Operator algebras from the discrete Heisenberg semigroup

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Outline

General Framework

$\mathbb{G}$ group , $\mathbb{S} \subseteq \mathbb{G}$ semigroup

A representation on a Hilbert space H :
$\sigma : \mathbb{S} \to B(H) : \; g \to S_g$ morphism, each S_g isometry.

$W(\sigma, \mathbb{S}) \subseteq B(H)$: w*-closed algebra generated by $\{S_g : g \in \mathbb{S}\}$.

$\mathrm{Lat}\, W(\sigma, \mathbb{S})$: all closed invariant subspaces: $M \subseteq H$ such that $S_g(M) \subseteq M$ for all $g \in \mathbb{S}$.

Example: $\mathbb{Z}_+$

$\mathbb{Z}_+$ acts on $\ell^2(\mathbb{Z}_+)$ [write $\mathbb{Z}_+ \curvearrowright \ell^2(\mathbb{Z}_+)$] by the left regular representation:

$$n \to \lambda(n) = S^n \quad \text{where} \quad S : e_k \to e_{k+1}.$$

Theorem (Beurling, 1949)

If $M \in \mathrm{Lat}(\lambda(\mathbb{Z}_+))$ (i.e. $S(M) \subseteq M$), there is $\phi \in H^2$ with $|\phi(z)| = 1$ for a.a. $z \in \mathbb{T}$ so that (after Fourier transform)

$$M = \phi H^2.$$

Remark

No analogous description for $\mathrm{Lat}(\lambda(\mathbb{Z}_+^2))$!

Reflexivity

Theorem (Sarason, 1966)

If $T \in B(\ell^2(\mathbb{Z}_+))$ satisfies $T(M) \subseteq M$ for all $M \in \mathrm{Lat}(\lambda(\mathbb{Z}_+))$ then T is in the w-closed algebra $W(\lambda, \mathbb{Z}_+)$ generated by $\{\lambda(n) : n \in \mathbb{Z}_+\}$ (: the analytic Toeplitz operators, $\simeq H^\infty$.)*

Thus the algebra $W(\lambda, \mathbb{Z}_+) \curvearrowright \ell^2(\mathbb{Z}_+)$ is reflexive:
Loginov - Shulman (1975): $\mathcal{W} = \mathrm{Ref}\,\mathcal{W} \equiv \{T : Tx \in \overline{\mathcal{W}x}\ \forall x\}$

Theorem (Bercovici (1994) / Li-McCarthy (1997))

For all $d \in \mathbb{N}$, the algebra $W(\lambda, \mathbb{Z}_+^d)$ is reflexive.

The Free Semigroup

Definition

Let $\mathbb{G} = \mathbb{F}_2 =< a, b >$ or generally $\mathbb{F}_n$.

- The semigroup $\mathbb{S}$ is generated by words in $a^n, b^m \, (n, m \geq 0)$.
- $\mathbb{S}$ acts on $\ell^2(\mathbb{S})$: left regular representation: $\lambda_a e_w = e_{aw}$ where $\{e_w : w \text{ word}\}$ is o.n. basis of $\ell^2(\mathbb{S})$.

Proposition (Popescu (1989))

Every $M \in \mathrm{Lat}\, W(\lambda, \mathbb{S})$ is a direct sum of cyclic subspaces, each of the form $M_w = U(\ell^2(\mathbb{S}))$ where U isometry commuting with all λ_w.

(Generalises Beurling.)

Proposition (Arias - Popescu (1995))

The algebra $W(\lambda, \mathbb{S})$ is reflexive. [in fact hyper-reflexive with constant 3 (Davidson-Pitts, Bercovici)]

(Generalises Sarason.)

The Heisenberg semigroup

Definition

The Heisenberg group $\mathbb{H}$ consists of all matrices

$$[x, y, z] \equiv \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{where}$$

if $x, y, z \in \mathbb{R}$ we call $\mathbb{H}$ continuous,
if $x, y, z \in \mathbb{Z}$ we call $\mathbb{H}$ discrete.

The semigroup $\mathbb{H}_+$ consists of all $[x, y, z]$ with $x, y \geq 0$ (but z free).

The Fourier binest algebra (S.C. Power & AK)

Here the continuous $\mathbb{H}_+$ acts on $L^2(\mathbb{R})$:

$$[s,t,\mu] = \begin{bmatrix} 1 & s & \mu \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow e^{i\mu} D_t M_s$$

where

$$(D_t f)(x) = f(x-t) \quad (f \in L^2(\mathbb{R})) \quad \text{Translations}$$
$$(M_s f)(x) = e^{isx} f(x) \quad (f \in L^2(\mathbb{R})) \quad \text{Multiplications}$$

The Weyl relations: $M_s D_t = e^{ist} D_t M_s$.

Theorem

•$\mathrm{Lat}\{M_s D_t : s,t \geq 0\}$*: is a topological manifold homeomorphic to a closed disc (the onion!).*
•*The w*-closed algebra generated by* $\{M_s D_t : s,t \geq 0\}$ *is reflexive.*

A class of representations for the discrete $\mathbb{H}_+$

The discrete $\mathbb{H}_+$ has generators
$u = [1,0,0]$, $v = [0,1,0]$, $w = [0,0,1]$; relation $uv = wvu$.
Represent $\mathbb{H}_+$ on $L^2(\mathbb{T},\nu)$ where ν is quasi-invariant and ergodic under rotations:
Fix $\lambda = e^{2\pi i\theta}$ where θ is irrational and

$$[k,m,n] = \begin{bmatrix} 1 & k & n \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \lambda^n V^m U^k$$

where

$$(Vf)(z) = \sqrt{r_\lambda(z)}f(\lambda z) \quad (f \in L^2(\mathbb{T},\nu)) \quad \text{Rotation}$$
$$(Uf)(z) = zf(z) \quad (f \in L^2(\mathbb{T},\nu)) \quad \text{Multiplication}$$

The Weyl relations $\quad UV = \lambda VU.$

A class of representations for the discrete $\mathbb{H}_+$

Theorem

Let $\mathcal{N} = \{\zeta_k H^2 : k \in \mathbb{Z}\}$ *(where* $\zeta_k(z) = z^k$*).*

1. *If* ν *is equivalent to Lebesgue measure, then the algebra* $\mathcal{W}(\pi, \mathbb{H}^+)$ *is unitarily equivalent to* $\mathrm{Alg}\,\mathcal{N}$ $\sim$ *lower triangular matrices.*
2. *If* ν *is singular to Lebesgue measure and not continuous, again one obtains* $\mathrm{Alg}\,\mathcal{N}$ *but 'with the generators reversed'.*
3. *If* ν *is singular to Lebesgue measure and is continuous, then* $\mathcal{W}(\pi, \mathbb{H}^+) = \mathcal{B}(L^2(\mathbb{T}, \nu))$.

Example

A non-reflexive representation: $\mathbb{H}_+$ acts on $H^2(\mathbb{T})$ and

$$(Uf)(z) = zf(z) \qquad (Vf)(z) = zf(\lambda z) \quad (f \in H^2(\mathbb{T})).$$

Even the WOT-closed algebra is not reflexive (fin. dim. diagonals).

The restricted left regular representation $\mathcal{T}_L(\mathbb{H}_+)$

$\mathbb{H}_+$ acts on
$\ell^2(\mathbb{H}_+) \simeq \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \simeq L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$ by

$$\begin{aligned} L_u(w^n \otimes u^k \otimes v^m) &= w^n \otimes u^{k+1} \otimes v^m \quad \text{(shift)} \\ L_v(w^n \otimes u^k \otimes v^m) &= w^{n-k} \otimes u^k \otimes v^{m+1} \quad \text{(shift)} \\ L_w(w^n \otimes u^k \otimes v^m) &= w^{n+1} \otimes u^k \otimes v^m \quad \text{(mult. by } w\text{)} \\ &(n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+. \end{aligned}$$

More generally for $f \in L^\infty(\mathbb{T})$,
$L_f(w^n \otimes u^k \otimes v^m) = fw^n \otimes u^k \otimes v^m$.

Algebraic properties of $\mathcal{T}_L(\mathbb{H}_+)$

- Tool: '2-dimensional Fourier' expansion
  $A \sim \sum_{k \geq 0, m \geq 0} L_{f_{k,m}} L_u^k L_v^m$ with centre-valued coefficients $L_{f_{k,m}}$.
- Diagonal and centre both equal to $\mathcal{M}(L^\infty(\mathbb{T})) \otimes \mathbf{1} \otimes \mathbf{1}$ (compare: Fourier binest algebra has trivial diagonal)
- No compacts (compare: Fourier binest algebra has approximate identity of compacts)
- No quasinilpotents, so semisimple
- Has the bicommutant property

Reflexivity of the restricted left regular representation

Now $[k, m, n] \to W^n V^m U^k$ acts on

$$\ell^2(\mathbb{H}_+) \simeq \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \simeq L^2(\mathbb{T}) \otimes H^2 \otimes H^2.$$

For reflexivity:
Diagonalise W as $M_\xi \otimes I \otimes I$ and on each "fiber" ξ the generators become:

$$(Uf)(z_1, z_2) = z_1 f(z_1, z_2)$$
$$(Vf)(z_1, z_2) = z_2 f(\bar{\xi} z_1, z_2), \qquad f \in H^2(\mathbb{T} \times \mathbb{T})$$

Theorem

The w-closed algebra $\mathcal{W}_\xi$ generated by U, V is reflexive for each ξ.*

Take the direct integral:

Theorem

The w-closed algebra $\mathcal{T}_L(\mathbb{H}_+)$ is reflexive.*

A Reflexivity proof

To prove reflexivity of $\mathcal{W}_\xi = \mathcal{W} \curvearrowright H^2 \otimes H^2$, generated by

$$U(v^m \otimes u^k) = v^m \otimes u^{k+1} \quad \text{and} \quad V(v^m \otimes u^k) = v^{m+1} \otimes (\lambda u)^k$$

(here $u(z_1) = z_1,\ v(z_2) = z_2$ and $\lambda = \bar{\xi}$) so

$$U = I \otimes S \quad \text{and} \quad V = S \otimes D$$

where S is the unilateral shift and $D = \mathrm{diag}(\lambda^k)$.

Take $T \in \mathrm{Ref}\,\mathcal{W}$. It has a formal Fourier series

$$T \sim \sum_{n \geq 0} S^n \otimes \widehat{T}_n \quad \text{(Fourier coeff. } \widehat{T}_n \in B(H^2)).$$

Using the dual action on $\{v^m\}$, show that $S^n \otimes \widehat{T}_n \in \mathrm{Ref}\,\mathcal{W}$.

Then show (Sarason) that we may write
$S^n \otimes \widehat{T}_n = S^n \otimes D^n f(S) = V^n f(U)$ and so $S^n \otimes \widehat{T}_n \in \mathcal{W}$.

By "Féjer", the Fourier series of T Cesaro-converges w* to T.

Hence $T \in \mathcal{W}$.

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