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The Fourier binest algebra

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Abstract

The Fourier binest algebra is defined as the intersection of the Volterra nest algebra on $L^2(\mathbb{R})$ with its conjugate by the Fourier transform. Despite the absence of nonzero finite rank operators this algebra is equal to the closure in the weak operator topology of the Hilbert–Schmidt bianalytic pseudo-differential operators. The (non-distributive) invariant subspace lattice is determined as an augmentation of the Volterra and analytic nests (the Fourier binest) by a continuum of nests associated with the unimodular functions $\exp(-isx^2/2)$ for $s > 0$. This multinest is the reflexive closure of the Fourier binest and, as a topological space with the weak operator topology, it is shown to be homeomorphic to the unit disc. Using this identification the unitary automorphism group of the algebra is determined as the semi-direct product $\mathbb{R}^2 \times_\kappa \mathbb{R}$ for the action $\kappa_t(\lambda, \mu) = (e^t\lambda, e^{-t}\mu)$.

A nest algebra is an algebra of operators on a complex Hilbert space consisting of all the bounded operators which leave invariant each subspace in a given chain of subspaces of the Hilbert space. In the present paper we write $\mathscr{N}_v$ for the *Volterra nest* in $L^2(\mathbb{R})$ consisting of the subspaces $L^2([\lambda, \infty))$, for $\lambda \in \mathbb{R}$, together with $\{0\}$ and $L^2(\mathbb{R})$, and we refer to the associated nest algebra $\mathscr{A}_v$ as the *Volterra nest algebra* on $L^2(\mathbb{R})$. To define the Fourier binest algebra we also require what we refer to as the *analytic nest* $\mathscr{N}_a$ which consists of $\{0\}$, $L^2(\mathbb{R})$ and the chain of subspaces $e^{isx}H^2(\mathbb{R})$, for $s \in \mathbb{R}$, where $H^2(\mathbb{R})$ is the usual Hardy space of boundary functions for the upper half plane. The *Fourier binest* is the subspace lattice

$$\mathscr{L} = \mathscr{N}_v \cup \mathscr{N}_a$$

and the *Fourier binest algebra* $\mathscr{A}$ is the non-self-adjoint algebra of operators which leave invariant each subspace of $\mathscr{L}$. Plainly $\mathscr{A} = \mathscr{A}_v \cap \mathscr{A}_a$, where $\mathscr{A}_a$ is the nest algebra for the analytic nest $\mathscr{N}_a$.

For the last 30 years, since their consideration by Ringrose, nest algebras have been studied intensely from a great many viewpoints. The monograph of Davidson [**4**] gives a survey of much of this theory. Their importance, even in finite-dimensions, lies in the fact that they provide the most fundamental class of noncommutative non-self-adjoint operator algebras. In the present paper, by focusing on perhaps the most natural continuous multiplicity one example, we initiate a study of binest algebras, by which we mean, simply, those algebras that are the intersection of two nest algebras. As we shall see, the Fourier binest algebra is intimately involved with analytic function theory and can be characterized in terms of the Weyl relations. Furthermore its naturalness is expressed by its intrinsic description as the weakly closed operator algebra generated by the Hilbert–Schmidt bianalytic pseudo-differential operators.

Amongst nest algebras the Volterra nest algebra plays a particularly distinguished role. Indeed, an elementary result of Kadison and Singer [**6**] asserts that each continuous nest algebra of uniform multiplicity one, which is separably acting, is unitarily equivalent to $\mathscr{A}_v$. Whilst for binests there is no direct parallel to this uniqueness, the Fourier binest algebra is nevertheless a distinguished example. In what follows we obtain the following three main results. The first is the intrinsic characterization in terms of pseudo-differential operators and the bianalytic Weyl algebra. The second result, which depends on the first, determines the lattice of invariant subspaces of $\mathscr{A}$ as a topological disc with disjoint ordering from a foliation by lines of longitude. This in turn enables the determination of the unitary automorphism group of $\mathscr{A}$ as the semi-direct product $\mathbb{R}^2 \times_\kappa \mathbb{R}$ for the action $\kappa_t(\lambda,\mu) = (e^t\lambda, e^{-t}\mu)$. That the unitary automorphism group is an elementary Lie group is in stark contrast to the situation for nest algebras themselves and is another reflection of the bianalytic nature of $\mathscr{A}$.

It will be convenient to define a pseudo-differential operator on $L^2(\mathbb{R})$ as an operator $\mathrm{Op}\,(a)$ such that

$$(\mathrm{Op}\,(a)f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(x,y)\,e^{-ixy}\tilde{f}(y)\,dy,$$

where $\tilde{f}$ is the inverse Fourier transform of $f \in L^2(\mathbb{R})$ and where $a(x,y)$, the symbol of $\mathrm{Op}\,(a)$, is a suitable function determining $\mathrm{Op}\,(a)$ as a bounded linear operator. (This is not quite the usual definition (cf. [**3**, **10**]) in that the roles of F and F^* have been exchanged.) If $\mathrm{Op}\,(a)$ is a Hilbert–Schmidt operator then the function $a(x,y)$ is necesssarily in $L^2(\mathbb{R}^2)$. Conversely, if $a(x,y)$ is such a function then $\mathrm{Op}\,(a)$ can be defined and the result is a Hilbert–Schmidt operator. It will be shown that the Fourier binest algebra is the weakly closed linear span of the Hilbert–Schmidt operators $\mathrm{Op}\,(a)$ which are bianalytic in the sense that the symbols $a(x,y)$ belong to the subspace $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ of $L^2(\mathbb{R}^2)$.

In the theory of nest algebras the finite rank operators often play a vital role. The Fourier binest algebra on the other hand contains no finite rank operators, other than zero. We expect that the operators above may nevertheless prove to be a useful substitute.

Another main tool in the study of nest algebras (and, more generally, CSL algebras) is the presence of selfadjoint operators and projections. By contrast, the Fourier binest algebra is antisymmetric in the sense that it contains no selfadjoint operators, other than scalar multiples of the identity.

The binest algebra $\mathscr{A}$ is the intersection of two reflexive algebras and so is a reflexive algebra. That is, with the usual notation, $\mathscr{A} = \mathrm{Alg}\,(\mathrm{Lat}\,\mathscr{A})$. However the binest $\mathscr{L}$ itself is not reflexive as a subspace lattice. Nevertheless we can identify its reflexive closure $\mathrm{Lat}\,(\mathrm{Alg}\,\mathscr{L})$, the lattice of invariant subspaces of $\mathscr{A}$. Curiously, as we alluded above, it turns out that $\mathrm{Lat}\,\mathscr{A}$, with the natural compact Hausdorff topology, is homeomorphic to the unit disc. As a set Lat $\mathscr{A}$ consists of $\mathscr{N}_v$ and $\mathscr{N}_a$, forming the topological boundary of Lat $\mathscr{A}$, together with a continuum of nests $\mathscr{N}_s$, indexed by a positive real parameter s, given by

$$\mathscr{N}_s = \{\phi_s K \colon K \in \mathscr{N}_a\},$$

where ϕ_s is the function $\phi_s(x) = e^{-isx^2/2}$. The order structure of Lat $\mathscr{A}$ is that of a multinest consisting of uncountably many copies of the partially ordered set $[0, 1]$ with the minimal points identified and the maximal points identified. Thus the supremum and infimum of proper elements of distinct nests are $L^2(\mathbb{R})$ and the zero subspace respectively.

We would like to thank Donald Sarason for the succinct cocycle argument used in the proof of Theorem 3·1. The characterization there of the closed subspaces of $L^2(\mathbb{R})$ which are simply invariant for the translation and multiplication semigroups also seems to be of some independent interest.

1. *Preliminaries*

We begin by setting out some useful terminology and notation and by recalling some well-known facts from the theory of Hardy spaces.

The Fourier–Plancherel transform is the unitary operator F on $L^2(\mathbb{R})$ which is the isometric extension of the linear operator on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ given by

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(y)\, dy.$$

Alternatively we can view Ff as the $\|\cdot\|_2$-limit of the sequence of functions given by integration over the intervals $[-n, n]$, for $n = 1, 2, \ldots$. The Paley–Wiener theorem implies that $F(H^2(\mathbb{R})) = L^2([0, \infty))$ and from this it follows that $FP_0F^* = Q_0$ where P_0 and Q_0 are the orthogonal projection onto $H^2(\mathbb{R})$ and $L^2([0, \infty))$ respectively. The adjoint of F is similarly defined, with e^{ixy} in place of e^{-ixy}. One can verify directly that F^2 is the symmetry induced by the reflection $x \to -x$.

Let M_λ, $\lambda \in \mathbb{R}$, be the operator of multiplication by the exponential function $e^{i\lambda x}$ on $L^2(\mathbb{R})$. Then $FM_\lambda F^*$ coincides with the translation unitary D_λ given by $(D_\lambda f)(y) = f(y - \lambda)$, $y \in \mathbb{R}$. In particular we have

$$\mathscr{N}_v = \{D_\lambda L^2([0, \infty)) \colon -\infty < \lambda < +\infty\} \cup \{0, L^2(\mathbb{R})\},$$

$$\mathscr{N}_a = \{M_\lambda H^2(\mathbb{R}) \colon -\infty < \lambda < +\infty\} \cup \{0, L^2(\mathbb{R})\}.$$

Since $FM_\lambda H^2(\mathbb{R}) = D_\lambda F H^2(\mathbb{R}) = D_\lambda L^2([0, \infty))$ we have $F\mathscr{N}_a = \mathscr{N}_v$ from which it follows that $F\mathscr{A}_a F^* = \mathscr{A}_v$. The set $\mathscr{L} = \mathscr{N}_v \cup \mathscr{N}_a$ is a complete lattice and it is straightforward to check that it has the order structure of a continuous binest.

For $\phi \in L^\infty(\mathbb{R})$, let M_ϕ be the corresponding multiplication operator on $L^2(\mathbb{R})$, and write D_ϕ for $FM_\phi F^*$. If $\lambda \geqslant 0$ then M_λ belongs to $\mathscr{A}_a$ and hence to the binest algebra $\mathscr{A}$. Similarly, if $s \geqslant 0$ then D_s belongs to $\mathscr{A}$. In particular if $\phi(x)$ and $\psi(x)$ are each a finite linear combination of exponentials $e^{i\mu x}$, with $\mu \geqslant 0$, then $M_\phi D_\psi$ is an operator in $\mathscr{A}$. This product coincides with the bianalytic pseudo-differential operator Op(a) with defining function $a(x, y) = \phi(x)\psi(y)$. In view of the Weyl commutation relations

$$M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda, \quad \text{for } \lambda, \mu \text{ in } \mathbb{R},$$

it follows that the linear span of these products is a complex operator algebra. This we refer to as the *bianalytic Weyl algebra*. As is well-known, this particular subalgebra of $\mathscr{A}$ contains no compact operators. See, for example, Coburn and Douglas [**2**].

More generally, if $\phi, \psi \in H^\infty(\mathbb{R})$ then M_ϕ, D_ψ belong to $\mathscr{A}$ because $\mathscr{A}$ is closed in the weak operator topology. If, additionally, ϕ and ψ are functions in $H^2(\mathbb{R}) \bigcap H^\infty(\mathbb{R})$ then $M_\phi D_\psi$ agrees with the pseudo-differential operator $\mathrm{Op}\,(a)$, and this has the form $(\mathrm{Int}\ k)F^*$ where $\mathrm{Int}\ k$ is the Hilbert–Schmidt integral operator with kernel function $k(x, y)$ with $k(x, y) = a(x, y)e^{-ixy}/\sqrt{2\pi} = \phi(x)\psi(y)e^{-ixy}/\sqrt{2\pi}$. It follows that if $a(x, y)$ belongs to the subspace $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ of $L^2(\mathbb{R}^2)$ then $\mathrm{Op}\,(a)$ is a Hilbert–Schmidt operator in $\mathscr{A}$. In the next section we show, conversely, that all Hilbert–Schmidt operators in the Fourier binest algebra have this particular form.

We need the following well-known version of Beurling's theorem for invariant subspaces of the shift which is due to Lax [**7**]. This may be obtained from the usual formulation for the disc by making use of the conformal equivalence with the upper half plane and the fact (already used in the last paragraph) that in the weak operator topology the set $\{M_\lambda\colon \lambda \geqslant 0\}$ has dense linear span in $\{M_\phi\colon \phi \in H^\infty(\mathbb{R})\}$. We refer the reader to Garnett [**5**] for the theory of inner functions.

Theorem. *Let K be a simply invariant subspace for the semigroup $\{M_\lambda\colon \lambda \geqslant 0\}$, so that $M_\lambda K \subseteq K$, for all $\lambda \geqslant 0$, and*

$$\bigcap_{\lambda \geqslant 0} M_\lambda K = \{0\}.$$

Then there is a unimodular function u in $L^\infty(\mathbb{R})$ such that $K = uH^2(\mathbb{R})$. In particular every simply invariant subspace which is contained in the Hardy space $H^2(\mathbb{R})$ has the form $uH^2(\mathbb{R})$ for some inner function u in $H^\infty(\mathbb{R})$.

2. *Characterizations of $\mathscr{A}$*

The first main result of this section is the following density theorem.

Theorem 2·1. *The Fourier binest algebra $\mathscr{A}$ coincides with each of the following spaces.*

(i) *The weak star closure of the bianalytic Weyl algebra.*

(ii) *The weak star closed linear span of the products $M_\phi D_\psi$ for $\phi, \psi \in H^\infty$.*

(iii) *The weak star closure of the algebra of Hilbert–Schmidt bianalytic pseudo-differential operators.*

That the weak star closures in (i) and (ii) coincide is an elementary consequence of the fact that the linear span of the analytic exponential functions is weak star dense in $H^\infty(\mathbb{R})$. Also, it is easy to see that these closures agree with the closure in (iii). Let $\mathscr{B}$ denote this closure. The proof will be completed by showing that $\mathscr{B} = \mathscr{A}$. This follows immediately from the next two lemmas.

Lemma 2·2. *Let $A \in \mathscr{A}$ be a Hilbert–Schmidt operator. Then $AF = \mathrm{Int}\, k$, where $k \in L^2(\mathbb{R}^2)$ has the following properties.*

(i) *For almost all $y \in \mathbb{R}$ the function $x \to e^{ixy}k(x, y)$ is in $H^2(\mathbb{R})$.*

(ii) *For almost all $x \in \mathbb{R}$ the function $y \to e^{ixy}k(x, y)$ is in $H^2(\mathbb{R})$.*

In particular $A = \mathrm{Op}\,(a)$ where $a \in H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$.

Proof. (i) There exists an element d in $L^2(\mathbb{R}^2)$, viewed as a function, such that $A = \operatorname{Int} d$. Since $A \in \mathscr{A}_v$ we have $A(L^2([\lambda,\infty))) \subseteq L^2([\lambda,\infty))$ for all $\lambda \in \mathbb{R}$ and so $d(x,y) = 0$ for almost all $x < y$. Redefining d on a set of two-dimensional Lebesgue measure zero, if necessary, we may assume that

$$d(x,y) = 0 \quad \text{for all } x < y. \tag{1}$$

There exists a null set $X \subseteq \mathbb{R}$ such that the function $y \to d(x,y)$ is in $L^2(\mathbb{R})$ for all $x \notin X$. For any fixed $x \notin X$, define

$$k_d(x,y) = \int_{-\infty}^{\infty} d(x,t)\, e^{-ity} dt. \tag{2}$$

More precisely, $k_d(x,\cdot)$ is the $\|\cdot\|_2$ limit of the functions $k_n(x,\cdot)$ where

$$k_n(x,y) = \int_{-n}^{n} d(x,t)\, e^{-ity} dt = \int_{-\infty}^{\infty} d(x,t)\, \chi_n(t)\, e^{-ity} dt$$

where χ_n denotes the characteristic function of the interval $[-n,n]$. Also,

$$\begin{aligned} e^{ixy} k_n(x,y) &= \int_{-\infty}^{\infty} d(x,t)\, \chi_n(t)\, e^{i(x-t)y} dt \\ &= \int_{-\infty}^{\infty} d(x,x-s)\, \chi_n(x-s)\, e^{isy} ds \\ &= \int_{0}^{\infty} d(x,x-s)\, \chi_n(x-s)\, e^{isy} ds \end{aligned}$$

by (1). Thus, by the Paley–Wiener theorem, the function $y \to e^{ixy} k_n(x,y)$ is in $H^2(\mathbb{R})$ for all $n \in \mathbb{N}$. Since $e^{ix\cdot} k_n(x,\cdot)$ converges to $e^{ix\cdot} k_d(x,\cdot)$ in $L^2(\mathbb{R})$, it follows that $y \to e^{ixy} k_d(x,y)$ is in $H^2(\mathbb{R})$ for all $x \notin X$.

(ii) Let $F^*AF = B = \operatorname{Int} b$. Since $A \in \mathscr{A}_a$ we have $AM_\lambda H^2(\mathbb{R}) \subseteq M_\lambda H^2(\mathbb{R})$ for each $\lambda \in \mathbb{R}$. Equivalently $AFL^2((-\infty,-\lambda]) \subseteq FL^2((-\infty,-\lambda])$ and so $F^*AFL^2((-\infty,-\lambda]) \subseteq L^2((-\infty,-\lambda])$. It follows that $b(x,y) = 0$ for almost all $x > y$. Redefining b we may assume that

$$b(x,y) = 0 \quad \text{for all } x > y. \tag{3}$$

As in (i), there is a null set $Y \subseteq \mathbb{R}$ such that for any fixed $y \notin Y$, we may define

$$k_b(x,y) = \int_{-\infty}^{\infty} b(t,y)\, e^{-itx} dt \tag{4}$$

in the usual sense. Then

$$\begin{aligned} e^{ixy} k_b(x,y) &= \int_{-\infty}^{\infty} b(t,y)\, e^{ix(y-t)}\, dt \\ &= \int_{-\infty}^{\infty} b(y-s,y)\, e^{isx}\, ds \\ &= \int_{0}^{\infty} b(y-s,y)\, e^{isx}\, ds \end{aligned}$$

by (3). This implies, as in (i), that the function $x \to e^{ixy} k_b(x,y)$ is in $H^2(\mathbb{R})$ for all $y \notin Y$.

Finally, it follows from (2) that $AF = \operatorname{Int} k_d$, and from (4) that $FB = \operatorname{Int} k_b$. Since $AF = FB$, the kernels k_d and k_b must be equal almost everywhere. The existence of the desired function k follows routinely from this and the final assertion follows from elementary functional analysis.

Lemma 2·3. *The Hilbert–Schmidt operators in the Fourier binest algebra are dense in the weak star topology.*

Proof. Let $h_n(x) = ni/(x + ni)$ so that $h_n \in H^\infty(\mathbb{R})$, $|h_n(x)| \leqslant 1$ for all real x and $h_n(x) \to 1$ uniformly on compact sets. Then $M_{h_n} \to I$ and $D_{h_n} \to I$ boundedly in the strong operator topology from which it follows that the Hilbert–Schmidt products $K_n = M_{h_n} D_{h_n}$, which are in $\mathscr{A}$, tend to the identity in the strong operator topology as $n \to \infty$. But now, if X is an operator in $\mathscr{A}$ then X is the strong operator topology limit of the Hilbert–Schmidt operators XK_n.

Remarks. (i) Recall that if $R \in \mathscr{A}_v$ is of finite rank then the range of R is contained in $L^2([t,\infty))$ for some real t. Similarly, if $R \in \mathscr{A}_a$ then its range is contained in $M_s H^2(\mathbb{R})$ for some real s. By the F. and M. Riesz Theorem, the intersection of these two subspaces is trivial. It follows that the Fourier binest algebra contains no nonzero finite-rank operators.

(ii) One can also see that the only selfadjoint operators in the binest algebra are trivial ones. Indeed, if $A = A^*$ is in $\mathscr{A}$, then the subspaces of the Volterra nest are reducing for $\mathscr{A}$ and so A is a multiplication operator M_f, where f is real-valued. But M_f must also leave $H^2(\mathbb{R})$ invariant, hence f must be in $H^\infty(\mathbb{R})$ and so is a constant function. In other words, the binest algebra has trivial diagonal: $\mathscr{A} \cap \mathscr{A}^* = \mathbb{C}I$.

The next theorem gives abstract characterizations of the Fourier binest algebra, and also expresses the bianalytic character of $\mathscr{A}$ in a different sense.

Let $\mathscr{U} = \{U_\lambda : \lambda \in \mathbb{R}\}$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathscr{H}$, and consider the spectral representation

$$U_\lambda = \int e^{it\lambda} dP_t.$$

By the *spectral nest* of $\mathscr{U}$ we will mean the complete nest $\mathscr{N}$ generated by $\{P_t^\perp : t \in \mathbb{R}\}$. Loebl and Muhly have characterized the nest algebra $\operatorname{Alg}\mathscr{N}$ as the set of all operators $A \in B(\mathscr{H})$ which are *$\mathscr{U}$-analytic* in the sense that, if $\alpha_\lambda = \operatorname{Ad}(U_\lambda)$ (where $\operatorname{Ad}(Z)$ denotes the map $X \to ZXZ^*$), the function $\lambda \to \operatorname{trace}(\alpha_\lambda(A)X)$ is in $H^\infty(\mathbb{R})$ for all trace class operators X. See theorem 4·2·3 of [**8**]. They also prove that every nest algebra arises in this way.

It follows that if $\mathscr{U}^1 = \{U_\lambda^{(1)}\}$ and $\mathscr{U}^2 = \{U_\lambda^{(2)}\}$ are two such groups with spectral nests $\mathscr{N}_1$ and $\mathscr{N}_2$, then the algebra $\operatorname{Alg}(\mathscr{N}_1 \cup \mathscr{N}_2)$ coincides with the set of all $(\mathscr{U}^1, \mathscr{U}^2)$-bianalytic operators, that is, all $A \in B(\mathscr{H})$ such that the functions

$$\lambda \to \operatorname{trace}\,(\alpha_\lambda^{(1)}(A)X)$$

and

$$\mu \to \operatorname{trace}\,(\alpha_\mu^{(2)}(A)X)$$

are in $H^\infty(\mathbb{R})$ for all trace class operators X.

Note that $\alpha^{(2)}$ acts *trivially* on $\mathscr{N}_2$: $\alpha_\lambda^{(2)}(P_t^\perp) = P_t^\perp$ for all t and λ. We say below that $\alpha^{(2)}$ acts *transitively* on some nest $\mathscr{N}_1$ if $\alpha_\lambda^{(2)}(\mathscr{N}_1) = \mathscr{N}_1$ for all $\lambda \in \mathbb{R}$ and if

$\alpha_\lambda^{(2)}(Q) \neq Q$ for all $\lambda \neq 0$ and all $Q \in \mathcal{N}_1$, $0 \neq Q \neq I$. In this situation (with just $\alpha^{(2)}$ and $\mathcal{N}_1$ given), it is shown in [**1**] that there exists a strongly continuous one-parameter unitary group $\{U_\lambda^{(1)}\}$, whose spectral nest is $\mathcal{N}_1$, such that the pair $\{U_\lambda^{(1)}\}, \{U_\mu^{(2)}\}$ satisfies the Weyl relations $U_\lambda^{(1)}U_\mu^{(2)} = e^{i\lambda\mu}U_\mu^{(2)}U_\lambda^{(1)}$. Conversely, if a pair of unitary one-parameter groups satisfies the Weyl relations, then each group acts transitively on the spectral nest of the other. Let us call a pair $(\mathcal{U}^1, \mathcal{U}^2)$ of strongly continuous one-parameter unitary groups satisfying the Weyl relations a *Weyl pair*.

THEOREM 2·4. *For a set $\mathcal{B}$ of bounded operators on a Hilbert space $\mathcal{H}$, the following are equivalent*:

(i) *There exists $n \in \{\infty, 1, 2, \dots\}$ such that $\mathcal{B}$ is unitarily equivalent to $\mathcal{A}\otimes B(\ell^2(n))$.*

(ii) *There exists a Weyl pair $(\mathcal{U}^1, \mathcal{U}^2)$ with spectral nests $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, such that $\mathcal{B} = \mathrm{Alg}(\mathcal{N}_1 \cup \mathcal{N}_2)$.*

(iii) *There exists a Weyl pair $(\mathcal{U}^1, \mathcal{U}^2)$ such that $\mathcal{B}$ consists of all $(\mathcal{U}^{(1)}, \mathcal{U}^{(2)})$-bianalytic operators on $\mathcal{H}$.*

(iv) *There exists a Weyl pair $(\mathcal{U}^1, \mathcal{U}^2)$ such that $\mathcal{B}$ is the weak star closed linear span of all products of the form $U_\lambda^{(1)*}U_\mu^{(2)}A$ where $\lambda \geqslant 0, \mu \geqslant 0$ and $A \in B(\mathcal{H})$ commutes with both groups.*

(v) *$\mathcal{B} = \mathrm{Alg}(\mathcal{N}_1 \cup \mathcal{N}_2)$ where the nest $\mathcal{N}_2$ admits a transitive action by a unitary one-parameter group whose spectral nest is $\mathcal{N}_1$.*

Proof. Since the automorphism groups $\{\mathrm{Ad}\, U_\lambda^{(1)}\}$ and $\{\mathrm{Ad}\, U_\mu^{(2)}\}$ commute (by the Weyl relations), the equivalence of (ii) and (iii) follows from the above observations.

If $\mathcal{B} = \mathcal{A}\otimes B(\ell^2(n))$ then setting $U_\lambda^{(1)} = D_\lambda^* \otimes I$ and $U_\mu^{(2)} = M_\mu \otimes I$, it is easy to see that the spectral nest of $U^{(1)}$ is $\mathcal{N}_1 = \mathcal{N}_a^{(n)} = \{(M_\lambda H^2(\mathbb{R}))\otimes \ell^2(n) : -\infty \leqslant \lambda \leqslant +\infty\}$ while that of $U^{(2)}$ is $\mathcal{N}_2 = \mathcal{N}_v^{(n)} = \{L^2([\mu,\infty))\otimes \ell^2(n) : -\infty \leqslant \mu \leqslant +\infty\}$, and clearly $\mathcal{B}$ is the intersection of the two nests algebras $\mathrm{Alg}(\mathcal{N}_1)$ and $\mathrm{Alg}(\mathcal{N}_2)$.

If (ii) holds, then, by the uniqueness of the Weyl relations [**9**], there exists $n = \infty, 1, 2, \dots$ and a unitary $W\colon \mathcal{H} \to L^2(\mathbb{R})^{(n)} = L^2(\mathbb{R})\otimes \ell^2(n)$ mapping $U_\lambda^{(1)}$ to $D_\lambda^*\otimes I$ and $U_\mu^{(2)}$ to $M_\mu \otimes I$. It is now clear that W will then map $\mathcal{B} = \mathrm{Alg}(\mathcal{N}_1 \cup \mathcal{N}_2)$ to $\mathcal{A}\otimes B(\ell^2(n))$.

If (iv) holds, then after a unitary equivalence we may write $U_\lambda^{(1)} = D_\lambda^* \otimes I$ and $U_\mu^{(2)} = M_\mu \otimes I$. Observing that the commutant of $\{D_\lambda^*\otimes I, M_\mu \otimes I\colon \lambda \in \mathbb{R}, \mu \in \mathbb{R}\}$ is $\mathbb{C}I \otimes B(\ell^2(n))$, we conclude that $\mathcal{B}$ is the weak star closed linear span of all products of the form $(D_\lambda M_\mu)\otimes T$ where $\lambda \geqslant 0, \mu \geqslant 0$ and $T \in B(\ell^2(n))$. Thus $\mathcal{B} = \mathcal{A}\otimes B(\ell^2(n))$ by Theorem 2·1. The converse is easy.

The equivalence of (v) and (ii) follows from [**1**] and our earlier remarks.

3. *The invariant subspace lattice of $\mathcal{A}$*

Let $\{M_\lambda\colon \lambda \geqslant 0\}$ and $\{D_\mu\colon \mu \geqslant 0\}$ be the multiplication and translation semigroups, as before. Let $s > 0$ and let K be a proper subspace of the nest $\mathcal{N}_s$. Then

$$K = M_{\phi_s}M_\lambda H^2(\mathbb{R})$$

for some real constant λ (recall that $\phi_s(x) = e^{-isx^2/2}$). Clearly K is simply invariant for the multiplication semigroup. Furthermore, for $\mu > 0$, $D_\mu M_{\phi_s} = \phi_s(\mu)M_{\phi_s}M_{\mu s}D_\mu$

and so

$$\begin{aligned} D_\mu K &= M_{\phi_s} M_{\mu s} D_\mu M_\lambda H^2(\mathbb{R}) \\ &= M_{\phi_s} M_{\mu s} M_\lambda H^2(\mathbb{R}) \\ &\subseteq K \end{aligned}$$

since $\mu s > 0$, and it follows that K is also simply invariant for the translation semigroup.

The converse is also true.

Theorem 3·1. *Every closed subspace K of $L^2(\mathbb{R})$ which is simply invariant for both the multiplication and translation semigroups is necessarily of the above form for some $s > 0$ and for some real constant λ.*

Proof. The following natural cocycle argument is due to Donald Sarason.

As K is simply invariant for the multiplication semigroup there is a unimodular function u such that $K = uH^2(\mathbb{R})$. It will be shown that $u(x) = ce^{-i(\rho\frac{1}{2}x^2+\sigma x)}$ for some $\rho > 0$ and $\sigma \in \mathbb{R}$, where c is a unimodular constant.

Since each subspace $D_t K$, for $t > 0$, is also of this form, with $D_t K \subseteq K$, it follows that $D_t K = w_t u H^2(\mathbb{R})$ for some inner function w_t. Thus w_t divides w_s, if $0 < t < s$. Moreover, $D_t K = u(x-t)H^2(\mathbb{R})$ and so we have $u(x-t) = c_t w_t(x) u(x)$ for some unimodular constant c_t which, by redefining w_t, we may take to be 1. Consider now the resulting cocycle identity

$$w_{s+t}(x) = \frac{u(x-s-t)}{u(x)} = \frac{u(x-s-t)}{u(x-s)}\frac{u(x-s)}{u(x)} = w_t(x-s)\, w_s(x).$$

This implies that for $0 < t < s$ the inner function $w_t(x-r)$ divides w_s for $0 < r < s-t$. Fix s and t with $0 < t < s$. If w_t has any zeros in the upper half-plane, then those zeros and all their translates by r with $0 < r < s-t$ must be zeros of w_s, which is impossible, since w_s is analytic in the upper half-plane. Thus w_t is a singular inner function and we can write, for some unimodular α, some real β and some singular measure μ,

$$w_t(z) = \alpha e^{i\beta z} \exp\left(i \int_{\mathbb{R}} \frac{sz+1}{s-z} \frac{1}{s^2+1} d\mu(s) \right) \quad (\operatorname{Im} z > 0).$$

Let α', β' and ν be the triple associated with w_s. Since $w_t(x-r)$ divides w_s for all r in an interval it follows that the associated r-translates of μ are dominated by ν and hence that $\mu = 0$.

To see this note first that $(w_s(z))/(w_t(z-r)) = e^{-u_r}$ where u_r is analytic in the upper half plane and Re $u_r \geqslant 0$. Also, calculation shows that

$$\text{Re } u_r(x+iy) = (\beta'-\beta)y + \int_R \frac{y}{(s-x)^2+y^2}(d\nu(s) - d\mu(s-r)).$$

Since u_r is harmonic, we obtain $\beta' - \beta \geqslant 0$ and the desired domination condition. (See, for example, theorem I·3·5c of [**5**].)

We now have $\mu(A-r) \leqslant \nu(A)$ for all Borel subsets A of $\mathbb{R}$ and so the measure μ_0 on $\mathbb{R}$ defined by

$$\mu_0(A) = \int_0^{s-t} \mu(A-r)\, dr$$

satisfies $\mu_0(A) \leqslant (s-t)\nu(A)$. Since ν is singular and μ_0 is absolutely continuous with respect to Lebesgue measure it follows that μ_0 is zero and hence so is μ.

It follows that

$$w_t(x) = \alpha(t)\, e^{i\beta(t)x}.$$

where $\alpha(t)$ is unimodular, $\beta(t)$ is nonnegative. Also, from the definition of w_t, β is strictly increasing.

Note that $\alpha(t)$ is measurable. This follows from the equation $u(x-t)/u(x) = \alpha(t)e^{i\beta(t)x}$. The quotient $u(x-t)/u(x)$ is measurable in (x,t) and is continuous in x for each fixed t. Thus for each x the quotient, and hence α, is measurable in t.

By the cocycle identity we have

$$\alpha(s+t)\, e^{i\beta(s+t)x} = \alpha(t)e^{i\beta(t)(x-s)}\, \alpha(s)\, e^{i\beta(s)x}$$

and so

$$\alpha(s+t) = \alpha(s)\alpha(t)\, e^{-i\beta(t)s}$$

and

$$\beta(s+t) = \beta(s) + \beta(t).$$

Since β is increasing it follows that β is continuous and hence that $\beta(t) = \rho t$ for some positive constant ρ.

Now define

$$\gamma(t) = \alpha(t)\, e^{i\rho\frac{1}{2}t^2}$$

Then γ is measurable and

$$\gamma(s+t) = \gamma(s)\gamma(t),$$

which implies that $\gamma(t) = e^{i\sigma t}$ for some real constant σ. Hence

$$\alpha(t) = e^{i(-\rho\frac{1}{2}t^2+\sigma t)}$$

and so

$$w_t(x) = e^{i(-\rho\frac{1}{2}t^2+\sigma t+\rho t x)} = \frac{u(x-t)}{u(x)}.$$

This equation holds for some $x = x_0$ and almost all $t > 0$ and so

$$u(x_0 - t) = u(x_0)e^{i(-\rho\frac{1}{2}t^2+\sigma t+\rho t x_0)}.$$

Equivalently

$$u(y) = ce^{i(-\rho\frac{1}{2}y^2-\sigma y)}$$

holds for almost every $y < x_0$, for some unimodular constant c depending on x_0. But in fact the last assertion holds for almost every x_0 from which we conclude that c is independent of x_0 and that the equality holds almost everywhere.

THEOREM 3·2. *The invariant projection lattice of the Fourier binest algebra is precisely the multinest consisting of the union of the binest $\mathcal{N}_v \cup \mathcal{N}_a$ with $\bigcup_{s>0}\mathcal{N}_s$ and this multinest is reflexive. Moreover, the supremum and infimum of proper elements of distinct nests are $L^2(\mathbb{R})$ and the zero subspace respectively.*

Proof. If a nonzero closed subspace of $L^2(\mathbb{R})$ is invariant for $\{M_\lambda\colon \lambda \geqslant 0\}$ and $\{D_\mu\colon \mu \geqslant 0\}$ but is not simply invariant for one of these semigroups then it is necessarily equal to $L^2(\mathbb{R})$. In view of what we have proven above it follows that the multinest, $\mathscr{M}$ say, coincides with the bi-invariant projections, that is,

$$\mathscr{M} = \mathrm{Lat}\{M_\lambda, D_\mu\colon \lambda, \mu \geqslant 0\}$$

This equality shows that $\mathscr{M}$ is a reflexive lattice. By the density of the bianalytic Weyl algebra given in Theorem 2·1(i), the bi-invariant lattice must coincide with $\mathrm{Lat}\,\mathscr{A}$, and so $\mathscr{M} = \mathrm{Lat}\,\mathscr{A}$.

The last assertion can be seen quickly from the following short argument, or from the more elementary argument of Lemma 4·3.

By the previous paragraph, the supremum and infimum of two elements of $\mathscr{M}$ must each belong to one of the subnests of $\mathscr{M}$. It suffices therefore to prove that if N and L are proper elements of $\mathscr{M}$ with $N \subseteq L$, then they must both belong to the same subnest. If one of them belongs to $\mathscr{N}_v$, then the assertion follows easily from the F. and M. Riesz theorem. Otherwise, by Lemma 3·1, $N = \phi_{s_1} K_1$ and $L = \phi_{s_2} K_2$ where s_1 and s_2 are positive and $K_1 = M_{\lambda_1} H^2(\mathbb{R}), K_2 = M_{\lambda_2} H^2(\mathbb{R})$ are in $\mathscr{N}_a$. Since $N \subseteq L$, it follows that the function

$$e^{-i(s_1 - s_2)\frac{1}{2}x^2} e^{i(\lambda_1 - \lambda_2)x}$$

is inner, and this implies that $s_1 = s_2$. (Recall (see, for example, [**5**]) that a continuous inner function on the line has the form $ce^{i\mu x}$ for some non-negative μ and unimodular constant c.)

Replacing the parametrization (s, λ), for $s > 0, \lambda \in \mathbb{R}$, with a parametrization indexed by $(-1, 1) \times (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we can identify Lat $\mathscr{A}$, as a set, with the closed unit disc $\mathbb{D}^-$ in $\mathbb{R}^2$. The left and right boundary semicircles correspond to $\mathscr{N}_a$ and $\mathscr{N}_v$ respectively. The other nests correspond to lines of longitude from the maximal element $(0, 1)$ to the minimal element $(0, -1)$, and the partial ordering on $\mathbb{D}^-$ is the corresponding disjoint ordering.

At the moment $\mathscr{N}_a$ and $\mathscr{N}_v$ have no distinguished identity in this realisation, since permutation of the parameter space $[-1, 1]$ induces a order isomorphism of $\mathbb{D}^-$. However, we see in the next section that $\mathrm{Lat}\,A$, with the topology induced by the strong operator topology, is homeomorphic to $\mathbb{D}^-$.

4. *Unitary automorphisms and the topology of Lat* ($\mathscr{A}$)

Consider the one parameter unitary group $\{V_t\colon t \in \mathbb{R}\}$ given by $(V_t f)(x) = e^{t/2} f(e^t x)$ for f in $L^2(\mathbb{R})$. In addition to the Weyl commutation relations

$$M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda$$

we have

$$V_t M_\lambda = M_{e^t \lambda} V_t,$$

$$V_t D_\mu = D_{e^{-t}\mu} V_t,$$

for t, μ, λ in $\mathbb{R}$. Let $\alpha_\lambda = \mathrm{Ad}(M_\lambda), \beta_\mu = \mathrm{Ad}(D_\mu)$ and $\gamma_t = \mathrm{Ad}(V_t)$, where $\mathrm{Ad}(Z)$ denotes the automorphism $X \to ZXZ^*$ of $B(L^2(\mathbb{R}))$. Computation shows that the commutation relations give the composition rule

$$(\alpha_{\lambda_1} \circ \beta_{\mu_1} \circ \gamma_{t_1}) \circ (\alpha_{\lambda_2} \circ \beta_{\mu_2} \circ \gamma_{t_2}) = \alpha_{\lambda_1 + e^{t_1}\lambda_2} \circ \beta_{\mu_1 + e^{-t_1}\mu_2} \circ \gamma_{t_1 + t_2}.$$

This means that the map

$$\rho\colon \begin{pmatrix} e^t & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & \mu & e^{-t} \end{pmatrix} \to \alpha_\lambda \circ \beta_\mu \circ \gamma_t$$

is a group isomorphism from the matrix group, G say, of such matrices to the group generated by the three one parameter automorphism groups. The matrix group can be viewed as the semidirect product $\mathbb{R}^2 \times_\kappa \mathbb{R}$ for the action κ of the additive group $\mathbb{R}$ on the additive group $\mathbb{R}^2$ given by

$$\kappa_t(\lambda, \mu) = (e^t\lambda, e^{-t}\mu).$$

Observe that for all t in $\mathbb{R}$ we have $V_t\mathscr{N}_v = \mathscr{N}_v, V_t\mathscr{N}_a = \mathscr{N}_a$, and also $V_t\mathscr{N}_{s_1} = \mathscr{N}_{s_2}$ where $s_2 = e^{2t}s_1$. In particular it follows that for each element g in $\mathbb{R}^2 \times_\kappa \mathbb{R}$ the automorphism $\rho(g)$ restricts to a unitary automorphism of the Fourier binest algebra. We shall show below that the converse also holds. For this we need the fact that $\mathrm{Lat}(\mathscr{A})$ is homeomorphic to the unit disc in order to identify the nests $\mathscr{N}_v$ and $\mathscr{N}_a$. From this it follows that a unitary automorphism either leaves these nests invariant or exchanges them. In fact the latter possibility cannot occur.

Lemma 4·1. *Let U be a unitary operator on $L^2(\mathbb{R})$ such that $\mathrm{Ad}\,(U)$ is an automorphism of $\mathscr{A}_a$ and of $\mathscr{A}_v$. Then $\mathrm{Ad}\,(U) = \rho(g)$ for some g in $\mathbb{R}^2 \times_\kappa \mathbb{R}$.*

Proof. Since $\mathrm{Ad}\,(U)$ gives an automorphism of $\mathscr{A}_v$, the unitary U must factor as $U = M_\phi V$, where ϕ is a unimodular function in $L^\infty(\mathbb{R})$ and V induces an order isomorphism of $\mathscr{N}_v$. (See [**4**], chapter 17.) Thus V is a composition operator C_g given by $(C_g f)(t) = (g'(t))^{\frac{1}{2}} f(g(t))$ where $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous order preserving bijection.

Consider now the order isomorphism θ of the multinest given by $\theta(M) = UM$. Since θ leaves invariant $\mathscr{N}_a$ and $\mathscr{N}_v$, we have $\theta\mathscr{N}_1 = \mathscr{N}_q$ for some positive number q. Replacing U by UV_t, for suitable t, we may assume, without loss of generality, that $\theta\mathscr{N}_1 = \mathscr{N}_1$. In particular this means that for some real number λ

$$M_\phi C_g M_{\phi_1} H^2(\mathbb{R})) = M_{\phi_1} M_\lambda H^2(\mathbb{R})$$

and hence

$$M_\phi M_{\phi_1 \circ g} C_g H^2(\mathbb{R}) = M_{\phi_1} M_\lambda H^2(\mathbb{R}).$$

On the other hand $\theta(\mathscr{N}_a) = \mathscr{N}_a$ and so, for some real number μ, $M_\phi C_g H^2(\mathbb{R}) = M_\mu H^2(\mathbb{R})$. It follows that the function $e^{-i(g(x)^2 - x^2)/2} e^{i(\mu - \lambda)x}$ must be constant, and hence that $g(x) = x - \lambda + \mu$ for all x. Therefore $C_g = D_{\lambda - \mu}$, and hence $\phi = ce^{i\mu x}$ for some unimodular constant c. Thus $U = M_\mu D_{\lambda - \mu}$, as required.

Lemma 4·2. *Let $\phi_s(x) = e^{-isx^2/2}, \psi_s(x) = e^{is^{-1}x^2/2}$. Then $F(\phi_s H^2(\mathbb{R})) = \psi_s(H^2(\mathbb{R}))^\perp$, for $s > 0$, and $F(\phi_s H^2(\mathbb{R})) = \psi_s H^2(\mathbb{R})$, for $s < 0$.*

Proof. Let $g, f \in L^2(\mathbb{R})$. Then

$$\begin{aligned} < M_{\bar{\psi}_s} F M_{\phi_s} g, f > &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-is^{-1}x^2/2} e^{-ixt} e^{-ist^2/2} g(t) \overline{f(x)}\, dt dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(xs^{-1}+t)^2 s/2} g(t) \overline{f(x)}\, dt dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iy^2 s/2} g(y - xs^{-1}) \overline{f(x)}\, dy dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y - xs^{-1}) \overline{f(x)}\, dx\, e^{-iy^2 s/2} dy. \end{aligned}$$

If $s < 0$ and $g \in H^2(\mathbb{R})$ then $g(y - xs^{-1})$, as a function of x, is in $H^2(\mathbb{R})$, and so the right hand side is zero if $f \in (H^2(\mathbb{R}))^\perp$. Thus $F(\phi_s H^2(\mathbb{R})) \subseteq \psi_s(H^2(\mathbb{R}))$ in this case. Similarly, $F^*(\psi_s H^2(\mathbb{R})) \subseteq \phi_s H^2(\mathbb{R})$, and so equality holds. The case $s > 0$ is similar. ∎

The following partial converse of the above lemma, which is a direct consequence of Lemma 3·1, may be of independent interest.

Corollary 4·3. *Let ϕ, ψ be unimodular functions in $L^\infty(\mathbb{R})$ such that $M_\phi^* F M_\psi$ maps $H^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. Then there exists $s > 0$ such that*

$$\phi(x) = a e^{i\lambda x} \phi_s(x) \quad \text{and} \quad \psi(x) = b e^{is^{-1}\lambda x} \psi_s(x)$$

for some real λ and unimodular constants a, b.

Proof. Let $K = M_\psi H^2(\mathbb{R})$ and $L = M_\phi H^2(\mathbb{R})$. By assumption $F(K) = L$. Clearly $F^*(L) = K$ is simply invariant under $\{M_\lambda : \lambda \geqslant 0\}$ and so L is simply invariant under $\{D_\lambda : \lambda \geqslant 0\}$. But L is also invariant under the multiplication semigroup. Thus, by Lemma 3·1, $L = M_{\phi_s} M_\lambda H^2(\mathbb{R})$ and ϕ has the asserted form.

On the other hand, by the previous lemma we have that $M_{\phi_s}^* F M_{\psi_s} H^2(\mathbb{R}) = H^2(\mathbb{R})$, since $s > 0$, i.e. $F M_{\psi_s} H^2(\mathbb{R}) = M_{\phi_s} H^2(\mathbb{R})$, so that

$$\begin{aligned} M_\lambda F M_{\psi_s} H^2(\mathbb{R}) &= M_\lambda M_{\phi_s} H^2(\mathbb{R}) \\ &= F M_\psi H^2(\mathbb{R}). \end{aligned}$$

It follows that

$$F D_{-\lambda} M_{\psi_s} H^2(\mathbb{R}) = F M_\psi H^2(\mathbb{R})$$

so that $\psi(x) = b\psi_s(x + \lambda)$, as required. ∎

The next simple lemma is useful for understanding the topology on the multinest which is induced by the strong operator topology. It also provides another way of seeing the disjoint order structure of the multinest.

Lemma 4·4. *Let g_n, g be functions in $H^2(\mathbb{R})$, with $g \neq 0$, and let s_n be positive real numbers for which $e^{-is_n x^2/2} g_n(x)$ converges to $g(x)$ in $L^2(\mathbb{R})$. Then $s_n \to 0$ as $n \to \infty$.*

Proof. Suppose that some subsequence $(s_{n_k}^{-1})$ of (s_n^{-1}) converges to l as $n \to \infty$. By Lemma 4·2, $F(\phi_{s_{n_k}} g_{n_k}) = \psi_{s_{n_k}} h_k$ where $h_k \in (H^2(\mathbb{R}))^\perp$. By our assumptions it follows that h_k is a Cauchy sequence with limit h in $H^2(\mathbb{R})^\perp$. But now $(Fg)(x) = e^{ilx^2/2} h(x)$ which is absurd, if g is nonzero, since Fg lies in $L^2[0, \infty)$.

THEOREM 4·5. *There exists a homeomorphism of the closed unit disc* $\mathbb{D}^-$ *onto the multinest* $\mathrm{Lat}\,\mathscr{A}$, *with the strong operator topology, which maps the boundary onto the binest* $\mathscr{N}_a \cup \mathscr{N}_v$. *In particular, the multinest is compact.*

Proof. Parametrise the set $\mathbb{D}^- \backslash \{\pm i\}$ as

$$\{t \cos\phi + i \sin\phi\colon t \in [-1, 1], \phi \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)\}.$$

Setting $s = (1+t)/(1-t) \in [0, \infty]$ and $\lambda = \tan\phi \in (-\infty, +\infty)$, write $P_{s,\lambda}$ for the projection onto the subspace

$$K_{s,\lambda} = D_\lambda M_{\phi_s} H^2(\mathbb{R}) = M_{\lambda s} M_{\phi_s} H^2(\mathbb{R}), \quad \text{when } s \in [1, \infty),$$

write $P_{s,\lambda}$ for the projection onto the subspace

$$K_{s,\lambda} = M_\lambda M_{\phi_s} H^2(\mathbb{R}), \quad \text{when } s \in [0, 1],$$

and write $P_{\infty,\lambda}$ for the projection onto

$$K_{\infty,\lambda} = L^2([\lambda, \infty)).$$

Mapping i to the identity and $-i$ to the zero operator, we obtain a bijection of the closed unit disc onto the multinest. It is clear that this bijection maps the left semicircle $\{-\cos\phi + i\sin\phi\colon \phi \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]\}$ onto the analytic nest $\mathscr{N}_a = \{K_{0,\lambda}\colon -\infty \leqslant \lambda \leqslant +\infty\}$ and the right semicircle $\{\cos\phi + i\sin\phi\colon \phi \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]\}$ onto the Volterra nest $\mathscr{N}_v = \{K_{\infty,\lambda}\colon -\infty \leqslant \lambda \leqslant +\infty\}$. Since $\mathbb{D}^-$ is compact, it remains to prove that this mapping is continuous.

First, it is immediate that as $s \to s_0 \in [1, \infty)$ and $\lambda \to \lambda_0 \in \mathbb{R}$, the projections $P_{s,\lambda} = D_\lambda M_{\phi_s} P_{0,0} M_{\phi_s}^* D_\lambda^*$ converge strongly to P_{s_0,λ_0}. The mapping is therefore continuous at (s_0, λ_0) for $1 \leqslant s_0 < \infty$. Similarly there is continuity if $s_0 \in [0, 1]$ and so the mapping is continuous on the union of $\mathbb{D}$ and the left semicircle.

Next, observe that for each real λ, the projections $P_{s,\lambda}$ converge strongly to $P_{\infty,\lambda}$ as $s \to +\infty$. Indeed, for $s > 1$,

$$\begin{aligned} FK_{s,\lambda} &= FD_\lambda M_{\phi_s} H^2(\mathbb{R}) \\ &= M_{-\lambda} F M_{\phi_s} H^2(\mathbb{R})) \\ &= M_{-\lambda} M_{\psi_s} (H^2(\mathbb{R}))^\perp \end{aligned}$$

by Lemma 4·2. Therefore as $s \to \infty$ the projections $FP_{s,\lambda}F^*$ converge strongly to the projection onto $M_{-\lambda}(H^2(\mathbb{R}))^\perp$. From this it follows that $P_{s,\lambda}$ converges to the projection $P_{\infty,\lambda}$ onto $L^2[\lambda, \infty)$.

As a consequence one easily sees, using the fact that $P_{s,\lambda_1} \leqslant P_{s,\lambda} \leqslant P_{s,\lambda_2}$ when $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, that the projections $P_{s,\lambda}$ converge weakly, hence strongly to P_{∞,λ_0} as $s \to +\infty$ and $\lambda \to \lambda_0 \in \mathbb{R}$. Indeed, given $g \in L^2(\mathbb{R})$ and $\epsilon > 0$ first choose $\lambda_1 < \lambda_0 < \lambda_2$ such that $|{<}(P_{\infty,\lambda} - P_{\infty,\lambda_0})g, g{>}| < \epsilon$ whenever $\lambda \in (\lambda_1, \lambda_2)$. Then choose $s_0 > 0$ such that $|{<}(P_{s,\lambda_i} - P_{\infty,\lambda_i})g, g{>}| < \epsilon$ for $i = 1, 2$ when $s > s_0$. It follows that $|{<}(P_{s,\lambda} - P_{\infty,\lambda_0})g, g{>}| < 2\epsilon$ whenever $\lambda \in (\lambda_1, \lambda_2)$ and $s > s_0$.

This proves that the map

$$t \cos\phi + i \cos\sin\phi \;\to\; P_{s(t),\lambda(\phi)}$$

is continuous on $\mathbb{D}^- \backslash \{\pm i\}$. To prove that it is continuous at $-i$ is equivalent to proving that as $\phi \to -\tfrac{1}{2}\pi$ the projections $P_{s(t),\lambda(\phi)}$ converge to 0 uniformly in $t \in$

$[-1,1]$. Fix $g \in L^2(\mathbb{R})$. By what has been proved so far, for each $\phi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, the function $t \to f_\phi(t) =< P_{s(t),\lambda(\phi)}g, g >$ is continuous on the compact set $[-1,1]$. Also for each fixed $t \in [-1,1]$, $f_\phi(t)$ decreases monotonically to 0 as $\phi \to -\frac{1}{2}\pi$. Hence $\lim_{\phi \to -\frac{1}{2}\pi} f_\phi(t) = 0$ uniformly in t by Dini's theorem. The proof of continuity at i is similar.

Theorem 4·6. *Let π be a unitary automorphism of the Fourier binest algebra. Then $\pi = \rho(g)$ for a unique $g \in G$.*

Proof. If $\rho(g)(A) = A$ for all $A \in \mathscr{A}$, then $\rho(g)(M_\lambda) = M_\lambda$ and $\rho(g)(D_\lambda) = D_\lambda$ for all $\lambda \in \mathbb{R}$. It easily follows from the commutation relations that g must be the identity of the group G. This settles uniqueness.

Now let $\pi = \mathrm{Ad}(U)$ determine an automorphism of $\mathscr{A}$. Then the map $\theta_U : K \to U(K)$ is a continuous order automorphism of the partially ordered compact Hausdorff space $\mathrm{Lat}\,\mathscr{A}$. Since $\mathrm{Lat}\,\mathscr{A}$ is homeomorphic to the unit disc, θ_U must map the boundary $\mathscr{N}_v \cup \mathscr{N}_a$ onto itself. Since θ_U also preserves order and maps the minimal and maximal elements to themselves, it follows that either $\theta_U(\mathscr{N}_a) = \mathscr{N}_v$ or $\theta_U(\mathscr{N}_a) = \mathscr{N}_a$. In the latter case, of necessity, $\theta_U(\mathscr{N}_v) = \mathscr{N}_v$ and Lemma 4·1 completes the proof.

It will be enough then to show that there is no unitary operator U with $\theta_U \mathscr{N}_a = \mathscr{N}_v$ and $\theta_U \mathscr{N}_v = \mathscr{N}_a$.

Assume then that U is a unitary operator which exchanges the Volterra nest and the analytic nest. Then $\mathscr{N}_v = \theta_U \mathscr{N}_a = \theta_{UF} \mathscr{N}_v$ and, moreover, θ_{UF} is an order preserving bijection of $\mathscr{N}_v$ onto itself. Consequently $UF = M_\phi C_g$ for some unitary multiplication operator M_ϕ and composition operator C_g determined by an increasing bijection $g\colon \mathbb{R} \to \mathbb{R}$. But now, since $\theta_{F^*} \mathscr{N}_v = \mathscr{N}_a^\perp$ our assumption that $\theta_U \mathscr{N}_v = \mathscr{N}_a$ leads to

$$\mathscr{N}_a = \theta_U \mathscr{N}_v = \theta_{UF} \mathscr{N}_a^\perp.$$

Thus, for some real numbers λ_1, λ_2, with $\lambda_2 < \lambda_1$,

$$H^2(\mathbb{R}) = M_\phi C_g M_{\lambda_1} H^2(\mathbb{R})^\perp$$

and

$$M_1 H^2(\mathbb{R}) = M_\phi C_g M_{\lambda_2} H^2(\mathbb{R})^\perp.$$

Hence, with $\mu = \lambda_2 - \lambda_1$, we have

$$M_1 H^2(\mathbb{R}) = M_\phi C_g (M_\mu M_{\lambda_1} H^2(\mathbb{R})^\perp) = M_{e^{i\mu g}} (M_\phi C_g M_{\lambda_1} H^2(\mathbb{R})^\perp) = M_{e^{i\mu g}} H^2(\mathbb{R}).$$

Thus $g(x) = \mu^{-1} x + c$ for some real constant c. However, since μ^{-1} is negative this contradicts the fact that g is increasing.

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Added in proof. In 'Completely contractive representations for some doubly generated antisymmetric operator algebras' to appear in *Proc. Amer. Math. Soc.*, the second author has shown that the unitary automorphisms of the Fourier binest algebra are precisely the isometric automorphisms.

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