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**Translation and dilation invariant subspaces of $L^2(\mathbb{R})$. (English summary)**
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FEATURED REVIEW.

The mere existence of invariant subspaces (IS) for a given linear operator means almost nothing. On the contrary, presence of a large number of IS, and especially of a *structured family*, tells you a lot about your operator. The classical example is an $n\times n$ matrix having a family of $n$ independent one-dimensional (or one-codimensional) IS. This is equivalent to saying that it is diagonalizable (and you easily know all functions of it, etc.). Something similar occurs in much more general situations. Roughly speaking, operator theory is based on two (not independent) techniques: perturbation theory (if $A$ is "good" and $K$ is "small" or "smooth", then $A+K$ is also "quite good") and model theory (in various senses: Hilbert-von Neumann, Friedrichs, Livshitz, Sz. Nagy and Foias, de Branges, . . . ). Any model theory lies upon a structured (parametrized) family of IS of a given operator. This model–invariant-subspaces approach was precursored by papers of A. Beurling [Acta Math. **81** (1948), 17 pp.; MR0027954 (10,381e)] and P. D. Lax [Acta Math. **101** (1959), 163–178; MR0105620 (21 #4359)]. Namely, it was shown that a cyclic one-parameter semigroup of pure Hilbert space isometries $S(t)\colon H\to H$, $t>0$, has a holomorphic family of basic one-codimensional IS $\{b_\lambda(A)H\colon \lambda\in\mathbb{C}^+\}$ where $\mathbb{C}^+=\{\lambda\in\mathbb{C}\colon \operatorname{Im}(\lambda)>0\}$, and all other IS are intersections of the basic ones and limits of those intersections; here $A$ is the generator of $S$ and $b_\lambda(z)=(z-\lambda)/(z-\overline{\lambda})$ stands for an elementary Blaschke factor. It follows that IS are in a one-to-one correspondence with $H^\infty$ functions unimodular on $\mathbb{R}$ (inner functions); the inclusion of subspaces corresponds to a factorization of corresponding inner functions, and any ordered chain of invariant subspaces gives rise to a kind of integral lower-triangular representation of $S$. Similar but more complicated models exist for every one-parameter Hilbert space contractive semigroup. Then, all of the Hardy-type function theory techniques can be employed to build a certain kind of spectral theory and/or its applications (say, to control or signal processing). For all details and references the reader is referred to the milestones of this "operator function theory" (or, "spectral function theory", depending on your taste): [M. S. Livshitz, Rec. Math. [Mat. Sbornik] N.S. **19(61)** (1946), 239–262; MR0020719 (8,588d); *Operators, oscillations, waves (open systems)*, Translated from the Russian by Scripta Technica, Ltd. English translation edited by R. Herden, Amer. Math. Soc., Providence, R.I., 1973; MR0347396 (49 #12116); H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964; MR0171178 (30 #1409); B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland, Amsterdam, 1970; MR0275190 (43 #947); L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York, 1966; MR0215065 (35 #5909); in *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965)*, 295–392, Wiley, New York, 1966; MR0244795 (39 #6109)]; for a more recent presentation see also [N. K. Nikol′skiĭ, *Treatise on the shift operator*, Translated from the Russian by Jaak Peetre, Springer, Berlin, 1986; MR0827223 (87i:47042); *Operators, functions, and systems: an easy reading. Vol. 1*, Translated from the French by Andreas Hartmann, Amer. Math. Soc., Providence, RI, 2002; MR1864396 (2003i:47001a); *Vol. 2*; MR1892647 (2003i:47001b)].

A technique aimed at a similar theory for non-singly generated semigroups of isometries has appeared in a seminal paper by H. Helson and D. Lowdenslager [in *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, 251–262, Jerusalem Academic Press, Jerusalem, 1961; MR0157251 (28 #487)]. They discovered that in this case, inner functions are not sufficient to describe invariant subspaces. An important case treated completely is a pair of semigroups satisfying the so-called Weyl commutation relations $S(t)V_\lambda = e^{it\lambda}V_\lambda S(t)$. Here, generic invariant subspaces of $S(t)$ and $V_\lambda$ are parametrized by unimodular cocycles, i.e. by families $(A_t)$ of unimodular functions on $\mathbb{R}$ such that $A_{t+u} - A_t S_t A_u = 0$, where $S_t$ stands for the group of right translations on $\mathbb{R}$, $S_t f(x) = f(x-t)$. It turns out that for a singly-generated continuous semigroup every such cocycle is a coboundary ($A_t = qS_t q^{-1}$, where $t \in \mathbb{R}$ and $|q| = 1$ a.e. on $\mathbb{R}$), and one gets the previous Beurling-Lax parametrization of invariant subspaces. Speaking this cohomology language suggests looking for a corresponding algebraic-geometric object whose topological nature is implicitly involved in the analysis of the corresponding semigroups. This is the idea developed in the paper under review, as well as in the previous paper by the same authors [Math. Proc. Cambridge Philos. Soc. **122** (1997), no. 3, 525–539; MR1466655 (98d:47097)].

In the latter paper, using the Helson-Lowdenslager approach, the authors showed that for $S(t) = M_t$ and $V_\lambda = S_\lambda$, where $M_t f(x) = e^{itx} f(x)$, $t > 0$, is a character multiplication operator, the lattice of jointly invariant subspaces $\mathrm{Lat}(M_t, S_\lambda)$ topologically is the closed unit disc in $\mathbb{R}^2$ consisting of two obvious nests, the Volterra nest $\mathcal{N}_v = \{L^2[t,\infty))\colon t \in \mathbb{R}\}$ and the analytic nest $\mathcal{N}_a = \{e^{i\lambda x}H^2(\mathbb{R})\colon \lambda \in \mathbb{R}\}$ (these form the topological boundary of $\mathrm{Lat}(M_t, S_\lambda)$), together with a continuum of nests $\mathcal{N}_s = \{e^{-isx^2/2}K\colon K \in \mathcal{N}_a\}$, $s > 0$, corresponding to the interior of the disc. Some interesting consequences were derived.

In the paper under review, the case of semigroups $S(t) = M_t$ and $V_\lambda f(x) = e^{\lambda/2} f(e^\lambda x)$, $\lambda > 0$, is considered. It is shown that in addition to the obvious subspaces $L^2[-a, b]$, $a, b \geq 0$, the lattice $\mathrm{Lat}(M_t, V_\lambda)$ consists of the four-parameter family $K_{s,\theta,\lambda,\mu}$ of subspaces,

$$K_{s,\theta,\lambda,\mu} = u_{s,\theta} e_{\lambda,\mu} H^2(\mathbb{R}),$$

where $u_{s,\theta}(x) = 1$ for $x > 0$ and $u_{s,\theta}(x) = \theta e^{s\pi}$ for $x \leq 0$ ($\theta \in \mathbb{R},\ |\theta| = 1$ and $s \in \mathbb{R}$), and $e_{\lambda,\mu}(x) = e^{i(\lambda x + \mu x^{-1})}$ ($\lambda, \mu \geq 0$). The main result says that the lattice of orthogonal projections onto the subspaces from $\mathrm{Lat}(M_t, V_\lambda)$ forms a compact connected 4-manifold in the strong operator topology (SOT).

To prove the SOT compactness of $[\mathrm{Lat}(M_t, V_\lambda)] = \{[K]\colon K \in \mathrm{Lat}(M_t, V_\lambda)\}$, where $[K]$ stands for the orthogonal projection onto $K$, the authors consider the $w^*$-closed nonselfadjoint algebra $\mathcal{A}_h = \mathrm{alg}(M_t, V_\lambda\colon\ t > 0,\ \lambda > 0)$ ($h$ stands for hyperbolic). This algebra is an analogue of the algebra $\mathcal{A}_{\mathrm{FB}} = \mathrm{alg}(M_t, S_\lambda\colon t > 0, \lambda > 0)$ (called by the authors the Fourier binest algebra) studied in the authors' previous paper [op. cit.]. It is shown, as in that paper, that $\mathcal{A}_h$ is antisymmetric and is generated by Hilbert-Schmidt operators contained in it (but contains no nontrivial finite-rank operators). Realizing the above Hilbert-Schmidt operators as pseudodifferential operators with bianalytic symbols, the authors show that $\mathcal{A}_h$ contains an approximate identity consisting of Hilbert-Schmidt operators. The compactness of $[\mathrm{Lat}(M_t, V_\lambda)]$ follows using a result of B. H. Wagner [Trans. Amer. Math. Soc. **304** (1987), no. 2, 515–535; MR0911083 (89h:47065)].

To prove the connectedness of $\mathrm{Lat}(M_t, V_\lambda)$ the authors establish various unusual SOT limits of projections of the form $[K_{s_n,\theta_n,\lambda_n,\mu_n}]$. In particular, $[K_{0,1,\lambda,\lambda}] \to [L^2(-1,1)]$ as $\lambda \to \infty$, and $[|x|^{is_n} e^{i\lambda_n x} H^2(\mathbb{R})] \to [L^2(-a,0)]$ for a choice of $(s_n, \lambda_n) \to (\infty, \infty)$ depending on $a > 0$. The arguments leading to these limits are rather complicated and contain the above compactness of $[\mathrm{Lat}(M_t, V_\lambda)]$ and a theorem of P. R. Halmos [Trans. Amer. Math. Soc. **144** (1969), 381–389; MR0251519 (40 #4746)] on two projection algebras. (Below, we propose a short elementary proof for the most painful detail of the authors' arguments where Halmos' spectral representation is used (Proposition 4.5); see reviewer's remark.)

The lattice structure of $[\mathrm{Lat}(M_t, V_\lambda)]$ and the isometric isomorphism group of $\mathcal{A}_h$ are also established. In particular, $e_{\lambda,\mu} H^2(\mathbb{R}) \wedge e_{\alpha,\beta} H^2(\mathbb{R}) = e_{\delta,\varepsilon} H^2(\mathbb{R})$ where $\delta = \max(\lambda, \alpha)$, $\varepsilon = \min(\mu, \beta)$, and $e_{\lambda,\mu} H^2(\mathbb{R}) \vee e_{\alpha,\beta} H^2(\mathbb{R}) = e_{\delta,\varepsilon} H^2(\mathbb{R})$ where $\delta = \min(\lambda, \alpha)$, $\varepsilon = \max(\mu, \beta)$.

Throughout the paper, the authors trace a challenging research program in the theory of nonselfadjoint operator algebra. Let us mention three points of this program.

First, the authors raise a question concerning a noncommutative two-variable analogue of a result of D. Sarason [Pacific J. Math. **17** (1966), 511–517; MR0192365 (33 #590)] on the reflexivity of the algebra $H^\infty(\mathbb{R})$. Namely, observing that the operator algebra $A = \mathrm{Alg}\,\mathrm{Lat}(M_t, V_\lambda)$ is reflexive (i.e. $\mathrm{Lat}\,A = \mathrm{Lat}(M_t, V_\lambda)$), they ask whether $A = \mathcal{A}_h$ (a similar result for the binest Fourier algebra $\mathcal{A}_{\mathrm{FB}}$ is proved in the authors' previous paper [op. cit.]).

Secondly, they discuss the very interesting problem of how to construct a theory of what they call *Euclidean lattice algebras*, meaning those operator algebras $A$ for which $[\mathrm{Lat}\,A]$, with the SOT, are Euclidean manifolds. In particular, it is of interest to know which manifolds are attainable by direct-sum-decomposable algebras $A \oplus \cdots \oplus A$ (for instance, for the Volterra nest algebra $\mathcal{A}_v$, $[\mathrm{Lat}(\mathcal{A}_v \oplus \cdots \oplus \mathcal{A}_v)]$ is homeomorphic to $[0,1]^n$).

Third, the authors observe that the operator algebras $\mathcal{A}_{\mathrm{FB}}$ and $\mathcal{A}_h$ that they studied are partial cases of what they call *Lie semigroup algebras*, i.e. weak operator topology closed operator algebras generated by the image of a Lie semigroup in a unitary representation of the corresponding Lie group. Namely, the algebra $\mathcal{A}_{\mathrm{FB}}$ is obtained from the Lie semigroup of the 3-dimensional Heisenberg group given as

$$\begin{pmatrix} 1 & \lambda & t \\ & 1 & \mu \\ & & 1 \end{pmatrix},\ \lambda \geq 0,\ \mu \geq 0.$$

The algebra $\mathcal{A}_h$ of the paper under review corresponds to the Lie semigroup $ax+b$, $a \geq 0$, $b \geq 0$ of the group $ax+b$. The program traced involves the eventual links between topological properties of the corresponding (semi)groups and the relevant indecomposable representations.

{Reviewer's remark: An alternative proof of a projection limit theorem.

{Proposition 4.5. Let $E$, $P$ be two projections on a Hilbert space $H$ such that $E \wedge P^\perp = E^\perp \wedge P = 0$. For $\delta > 0$, denote by $Q_\delta$ the orthoprojection onto the range of the operator $(E + \delta I)P$. Then $\lim_{\delta\to 0} Q_\delta x = Ex$ for every $x \in H$.

{Proof. Let $x \in EP(H)$, that is, $x = EPh$, $h \in H$. Then $\|x - Q_\delta x\| = \mathrm{dist}(x, Q_\delta H) \leq \|EPh - (E+\delta I)Ph\| = \delta\|Ph\|$, which tends to zero as $\delta \to 0$. Since by assumption $EP(H)$ is dense in $E(H)$, it follows that $\lim_{\delta\to 0} Q_\delta x = x$ for every $x \in E(H)$.

{Now let $x \in E^\perp P^\perp(H)$, that is, $x = E^\perp P^\perp h$, $h \in H$. Then $\|Q_\delta x\| = \mathrm{dist}(x, Q_\delta^\perp H)$, where $Q_\delta^\perp$ is the projection onto $((E+\delta I)P(H))^\perp = \mathrm{Ker}(P(E+\delta I)) = (E+\delta I)^{-1}P^\perp(H)$. Let $x_\delta = (E+\delta I)^{-1}P^\perp h \in Q_\delta^\perp H$. Since $(E+\delta I)^{-1} = \frac{1}{1+\delta}E + \frac{1}{\delta}E^\perp$, we have $\|Q_\delta x\| \le \|x - \delta x_\delta\| = \|\frac{\delta}{1+\delta}EP^\perp h\|$, which obviously tends to zero as $\delta \to 0$. Since by assumption $E^\perp P^\perp(H)$ is dense in $E^\perp(H)$, it follows that $\lim_{\delta\to 0} Q_\delta x = 0$ for every $x \in E^\perp(H)$.

{The result follows.

{It seems that this reasoning does not depend on the nature of operators $T = E$ and $T_\delta = (E+\delta I)P$ and should work well if one assumes that $T_\delta$ is uniformly normally solvable and that $T_\delta \to T, T_\delta^* \to T^*$.}

Reviewed by *N. K. Nikolski*

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*