Rank One Subspaces of Bimodules over Maximal Abelian Selfadjoint Algebras

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Abstract

Spaces of operators that are left and right modules over maximal abelian selfadjoint algebras (masa bimodules for short) are natural generalizations of algebras with commutative subspace lattices. This paper is concerned with density properties of finite rank operators and of various classes of compact operators in such modules. It is shown that every finite rank operator of a norm closed masa bimodule $\mathcal{M}$ is in the trace norm closure of the rank one subspace of $\mathcal{M}$. An important consequence is that the rank one subspace of a strongly reflexive masa bimodule (that is, one which is the reflexive hull of its rank one operators) is dense in the module in the weak operator topology. However, in contrast to the situation for algebras, it is shown that such density need not hold in the ultraweak topology.

A new method of representing masa bimodules is introduced. This uses a novel concept of an ω-topology. With the appropriate notion of ω-support, a correspondence is established between reflexive masa bimodules and their ω-supports. It is shown that, if a $\mathcal{C}_2$-closed masa bimodule contains a trace class operator then it must contain rank one operators; indeed, every such operator is in the $\mathcal{C}_2$-norm closure of the rank one subspace of the module. Consequently the weak closure of any masa bimodule of trace class operators is strongly reflexive. However, the trace norm closure of the rank one subspace need not contain all trace class operators of the module. Also, it is shown that there exists a CSL algebra which contains no trace class operators yet contains an operator belonging to $\mathcal{C}_p$ for all $p > 1$. From this it follows that a transitive bimodule spanned by the rank one operators it contains need not be dense in $\mathcal{C}_p$ for $1 \leq p < \infty$.

As an application, it is shown that there exists a commutative subspace lattice $\mathcal{L}$ such that $\mathcal{L}$ is non-synthetic but every weakly closed algebra which contains a masa and has invariant lattice $\mathcal{L}$ coincides with $\operatorname{Alg}\mathcal{L}$.

1 Introduction

The sets of compact operators contained in reflexive algebras have been widely investigated. Attention has been focused on strongly reflexive algebras, that is, on algebras whose invariant subspace lattices are determined by the rank one operators they contain. Questions concerning the density properties of the rank one subspaces of these algebras, that is, the linear span of the rank one operators, remained open for many years. Although the general (non-commutative) case will not be considered in the present paper, we mention that the question whether the rank one subspace of a strongly reflexive algebra is weakly dense in the algebra has been settled in the negative. This is shown in the Addendum of [3] using the main example of [16] (see also [14]). Therefore the unit ball of the rank one subspace need not be weakly dense in the unit ball of the algebra. Also, a Banach space example due to M.S. Lambrou [19] shows that for any $r > 1$ one can have weak density of the s-ball of the rank one subspace in the unit ball holding for $s > r$ and failing for $s < r$. This cannot occur in Hilbert space. In fact, ultraweak density of the rank-one subspace $\mathcal{R}$ in any ultraweakly closed subspace X of $\mathcal{B}(\mathcal{H})$ implies (hence is equivalent to) weak density of the unit ball $\mathcal{R}_1$ in the unit ball of X [2]. In this paper, we provide the first Hilbert space example where weak density of the rank-one subspace does not imply weak density of $\mathcal{R}_1$ in X_1. These properties were known to be inequivalent in Banach spaces [19]. In the commutative case a number of positive results are known. For example, the rank one subspace of a strongly reflexive CSL algebra (for definitions, see below) is dense in the algebra ultraweak topology (hence also in the weak topology) [17].

A generalized notion of reflexivity for linear spaces of operators was introduced in [20]. A suitable analogue of a subspace lattice, namely a subspace map, was formulated in [8] and concepts corresponding to reflexivity, commutativity and strong reflexivity were introduced (the basic definitions are given below). Since these objects can be embedded in a natural way in algebras and lattices respectively, it might be expected that they would produce nothing new. Indeed, many results carry over to the new situation. However, the embedding of a strongly reflexive space of operators does not result in a strongly reflexive algebra and so the density of rank one subspaces cannot be deduced from the algebra results mentioned above.

A reflexive space of operators with commutative subspace map (a 'CSM space' in the terminology of [8]) turns out to be a particular case of a left and a right module over maximal abelian selfadjoint algebras; that is, a masa bimodule. Such bimodules have been studied in [4],[7] and [9]. In the present paper, we continue this study. We investigate the rank one subspaces of strongly reflexive masa bimodules or, in other words, spaces of operators with strongly reflexive, commutative subspace maps. We prove that finite rank operators in a norm-closed masa bimodule can be approximated in trace norm by elements of the rank one subspace. This result is new also for algebras. However, it turns out that some density properties for masa bimodules are not the same as for algebras. We show that in the bimodule case the rank one subspace is weakly dense in the whole space but that, in general, density in the ultraweak topology fails. Consequently, weak density of the unit ball of the rank one subspace in the unit ball of the whole space also fails. These results correct Theorem 9.7 of [8]. They are the first results where reflexive subspaces exhibit phenomena that differ significantly from the analogous algebra case.

A number of further results concerning compact and finite rank operators in CSM spaces and CSL algebras are proved. These concern trace norm and Hilbert-Schmidt norm approximation of operators by elements of the rank one subspace. In the process, we prove new representation results for CSM spaces. We introduce a notion of ω-topology and use this to provide criteria for the presence of rank one operators.

We show that if a $\mathcal{C}_2$-closed masa bimodule $\mathcal{M}$ contains a trace class operator T then it must contain rank one operators; specifically, every such T is in the $\mathcal{C}_2$-norm closure of the rank one subspace of $\mathcal{M}$. Consequently the weak closure of any masa bimodule of trace class operators is strongly reflexive. It is shown that many plausible stronger statements cannot hold. For instance a trace class operator of a module need not be in the $\mathcal{C}_1$-norm closure of the rank one subspace.

We provide an example of a CSL algebra which contains no trace class operators yet contains an operator belonging to $\mathcal{C}_p$ for all $p > 1$. From this we show that a transitive bimodule that is spanned by rank one operators need not be dense in $\mathcal{C}_p$ for any p with $1 \leq p < \infty$. Using our result, we settle a question concerning commutative subspace lattices. We show that there exists a commutative subspace lattice $\mathcal{L}$ such that $\mathcal{L}$ is non-synthetic (that is, there exist masa containing, ultraweakly closed algebras with lattice $\mathcal{L}$ which are strictly contained in $\operatorname{Alg}\mathcal{L}$) but every weakly closed algebra which contains a masa and has invariant lattice $\mathcal{L}$ coincides with $\operatorname{Alg}\mathcal{L}$.

The authors are indebted to Dr. D. Salinger for providing the reference [6].

In general we shall use standard terminology and notation as used, for example in [7] and we refer the reader to this text for more detailed background. We denote the set of all bounded linear operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and write $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}=\mathcal{K}$. The usual trace on $\mathcal{B}(\mathcal{H})$ will be denoted by τ. A maximal abelian selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ will be termed a masa.

We briefly review the reflexivity concepts from [20] and [8]. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $\mathcal{P}$ and $\mathcal{Q}$ be the sets of orthogonal projections onto their subspaces. We shall occasionally identify subspaces with the orthogonal projections onto them. Let $\mathcal{S}\subseteq\mathcal{B}(\mathcal{H},\mathcal{K})$. Define $\phi=\operatorname{Map}\mathcal{S}:\mathcal{P}\to\mathcal{Q}$ by

$$\phi(P)=\bigvee\{SP:S\in\mathcal{S}\}.$$

Dually, if $\phi:\mathcal{P}\to\mathcal{Q}$ is any map, define $\operatorname{Op}\phi\subseteq\mathcal{B}(\mathcal{H},\mathcal{K})$ by

$$\operatorname{Op}\phi=\{T\in\mathcal{B}(\mathcal{H},\mathcal{K}):\operatorname{ran}TP\subseteq\operatorname{ran}\phi(P)\text{ for all }P\in\mathcal{P}\}\,.$$

The set $\mathcal{S}$ is said to be *reflexive* if $\mathcal{S}=\operatorname{Op}\operatorname{Map}\mathcal{S}$. Alternatively (as in [20]) define the *reflexive hull* $\operatorname{Ref}\mathcal{S}$ of $\mathcal{S}$ by

$$\operatorname{Ref}\mathcal{S}=\{T:Tx\in\overline{\operatorname{span}_{S\in\mathcal{S}}(Sx)}\text{ for all }x\in\mathcal{H}\}\,.$$

Then it is easy to show that $\operatorname{Ref}\mathcal{S}=\operatorname{Op}\operatorname{Map}\mathcal{S}$ and so $\mathcal{S}$ is reflexive if $\operatorname{Ref}\mathcal{S}=\mathcal{S}$. A set of operators is reflexive if and only if it is the annihilator of some set of rank one operators ([8], Theorem 9.2). For $\mathcal{S}$ to be reflexive it is necessary that it be a linear subspace that is closed in the weak operator topology.

When $\mathcal{H}=\mathcal{K}$ and $\mathcal{S}$ is an algebra containing the identity operator, the range of $\phi=\operatorname{Map}\mathcal{S}$ is a lattice and each element of this lattice is a fixed element of ϕ. The lattice is $\operatorname{Lat}\mathcal{S}$, the set of all subspaces invariant under $\mathcal{S}$ and $\operatorname{Ref}\mathcal{S}=\operatorname{Op}\operatorname{Map}\mathcal{S}=\operatorname{Alg}\operatorname{Lat}\mathcal{S}$. Reflexivity then reduces to the well-known concept originally defined by Halmos in [10].

For algebras, much work has been done for the case when the projections onto the elements of its lattice commute. This is equivalent to saying that the reflexive hull contains a masa. For the more general case of a reflexive subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$, the analogous condition is for there to exist maximal abelian subalgebras $\mathcal{A},\mathcal{B}$ of $\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{K})$ respectively such that $\mathcal{S}$ is a right $\mathcal{A}$ and a left $\mathcal{B}$ module. This condition can be imposed independently of reflexivity and it is such modules that are studied in this paper. We shall refer to them as $(\mathcal{A},\mathcal{B})$-bimodules or, when the masas do not need to be specified, as masa bimodules. Reflexive masa bimodules can also be characterized in terms of their maps: the projections onto the range of the map must commute as must the projections onto the range of the so-called "co-map" (see [8]).

A subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be *strongly reflexive* if for some set $\mathcal{R}$ of rank one operators $\mathcal{S} = \operatorname{Ref} \mathcal{R}$ (or equivalently $\operatorname{Map} \mathcal{S} = \operatorname{Map} \mathcal{R}$). This concept was introduced for algebras by Longstaff in [18] where it was shown that a sufficient condition for $\operatorname{Alg} \mathcal{L}$ to be strongly reflexive is that $\mathcal{L}$ be completely distributive.

2 Approximation of Finite Rank Operators

It is known ([7] Theorem 23.16) that every finite rank operator of a CSL algebra is in the norm closure of the rank one subspace of the algebra. This happens in spite of the fact ([13]) that in general it is not possible to decompose every rank 2 operator of a CSL algebra into a finite sum of rank 1 operators within the algebra. In Theorem 2.2 below, we use essentially the same methods as in [7] and show that the result can be extended in a number of directions: the algebra can be replaced by a norm closed masa bimodule which need not be reflexive and, most significantly, the *norm* closure of the rank 1 subspace can be replaced by the *trace* norm closure. This enables us to apply the result to prove that the rank 1 subspace of a strongly reflexive CSM space is dense in the CSM space in the weak operator topology. This result is known for algebras and follows from the stronger result ([17]) that density holds in the ultraweak topology. However, we shall see later (Theorem 5.3) that ultraweak density can fail for bimodules.

Let $\mathcal{A}$ and $\mathcal{B}$ be masas acting on Hilbert spaces $\mathcal{H}, \mathcal{K}$. We say that a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is *transitive* if the linear span of $\mathcal{S}x = \{Sx : S \in \mathcal{S}\}$ is dense in $\mathcal{K}$ for every non-zero x in $\mathcal{H}$. If $\mathcal{B}\mathcal{S}\mathcal{A}$ is transitive then we say that $\mathcal{S}$ is *transitive relative to* $\mathcal{A}, \mathcal{B}$. Finally we call $\mathcal{S}$ *essentially transitive* relative to $\mathcal{A}$ and $\mathcal{B}$ if there are projections $E \in \mathcal{A}$ and $F \in \mathcal{B}$ such that $\mathcal{S} = F\mathcal{S}E$ and the compression of $\mathcal{S}$ to the ranges of E and F is transitive relative to the restrictions $\mathcal{A}_E$ and $\mathcal{B}_F$ of $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be masas and let X be a finite rank operator. Then either $\{X\}$ is essentially transitive relative to $\mathcal{A}$ and $\mathcal{B}$ or there is a projection $Q \in \mathcal{B}$ such that*

$$0 < \operatorname{rank} QX < \operatorname{rank} X.$$

Proof. Let $X = \sum_{i=1}^n x_i \otimes y_i$, where $\{x_i\}_1^n$ and $\{y_i\}_1^n$ are linearly independent and let the projections onto the closed spans of $\{By_i : B \in \mathcal{B}, i = 1, 2, \cdots, n\}$ and $\{Ax_i : A \in \mathcal{A}, i = 1, 2, \cdots, n\}$ be denoted by $F \in \mathcal{B}$ and $E \in \mathcal{A}$ respectively. If $Q \in \mathcal{B}$ and $QX = \sum_{i=1}^n x_i \otimes Qy_i = 0$ then, since $\{x_i\}_1^n$ are linearly independent, $Qy_i = 0$ for $i = 1, 2, \cdots, n$, so that $QF = 0$. Thus if $0 \neq Q \leq F$ then $QX \neq 0$. Hence either $0 < \operatorname{rank} QX < \operatorname{rank} X$ for some $Q < F$ or $\operatorname{rank} QX = \operatorname{rank} X$ for all $0 \neq Q \leq F$. In the latter case $\{Qy_i\}_1^n$ are linearly independent for any $0 \neq Q \leq F$, and so if also $0 \neq P \leq E$ then $QXP = \sum Px_i \otimes Qy_i \neq 0$. This shows that $\{X\}$ is essentially transitive relative to $\mathcal{A}$ and $\mathcal{B}$. ■

Theorem 2.2 *Every finite rank operator of a norm closed masa bimodule $\mathcal{M}$ is in the trace norm closure of the rank one subspace of $\mathcal{M}$.*

Proof. The proof is by induction on the rank. Let $X \in \mathcal{M}$ have rank n and let $\mathcal{Q}$ be the set of projections in the masa $\mathcal{B}$ such that $\operatorname{rank} QX < n$. Choose a maximal family $\{Q_k\}$ of pairwise orthogonal projections in $\mathcal{Q}$. Note that each $Q_k X$ is in the stated closure by the induction hypothesis. If $Q_0 = \sum Q_k$, then $Q_0 X = \sum_k (Q_k X)$ and the sum converges in the trace norm since X has finite rank. Thus $Q_0 X$ lies in the stated closure.

Let $Y = (I - Q_0)X$. By the maximality of $\{Q_k\}$, it is clear that $\operatorname{rank} QY = n$ whenever $QY \neq 0$. Thus from Lemma 2.1, $\{Y\}$ is essentially transitive. Hence there are projections $E \in \mathcal{A}$ and $F \in \mathcal{B}$ such that $Y = FYE$ and the linear span $\mathcal{C}$ of $\mathcal{B}_F Y \mathcal{A}_E$ is transitive as a subset of $\mathcal{B}(\mathcal{H}_E, \mathcal{K}_F)$. Since $\mathcal{C}$ is a masa bimodule, it follows from [4], Section 2.3 that $\mathcal{C}$ is weak* (that is, ultraweakly) dense in $\mathcal{B}(\mathcal{H}_E, \mathcal{K}_F)$ (see also [8] Corollary 9.6, [7] Theorem 15.9 and [22] Lemma 1.1). Also, $\mathcal{C} \subseteq \mathcal{C}_\infty(\mathcal{H}_E, \mathcal{K}_F)$ (the compact operators from $\mathcal{H}_E$ to $\mathcal{K}_F$) and since $\overline{\mathcal{C}}^{uw} = (\mathcal{C}^\perp)^\perp$, we conclude that $\mathcal{C}^\perp = (0)$. Thus $\mathcal{C}$ is dense in $\mathcal{C}_\infty(\mathcal{H}_E, \mathcal{K}_F)$ in the weak (Banach space) topology and hence norm dense in $\mathcal{C}_\infty(\mathcal{H}_E, \mathcal{K}_F)$. Since $\mathcal{M}$ is norm closed, it follows that $\mathcal{M}$ contains $\mathcal{C}_\infty(\mathcal{H}_E, \mathcal{K}_F)$ and so the finite rank operator Y is a sum of a finite number of rank one operators of $\mathcal{M}$. ■

The above proof is "co-ordinate free"; that is, it is independent of the measure theoretic representation of commutative lattices. The original proof that $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is the only weak* closed transitive masa bimodule of operators from $\mathcal{H}$ to $\mathcal{K}$ used the co-ordinate representation but a co-ordinate free proof was found by the third named author. A brief indication of this proof was given in [24] page 251. We present a complete version in the Appendix to the present paper.

In the special case that $\mathcal{M}$ is reflexive in Theorem 2.2, the last paragraph of the proof is superfluous since in this case it follows that $F\mathcal{M}E$ contains every operator of $\mathcal{B}(\mathcal{H}_\mathcal{E}, \mathcal{K}_\mathcal{F})$ and so Y is the sum of n rank one elements of $\mathcal{M}$. Of course in this case the norm closure condition is also automatically satisfied. This simpler proof of the weaker statement is obviously co-ordinate free.

Our result, Theorem 2.2 above, strengthens Proposition 1.9 of [22] which states that every operator of rank 2 of a CSL algebra is in the trace norm closure of its rank one subspace.

We shall show in due course (Corollary 5.4) that Theorem 2.2 cannot be extended to all trace class operators in $\mathcal{M}$. Similarly, the fact that $\mathcal{M}$ is norm closed is essential and cannot be replaced by trace norm closure. Indeed, in providing the example of a non-reflexive operator algebra containing a masa, that is, where $\mathcal{A}_{\min} \neq \mathcal{A}_{\max}$ (see [4] Section 2.5, particularly Proposition 2.5.5), Arveson constructs an operator of rank 10 (call it R) which annihilates $\mathcal{A}_{\min}$ but not $\mathcal{A}_{\max}$. Since $\mathrm{Ref}(\mathcal{A}_{\min}) = \mathcal{A}_{\max}$, the same set of rank one operators annihilate $\mathcal{A}_{\min}$ and $\mathcal{A}_{\max}$; this shows that in the pre-annihilator of $\mathcal{A}_{\min}$ the finite rank operator R is not in the trace norm closed span of the rank one operators in that module.

If every transitive masa bimodule of finite rank operators were trace norm dense, the proof of Theorem 2.2 would hold for trace norm closed modules, contradicting the remark above. Thus we have the following result.

Corollary 2.3 *There exists a transitive masa bimodule of finite rank operators that is not dense with the trace norm in the set of all trace class operators.*

We shall prove a more general result in due course (Theorem 6.10): there exists a transitive bimodule of finite rank operators that is not dense in $\mathcal{C}_p$ in $\mathcal{C}_p$ norm for any $1 \leq p < \infty$.

Theorem 2.4 *If a masa bimodule consists of trace class operators then its closure in the weak operator topology coincides with its reflexive hull.*

Proof. Let $\mathcal{M}_1 \subseteq \mathcal{C}_1(\mathcal{H}, \mathcal{K})$ be a masa bimodule consisting of trace class operators. The set $\mathcal{T} = \{T \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \tau(TM) = 0 \text{ for all } M \in \mathcal{M}_1\}$ is clearly a norm closed masa bimodule. The reflexive hull $\mathcal{M}$ of $\mathcal{M}_1$ is the annihilator of the rank one elements of $\mathcal{T}$. However, the wot-continuous linear functionals annihilating $\mathcal{M}_1$ are of the form $X \mapsto \tau(TX)$ where T is a finite rank operator of $\mathcal{T}$. From Theorem 2.2, T is in the trace norm closure of the rank one subspace of $\mathcal{T}$ and so each such linear functional also annihilates $\mathcal{M}$. Thus $\mathcal{M}_1$ is wot-dense in $\mathcal{M}$. ■

The following corollary which is the positive solution of the rank one density problem for CSM spaces in the weak operator topology is an immediate consequence.

Corollary 2.5 *The rank one subspace of a strongly reflexive CSM space $\mathcal{M}$ is dense in $\mathcal{M}$ in the weak operator topology.*

Proof. Apply the theorem to the rank one subspace of $\mathcal{M}$. ■

Corollary 2.6 *If a masa bimodule consists of finite rank operators then its closure in the weak operator topology is strongly reflexive.*

Proof. Let $\mathcal{R}$ be a masa bimodule consisting of finite rank operators and let $\mathcal{M}$ be its closure in the weak operator topology. Then from the theorem, $\mathcal{M} = \operatorname{Ref} \mathcal{R}$. But, from Theorem 2.2 each member of $\mathcal{R}$ is in the trace norm closure of the rank one subspace $\mathcal{M}_0$ of $\mathcal{M}$. Hence $\operatorname{Ref} \mathcal{M}_0 = \operatorname{Ref} \mathcal{R} = \mathcal{M}$. ■

In due course we shall prove a somewhat stronger result of this type (Corollary 6.9).

3 Omega Topology

In the remainder of this paper we shall, for the most part, consider (separable) Hilbert spaces concretely represented as $\mathcal{H} = L^2(X, \mu)$ and $\mathcal{K} = L^2(Y, \nu)$. This is the broad context of Arveson's paper [4] which is the foundation of this area. However, in that paper the compact metric spaces X, Y with regular Borel measures μ, ν are dependent on the one particular algebra (or module) under consideration. The notion of support is defined so as to be a closed set and therefore depends on the particular topologies. Consequently the connection between general reflexive masa bimodules and their supports is somewhat loose; specifically the support of a reflexive masa bimodule does not, in any way, determine the set. To illustrate with a trivial example, let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a continuous masa and let E be a projection of $\mathcal{A}$. Then $\mathcal{M} = \{A + EXE : A \in \mathcal{A}, X \in \mathcal{B}(\mathcal{H})\}$ is reflexive (in fact it is a von Neumann algebra). For any subset α of $[0, 1]$ whose measure is neither 0 or 1, a representation of $\mathcal{H}$ as $L^2[0, 1]$ can be chosen so that $E = E(\alpha)$. But for different choices of α the support of $\mathcal{M}$ is very different. For example, if $\alpha = [\frac{1}{2}, 1]$ then the support of $\mathcal{M}$ is $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \cup D$ where $D = \{(x, x) : x \in [0, 1]\}$. However, we may take α to be a set such that both α and its complement intersect every open interval in a set of non-zero measure. In this case the support of $\mathcal{M}$ is the whole unit square and then both $\mathcal{M}$ and $\mathcal{B}(\mathcal{H})$ have the same support.

In the present paper, we define a new notion of support so that there is a bijection between masa bimodules and a certain class of subsets. This is accomplished in the representations theorems 4.2 and 4.3. The approach we adopt is measure theoretic rather than topological and is essentially a countable version of topology. This is better suited for interaction with measure theory. The technical tools developed in this section, in particular Lemma 3.4, are essential for establishing the representation theorems. All the subsequent developments in the paper follow from these.

It has recently come to the authors' attention that Beggs in [5] has used a similar countable version of topology in order to obtain a generalized version of the Riesz representation theorem.

An *ω-topological space* is a set X and a family $\mathcal{G}$ of subsets which contains $\emptyset$ and X and is closed with respect to finite intersections and countable unions. We call the members of $\mathcal{G}$ *ω-open* sets. The complements of ω-open sets are called *ω-closed*.

Any topological space can be considered as ω-topological. For our purposes, a more important example is a measure space (X, μ) where the set $\mathcal{G}$ is the set of μ-measurable sets.

Given two ω-topological spaces $(X_1, \mathcal{G}_1)$, $(X_2, \mathcal{G}_2)$ we define the direct product as the ω-topological space $(X_1 \times X_2, \mathcal{G})$ where $\mathcal{G}$ consists of all countable unions of sets of the form $A \times B$ where $A \in \mathcal{G}_1, B \in \mathcal{G}_2$. Analogously to continuous maps, we define ω-continuous maps.

Lemma 3.1 *If $(Y_1, \mathcal{G}_1), (Y_2, \mathcal{G}_2)$ and $(X, \mathcal{G})$ are ω-topological spaces and the maps $f_i : X \to Y_i$ are ω-continuous, then $f_1 \times f_2 : X \to Y_1 \times Y_2$ defined by $(f_1 \times f_2)(x) = (f_1(x), f_2(x))$ is ω-continuous.*

Proof. Trivial from $(f_1 \times f_2)^{-1}(\cup_{i=1}^{\infty} A_i \times B_i) = \cup_{i=1}^{\infty}(f_1^{-1}(A_i) \cap f_2^{-1}(B_i))$. ■

Corollary 3.2 *Let f_1, f_2 be ω-continuous functions from an ω-topological space X to $\mathbb{C}$ (or any topological ring). Then $f_1 + f_2$ and $f_1 f_2$ are ω-continuous.*

Proof. Each of these functions is a composition of $f_1 \times f_2$ with a continuous map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$. ■

Lemma 3.3 *Let X be an ω-topological space and let $f_n : X \to \mathbb{C}$ and $\phi_n : X \to \mathbb{R}^+$ be ω-continuous functions with $\inf_n \phi_n(x) = 0$ for each $x \in X$. If $f : X \to \mathbb{C}$ satisfies*

$$|f_n(x) - f(x)| \leq \phi_n(x)$$

then f is ω-continuous.

Proof. It is enough to assume that f_n and f are real-valued. For any real a, if $f(x) > a$ choose $r \in \mathbb{N}$ such that $r > \frac{2}{f(x)-a}$ and then choose $n \in \mathbb{N}$ with $\phi_n(x) < \frac{1}{r}$. It follows that

$$f^{-1}\left((a, \infty)\right) = \bigcup_{r,n=1}^{\infty} \left(f_n^{-1}\left((a + \frac{1}{r}, \infty) \right) \cap \phi_n^{-1}\left((0, \frac{1}{r}) \right) \right).$$

Similarly one shows that $f^{-1}\left((-\infty, a)\right)$ is ω-open and so f is ω-continuous. ■

Given two measure spaces $(X, \mu), (Y, \nu)$, recall that a subset ζ of $X \times Y$ is said to be *marginally null* if $\zeta \subseteq (\chi \times Y) \cup (X \times \psi)$ where $\mu(\chi) = 0 = \nu(\psi)$. If ζ and η are marginally equivalent (i.e. their symmetric difference is marginally null) we write $\zeta \cong \eta$. We define an ω-topology on $X \times Y$ such that the ω-open sets are, modulo marginally null sets, countable unions of measurable rectangles; that is sets δ satisfying $\delta \cong \cup_{i=1}^{\infty} \alpha_i \times \beta_i$ for measurable sets α_i, β_i.

Definition. We shall call a finite measure m on a set X a *standard* measure if there exists a topology on X that makes X a compact space and m a regular Borel measure.

Note that there is no need to fix a particular topology on X in order to satisfy this condition for a measure m; mere existence of such a topology is sufficient. It is clear that in the "co-ordinate" representations mentioned at the beginning of this section, the measures are standard. The only places where the condition of standardness is used is in the following lemma and in Theorem 4.3.

Lemma 3.4 *Let μ and ν be standard measures on X and Y respectively. Suppose $\{\gamma_n\}$ are ω-open sets and κ is an ω-closed subset of $\bigcup_{n=1}^{\infty} \gamma_n$. Then for any $\epsilon > 0$ there exist $X_\epsilon \subset X,\ Y_\epsilon \subset Y$ with $\mu(X \setminus X_\epsilon) < \epsilon,\ \nu(Y \setminus Y_\epsilon) < \epsilon$ such that the set $\kappa_\epsilon = \kappa \cap (X_\epsilon \times Y_\epsilon)$ is contained in the union of a finite subset of $\{\gamma_n\}$.*

Proof. Since ω-open sets are marginally equivalent to countable unions of rectangles, we may assume that the sets γ_n are themselves rectangles. Adding to $\{\gamma_n\}$ those rectangles whose union is κ^c (up to marginal equivalence), we reduce the general case to the case $\kappa = X \times Y$. Thus we should prove that if

$$X \times Y = \bigcup_{n=1}^{\infty} \alpha_n \times \beta_n,$$

then, modulo subsets of X and Y with measure less than ϵ, we can take a finite subcovering.

Now we use the condition that X and Y are compact spaces and μ, ν are regular measures. Let $\tilde{\alpha}_n,\ \tilde{\beta}_n$ be open subsets in X, Y with $\alpha_n \subset \tilde{\alpha}_n, \beta_n \subset \tilde{\beta}_n$, such that $\mu(\tilde{\alpha}_n \backslash \alpha_n) < \frac{\epsilon}{2^n},\ \nu(\tilde{\beta}_n \backslash \beta_n) < \frac{\epsilon}{2^n}$. Then $\{\tilde{\alpha}_n \times \tilde{\beta}_n\}_{n=1}^{\infty}$ is an open covering of $X \times Y$; taking a finite subcovering, we obtain

$$X \times Y \subset \bigcup_{n=1}^{N} \tilde{\alpha}_n \times \tilde{\beta}_n$$

Now just put

$$X_\epsilon = X \setminus \bigcup_{n=1}^{N} (\tilde{\alpha}_n \backslash \alpha_n) \qquad Y_\epsilon = Y \setminus \bigcup_{n=1}^{N} (\tilde{\beta}_n \backslash \beta_n)$$

and notice that $\mu(X \setminus X_\epsilon) < \epsilon,\ \nu(Y \setminus Y_\epsilon) < \epsilon$. ■

4 Supports and Reflexivity

We investigate masa bimodules $\mathcal{M}$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$ with the (separable) Hilbert spaces represented as $\mathcal{H} = L^2(X,\mu)$ and $\mathcal{K} = L^2(Y,\nu)$ for standard measures μ,ν in such a way that $\mathcal{M}$ is an $(\mathcal{A},\mathcal{B})$-bimodule where the masas $\mathcal{A}$ and $\mathcal{B}$ are $L^\infty(X,\mu)$ and $L^\infty(Y,\nu)$. We denote the projections in $\mathcal{A}$ and $\mathcal{B}$ corresponding to multiplication by the characteristic functions of α,β by $E(\alpha), F(\beta)$ respectively. Let $\mathcal{P}$ and $\mathcal{Q}$ denote the set of all projections on $\mathcal{H}$ and $\mathcal{K}$ respectively. In this section (for all but the last result, Theorem 4.6) we consider the representation as fixed and study the properties of all $(\mathcal{A},\mathcal{B})$-bimodules simultaneously. This is in contrast to the work in [4] and its subsequent developments (see [7]) where the representation is chosen so as to give a specific algebra or bimodule a particular structure in the representation.

Let κ be any subset of $X \times Y$. We say that $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is *supported* by κ if $F(\beta)TE(\alpha) = 0$ for each Borel rectangle $\alpha \times \beta$ such that $(\alpha \times \beta) \cap \kappa \cong \emptyset$.

We define $\mathcal{M}_{\max}(\kappa)$ by

$$\mathcal{M}_{\max}(\kappa) = \{T \in \mathcal{B}(\mathcal{H},\mathcal{K}) : T \text{ is supported by } \kappa\}.$$

Note that in the case when X and Y are topological spaces and κ is closed, the definition of $\mathcal{M}_{\max}(\kappa)$ is equivalent to Arveson's given originally in [4].

Theorem 4.1 *For any subset κ of $X \times Y$, the set $\mathcal{M}_{\max}(\kappa)$ of operators is a reflexive masa bimodule.*

Proof. It is clear that $\mathcal{M}_{\max}(\kappa)$ is a masa bimodule. To prove reflexivity, we exhibit a set $\mathcal{R} \subseteq \mathcal{B}(\mathcal{K},\mathcal{H})$ consisting of rank one operators such that $\mathcal{M}_{\max}(\kappa) = \mathcal{R}^\perp$. Let

$$\mathcal{R} = \{E(\alpha)RF(\beta) : \operatorname{rank} R = 1, \alpha \times \beta \text{ Borel }, \alpha \times \beta \subseteq \kappa^c\}.$$

If $R = E(\alpha)RF(\beta)$ is in $\mathcal{R}$ then for all M in $\mathcal{M}_{\max}(\kappa)$

$$\tau(RM) = \tau(E(\alpha)RF(\beta)M) = \tau(RF(\beta)ME(\alpha)) = 0$$

since $F(\beta)ME(\alpha) = 0$. Thus $\mathcal{M}_{\max}(\kappa)$ annihilates $\mathcal{R}$. Conversely if T annihilates $\mathcal{R}$ then for every Borel rectangle $\alpha \times \beta$ disjoint from κ and every rank one operator R,

$$\tau(F(\beta)TE(\alpha)R) = \tau(TE(\alpha)RF(\beta)) = 0$$

and so $F(\beta)TE(\alpha) = 0$. This shows that T is supported by κ. ■

The map ϕ of the reflexive bimodule $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$ can be found directly from the set κ. It is uniquely determined by

$$\phi(E(\alpha))^{\perp} = \bigvee\{F(\beta) : (\alpha \times \beta) \cap \kappa \cong \emptyset\}$$

for every Borel set $\alpha \subseteq X$.

Theorem 4.2 *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a reflexive $(\mathcal{A}, \mathcal{B})$-bimodule. Then there exists an ω-closed set $\kappa \subseteq X \times Y$ such that $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$.*

Proof. Since $\mathcal{M}$ is reflexive, $\mathcal{M} = \mathcal{S}^{\perp}$ for some set $\mathcal{S}$ of rank one operators. Let $\mathcal{S}_0 = \{S_n : n = 1, 2, \ldots\}$ be a countable trace norm dense subset of $\mathcal{S}$ and, for each n, let $F_n \in \mathcal{B}$ and $E_n \in \mathcal{A}$ be projections that are minimal subject to $S_n = E_n S_n$ and $S_n = S_n F_n$. Now $T \in \mathcal{M}$ if and only if $\tau(TS_n) = 0$ for all n. Writing $S_n = x_n \otimes y_n$ we have that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\tau(B^* TAS_n) = \langle F_n T E_n \, A x_n, B y_n\rangle$$

and as $\{Ax_n : A \in \mathcal{A}\}$ and $\{By_n : B \in \mathcal{B}\}$ are dense in the ranges of E_n and F_n it follows easily that $T \in \mathcal{M}$ if and only if $F_n T E_n = 0$ for all n. Let $\alpha_n \subseteq X$ and $\beta_n \subseteq Y$ be Borel sets such that $E_n = E(\alpha_n), F_n = F(\beta_n)$. Define

$$\kappa = [\cup_n(\alpha_n \times \beta_n)]^c \ .$$

Clearly κ is ω-closed. If an operator T is supported by κ then since $(\alpha_n \times \beta_n) \cap \kappa = \emptyset$ we have $F(\beta_n)TE(\alpha_n) = 0$ for each n and so $T \in \mathcal{M}$. Hence $\mathcal{M}_{\max}(\kappa) \subseteq \mathcal{M}$.

For the reverse inclusion, it suffices to show that if $\alpha \times \beta$ is a Borel rectangle with $(\alpha \times \beta) \cap \kappa \cong \emptyset$ then $F(\beta)T(E(\alpha)) = 0$ for all $T \in \mathcal{M}$. For any projection $P \in \mathcal{A}$ we write $\phi(P)$ for the projection (in $\mathcal{B}$) onto the closure of $\{MPx : M \in \mathcal{M}, x \in \mathcal{H}\}$, (that is, $\phi = \operatorname{Map} \mathcal{M}$). Thus we need to show that $\phi(E(\alpha)) \leq F(\beta^c)$.

We first establish this in the case when $\alpha \times \beta$ is a subset of the finite union $\bigcup_{n=1}^{N}(\alpha_n \times \beta_n)$. The argument is by induction on N (the case $N = 1$ being obvious from the choice of E_1, F_1). Then

$$\begin{aligned}(\alpha \setminus \alpha_1) \times \beta \ &\subseteq \ \bigcup_{n=2}^{N}(\alpha_n \times \beta_n) \\ (\alpha \cap \alpha_1) \times (\beta \cap \beta_1^c) \ &\subseteq \ \bigcup_{n=2}^{N}(\alpha_n \times \beta_n)\end{aligned}$$

and so, by the induction hypothesis,

$$\begin{aligned}\phi(E(\alpha\setminus\alpha_1)) &\leq F(\beta^c)\\ \phi(E(\alpha\cap\alpha_1)) &\leq F((\beta\cap\beta_1^c)^c)=F(\beta^c\cup\beta_1)\,.\end{aligned}$$

The second equation, together with $\phi(E(\alpha_1))\leq F(\beta_1^c)$ shows that $\phi(E(\alpha\cap\alpha_1))\leq F(\beta^c)$ and the assertion follows.

For the general case, deleting null sets from α and β if necessary, we may assume that $\alpha\times\beta\subseteq\kappa^c=\bigcup_{n=1}^{\infty}(\alpha_n\times\beta_n)$. Now, given $\epsilon>0$ we can apply Lemma 3.4 to find an integer N and sets $\alpha_\epsilon,\beta_\epsilon$ such that $\mu(\alpha\setminus\alpha_\epsilon)<\epsilon$, $\nu(\beta\setminus\beta_\epsilon)<\epsilon$ and

$$\alpha_\epsilon\times\beta_\epsilon\subseteq\bigcup_{n=1}^{N}(\alpha_n\times\beta_n)\,.$$

The result of the previous paragraph shows that $\phi(E(\alpha_\epsilon))\leq F(\beta_\epsilon^c)$. Thus $F(\beta_\epsilon)TE(\alpha_\epsilon)=0$ for $T\in\mathcal{M}$ and the result follows, since $E(\alpha_\epsilon)\to E(\alpha)$ and $F(\beta_\epsilon)\to F(\beta)$ strongly as ϵ tends to 0. ■

The condition of standardness is used implicitly in the theorem below since it is required for Arveson's null set theorem. Note that a more general version of Arveson's theorem was obtained in [12].

Theorem 4.3 *Let $\mathcal{M}\subseteq\mathcal{B}(\mathcal{H},\mathcal{K})$ be a reflexive $(\mathcal{A},\mathcal{B})$-bimodule. For any representation of $\mathcal{H}$ and $\mathcal{K}$ as $L^2(X,\mu)$ and $L^2(Y,\nu)$ respectively, the ω-closed set κ such that $\mathcal{M}=\mathcal{M}_{\max}(\kappa)$ is uniquely determined by $\mathcal{M}$ up to marginal equivalence and is the smallest ω-closed set supporting $\mathcal{M}$.*

Proof. We first show that if $F(\beta)\mathcal{M}E(\alpha)=\{0\}$, then the set $\sigma=(\alpha\times\beta)\cap\kappa$ is marginally null. Indeed if T is supported by σ, then on the one hand T is supported by κ, so $T\in\mathcal{M}$. On the other hand T is supported by $\alpha\times\beta$ and so, since $(X\times\beta^c)\cap(\alpha\times\beta)=\emptyset$, we have $F(\beta^c)TE(X)=0$. So $T=F(\beta)T$ and similarly $T=TE(\alpha)$. Therefore $T=F(\beta)TE(\alpha)$ and hence $T=0$. Using the fact that σ^c is a countable union of (Borel) rectangles, it follows from Arveson's Null Set Theorem ([4] Theorem 1.4.3) that σ is marginally null (otherwise it would support a nonzero pseudointegral operator).

Now if $\kappa'\subseteq X\times Y$ is an ω-closed set which supports $\mathcal{M}=\mathcal{M}_{\max}(\kappa)$, then $\mathcal{M}_{\max}(\kappa)\subseteq\mathcal{M}_{\max}(\kappa')$. Then if $\alpha\times\beta$ is a Borel rectangle disjoint from κ' we will have $F(\beta)TE(\alpha)=0$ for all $T\in\mathcal{M}_{\max}(\kappa')$, hence for all $T\in\mathcal{M}_{\max}(\kappa)$. From above, we now have $(\alpha\times\beta)\cap\kappa\cong\emptyset$. Since κ' is ω-closed, its complement is (up to marginally null sets) a countable union of rectangles. It follows that

$\kappa \cap \kappa'^c \cong \emptyset$. Now if $\mathcal{M}_{\max}(\kappa) = \mathcal{M}_{\max}(\kappa')$, then by symmetry we have that $\kappa \cong \kappa'$, thus proving uniqueness. ■

For the definition that follows, note that if a masa bimodule $\mathcal{S}$ is supported by a set κ, then so is $\mathrm{Ref}(\mathcal{S})$. Indeed, if $\alpha \times \beta$ is a Borel rectangle disjoint from κ then for any rank one $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ the operator $E(\alpha)RF(\beta)$ will annihilate $\mathcal{S}$ and hence also $\mathrm{Ref}(\mathcal{S})$. Thus $F(\beta)(\mathrm{Ref}(\mathcal{S}))E(\alpha) = \{0\}$.

Definition. Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be any set, and let $\mathcal{M} = \mathrm{Ref}(\mathcal{BSA})$. The ω-closed set κ satisfying $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$ constructed in Theorem 4.2 is called the ω-*support* of $\mathcal{S}$ and denoted by $\mathrm{supp}_\omega \mathcal{S}$.

A disadvantage of ω-topologies is the fact that in general there is no notion of closure. The following result shows that in the ω-topology on the product of two standard measure spaces a version of closure modulo marginally null sets can be defined.

Lemma 4.4 *Given any subset $\sigma \subseteq X \times Y$, there exists an ω-closed set κ containing σ such that for any ω-closed set ρ containing σ, the set $\kappa \setminus \rho$ is marginally null.*

Proof. Let $\mathcal{M} = \mathcal{M}_{\max}(\sigma)$. Then, from Theorem 4.1 $\mathcal{M}$ is reflexive and, from Theorem 4.2 and its proof, $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$ where κ is the complement of a countable union of Borel rectangles disjoint from σ. Thus $\sigma \subseteq \kappa$ and, if ρ is any ω-closed set containing σ, $\mathcal{M}_{\max}(\kappa) = \mathcal{M}_{\max}(\sigma) \subseteq \mathcal{M}_{\max}(\rho)$. Thus it follows from Theorem 4.3 that, modulo marginally null sets, $\kappa \subseteq \rho$. The set κ is clearly unique modulo marginally null sets. ■

In the sequel we shall usually identify sets that are marginally equivalent. The set found (uniquely up to marginal equivalence) in the above lemma is called the ω-closure of σ and is denoted by $\mathrm{cl}_\omega \sigma$. It follows from the definition that an ω-closed set κ is an ω-closure of a set γ if and only if, whenever $\rho \cap \kappa$ is not marginally null for a rectangle ρ then $\rho \cap \gamma$ is also not marginally null.

The definition of ω-closure enables to define the ω-interior of a set in the usual way, namely as the complement of the ω-closure of the complement. Clearly the ω-interior of κ contains all ω-open subsets of κ.

Lemma 4.5 *For an ω-closed set κ the following conditions are equivalent.*
(a) κ is the ω-closure of some ω-open set,
(b) κ is the ω-closure of its ω-interior,
(c) for any rectangle ρ, either $\kappa \cap \rho$ contains a non-null rectangle or it is marginally null.

Proof. That (b) implies (a) is clear. To show that (a) implies (c), let κ be the ω-closure of $\gamma = \cup_n \gamma_n$, where the γ_n are all rectangles. If $\kappa \cap \rho$ contains no non-null rectangles then $\rho \cap \gamma_n$ are marginally null. It follows that $\rho \cap \gamma$ is marginally null and hence $\rho \cap \kappa$ is marginally null.

For (c) implies (a), let $\gamma = \mathrm{int}_\omega\, \kappa$. If κ is not the ω-closure of γ then there exists a rectangle ρ such that $\rho \cap \kappa$ is not marginally null but $\rho \cap \gamma$ is marginally null. Then (c) implies that $\rho \cap \kappa$ contains a non-null rectangle. But this is impossible since any subrectangle of κ is contained in γ. ■

For algebras, the result below is a consequence of Arveson's representation of a commutative subspace lattice; the extension to modules does not involve anything new. However, we formulate it explicitly in order to highlight the connection between this paper and the work in [4] and [7].

Theorem 4.6 *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a reflexive masa bimodule. Then there exist compact metric spaces X, Y with regular Borel measures μ, ν and a* closed *subset $\kappa \subseteq X \times Y$ such that, up to unitary equivalence, $\mathcal{H} = L^2(X, \mu), \mathcal{K} = L^2(Y, \mu)$ and $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$.*

Proof. Let $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule where $\mathcal{A}$ and $\mathcal{B}$ are masas. Choose, as in the proof of Theorem 4.2, two countable families $\{N_n\} \subseteq \mathcal{A}$, $\{M_n\} \subseteq \mathcal{B}$ of projections such that $T \in \mathcal{M}$ if and only if $M_n^{\perp} T N_n = 0$ for all $n \in \mathbb{N}$. Let $\{E_n\}, \{F_n\}$ be sequences of projections in $\mathcal{A}, \mathcal{B}$ respectively which are strongly dense in the projection lattices of the masas. Let $\mathcal{A}_0, \mathcal{B}_0$ be the C*-algebras generated by N_n, E_n and M_n, F_n; these are separable C*-algebras which are strongly dense in the masas. If X and Y denote the spectra of $\mathcal{A}_0$ and $\mathcal{B}_0$ respectively, then it is well known that there are regular Borel measures μ, ν on X, Y such that $\mathcal{A}, \mathcal{B}$ are unitarily equivalent to the multiplication algebras of $L^\infty(X, \mu)$ and $L^\infty(Y, \nu)$ acting on $L^2(X, \mu)$ and $L^2(Y, \nu)$. In the sequel we assume that the unitary transformation has been carried out.

The elements of $\mathcal{A}_0, \mathcal{B}_0$ are mapped onto (multiplication by) continuous functions and so the projections $\{N_n\}$ and $\{M_n\}$ are of the form $N_n = E(\alpha_n)$ and $M_n = F(\beta_n^c)$ where $(\alpha_n), (\beta_n)$ are both open and closed. It follows that κ, defined by
$$\kappa^c = \bigcup_n (\alpha_n \times \beta_n).$$
is a closed set. The fact that $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$ follows from Theorem 4.2. However, in this situation there is an easier argument. Indeed if $T \in \mathcal{M}$ then $F(\beta_n) T E(\alpha_n) = 0$ and hence the open rectangles $\alpha_n \times \beta_n$ are disjoint

from the closed support of T for all n. Therefore their union κ^c is also disjoint from the support of T. It follows that if $\alpha \times \beta$ is an *open* rectangle disjoint from κ then $F(\beta)TE(\alpha) = 0$. Now a regularity argument (see e.g. [4] proposition 2.2.5) shows that the same holds for any *Borel* rectangle disjoint from κ. The converse is easy. ■

5 Strong Reflexivity and Failure of Weak*-density.

In this section we characterize strongly reflexive bimodules in terms of their ω-supports. We then show that for CSM spaces the rank one density problem for the ultraweak topology has a negative solution. This should be compared with the positive result of [17] for CSL algebras. It shows that the structure of masa bimodules differs significantly from that of unital algebras of the analogous type. As a consequence of our example we show the existence of a certain type of non-synthetic commutative subspace lattice.

Lemma 5.1 *Let $\mathcal{R}$ be a set of rank one operators. Then the ω-support of $\mathcal{R}$ is the ω-closure of an ω-open set.*

Proof. Let $\{R_n = x_n \otimes y_n : n = 1, 2, \cdots\}$ be a dense subset of $\mathcal{R}$. Define $E_n = \bigwedge\{E \in \mathcal{A} : Ex_n = x_n\}$ and let $\alpha_n \subseteq X$ satisfy $E(\alpha_n) = E_n$. Define $\beta_n \subseteq Y$ in an analogous way. Let

$$\kappa = \mathrm{cl}_\omega[\bigcup_{n=1}^{\infty} \alpha_n \times \beta_n]\,.$$

Clearly κ is the ω-closure of an ω-open set and supports $\mathcal{R}$.

Suppose σ is ω-closed and supports $\mathcal{R}$. Then if $\gamma \times \delta \subseteq \sigma^c$, we have that $F(\delta)RE(\gamma) = 0$ for all $R \in \mathcal{R}$. It follows that for each n, $F(\delta \cap \beta_n)R_nE(\gamma \cap \alpha_n) = 0$ and therefore, for each n, either $\gamma \cap \alpha_n$ or $\delta \cap \beta_n$ is null. Thus (always working modulo marginally null sets) we have $(\gamma \times \delta) \cap \bigcup_n(\alpha_n \times \beta_n) = \emptyset$ and so $\gamma \times \delta \subseteq \kappa^c$. This shows that $\sigma \supseteq \kappa$ and so κ is the ω-support of $\mathcal{R}$. ■

Theorem 5.2 *For an ω-closed set κ, the bimodule $\mathcal{M}_{\max}(\kappa)$ is strongly reflexive if and only if κ is the ω-closure of an ω-open set.*

Proof. If $\mathcal{M}_{\max}(\kappa)$ is strongly reflexive then $\mathcal{M}_{\max}(\kappa) = \mathrm{Ref}(\mathcal{R})$ where $\mathcal{R}$ is a set of rank one operators. Then, since κ is ω-closed, it is the ω-support of $\mathcal{R}$ and so, by Lemma 5.1, it is the ω-closure of an ω-open set.

Conversely, suppose that $\mathcal{M}$ is a reflexive bimodule of the form $\mathcal{M}_{\max}(\kappa)$ where $\kappa = \mathrm{cl}_\omega(\cup_n(\gamma_n \times \delta_n))$. Write $\kappa_0 = \cup_n(\gamma_n \times \delta_n)$. ¿From Theorem 4.3, κ is the smallest ω-closed set supporting $\mathcal{M}_{\max}(\kappa)$, and so $\mathcal{M}_{\max}(\kappa_0) = \mathcal{M}_{\max}(\kappa)$. Define

$$\mathcal{S} = \{F(\delta_n)SE(\gamma_n) : \mathrm{rank}\, S = 1, n \in \mathbb{N}\}.$$

It is clear that $\mathcal{S}$ is supported by κ_0, so $\mathcal{S} \subseteq \mathcal{M}_{\max}(\kappa_0)$, and it suffices to prove that any rank one operator annihilating $\mathcal{S}$ also annihilates $\mathcal{M}_{\max}(\kappa_0)$. Now suppose $E(\alpha)RF(\beta)$ annihilates $\mathcal{S}$. Then it follows as before that each $(\alpha \times \beta) \cap (\gamma_n \times \delta_n)$ is marginally null and hence so is $(\alpha \times \beta) \cap \kappa_0$. Therefore $E(\alpha)RF(\beta)$ annihilates $\mathcal{M}_{\max}(\kappa_0)$ as required. ■

Theorem 5.3 *There exists a strongly reflexive masa bimodule such that the linear span of its rank one operators is not ultraweakly dense in the module.*

Proof. Let γ be a subset of the unit circle $\mathbb{T}$ that is the closure of its interior and let $\kappa = \{(t,s) : t - s \in \gamma\}$ and $\lambda = \{(t,s) : t - s \in \operatorname{int}\gamma\}$. Then it is easy to see that λ is open and $\overline{\lambda} = \kappa$.

Consider $\mathbb{T} \times \mathbb{T}$ with the ω-topology derived from the usual Borel structure on each component. We show that the ω-closure of λ is the same as the topological closure. For this we show that if $\alpha \subseteq \mathbb{T}, \beta \subseteq \mathbb{T}$ are Borel sets with $(\alpha \times \beta) \cap \lambda \cong \emptyset$, then $(\alpha \times \beta) \cap \kappa \cong \emptyset$. Deleting null sets from α, β if necessary, we may assume that $(\alpha \times \beta) \cap \lambda = \emptyset$; and deleting further null sets from α, β, we may assume that each point of α and β is a point of density. It is an exercise in measure theory (see e.g. [23] Chapter 8, Ex 12 and 13) to show that the set $\beta - \alpha$ is open. If $(a,b) \in (\alpha \times \beta) \cap \kappa$, then $b - a \in \gamma$. Thus $\beta - \alpha$ meets γ and, since it is open, it meets $\operatorname{int}\gamma$. Hence there are $x \in \alpha, y \in \beta$ such that $y - x \in \operatorname{int}\gamma$, that is $(x,y) \in (\alpha \times \beta) \cap \lambda$, contradicting that $(\alpha \times \beta) \cap \lambda = \emptyset$.

Thus κ is the ω-closure of λ and, since open sets are ω-open, it follows from Theorem 5.2 that $\mathcal{M}_{\max}(\kappa)$ is strongly reflexive. We now take γ to be a set which is not a set of spectral synthesis and is the closure of its interior. It is the result of [6] that such a set exists. It is shown in [9] (Theorem 2.1.7) that, with the above construction, the fact that γ is not a set of spectral synthesis implies that there exists an ultraweakly closed module $\mathcal{M}$ with support κ such that $\mathcal{M}$ is strictly contained in $\mathcal{M}_{\max}(\kappa)$. But then, for any rectangle $\xi \times \eta \subseteq \kappa$ the module $F(\eta)\mathcal{M}E(\xi)$ is essentially transitive and so contains all operators supported on the rectangle. Since all rank one operators of $\mathcal{M}_{\max}(\kappa)$ are clearly of this form, it follows that $\mathcal{M}$ contains the ultraweak closure of the rank one subspace of $\mathcal{M}_{\max}(\kappa)$. ■

Corollary 5.4 *There exists a strongly reflexive masa bimodule containing a trace class operator which is not the limit in $\mathcal{C}_1$ norm of finite rank operators in the module.*

Proof. Let $\mathcal{T}$ be the bimodule constructed in the previous theorem and let $\mathcal{S}$ be the annihilator of the rank one subspace $\mathcal{T}_1$ of $\mathcal{T}$. Then $\mathcal{S}$ is a reflexive masa bimodule. Since $\mathcal{T}$ is reflexive, it coincides with the annihilator of the rank one subspace $\mathcal{S}_1$ of $\mathcal{S}$. Now $\mathcal{S}$ and $\mathcal{S}_1$ are annihilated by the same set of rank one operators, namely $\mathcal{T}_1$, and hence $\mathcal{S} = \operatorname{Ref} \mathcal{S}_1$ is strongly reflexive.

Since $\mathcal{T}_1$ is not ultraweakly dense in $\mathcal{T}$, there exists a trace class operator S which annihilates $\mathcal{T}_1$ but not $\mathcal{T}$. But every finite rank operator annihilating $\mathcal{T}_1$ is in the trace norm closure of the rank one subspace of $\mathcal{S}$ (Theorem 2.2) and thus annihilates $\mathcal{T}$. Thus $S \in \mathcal{S}$ is not the trace norm limit of finite rank operators in $\mathcal{S}$. ■

Corollary 5.5 *There exists a CSL algebra containing a trace class operator which is not the limit in $\mathcal{C}_1$ norm of finite rank operators in the algebra.*

Proof. Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be the $\mathcal{A}$-bimodule found in the previous corollary, and define $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ to be

$$\mathcal{U} = \begin{pmatrix} \mathcal{A} & \mathcal{S} \\ 0 & \mathcal{A} \end{pmatrix}.$$

This is a unital algebra containing the masa $\mathcal{A} \oplus \mathcal{A}$, and is reflexive since $\mathcal{S}$ is. Since $\mathcal{A}$, the multiplication algebra of $L^\infty(\mathbb{T})$, contains no finite rank operators, the result follows from the previous corollary. ■

Note that the CSL algebra $\mathcal{U}$ of the last corollary cannot be strongly reflexive. Indeed if it were, then by [17] it would contain a bounded approximate identity (R_n) for the ultraweak topology consisting of sums of rank one operators, and hence every trace class operator $T \in \mathcal{U}$ would be the limit, in trace norm, of the sequence (TR_n) from the rank one subalgebra of $\mathcal{U}$.

In Theorem 5.3 we prove, for a specific class of subsets of the product of two standard measure spaces, that if a set is the topological closure of its interior then it is the ω-closure of its ω-interior. The example below shows that this is not true in general.

Example. Consider the unit interval with Lebesgue measure. Let C be a Cantor subset of positive measure (formed, for example, by removing open intervals of length 5^{-k} at the k-th stage). Let $\{I_n : n \in \mathbb{N}\}$ be the intervals removed and let $S = D \cup \bigcup_n R_n$ where $D = \{(x,x) : x \in [0,1]\}$ is the diagonal and $R_n = \overline{I_n \times I_n}$ is the closed square with side I_n. The closure of $\bigcup_n I_n \times I_n$ contains D since C contains no interval, so it is clear that S is the closure of its interior. However, $C \times C$ is ω-open and its complement contains, modulo marginally null sets, each rectangle contained in S. Thus the ω-closure of the ω-interior of S is contained in the complement of $C \times C$ and so it is a proper subset of S.

If S is as in the Example, the reflexive bimodule $\mathcal{M}_{\max}(S)$ is not strongly reflexive. However, we show that any strongly reflexive masa bimodule can be represented as $\mathcal{M}_{\max}(\kappa)$ where κ is the closure of its interior.

Theorem 5.6 *Let $\mathcal{M}$ be a strongly reflexive masa bimodule. Then there exist compact metric totally disconnected spaces X, Y equipped with regular Borel probability measures μ and ν and sequences of sets $\alpha_n \subseteq X$, $\beta_n \subseteq Y$ which are all both open and closed such that $\mathcal{M}$ is unitarily equivalent to $\mathcal{M}_{\max}(\kappa)$ where*

$$\kappa = \overline{\cup_n \{\alpha_n \times \beta_n\}}\,.$$

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are masas that make $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule. Then $\mathcal{M} = \mathcal{S}^{\perp}$ for some set of rank one operators and we can find, as in Theorem 4.2, via a countable dense subset $\{S_n\}$ of $\mathcal{S}$, sequences (E_n) of $\mathcal{A}$ and (F_n) of $\mathcal{B}$ such that $T \in \mathcal{M}$ if and only if $F_n T E_n = 0$.

Let $\mathcal{R}$ be the set of rank one operators in $\mathcal{M}$ and let $\mathcal{R}_0 = \{R_n : n = 1, 2, \ldots\}$ be a countable dense subset of $\mathcal{R}$. Again as in Theorem 4.2 let $\{P_n\}$ and $\{Q_n\}$ be projections that are minimal subject to $R_n = R_n P_n$ and $R_n = Q_n R_n$. As in the proof of Theorem 4.6, we choose separable C*-algebras $\mathcal{A}_0$ and $\mathcal{B}_0$ that are weakly dense in $\mathcal{A}$ and $\mathcal{B}$ respectively such that $\mathcal{A}_0$ contains $\{E_n\}, \{P_n\}$ and $\mathcal{B}_0$ contains $\{F_n\}, \{Q_n\}$. Let X and Y be the spectra of $\mathcal{A}$ and $\mathcal{B}$. Then just as in Theorem 4.6 we we may assume that, after a suitable unitary transformation, $\mathcal{A}, \mathcal{B}$ are the multiplication algebras of $L^\infty(X, \mu)$ and $L^\infty(Y, \nu)$ acting on $L^2(X, \mu)$ and $L^2(Y, \nu)$ where μ, ν are regular Borel measures on the compact spaces X, Y. Further, we have that $E_n = E(\rho_n), F_n = F(\sigma_n)$ and $P_n = E(\alpha_n), Q_n = F(\beta_n)$ where all the sets $\alpha_n, \beta_n, \rho_n, \sigma_n$, are both open and closed. Consequently, since $\tau(S_m B R_n A) = 0$ for all $m, n \in \mathbb{N}, A \in \mathcal{A}$ and $B \in \mathcal{B}$, an easy argument (cf Theorem 4.2) shows that

$$(\alpha_n \times \beta_n) \cap (\rho_m \times \sigma_m) = \emptyset$$

for all $m, n \in \mathbb{N}$.

Let

$$\lambda = \left[\bigcup_n (\rho_n \times \sigma_n)\right]^c .$$

and let κ be as in the statement of the theorem. Then for all n we have $\kappa \cap (\rho_n \times \sigma_n) = \emptyset$ and so $\kappa \subseteq \lambda$.

¿From Theorem 4.2, $\mathcal{M} = \mathcal{M}_{\max}(\lambda)$ so since each element of $\mathcal{R}_0$ is supported by κ, we have that $\mathcal{R}_0 \subseteq \mathcal{M}_{\max}(\kappa) \subseteq \mathcal{M}$. But from the strong reflexivity condition, and the density of $\mathcal{R}_0$ in $\mathcal{R}$, $\mathcal{M} = \operatorname{Ref} \mathcal{R}_0$ and so, since $\mathcal{M}_{\max}(\kappa)$ is reflexive, $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$. ■

We conclude this section with an application of Theorem 5.3 to a problem concerning commutative subspace lattices. We shall call any ultraweakly closed operator algebra containing a masa an *Arveson* algebra and, for any CSL $\mathcal{L}$, denote by $\mathcal{U}(\mathcal{L})$ the set of all Arveson algebras with invariant lattice $\mathcal{L}$. Such algebras were studied in [4] and we refer to that paper (or to [7]) for the definitions of pseudointegral operators and of $\mathcal{A}_{\min}(\mathcal{L})$. It was shown in [4] (at least in separable Hilbert space) that for any $\mathcal{L}$ the set $\mathcal{U}(\mathcal{L})$ contains elements $\mathcal{A}_{\min}(\mathcal{L})$ and $\mathcal{A}_{\max}(\mathcal{L})$ such that

$$\mathcal{A}_{\min}(\mathcal{L}) \subseteq \mathcal{A} \subseteq \mathcal{A}_{\max}(\mathcal{L})$$

for any $\mathcal{A} \in \mathcal{U}(\mathcal{L})$. In fact, $\mathcal{A}_{\max}(\mathcal{L}) = \operatorname{Alg} \mathcal{L}$. If $\mathcal{A}_{\min}(\mathcal{L})$ and $\mathcal{A}_{\max}(\mathcal{L})$ coincide (that is if $\mathcal{U}(\mathcal{L})$ is a singleton) then the lattice $\mathcal{L}$ is said to be *synthetic*.

It is shown in [4], Section 2.5, that there exists a weakly closed Arveson algebra that is not reflexive and so there exists a non-synthetic lattice. The question remains whether a lattice $\mathcal{L}$ can be non-synthetic when there is only one weakly closed element in $\mathcal{U}(\mathcal{L})$. This is settled below.

Theorem 5.7 *There exists a non-synthetic commutative subspace lattice $\mathcal{L}$ such that all weakly closed algebras in $\mathcal{U}(\mathcal{L})$ coincide with $\mathcal{A}_{\max}(\mathcal{L})$.*

Proof. Let $\mathbb{T}$ be the unit circle and let $\kappa \subseteq \mathbb{T} \times \mathbb{T}$ be as in Theorem 5.3. Let $X = \mathbb{T}_1 \cup \mathbb{T}_2$ be the disjoint union of two copies of $\mathbb{T}$. Put $\Delta = \{(x, x) : x \in X\}$, $\tilde{\kappa} = \{(x, y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2, (x, y) \in \kappa\}$ and $K = \Delta \cup \tilde{\kappa}$. Then

$$\mathcal{M}_{\max}(K) = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : A, B \in L^\infty(\mathbb{T}), C \in \mathcal{M}_{\max}(\kappa) \right\}$$

is a reflexive Arveson algebra in $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$. Writing $\mathcal{L} = \operatorname{Lat} \mathcal{M}_{\max}(K)$ we then have

$$\mathcal{M}_{\max}(K) = \operatorname{Alg} \mathcal{L} = \mathcal{A}_{\max}(\mathcal{L})\,.$$

Let $\mathcal{M}_0$ be the set of all finite rank operators in $\mathcal{M}_{\max}(\kappa)$ and let

$$\mathcal{M}_0(K) = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : A, B \in L^\infty(\mathbb{T}), C \in \mathcal{M}_0 \right\}.$$

It follows from Theorem 2.4 that the closure of $\mathcal{M}_0(K)$ in the weak operator topology is $\mathcal{M}_{\max}(K)$. Therefore $\operatorname{Lat}\mathcal{M}_0(K) = \mathcal{L}$ and the ultraweak closure of $\mathcal{M}_0(K)$ is in $\mathcal{U}(\mathcal{L})$. But, since all finite rank operators of $\operatorname{Alg}\mathcal{L}$ and also all operators of $L^\infty(\mathbb{T})$ are pseudo-integral, we have that $\mathcal{M}_0(K) \subseteq \mathcal{A}_{\min}(\mathcal{L})$ so that the weak closure of $\mathcal{A}_{\min}(\mathcal{L})$ is $\mathcal{A}_{\max}(\mathcal{L})$. Therefore $\mathcal{U}(\mathcal{L})$ contains exactly one algebra that is closed in the weak operator topology. However, Theorem 5.3 shows that the ultraweak closure of $\mathcal{M}_0(K)$ is not $\mathcal{M}_{\max}(K)$ and hence $\mathcal{L}$ is not synthetic. $\blacksquare$

6 Operators of $\mathcal{C}_p$ class in bimodules.

We shall be concerned with $(\mathcal{A},\mathcal{B})$-bimodules $\mathcal{M}$ of $\mathcal{B}(\mathcal{H},\mathcal{K})$ where $\mathcal{A}$ and $\mathcal{B}$ are, as before, the multiplication masas on $\mathcal{H}=L^2(X,\mu)$ and $\mathcal{K}=L^2(Y,\nu)$ respectively. Write $\mathcal{M}_0$ for the rank one subspace of $\mathcal{M}$. For $\mathcal{M}\subseteq\mathcal{C}_p$, we say that $\mathcal{M}$ is $\mathcal{C}_p$-reflexive if $\mathcal{M}=\mathcal{C}_p\cap\operatorname{Ref}\mathcal{M}$ and that $\mathcal{M}$ is strongly $\mathcal{C}_p$-reflexive if $\mathcal{M}=\mathcal{C}_p\cap\operatorname{Ref}\mathcal{M}_0$.

The next set of results are concerned with the class $\mathcal{C}_2$ of Hilbert-Schmidt operators. For $\kappa\subseteq X\times Y$, we write $\mathcal{M}_2(\kappa)$ for the set of $\mathcal{C}_2$ operators whose kernels vanish $\mu\times\nu$ a.e. outside κ. Identifying $\mathcal{C}_2$ with $L^2(X\times Y,\mu\times\nu)$, $\mathcal{M}_2(\kappa)$ is precisely the range of the projection $P(\kappa)\in\mathcal{A}\otimes\mathcal{B}$ corresponding to multiplication by the characteristic function of κ. Thus $\mathcal{M}_2(\kappa)$ is $\mathcal{A}\otimes\mathcal{B}$-invariant, i.e. an $(\mathcal{A},\mathcal{B})$-bimodule, and every closed subset of $\mathcal{C}_2$ that is an $(\mathcal{A},\mathcal{B})$-bimodule is of this form.

Lemma 6.1 *For an ω-closed set κ,*

$$\mathcal{M}_2(\kappa)=\mathcal{C}_2\cap\mathcal{M}_{\max}(\kappa)\,.$$

Proof. Let $A\in\mathcal{C}_2\cap\mathcal{M}_{\max}(\kappa)$. Since the complement of κ is $\cup_{n=1}^{\infty}\alpha_n\times\beta_n$ and since $F(\beta_n)AE(\alpha_n)=0$ we have that the kernel $a(x,y)$ of A vanishes almost everywhere on $\alpha_n\times\beta_n$ and hence on κ^c. Thus $A\in\mathcal{M}_2(\kappa)$.

For the converse, if $A\in\mathcal{M}_2(\kappa)$ and $\kappa\cap(\alpha\times\beta)=\emptyset$, then $a(x,y)=0$ almost everywhere on $\alpha\times\beta$, where a is the kernel of A. Hence $F(\beta)AE(\alpha)=0$ and $A\in\mathcal{M}_{\max}(\kappa)$. ■

Corollary 6.2 *A set κ supports a non-zero Hilbert-Schmidt operator if and only if the ω-closure of κ has non-zero measure.*

If κ and λ are subsets of $X\times Y$ we shall write $\kappa\sim\lambda$ if the symmetric difference $\kappa\Delta\lambda$ is $\mu\times\nu$-null and continue to write $\kappa\cong\lambda$ if $\kappa\Delta\lambda$ is marginally null.

Theorem 6.3 *Let $\mathcal{M}=\mathcal{M}_2(\kappa)$ be a $\mathcal{C}_2$-closed bimodule. Then*
(a) $\mathcal{M}$ is $\mathcal{C}_2$-reflexive if and only if $\kappa\sim\lambda$ for some ω-closed set λ.
(b) $\mathcal{M}$ is strongly $\mathcal{C}_2$-reflexive if and only if $\kappa\sim\lambda$ where λ is the ω-closure of an ω-open set.
(c) $\mathcal{M}$ is the $\mathcal{C}_2$-closure of its rank one subspace if and only if $\kappa\sim\sigma$ for some ω-open set σ.

Proof. (a) If $\kappa \sim \lambda$ and λ is ω-closed then, using Lemma 6.1 $\mathcal{M}_2(\kappa) = \mathcal{M}_2(\lambda) = \mathcal{C}_2 \cap \mathcal{M}_{\max}(\lambda)$. Clearly every member of $\mathcal{M}_2(\lambda)$ is supported by λ so $\operatorname{Ref} \mathcal{M} \subseteq \mathcal{M}_{\max}(\lambda)$. Hence

$$\mathcal{M} = \mathcal{M} \cap \operatorname{Ref} \mathcal{M} = \mathcal{C}_2 \cap \mathcal{M}_{\max}(\lambda) \cap \operatorname{Ref} \mathcal{M} = \mathcal{C}_2 \cap \operatorname{Ref} \mathcal{M}$$

so $\mathcal{M}$ is $\mathcal{C}_2$-reflexive.

Conversely suppose $\mathcal{M}_2(\kappa)$ is $\mathcal{C}_2$-reflexive. Then $\operatorname{Ref} \mathcal{M}_2(\kappa) = \mathcal{M}_{\max}(\lambda)$ for some ω-closed set λ and

$$\mathcal{M}_2(\kappa) = \mathcal{C}_2 \cap \operatorname{Ref} \mathcal{M}_2(\kappa) = \mathcal{C}_2 \cap \mathcal{M}_{\max}(\lambda) = \mathcal{M}_2(\lambda)$$

by Lemma 6.1. Hence $\kappa \sim \lambda$.

(b) If $\mathcal{M}$ is strongly $\mathcal{C}_2$-reflexive then from Theorem 4.2 and Lemma 5.1, $\operatorname{Ref} \mathcal{M}_0 = \mathcal{M}_{\max}(\lambda)$ where λ is the ω-closure of an ω-open set. Hence $\mathcal{M} = \mathcal{C}_2 \cap \operatorname{Ref} \mathcal{M}_0 = \mathcal{M}_2(\lambda)$ and $\lambda \sim \kappa$.

For the converse, if $\mathcal{M} = \mathcal{M}_2(\lambda)$ where λ is the ω-closure of an ω-open set then $\mathcal{M}_{\max}(\lambda) = \operatorname{Ref} \mathcal{M}_0$ where $\mathcal{M}_0$ is the rank one subspace of $\mathcal{M}_{\max}(\lambda)$ (hence of $\mathcal{M}$). By Lemma 6.1 we have

$$\mathcal{M}_2(\lambda) = \mathcal{C}_2 \cap \mathcal{M}_{\max}(\lambda) = \mathcal{C}_2 \cap \operatorname{Ref} \mathcal{M}_0$$

as required.

(c) Fix a $\mathcal{C}_2$-dense subset $\mathcal{N}$ of the rank one subspace of $\mathcal{M}$. As usual we may assume that $\mathcal{N}$ is of the form $\{F(\beta_n)RE(\alpha_n) : \operatorname{rank} R = 1, n \in \mathbb{N}\}$ where the union σ of $\alpha_n \times \beta_n$ is contained in κ. Now a Hilbert-Schmidt operator $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ annihilates the $\mathcal{C}_2$-closed linear span of $\mathcal{N}$ if and only if its kernel τ satisfies $\tau(y,x) = 0$ for $\mu \times \nu$-almost all (x,y) in σ, whereas it annihilates $\mathcal{M}_2(\kappa)$ if and only if $\tau(y,x) = 0$ for $\mu \times \nu$-almost all (x,y) in κ. Thus if the linear span of the rank one subspace of $\mathcal{M}$ is $\mathcal{C}_2$-dense in $\mathcal{M}$ then $\kappa \sim \sigma$.

Conversely if $\kappa \sim \lambda$ for some ω-open set λ then $\mathcal{M}_2(\kappa) = \mathcal{M}_2(\lambda)$ and, by the argument just given, $\mathcal{M}_2(\lambda)$ is the $\mathcal{C}_2$-closure of its rank one subspace. ■

It is clear that if a $\mathcal{C}_2$-reflexive bimodule $\mathcal{M}$ is the $\mathcal{C}_2$-closure of its rank one subspace then it is strongly $\mathcal{C}_2$-reflexive. To show that the converse is false it suffices, by (c) of the preceding Theorem, to produce a subset κ of $X \times Y$ which is the ω-closure of its ω-interior κ^0 such that $\kappa \setminus \kappa^0$ has non-zero measure.

Example. Take the standard construction of a Cantor subset of $[0,1]$ with non-zero measure, removing intervals of length 5^{-k} at the k-th stage; but, in each removed interval insert a smaller nontrivial closed interval. (The exact size or positioning is immaterial.) Let γ be the resulting set. Then the interior of γ is clearly the union of the interiors of the added intervals and $\gamma \setminus \operatorname{int} \gamma$ contains the standard Cantor set of non-zero measure. Clearly γ is closed and $\gamma = \overline{\operatorname{int} \gamma}$. Let $\kappa = \{(x, y) : x - y \in \gamma\}$. Then, as in the proof of Theorem 5.3, it follows that κ is the ω-closure of its ω-interior κ^0. Also $\kappa \setminus \kappa^0$ has non-zero measure and so the example is established.

The above construction also gives an example of a $\mathcal{C}_2$-closed bimodule $\mathcal{M}$ which is not $\mathcal{C}_2$-reflexive, although it is the $\mathcal{C}_2$-closure of its rank one subspace. Indeed, $\mathcal{M}_2(\kappa^0)$ has the claimed properties.

Theorem 6.4 *For an ω-closed set κ the following are equivalent.*
(a) κ is the ω-support of a Hilbert-Schmidt operator;
(b) κ is the ω-support of a set of Hilbert-Schmidt operators;
(c) $\mathcal{M}_{\max}(\kappa) = \operatorname{Ref} \mathcal{M}_2(\kappa)$;
(d) for any rectangle γ either $\gamma \cap \kappa$ is marginally null or it has non-zero measure.

Proof. That (a) implies (b) is trivial. If (b) holds so that κ is the ω-support of a set $\mathcal{S}$ of Hilbert-Schmidt operators then, using Lemma 6.1

$$\mathcal{M}_{\max}(\kappa) = \operatorname{Ref} \mathcal{S} \subseteq \operatorname{Ref} \mathcal{M}_2(\kappa) \subseteq \mathcal{M}_{\max}(\kappa)$$

and (c) follows.

Suppose (c) holds and $(\mu \times \nu)(\kappa \cap \gamma) = 0$ where $\gamma = \alpha \times \beta$ is a rectangle. Then, for any $T \in \mathcal{M}_2(\kappa)$ we have that $F(\beta)TE(\alpha) = 0$. Hence $\gamma \cap \operatorname{supp}_\omega \mathcal{M}_2(\kappa)$ and so also $\gamma \cap \kappa$ are marginally null.

To show that (d) implies (a), let T be the integral operator whose kernel is the characteristic function of κ. Clearly $\operatorname{supp}_\omega T \subseteq \kappa$. If $F(\beta)TE(\alpha) = 0$ then $(\alpha \times \beta) \cap \kappa$ has measure 0 and so, by (d) it is marginally null. This shows that $\kappa \subseteq \operatorname{supp}_\omega T$ modulo marginally null sets and so κ is the ω-support of T. ■

We now turn to $\mathcal{C}_1$, the trace class operators. Let $\Gamma(X, Y) = L^2(X) \hat{\otimes} L^2(Y)$ be the projective tensor product. Thus $f \in \Gamma(X, Y)$ if

$$f(x, y) = \sum_1^\infty a_n(x) b_n(y)$$

with

$$\sum_n \int_X |a_n(x)|^2 \, d\mu(x) \qquad \text{and} \qquad \sum_n \int_Y |b_n(y)|^2 \, d\nu(y)$$

both convergent. Note that f is defined marginally almost everywhere in that, if a_n, b_n are changed on null sets then f will change on a marginally null set. The elements of $\Gamma(X, Y)$ are exactly the kernels of the trace class operators from $\mathcal{H}$ to $\mathcal{K}$. For full details of the above, see [4], especially Proposition 2.2.7.

Theorem 6.5 *All functions in* $\Gamma(X, Y)$ *are* ω*-continuous.*

Proof. Let $f(x, y) = \sum a_n(x)b_n(y)$ and put $f_N(x, y) = \sum_{n=1}^{N} a_n(x)b_n(y)$ and $\phi_N(x, y) = \sum_{n=N+1}^{\infty} (|a_n(x)|^2 + |b_n(y)|^2)$. Clearly $a_n(x)$ and $b_n(y)$ are ω-continuous as functions on $X \times Y$ and so f_N and ϕ_N are ω-continuous (Corollary 3.2). It follows from Fatou's lemma, that $\phi_N(x, y) \to 0$ marginally almost everywhere. Since $|f(x, y) - f_N(x, y)| \leq \phi_N(x, y)$, the result follows using Lemma 3.3 (if we previously remove subsets of null measure in X and Y on which the series diverge). ∎

Corollary 6.6 *A set* κ *supports a trace class operator if and only if the* ω*-closure of* κ *contains a non-null rectangle.*

Proof. If $\mathrm{cl}_\omega \kappa$ contains a rectangle then it supports a rank one operator. Conversely, if κ supports a trace class operator T then (modulo marginally null sets) the integral kernel ϕ of T is zero on the complement of $\mathrm{cl}_\omega \kappa$. Hence $\mathrm{cl}_\omega \kappa$ contains the ω-open set $\{(x, y) : \phi(x, y) \neq 0\}$ which contains a rectangle. ∎

Theorem 6.7 *The following are equivalent:*
(a) κ *is the* ω*-support of a trace class operator;*
(b) κ *is the* ω*-support of a set of trace class operators;*
(c) κ *is the* ω*-closure of an* ω*-open set.*

Proof. If (b) holds, let $\kappa = \mathrm{supp}_\omega(\mathcal{T})$ where $\mathcal{T}$ is a set of trace class operators. Using separability, we may suppose that $\mathcal{T} = \{T_n\}$ is countable. Suppose T_n has integral kernel ϕ_n and let $\kappa_n = \{(x, y) : \phi_n(x, y) \neq 0\}$. From Theorem 6.5 each κ_n is ω-open and hence so is $\sigma = \bigcup_n \kappa_n$. We show that the ω-closure of σ is κ. If a rectangle $\gamma = \alpha \times \beta$ has null intersection with κ_n then $F(\beta)T_nE(\alpha) = 0$ and so $\mathrm{supp}_\omega T_n \subseteq \mathrm{cl}_\omega \kappa_n$. Hence $\kappa \subseteq \mathrm{cl}_\omega \sigma$. For the reverse inclusion, if γ has null intersection with κ then $F(\beta)T_nE(\alpha) = 0$ so that ϕ_n is null on γ and so $\gamma \cap \kappa_n$ is null. Since κ_n is ω-open, this shows that $\gamma \cap \kappa_n$ is marginally null and hence so is $\gamma \cap \sigma$. Thus (c) holds.

To show that (c) implies (a), suppose $\kappa = \mathrm{cl}_\omega \gamma$ where $\gamma = \bigcup_n \gamma_n$ is a union of rectangles γ_n. As usual, we may arrange that the sets γ_n are pairwise disjoint. Choose unit vectors x_n, y_n such that the operators $T_n = x_n \otimes y_n$ have supports γ_n. For a suitable sequence (λ_n) of positive real numbers, the operator $T = \sum_n \lambda_n T_n$ is in trace class. Clearly T is supported by γ. However, if $F(\beta)TE(\alpha) = 0$ then for each n we have that $\langle F(\beta)TE(\alpha)x_n, y_n\rangle = 0$ so that either $F(\beta)x_n = 0$ or $E(\alpha)y_n = 0$. Thus $\alpha \times \beta$ has null intersection with γ_n and since γ_n is a rectangle, this intersection is marginally null. Therefore $(\alpha \times \beta) \cap \gamma$ is marginally null and so $\kappa = \mathrm{cl}_\omega \gamma = \mathrm{supp}_\omega T$. Since (a) trivially implies (b) the theorem is proved. ■

Using Lemma 4.5 and the comments preceding it, makes it clear that the above theorem shows that a set κ can support a nonzero trace class operator if and only if there exists a rectangle $\alpha \times \beta$, such that the intersection of every sub-rectangle $\alpha_1 \times \beta_1$ with κ is not marginally null.

Theorem 6.8 *Let $\mathcal{M} \subseteq \mathcal{C}_2(\mathcal{H}, \mathcal{K})$ be a $\mathcal{C}_2$-closed masa bimodule. Then any trace class operator in $\mathcal{M}$ is a limit in $\mathcal{C}_2$-norm of sums of rank one operators from $\mathcal{M}$.*

Proof. We assume $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. Then $\Gamma(X, Y)$ corresponds to nuclear operators and $L^2(X \times Y)$ to Hilbert-Schmidt operators. We have that $\mathcal{M} = \mathcal{M}_2(\kappa)$, where $\kappa \subseteq X \times Y$ is a measurable set. If $a(x, y)$ is the kernel of an operator $A \in \mathcal{M} \cap \mathcal{C}_1$, then the ω-open set

$$\sigma = \{(x, y) : a(x, y) \neq 0\}$$

is contained in κ up to a null set.

Since $\sigma = \cup_{j=1}^{\infty} \alpha_j \times \beta_j$ we have, modulo marginally null sets, $\alpha_j \times \beta_j \subseteq \kappa$ for each j. Hence $\mathcal{M}_2(\alpha_j \times \beta_j) \subseteq \mathcal{M}_2(\kappa)$. Since $\mathcal{M}_2(\alpha_j \times \beta_j)$ is generated by rank-one operators, $\mathcal{M}_2(\alpha_j \times \beta_j)$ is contained in the $\mathcal{C}_2$-closure $\overline{\mathcal{M}_0}^2$ of the rank one subspace $\mathcal{M}_0$ of $\mathcal{M}_2(\kappa)$. But $A \in \mathcal{M}_2(\cup_j \alpha_j \times \beta_j)$ which is the $\mathcal{C}_2$-closed linear span of $\mathcal{M}(\alpha_j \times \beta_j) \subseteq \overline{\mathcal{M}_0}^2$.

■

Corollary 5.4 shows that the above statement cannot be improved to "any trace class operator in $\mathcal{M}$ is a limit in $\mathcal{C}_1$-norm of sums of rank one operators from $\mathcal{M}$".

Corollary 6.9 *If a masa bimodule consists of trace class operators then its closure in the weak operator topology is strongly reflexive.*

Proof. Since the weak operator topology is weaker than the $\mathcal{C}_2$ topology this follows immediately from the theorem and Corollary 2.6. ■

In [9] an example is given of a CSL agebra containing no Hilbert-Schmidt operators but containing an operator belonging to $\mathcal{C}_p$ for all $p > 2$. We improve this in the following example. This example exhibits a surprising obstacle to certain types of density in $\mathcal{C}_p$ norms and is the basis of Theorem 6.10 below.

Example. There is a CSL algebra which contains no trace class operators yet contains an operator belonging to $\mathcal{C}_p$ for all $p > 1$.

It is sufficient to find a reflexive masa bimodule with the required properties since it can be embedded in an algebra in the standard way. This example is based on the existence of a subset K of the circle group $\mathbb{T}$ which is closed and nowhere dense and a nonzero function $f \in L^2(\mathbb{T})$ supported on K whose Fourier coefficients satisfy $\sum_{-\infty}^{\infty} n|\hat{f}(n)|^2 < \infty$. This is due to Ahlfors and Beurling [1].

Define

$$(Tx)(t) = \int_{\mathbb{T}} f(t-s)x(s)\, ds\,.$$

Then $T \in \mathcal{C}_p$ for all $p \geq 2$ and, for $1 < p < 2$, an easy calculation shows that

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^p \leq \left(\sum n|\hat{f}(n)|^2\right)^{\frac{p}{2}} \left(\sum \frac{1}{n^r}\right)^{\frac{1}{r}}$$

where $r = \frac{2}{2-p} > 1$. Therefore $T \in \mathcal{C}_p$ for all $p > 1$. Also, T is supported by $\{(t,s) \in \mathbb{T}^2 : t - s \in K\} = \kappa$ and this contains no rectangles. Thus $\mathcal{M}_{\max}(\kappa)$ contains T but does not contain any trace class operators. ■

Theorem 6.10 *There exists a transitive bimodule $\mathcal{M}$ spanned by rank one operators such that $\mathcal{M}$ is not dense in $\mathcal{C}_p$ for any p, $1 \leq p < \infty$.*

Proof. Let $T \in \bigcup_{p>1} \mathcal{C}_p$ be as in the the example above and let κ be the ω-support of T. Define

$$\mathcal{M} = \operatorname{span}\{R : \operatorname{rank} R = 1, \operatorname{supp}_\omega R \cap \kappa = \emptyset\}.$$

Then the complement of the ω-support of $\mathcal{M}$ contains no non-null rectangles and so no rank 1 operators annihilate $\mathcal{M}$. Therefore $\operatorname{Ref} \mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is transitive.

However, for every rank one operator X of $\mathcal{M}$, we have that $X = F(\beta)XE(\alpha)$ with $(\alpha \times \beta) \cap \kappa = \emptyset$. Thus $\tau(X^*T) = \tau(E(\alpha)X^*F(\beta)T) = 0$. If $1 \leq p < \infty$

and Y is the $\mathcal{C}_p$-limit of finite sums of elements of $\mathcal{M}$, since $T \in \mathcal{C}_q$ where $\frac{1}{p} + \frac{1}{q} = 1$, it follows that

$$\tau(Y^*T) = 0\,.$$

Therefore $\mathcal{M}$ is not dense in $\mathcal{C}_p$. ∎

The work in this section on trace class operators enables us to complete an extensive set of characterisations of strong reflexivity. These are summarised below.

Theorem 6.11 *Let $\mathcal{M}$ be a masa bimodule which is closed in the weak operator topology. The following are equivalent:*
(a) $\mathcal{M}$ is strongly reflexive;
(b) $\mathcal{M}$ is the reflexive hull of some set $\mathcal{T}$ of trace class operators;
(c) $\mathcal{M}$ is the weak operator closure of its rank one subspace;
(d) $\mathcal{M}$ is the weak operator closure of $\mathcal{C}_1 \cap \mathcal{M}$;
(e) $\mathcal{M}$ is reflexive and its ω-support is the ω-closure of an ω-open set;
(f) $\mathcal{M} = \mathcal{M}_{\max}(\kappa)$ where κ is ω-closed and satisfies the condition that for any Borel rectangle $\alpha \times \beta$, either $(\alpha \times \beta) \cap \kappa$ is marginally null or it contains a rectangle of positive measure.

Proof. If (b) holds then, by Theorem 2.4, $\mathcal{M} = \operatorname{Ref} \mathcal{T}$ is the weak operator closure of $\mathcal{T}$. Since the latter is strongly reflexive by Corollary 6.9, (a) follows. As it is obvious that (a) implies (b), the equivalence of (a) and (b) is proved. Now (a) implies (c) by Corollary 2.5, (c) obviously implies (d) and Corollary 6.9 shows that (d) implies (a). The equivalence of (a) and (e) is Theorem 5.2.

That (a) implies (f) follows from the observation that, if $(\alpha \times \beta) \cap \kappa$ is not marginally null, then it supports a nonzero operator in $\mathcal{M}_{\max}(\kappa)$. Thus if $\mathcal{M}$ is strongly reflexive, then $(\alpha \times \beta) \cap \kappa$ must support a nonzero rank one operator, and hence must contain a rectangle.

Finally we show that (f) implies (a). Suppose $F(\beta)RE(\alpha) = 0$ for all rank one operators $R \in \mathcal{M}$. Then $(\alpha \times \beta) \cap \kappa$ contains no rectangle and so by (e) it is marginally null; hence $F(\beta)TE(\alpha) = 0$ for all $T \in \mathcal{M}$. This means that $\mathcal{M}$ is the reflexive hull of its rank one subspace. ∎

7 Appendix.

We present a co-ordinate free proof of the following result.

Theorem 7.1 (Arveson) *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}), \mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ masas. Then $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is the only transitive* w**-closed $(\mathcal{A}, \mathcal{B})$-bimodule.*

The proof is divided into a series of lemmas. The first is essentially due to Arveson (see Lemma 2.1.4 of [4] and the subsequent remark).

Lemma 7.2 *Let $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, $A \in \mathcal{A}, B \in \mathcal{B}$ positive contractions. Suppose that each contraction $T \in \mathcal{T}$ satisfies*

$$T^* B T \leq A.$$

Then each $T \in \mathcal{T}$ satisfies

$$T E_A[0, a](\mathcal{H}) \subseteq E_B[0, a](\mathcal{K}) \quad \text{for all } a \geq 0$$

where $E_B(.)$ (resp. $E_A(.)$) denotes the spectral measure of B (resp. A).

Proof. Given $\epsilon > 0$ we have, for any contraction $T \in \mathcal{T}$,

$$T^*(B + \epsilon I)T \leq (A + \epsilon I)$$

equivalently

$$(A + \epsilon I)^{-1/2} T^* (B + \epsilon I)^{1/2} (B + \epsilon I)^{1/2} T (A + \epsilon I)^{-1/2} \leq I$$

that is $T_1^* T_1 \leq 1$, where $T_1 = (B + \epsilon I)^{1/2} T (A + \epsilon I)^{-1/2} \in \mathcal{T}$. Thus the previous inequality is valid with T_1 replacing T, and yields $\|T_2\| \leq 1$, where $T_2 = (B + \epsilon I) T (A + \epsilon I)^{-1}$. Continuing, we find

$$\|(B + \epsilon I)^n T (A + \epsilon I)^{-n}\| \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

This implies that $\|(B + \epsilon I)^n T \xi\| \leq \|(A + \epsilon I)^n \xi\|$ for all $\xi \in \mathcal{H}$ and $n \in \mathbb{N}$, and, letting $\epsilon \longrightarrow 0$,

$$\|B^n T \xi\| \leq \|A^n \xi\| \text{ for all } \xi \in \mathcal{H} \text{ and } n \in \mathbb{N}. \tag{1}$$

Now a standard argument shows that $\xi \in E_A[0, a](\mathcal{H})$ if and only if there exists a constant c such that $\|A^n \xi\| \leq c a^n$ for all $n \in \mathbb{N}$ (see e.g. [11], Problem 194). Thus if $\xi \in E_A[0, a](\mathcal{H})$ then $\|A^n \xi\| \leq c a^n$ and so, by (1), $\|B^n T \xi\| \leq c a^n$ for all $n \in \mathbb{N}$ which implies that $T\xi \in E_B[0, a](\mathcal{K})$. ■

Given an operator $A \in \mathcal{B}(\ell^2 \otimes \mathcal{H})$ and an $e \in \ell^2$ define $A^e \in \mathcal{B}(\mathcal{H})$ by

$$\langle A^e \xi, \eta \rangle = \langle A(e \otimes \xi), (e \otimes \eta) \rangle.$$

It is immediate to check that, for each $T \in \mathcal{B}(\mathcal{H})$ we have

$$((I \otimes T)A)^e = T A^e \text{ and } (A(I \otimes T))^e = A^e T$$

and that if A is positive then so is each A^e.

Lemma 7.3 *Let $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a transitive $(\mathcal{A}, \mathcal{B})$-bimodule. Let P and Q be projections in $\mathcal{B}(\ell^2 \otimes \mathcal{H})$ and $\mathcal{B}(\ell^2 \otimes \mathcal{K})$ respectively such that*

$$Q(I \otimes \mathcal{T})P = \{0\}\,.$$

Then for all vectors $e \in \ell^2$, $\xi \in \mathcal{H}, \eta \in \mathcal{K}$ of norm at most 1,

$$\langle P(e \otimes \xi), (e \otimes \xi)\rangle + \langle Q(e \otimes \eta), (e \otimes \eta)\rangle \leq 1.$$

Proof. For each $T \in \mathcal{T}$ we have $Q(I \otimes T)P = 0$ and so $Q(I \otimes T) = Q(I \otimes T)P^{\perp}$. Thus if $T \in \mathcal{T}$ is a contraction,

$$(I \otimes T^*)Q(I \otimes T) = P^{\perp}(I \otimes T^*)Q(I \otimes T)P^{\perp} \leq P^{\perp}.$$

Hence if $e \in \ell^2$,

$$T^*Q^eT = ((I \otimes T^*)Q(I \otimes T))^e \leq (P^{\perp})^e.$$

Putting $B = Q^e$ and $A = (P^{\perp})^e$ and using Lemma 7.2, we obtain

$$TE_A[0,a](\mathcal{H}) \subseteq E_B[0,a](\mathcal{K}) \quad \text{for all } a \geq 0$$

and all $T \in \mathcal{T}$. Since $\mathcal{T}$ is transitive, it follows that for each $a \geq 0$ either $E_A[0,a] = 0$ or $E_B[0,a] = I$. Let $b = \inf\{a > 0 : E_B[0,a] = I\}$. Thus $E_A[0,a] = 0$ for $a < b$, and so

$$(P^{\perp})^e = A = \int_0^1 t dE_A(t) = \int_b^1 t dE_A(t) \geq bI.$$

Therefore,

$$\langle P^{\perp}(e \otimes \xi), (e \otimes \xi)\rangle = \langle (P^{\perp})^e \xi, \xi\rangle \geq b$$

so that

$$\langle P(e \otimes \xi), (e \otimes \xi)\rangle \leq \|e\|^2\|\xi\|^2 - b \leq 1 - b. \tag{2}$$

Also $E_B[0,a] = I$ for $a > b$ and so

$$(Q)^e = B = \int_0^1 t dE_B(t) = \int_0^b t dE_B(t) \leq bI$$

from which we conclude that

$$\langle Q(e \otimes \eta), (e \otimes \eta)\rangle = \langle Q^e \eta, \eta\rangle \leq b. \tag{3}$$

Adding (2) and (3), we obtain the result. ∎

Lemma 7.4 *Given a projection P, define*

$$P' = \bigwedge\{Q \in \mathcal{B}(\ell^2) \otimes I_{\mathcal{H}} : Q \text{ projection, } Q \geq P\}.$$

If $A \in \mathcal{B}(\ell^2) \otimes I_{\mathcal{H}}$ and $P \leq A \leq I$, then $P' \leq A$.

Proof. For any unit vector ξ in the range of P we have

$$1 = \langle P\xi, \xi\rangle \leq \langle A\xi, \xi\rangle \leq \|A\xi\|.\|\xi\| = 1$$

and so $\langle A\xi, \xi\rangle = \|A\xi\|.\|\xi\|$. Thus $A\xi = e^{i\theta}\xi$ and, since $A \geq 0$, $A\xi = \xi$. This shows that the range of P is contained in the eigenspace of A corresponding to the eigenvalue 1. Thus if we let P_1 be the projection on that eigenspace, then $P \leq P_1$. But $P_1 \in \mathcal{B}(\ell^2) \otimes I_{\mathcal{H}}$ since $A \in \mathcal{B}(\ell^2) \otimes I_{\mathcal{H}}$, and so $P' \leq P_1$ by the definition of P'. On the other hand it is clear that $P_1 \leq A$ [indeed $A = AP_1^{\perp} + AP_1 \geq AP_1 = P_1$], hence $P' \leq A$. ■

In the lemma below we shall call two vectors ξ_1, ξ_2 *very orthogonal* with respect to a masa $\mathcal{A}$ if $\mathcal{A}\xi_1 \perp \mathcal{A}\xi_2$.

Lemma 7.5 *(i) Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Then the set of all finite sums of the form $\sum e_i \otimes \xi_i$ where $\xi_i \in \mathcal{H}$ are pairwise very orthogonal with respect to $\mathcal{A}$ and $e_i \in \ell^2$, is dense in $\ell^2 \otimes \mathcal{H}$.*

(ii) Let $T \in \mathcal{B}(\ell^2) \otimes \mathcal{A}$. If $\langle T(e \otimes \xi), (e \otimes \xi)\rangle \geq 0$ for $e \in \ell^2$ and $\xi \in \mathcal{H}$, then $T \geq 0$.

Proof. (i) Given $\epsilon > 0$ and

$$\eta = \sum_{i=1}^{n} f_i \otimes \eta_i,$$

it suffices to find $e_i \in \ell^2$ and very orthogonal $\xi_i \in \mathcal{H}$ with $\|\eta - \sum e_i \otimes \xi_i\| < \epsilon$.

Let $\delta = \epsilon(2\sum \|f_i\|)^{-1}$. If $\eta \in \mathcal{H}$ is a cyclic vector for the masa $\mathcal{A}$, we can find $A_i \in \mathcal{A}$ such that $\|A_i\eta - \eta_i\| < \delta$ for $1 \leq i \leq n$. By the spectral theorem, for each i there exist finitely many projections $P_{ij} \in \mathcal{A}$ $(1 \leq j \leq n_i)$ and scalars a_{ij} such that $\|A_i\eta - \sum_j a_{ij}P_{ij}\eta\| < \delta$. The finite set $\{P_{ij} : 1 \leq j \leq n_i, 1 \leq i \leq n\}$ generates a finite, hence atomic, Boolean lattice, with atoms $Q_1, \ldots, Q_m$, say. Replacing each P_{ij} by an appropriate linear combination of atoms, we obtain $\|A_i\eta - \sum_j b_{ij}Q_j\eta\| < \delta$ for appropriate scalars b_{ij} . Now define

$$\xi = \sum_{i=1}^{n} f_i \otimes \left(\sum_{j=1}^{m} b_{ij}Q_j\eta\right).$$

Note that $\|\eta - \xi\| \leq \sum_i \|f_i\|.\|\eta_i - \sum_j b_{ij} Q_j \eta\| \leq \sum_i \|f_i\|.2\delta = \epsilon$. But also

$$\xi = \sum_{i=1}^{n} f_i \otimes \sum_{j=1}^{m} b_{ij} Q_j \eta = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} b_{ij} f_i \right) \otimes Q_j \eta = \sum_{j=1}^{m} e_j \otimes \xi_j$$

and the vectors $\xi_j = Q_j \eta$ are very orthogonal.

(ii) Let $\xi = \sum e_i \otimes \xi_i$ where $\xi_i \in \mathcal{H}$ are very orthogonal and $e_i \in \ell^2$. Then

$$\langle T\xi, \xi \rangle = \sum_{i,j} \langle T(e_i \otimes \xi_i), (e_j \otimes \xi_j) \rangle = \sum_{i,j} \langle (I \otimes P_j) T (I \otimes P_i)(e_i \otimes \xi_i), (e_j \otimes \xi_j) \rangle$$

where P_i denotes the projection onto $[\mathcal{A}\xi_i]$. Since $(I \otimes P_i) \in I \otimes \mathcal{A}$ and $T \in \mathcal{B}(\ell^2) \otimes \mathcal{A}$, we have $(I \otimes P_i)T = T(I \otimes P_i)$. Now $P_i P_j = \delta_{ij} P_i$, and so

$$\langle T\xi, \xi \rangle = \sum_{i} \langle T(e_i \otimes \xi_i), (e_j \otimes \xi_j) \rangle \geq 0$$

by assumption. Since vectors of this form are dense in $\ell^2 \otimes \mathcal{H}$, the result follows. ■

Proof of the Theorem. Let $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a transitive $(\mathcal{A}, \mathcal{B})$-bimodule and let ϕ be a w*- continuous linear form on $\mathcal{B}(\mathcal{H}, \mathcal{K})$ annihilating $\mathcal{T}$. We will show that $\phi = 0$.

Let $\phi(A) = \sum_n \langle A\xi_n, \eta_n \rangle$ $(A \in \mathcal{B}(\mathcal{H}, \mathcal{K}))$, where $\xi = (\xi_n), \eta = (\eta_n)$ are in $\ell^2 \otimes \mathcal{H}$ and $\ell^2 \otimes \mathcal{K}$ respectively. Then $\phi(A) = \langle (I \otimes A)\xi, \eta \rangle$.Thus if $E \in \mathcal{B}(\ell^2 \otimes \mathcal{H}), F \in \mathcal{B}(\ell^2 \otimes \mathcal{K})$ are the projections on $\overline{[(I \otimes \mathcal{A})\xi]}$ and $[(I \otimes \mathcal{B})\eta]^\perp$ respectively, then

$$F(I \otimes \mathcal{T})E = \{0\}$$

and we have to show that

$$F(I \otimes \mathcal{B}(\mathcal{H}, \mathcal{K}))E = \{0\}.$$

By Lemma 7.3, for all unit vectors $e \in \ell^2$, $\xi \in \mathcal{H}, \eta \in \mathcal{K}$,

$$\langle E(e \otimes \xi), (e \otimes \xi) \rangle + \langle F(e \otimes \eta), (e \otimes \eta) \rangle \leq 1$$

equivalently

$$\langle E(e \otimes \xi), (e \otimes \xi) \rangle \leq \langle F^\perp(e \otimes \eta), (e \otimes \eta) \rangle. \tag{4}$$

For $y \in \mathcal{K}$, define $C_y \in \mathcal{B}(\ell^2)$ by

$$\langle C_y e, f \rangle = \langle F^\perp(e \otimes y), (f \otimes y) \rangle.$$

It is clear that C_y is a positive contraction, and relation (4) gives

$$\langle E(e\otimes\xi),(e\otimes\xi)\rangle \le \langle C_\eta e,e\rangle = \langle (C_\eta\otimes I_{\mathcal{H}})(e\otimes\xi),(e\otimes\xi)\rangle$$

for any unit vector $\eta\in\mathcal{K}$. Thus the operator $T=(C_\eta\otimes I_{\mathcal{H}}-E)\in\mathcal{B}(\ell^2)\otimes\mathcal{A}$ satisfies

$$\langle T(e\otimes\xi),(e\otimes\xi)\rangle \ge 0$$

for all unit vectors $e\in\ell^2$ and $\xi\in\mathcal{H}$. By Lemma 7.5(ii), this means that $T\ge 0$, that is, $E\le C_\eta\otimes I_{\mathcal{H}}$. Since $C_\eta\otimes I_{\mathcal{H}}$ is a positive contraction in $\mathcal{B}(\ell^2)\otimes\mathcal{A}$, Lemma 7.4 implies that $E'\le C_\eta\otimes I_{\mathcal{H}}$ and so

$$\langle E'(e\otimes\xi),(e\otimes\xi)\rangle \le \langle (C_\eta\otimes I_{\mathcal{H}})(e\otimes\xi),(e\otimes\xi)\rangle = \langle C_\eta e,e\rangle = \langle F^{\perp}(e\otimes\eta),(e\otimes\eta)\rangle.$$

Thus

$$\langle E'(e\otimes\xi),(e\otimes\xi)\rangle + \langle F(e\otimes\eta),(e\otimes\eta)\rangle \le 1.$$

Applying the same argument to the pair (F,E') instead of (E,F), we obtain

$$\langle E'(e\otimes\xi),(e\otimes\xi)\rangle + \langle F'(e\otimes\eta),(e\otimes\eta)\rangle \le 1.$$

But now $E'\in\mathcal{B}(\ell^2)\otimes I_{\mathcal{H}}$ and $F'\in\mathcal{B}(\ell^2)\otimes I_{\mathcal{K}}$, so that we can write $E'=P\otimes I_{\mathcal{H}}, F'=Q\otimes I_{\mathcal{K}}$, where $P,Q\in\mathcal{B}(\ell^2)$ are projections. Then the previous inequality becomes

$$\langle Pe\otimes\xi, e\otimes\xi\rangle + \langle Qe\otimes\eta, e\otimes\eta\rangle \le 1$$

or

$$\langle Pe,e\rangle\|\xi\|^2 + \langle Qe,e\rangle\|\eta\|^2 \le 1.$$

It follows that $0\le P+Q\le I$, and this implies that $QP=0$.

But then for all $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ we have

$$F'(I\otimes A)E' = (Q\otimes I_{\mathcal{K}})(I\otimes A)(P\otimes I_{\mathcal{H}}) = QP\otimes A = 0$$

that is $F'(I\otimes\mathcal{B}(\mathcal{H},\mathcal{K}))E'=\{0\}$. Since $E\le E'$ and $F\le F'$, we finally obtain

$$F(I\otimes\mathcal{B}(\mathcal{H},\mathcal{K}))E=\{0\}$$

as required. ■

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