_{s,\theta,t}$ is an abelian group isomorphic to $\mathbb{R}^2 \times S^1$.

Theorem 6.1. *If U is a unitary operator which induces a bijection β: $\mathcal{L} \to \mathcal{L}$ then $U = \eta U_{s,\theta,t}$ for some quadruple η, s, θ, t with $\eta, \theta \in S^1, s, t \in \mathbb{R}$.*

Proof. We prove first that β maps the commutative lattice $\mathcal{L}_M$ to itself. Suppose this is not the case. Then $\beta(\mathcal{L}_M)$ must be contained in a single lattice $\mathcal{L}_{s,\theta}$. Indeed, suppose that there are two elements K_1, K_2 of $\mathscr{L}_M$, with nontrivial intersection, and $U(K_1)$ is not in $\mathscr{L}_M$. As the intersection $U(K_1) \cap U(K_2)$ is nontrivial it follows from Proposition 5.2 that $U(K_1)$ and $U(K_2)$ are both contained in the same lattice $\mathscr{L}_{s,\theta}$. It follows from this that $U(\mathscr{L}_M)$ is contained in $\mathscr{L}_{s,\theta}$. Now note that the elements $L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R}_+)$ of $\mathscr{L}_M$ are orthogonal. Hence their images must be orthogonal and nontrivial. But $\mathscr{L}_{s,\theta}$ can contain no nontrivial orthogonal elements (again by Proposition 5.2), so we have the desired contradiction.

It follows now that β maps projections in $\mathscr{L}_M''$ (the double commutant) to projections in $\mathscr{L}_M''$, and hence U normalizes the multiplication algebra of $L^\infty(\mathbb{R})$. Since an automorphism of $L^\infty(\mathbb{R})$ is induced by an (a.e. defined) Borel isomorphism of $\mathbb{R}$ [20], it follows that U is of the form $U = M_\phi C_\gamma$ where M_ϕ is multiplication by some unimodular function ϕ and C_γ is the unitary composition operator induced by γ.

Observe now that C_γ induces an automorphism of $H^\infty(\mathbb{R})$. Indeed, for each h in $H^\infty(\mathbb{R})$ the multiplication operator M_h belongs to Alg $\mathscr{L}$ and so UM_hU^* belongs to Alg $\mathscr{L}$. Also

$$UM_hU^* = M_\phi C_\gamma M_h C_\gamma^* M_\phi^* = M_\phi M_{h\circ\gamma} M_\phi^* = M_{h\circ\gamma}.$$

Since $M_{h\circ\gamma}$ must leave $H^2(\mathbb{R})$ invariant, it follows that $h\circ\gamma$ is in $H^\infty(\mathbb{R})$. Since β is a bijection it follows that the map $h \to h\circ\gamma$ is an automorphism of $H^\infty(\mathbb{R})$ and so γ is a conformal map of the upper half plane onto itself.

There are two cases: either $\gamma(x) = ax + b$ with $a > 0$ and $b \in \mathbb{R}$ or $\gamma(x) = a - b(x+c)^{-1}$ with $a, b, c \in \mathbb{R}$ and $b > 0$.

In fact the latter case cannot occur. Indeed, suppose so and consider the subspace $L^2([-s,\infty)) \in \mathscr{L}_M$ where $s > -a$. Then U maps this subspace to

$$L^2\big((-\infty,-c] \cup [b(s+a)^{-1} - c, \infty)\big)$$

which is not in $\mathscr{L}_M$ because $b(s+a)^{-1} > 0$.

In case $\gamma(x) = ax + b$, if $b > 0$, then $L^2([-s,0])$ is mapped to $L^2\left(\left[-\frac{s+b}{a}, -\frac{b}{a}\right]\right)$ which is not in $\mathscr{L}_M$; considering $L^2([0,s])$ shows that $b < 0$ cannot occur either. We conclude that $\gamma(x) = ax$ or

$$\gamma(x) = e^t x$$

for some $t \in \mathbb{R}$, since $a > 0$.

So we now have $U = M_\phi V_t$, and it remains to find the form of the unimodular factor ϕ. Since V_{-t} defines a bijection of $\mathscr{L}$, it is enough to consider the case $U = M_\phi$. Now $H^2(\mathbb{R})$ lies in $\mathscr{L}\backslash\mathscr{L}_M$, and so we have $M_\phi(H^2(\mathbb{R})) = g_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R})$ for some $s \in \mathbb{R}$, $\theta \in S^1$, $\lambda \geqq 0$ and $\mu \geqq 0$. Hence $\phi = \eta g_{s,\theta} e_{\lambda,\mu}$ where η is a unimodular constant. The operator $M = M_\phi^* M_{g_{s,\theta}}$ also defines a bijection of $\mathscr{L}$. Since M is the operator of multiplication by $\bar{\eta}\bar{e}_{\lambda,\mu}$, we have $M(H^2(\mathbb{R})) = e_{-\lambda,-\mu} H^2(\mathbb{R})$. This subspace is in $\mathscr{L}$ only if $\lambda \leqq 0$ and $\mu \leqq 0$ and so $\lambda = \mu = 0$ and $\phi = \eta g_{s,\theta}$, as required. □

To obtain the isometric automorphism group of $\mathscr{A}_h$ we make use of the following result which is perhaps of independent interest.

Theorem 6.2. *Let ρ be a weak* continuous contractive representation of $\mathcal{A}_h$ on the Hilbert space $\mathcal{H}$. Then ρ is completely contractive. Furthermore there is a weak* continuous *-representation $\pi: \mathcal{B}(L^2(\mathbb{R})) \to \mathcal{B}(\mathcal{H}_1)$, with $\mathcal{H}_1 \supseteqq \mathcal{H}$, such that $\rho(A) = P_{\mathcal{H}}\pi(A)|_{\mathcal{H}}$ for all A in $\mathcal{A}_h$.*

Proof. Let $A(\mathbb{R})$ be the algebra of continuous functions in $H^\infty(\mathbb{R})$ which have equal limits at $+\infty$ and $-\infty$ and, for fixed $t > 0$, let $\mathcal{B}_t$ be the norm closed subalgebra of $\mathcal{A}_h$ generated by the multiplication operators M_f, $f \in A(\mathbb{R})$ and the unitary V_t. Now for the indices $t_n = 1/2^n$ we have the subalgebra chain

$$\mathcal{B}_{t_1} \subseteqq \mathcal{B}_{t_2} \subseteqq \mathcal{B}_{t_3} \subseteqq \cdots$$

whose union is weak star dense in $\mathcal{A}_h$. Indeed, observing that $A(\mathbb{R})$ is weak star dense in $H^\infty(\mathbb{R})$ and that the dilation group is weak star continuous, we see that the weak star closure of this union is an algebra containing the multiplication algebra of $H^\infty(\mathbb{R})$ and $\{V_t : t \geqq 0\}$.

To obtain the first assertion of the theorem it will be sufficient to show that the restriction of ρ to each subalgebra $\mathcal{B}_t$ is completely contractive. To see this, we identify $\mathcal{B}_t$ (completely isometrically) with a semicrossed product as follows:

Consider the C*-algebra $\mathcal{C}_t \subseteqq \mathcal{B}(L^2(\mathbb{R}))$ generated by $\mathcal{B}_t$, namely the C*-algebra generated by multiplications by $f \in C(\mathbb{R} \cup \{\infty\})$ and V_t (note that $A(\mathbb{R}) + A(\mathbb{R})^*$ is norm dense in $C(\mathbb{R} \cup \{\infty\})$). Since $M_f V_t = V_t M_{\alpha(f)}$ (where $\alpha(f)(x) = f(e^t x)$), $\mathcal{C}_t$ is *-isomorphic to the crossed product $C(\mathbb{R} \cup \{\infty\}) \times_{\psi_t} \mathbb{Z}$ arising from the hyperbolic automorphism $\psi_t: \mathbb{R} \to \mathbb{R}$ given by $x \to e^t x$. Hence $\mathcal{B}_t$ is completely isometrically isomorphic to the semicrossed product $A(\mathbb{R}) \times_{\psi_t} \mathbb{Z}_+$. Now Theorem 2 of [16] shows that every contractive representation of this semicrossed product, and hence of $\mathcal{B}_t$, is completely contractive (actually, [16] deals with the disc algebra in place of $A(\mathbb{R})$, but a conformal equivalence of the disc to the upper half plane allows one to identify the two algebras).

The second assertion of the theorem follows from Arveson's dilation theorem [1] for completely contractive maps and general dilation theory, as in [15], for example. □

The proof of the next theorem requires the following lemma. Let $\mathcal{K}, \mathcal{K}_+, \mathcal{K}_-$ be the spaces of compact operators on the Hilbert spaces $L^2(\mathbb{R}), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-)$ respectively.

Lemma 6.3. $C^*(\mathcal{A}_h \cap \mathcal{K}) = \mathcal{K}_+ \oplus \mathcal{K}_-$.

Proof. Let (K_n) be the bounded approximate identity in $\mathcal{A}_h \cap \mathcal{K}$ given by Proposition 3.2 and let N be a proper reducing subspace for this subalgebra. Since N is reducing for $K_n M_\lambda$, for $\lambda > 0$, it is also reducing for M_λ. Similarly N is reducing for V_t for $t > 0$. But the multiplication and dilation operators are only jointly reduced by $L^2(\mathbb{R}_+)$ or $L^2(\mathbb{R}_-)$. Thus $C^*(\mathcal{A}_h \cap \mathcal{K})$ is an irreducible algebra of compact operators on each of these spaces and so the lemma follows. □

Theorem 6.4. *The isometric automorphism group of the hyperbolic algebra $\mathcal{A}_h$, with the point weak* topology, is the Lie group consisting of the automorphisms $\alpha_{s,\theta,t} = \mathrm{Ad}(U_{s,\theta,t})$, for $(s, \theta, t) \in \mathbb{R} \times S^1 \times \mathbb{R}$.*

Proof. Let α be an isometric automorphism. Since $\mathscr{A}_h$ has a bounded approximate identity of compact operators it follows from Theorem 4 of Power [16] that α is weak star continuous and maps compact operators to compact operators. By Theorem 6.2, α is completely contractive and so induces a completely positive bijection

$$\tilde{\alpha}: (\mathscr{A}_h \cap \mathscr{K}) + (\mathscr{A}_h \cap \mathscr{K})^* \to (\mathscr{A}_h \cap \mathscr{K}) + (\mathscr{A}_h \cap \mathscr{K})^*.$$

By the universal property of the C*-envelope this bijection has a unique extension to a C*-algebra automorphism of the C*-envelope. This envelope is a quotient of $C^*(\mathscr{A}_h \cap \mathscr{K})$ and so, by the lemma coincides with $\mathscr{K}_+ \oplus \mathscr{K}_-$. Thus the extension is unitarily implemented and hence so is α. □

7. Subspace manifolds

We now identify the hyperbolic lattice $\mathscr{L}$ as a Euclidean manifold and identify the inclusions $\mathscr{L} \subset \tilde{\mathscr{L}}$ and $\tilde{\mathscr{L}} \subset \hat{\mathscr{L}}$ obtained by extending the parameter range and by taking orthogonal complements.

Recall that $P_{s,\theta,\lambda,\mu} = [K_{s,\theta,\lambda,\mu}]$ and for fixed θ consider the θ-section

$$\mathscr{L}_\theta = \{P_{s,\theta,\lambda,\mu} : (s, \lambda, \mu) \in \mathbb{R} \times \mathbb{R}_+^2\}.$$

In view of the results of Section 4 the strong operator topology closure of $\mathscr{L}_\theta$ is homeomorphic to the union of two triangular cones as indicated in Figure 2.

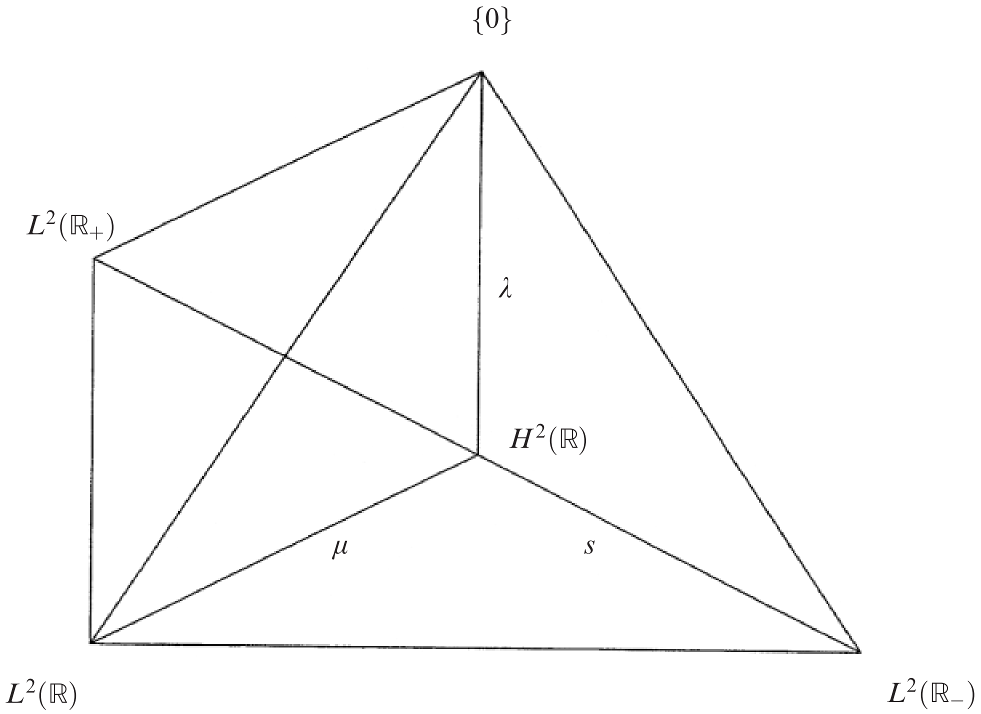

Figure 2. Section of $\mathscr{L}$ for $\theta = 1$.

We can now identify the strong operator topology closure, denoted $\tilde{\mathscr{L}}$, of the set of orthogonal projections $P_{s,\theta,\lambda,\mu}$ with extended parameter range $\lambda, \mu \in \mathbb{R}$, $s \in \mathbb{R}$, $\theta \in S^1$. Note that $P_{s,\theta,\lambda,\mu} \to 0$ as $\mu \to -\infty$ for fixed s, θ, λ. To see this observe that the subspace $K_{s,\theta,\lambda,\mu}$ is transformed to $e_{\lambda,\mu}H^2(\mathbb{R})$ by the unitary operator of multiplication by $\bar{g}_{s,\theta}$ and the latter is transformed to $e_{\mu,\lambda}\overline{H^2(\mathbb{R})} = \overline{e_{-\mu,-\lambda}H^2(\mathbb{R})}$ by the unitary operator U given by $(Uf)(x) = x^{-1}f(x^{-1})$. Since $\lim_{\mu \to -\infty}[e_{-\mu,-\lambda}H^2(\mathbb{R})] = 0$ strongly, the assertion follows. Also $P_{s,\theta,\lambda,\mu} \to 0$ as $\lambda \to \infty$, with the other parameters fixed and $P_{s,\theta,\lambda,\mu} \to I$ as $\lambda \to -\infty$ or $\mu \to \infty$. It follows that the section of $\tilde{\mathscr{L}}$ for $s = 0$, $\theta = 1$ is homeomorphic to a topological disc, $\mathscr{Q}$ say, as in Figure 3, where the transformed axes for λ, μ are indicated as the lemniscate.

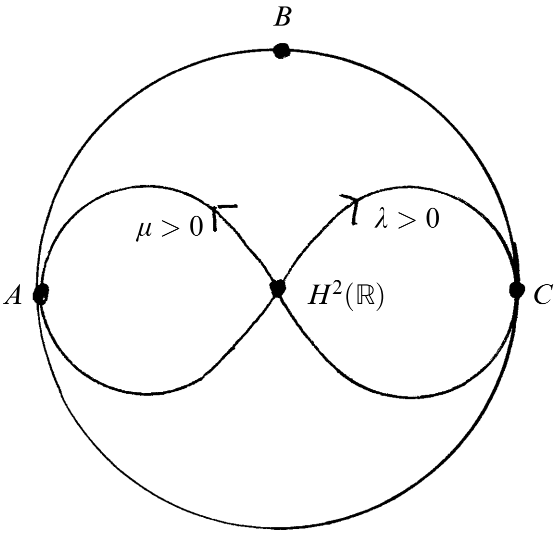

Figure 3. The disc $\mathscr{Q}$. $A = L^2(\mathbb{R})$, $B = L^2[-1,1]$, $C = \{0\}$.

The triangular region of Figure 1 corresponds to the upper region of Figure 3 with boundary consisting of the upper semicircle and the semiaxes $\mu \geqq 0$, $\lambda \geqq 0$. We can now identify the $\theta = 1$ section of $\tilde{\mathscr{L}}$ is a double cone over $\mathscr{Q}$ whose apexes are $[L^2(\mathbb{R}_-])$ at $s = \infty$ and $[L^2(\mathbb{R}_+)]$ at $s = -\infty$. Topologically this section is a closed 3-ball B^3 which is an (unnatural) compactification of $\mathbb{R}^3$ by a sphere. We have indicated a perspective view of this in Figure 4. Its surface is the union of the square lattice $\mathscr{L}_M$ (the northern region), its orthocomplement $\mathscr{L}_M^\perp$ (the southern region) and four equatorial lens-like regions, labeled C, D, E, F whose projections on the plane $s = 0$ are the lobe regions of the lemniscate. These regions are the sets

$$\mathscr{L}_E = \{[L^2[b^{-1}, a]] : 0 \leqq b^{-1} \leqq a \leqq \infty\},$$

$$\mathscr{L}_D = \{[L^2[-b, -a^{-1}]] : 0 \leqq a^{-1} \leqq b \leqq \infty\},$$

$$\mathscr{L}_F = \mathscr{L}_D^\perp,$$

$$\mathscr{L}_C = \mathscr{L}_E^\perp.$$

Note that the λ and μ coordinates partition the interior of the ball into four parts which meet the sphere at $\mathscr{L}_M$ (for $\lambda \geqq 0$, $\mu \geqq 0$), $\mathscr{L}_M^\perp$ (for $\lambda \leqq 0$, $\mu \leqq 0$), $\mathscr{L}_E \cup \mathscr{L}_D$ (for $\lambda \geqq 0$, $\mu \leqq 0$), $\mathscr{L}_F \cup \mathscr{L}_C$ (for $\lambda \leqq 0$, $\mu \geqq 0$). The double cone region of Figure 2 (with $\theta = 1$) corresponds to the first (upper) part.

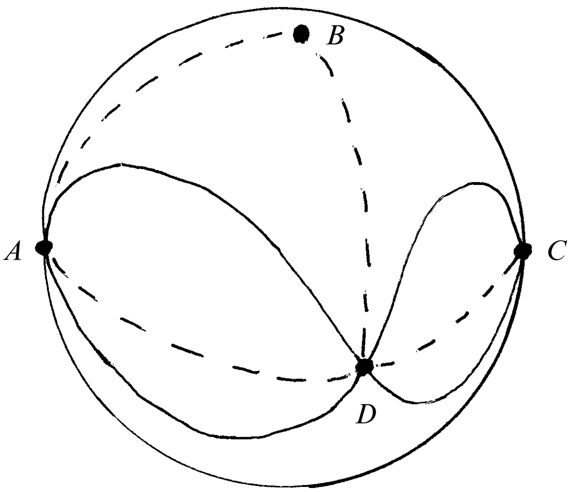

Figure 4. The $\theta = 1$ section of $\tilde{\mathscr{L}}$; $D = L^2(\mathbb{R}_-)$.

The topological space $\tilde{\mathscr{L}}$ is thus homeomorphic to the identification space $B^3 \times S^1/\sim$ where

$$(x, \theta) \sim (x', \theta') \quad \text{if and only if} \quad x \in \partial B^3,\ x = x'.$$

The closed subset $\mathscr{L}$ is the subset determined by the upper part of B^3.

The topological space $\tilde{\mathscr{L}}^\perp$ has a similar description with $\overline{H^2(\mathbb{R})}$ taking the role of $H^2(\mathbb{R})$ and, topologically, the union $\tilde{\mathscr{L}} \cup \tilde{\mathscr{L}}^\perp$ is equal to two copies of $\tilde{\mathscr{L}}$ joined at their common spherical boundaries. Thus

$$\hat{\mathscr{L}} = \tilde{\mathscr{L}} \cup \tilde{\mathscr{L}}^\perp = \big((B^3 \times S^1)/\sim\big) \cup_{\partial B^3} \big((B^3 \times S^1)/\sim\big).$$

In particular the $\theta = 1$ section of $\hat{\mathscr{L}}$, being the join of two 2-spheres at their surfaces, is a 3-sphere.

The Fourier-Plancherel sphere $\hat{\mathscr{L}}_{FB}$. One can readily observe that there is a natural action of the four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\hat{\mathscr{L}}$ which is induced by the unitary operators for the maps $x \to -x$ and $x \to x^{-1}$. For comparison we now identify the parabolic analogue $\hat{\mathscr{L}}_{FB}$. This is a 2-sphere and the analogous group action is a $\mathbb{Z}_4$ action implemented by the Fourier-Plancherel transform as rotation of this sphere.

Recall that the Paley-Wiener theorem ensures that $FH^2(\mathbb{R}) = L^2[0, \infty)$ and from this one sees that $Fe^{i\lambda x}H^2(\mathbb{R}) = L^2[\lambda, \infty)$ for λ in $\mathbb{R}$. The chain of projections $[e^{i\lambda x}H^2(\mathbb{R})]$ together with 0 and I comprise the analytic nest, denoted $\mathcal{N}_a$, whilst the chain of projections $[L^2[\lambda, \infty)]$ is the Volterra nest $\mathcal{N}_v$. Let $\phi_s(x) = e^{-isx^2/2}$. It was shown in [9] that as $s \to \infty$ we have

$$[e^{-isx^2/2}e^{i\lambda sx}H^2(\mathbb{R})] \to [L^2[\lambda, \infty)]$$

in the strong operator topology, and from this it follows that the family of projection nests $\phi_s\mathcal{N}_a$, $0 < s < \infty$, forms a continuous interpolation between $\mathcal{N}_a$ and $\mathcal{N}_v$. More precisely, their union, which, as we remarked in the introduction, is $\mathcal{L}_{FB}$, is homeomorphic to a closed disc, with $\mathcal{N}_a$ and $\mathcal{N}_v$ as bounding semicircles.

The family $\hat{\mathcal{L}}_{FB}$ is obtained from $\mathcal{L}_{FB}$ by extending the parameter range to $-\infty < s < \infty$ and by admitting orthogonal complements. Alternatively, and more intrinsically, it may be defined as the union of the invariant projection lattices for the four natural pairs of 1-parameter semigroups arising from translations and the Fourier translations. Explicitly we have

$$\hat{\mathcal{L}}_{FB} = \left(\bigcup_{s\in\mathbb{R}} e^{-isx^2/2}\mathcal{N}_a\right) \cup \mathcal{N}_v \cup \left(\bigcup_{s\in\mathbb{R}} e^{-isx^2/2}\mathcal{N}_a^{\perp}\right) \cup \mathcal{N}_v^{\perp}.$$

The Fourier Plancherel transform has period 4 and gives a cyclic permutation of the spaces $H^2(\mathbb{R}), L^2[0,\infty), H^2(\mathbb{R})^{\perp}, L^2[0,\infty)^{\perp}$. The next theorem whose proof follows immediately from Lemma 4.2 of [9], gives the detail of the rotation action of the Fourier Plancherel transform on the sphere $\hat{\mathcal{L}}_{FB}$. This is illustrated in Figure 5. The usual partial ordering of projections gives a foliation of $\hat{\mathcal{L}}_{FB}$ by lines of longitude.

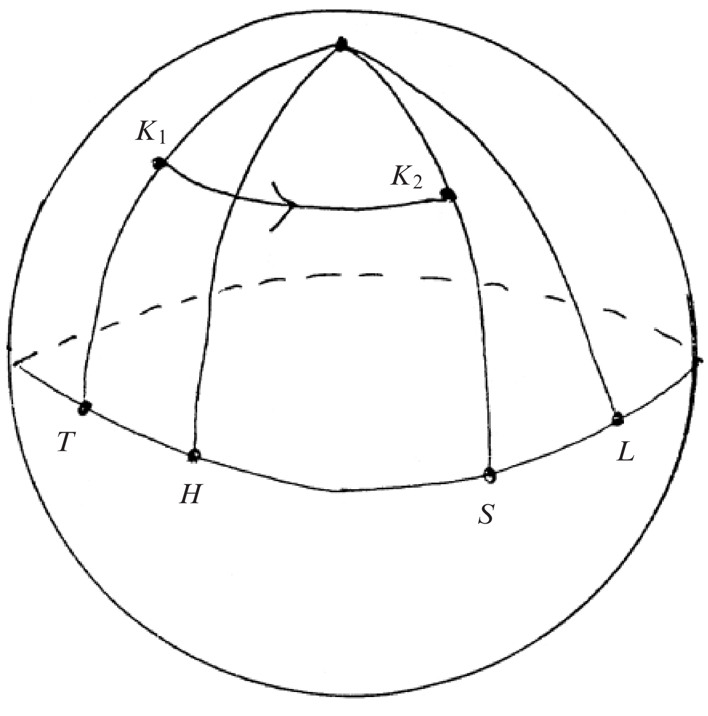

Figure 5. The Fourier Plancherel sphere $\tilde{\mathcal{L}}$; $H = H^2(\mathbb{R})$, $L = L^2(\mathbb{R}_+)$, $S = e^{-isx^2/2}H^2(\mathbb{R})$, $T = e^{is^{-1}x^2/2}H^2(\mathbb{R})$, $K_1 = e^{is^{-1}\lambda x}T$, $K_2 = e^{i\lambda x}S = FK_1$.

Theorem 7.1. *For $\lambda \in \mathbb{R}$ and $s \in \mathbb{R}_+$ we have*

$$e^{i\lambda x}\phi_s H^2(\mathbb{R}) \xrightarrow{F} e^{-is^{-1}\lambda x}\phi_{-s^{-1}} H^2(\mathbb{R})^\perp \xrightarrow{F} e^{-i\lambda x}\phi_s H^2(\mathbb{R})^\perp$$

and

$$e^{-i\lambda x}\phi_s H^2(\mathbb{R})^\perp \xrightarrow{F} e^{is^{-1}\lambda x}\phi_{-s^{-1}} H^2(\mathbb{R}) \xrightarrow{F} e^{i\lambda x}\phi_s H^2(\mathbb{R}).$$

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