Notes on the C^*-envelope and the Šilov Ideal

Evgenios Kakariadis

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1 Definitions

A (concrete) operator space is a (usually) closed linear subspace X of $\mathcal{B}(K,H)$, for Hilbert spaces H, K (indeed the case $H = K$ usually suffices, via the canonical inclusion $\mathcal{B}(K,H) \subset \mathcal{B}(H \oplus K)$). However, sometimes we want to keep track too of the norm $\| \cdot \|_{n,m}$ that $\mathcal{M}_{n,m}(X)$ inherits from $\mathcal{M}_{n,m}(\mathcal{B}(H,K))$, for all $n, m \in \mathbb{N}$. An abstract operator space is a pair $(X, \{\| \cdot \|_n\}_{n \geq 1})$, consisting of a vector space, and a norm on $\mathcal{M}_n(X)$ for all $n \in \mathbb{N}$, such that there exists a complete isometry $u : X \to \mathcal{B}(K,H)$. In this case we call the sequence $\{\| \cdot \|_n\}_{n \geq 1}$ an operator space structure on the vector space X. An operator structure on a normed space $(X, \| \cdot \|)$, will usually mean a sequence of matrix norms as above, but with $\| \cdot \| = \| \cdot \|_1$ ([3, 1.2.2]).

If X is a linear subspace of a C^*-algebra $\mathcal{C}$, then X is an operator space with the matrix norm structure inherited by a faithful representation of $\mathcal{C}$.

Let X be an operator space and $\phi : X \to \mathcal{B}(H)$ a linear map. We define $\phi_n := id_n \otimes \phi : \mathcal{M}_n(X) \to \mathcal{B}(H^n)$ by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. We call ϕ *completely positive, completely contractive or completely isometry* if ϕ_n is positive, contractive or isometry, for every $n \in \mathbb{N}$.

An *operator system* is a selfadjoint linear subspace S of a unital C^*-algebra, that contains the unit. We usually require that the C^*-algebra is generated by S.

We can use the decomposition of an element $x \in S$ in the sum of two positive elements in S, i.e. $x = (1\|x\| + x)/2 + (1\|x\| - x)/2$, to prove that

> a unital linear map $\phi : S \to \mathcal{B}(H)$ is completely positive iff it is completely contractive.

A *concrete operator algebra* $\mathcal{A}$ is a closed subalgebra of some $\mathcal{B}(H)$. Then an operator algebra is both an operator space (with the operator structure inherited by $\mathcal{B}(H)$) and a Banach algebra. Conversely, if $\mathcal{A}$ is both an (abstract) operator space and a Banach algebra, then we call $\mathcal{A}$ an *(abstract) operator algebra* if there exist a Hilbert space H and a complete isometric homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$([3]).

We will consider only unital operator algebras. Note that if $\mathcal{A}$ is an operator algebra then $\mathcal{A} + \mathcal{A}^*$ is an operator system. Also, If $\mathcal{C} = C^*(\mathcal{A})$, then $\mathcal{C} = \mathcal{C}^*(\mathcal{A} + \mathcal{A}^*)$.

Let $\mathcal{X} \subseteq C^*(\mathcal{X})$ be an operator space. Given two unital completely contractive maps $\phi_k : \mathcal{X} \to \mathcal{B}(H_k)$, $k = 1, 2$, we write $\phi_1 \leq \phi_2$ if $H_1 \subseteq H_2$ and $P_{H_1}\phi_2(x)|_{H_1} = \phi_1(x)$, $x \in \mathcal{X}$; ϕ_2 is called a *dilation* of ϕ_1 and ϕ_1 is called a *compression* ϕ_2. The relation $\leq$ is transitive and one has $\phi_1 \leq \phi_2$ and $\phi_2 \leq \phi_1$ iff $(H_1, \phi_1) = (H_2, \phi_2)$. Thus $\leq$ defines a partial ordering of ucc maps of $\mathcal{X}$. Of course there is always a trivial way in dilating a ucc simply by taking the direct sum with any other ucc map.

A dilation ϕ_2 of ϕ_1 need not satisfy $H_2 = [C^*(\phi_2(\mathcal{X}))H_1]$, but it can always be replaced with a smaller dilation of ϕ_1 that has this property; in consequence the dimension of H_2 has an upper bound in terms of the dimension of H_1 and the cardinality of $\mathcal{X}$.

In general, we can have the following scheme. Let $\mathcal{X} \subseteq C^*(\mathcal{X})$ be a unital operator space and $\phi : \mathcal{X} \to \mathcal{B}(H)$ a completely contractive map. Then ϕ extends uniquely to a ucp map $\tilde{\phi}$ of the operator system $S = \mathcal{X} + \mathcal{X}^*$ (see [2]). *Arveson's Extension Theorem* implies that there is a completely positive (thus completely contractive) map $\psi : C^*(S) = C^*(\mathcal{X}) \to \mathcal{B}(H)$ extending $\tilde{\phi}$. Now, we can apply *Stinespring's Dilation Theorem* on ψ, so that there is a Hilbert space $K \supseteq H$ and a unital representation $\pi : C^*(\mathcal{X}) \to \mathcal{B}(K)$ such that $\psi(c) = P_H\pi(c)|_H$, for every $c \in C^*(\mathcal{X})$. When $K = [\pi(C^*(S))H]$, π is called *minimal Stinespring dilation* and it is unique up to unitary equivalence. Hence, $\pi|_{\mathcal{X}}$ is a dilation of ϕ.

Remark 1.1 Note that if $\phi : \mathcal{X} \to \mathcal{B}(H)$ is a ucis map and $\psi : \mathcal{X} \to \mathcal{B}(K)$ is a ucc dilation of ϕ, then ψ is also ucis. This happens because

$$\begin{aligned}\|[x_{ij}]\| &= \|[\phi(x_{ij})]\| = \|[P_H\psi(x_{ij})|_H]\| \\ &= \|(1_\nu \otimes P_H)[\psi(x_{ij})]|_{H^{(\nu)}}\| \leq \|[\psi(x_{ij})]\| \leq \|[x_{ij}]\|,\end{aligned}$$

for every $x_{ij} \in \mathcal{X}$.

Definitions 1.2 1. ([1]) A ucc map $\phi : \mathcal{X} \to \mathcal{B}(H)$ is said to be *maximal* if it has no nontrivial dilations, i.e. $\phi' \geq \phi \Rightarrow \phi' = \phi \oplus \psi$, for some ucc map ψ.
2. ([1]) A ucc map $\pi : \mathcal{X} \to \mathcal{B}(H)$ is said to have the *unique extension property* if

i. π has a unique completely positive extension $\tilde{\pi} : \mathcal{C}^*(\mathcal{X}) \to \mathcal{B}(H)$,
ii. $\tilde{\pi} : C^*(\mathcal{X}) \to \mathcal{B}(H)$ is a representation of $C^*(\mathcal{X})$ on $\mathcal{H}$.

Remark 1.3 The unique extension property for $\pi : \mathcal{X} \to \mathcal{B}(H)$ is equivalent to the assertion that every extension of π to a ucp map $\phi : C^*(\mathcal{X}) \to \mathcal{B}(H)$ should be a *-homomorphism of $C^*(\mathcal{X})$.

Proposition 1.4 *A ucc map $\phi : \mathcal{X} \to \mathcal{B}(H)$ is maximal if, and only if, it has the unique extension property.*

Proof. Assume first that ϕ is maximal and let $\widetilde{\phi} : C^*(\mathcal{X}) \to B(H)$ be a completely positive extension of it. We have to show that $\widetilde{\phi}$ is a *-homomorphism. By Stinesprings theorem, there is a representation $\pi : C^*(\mathcal{X}) \to B(K)$ on a Hilbert space $K \supseteq H$ such that $\widetilde{\phi}(x) = P_H \pi(x)|_H, x \in C^*(\mathcal{X})$. We can assume that the dilation is minimal in that $K = [\pi(C^*(\mathcal{X}))H] = [C^*(\pi(\mathcal{X}))H]$. By maximality of ϕ, $K = H$ and $\widetilde{\phi} = \pi$ is a *-homomorphism.

Conversely, suppose ϕ has the unique extension property and let $\psi : \mathcal{X} \to \mathcal{B}(K)$ be a dilation of ϕ, such that $K = [C^*(\psi(\mathcal{X}))H]$. It suffices to show that $K = H$ and $\psi = \phi$. By the Arveson's extension theorem, ψ can be extended to a ucp map $\tilde{\psi} : C^*(\mathcal{X}) \to \mathcal{B}(K)$. Since the compression of $\tilde{\psi}$ to H defines a ucp map of $C^*(\mathcal{X})$ to $\mathcal{B}(H)$ that restricts to ϕ on $\mathcal{X}$, the unique extension property implies that $P_H \tilde{\psi}(\cdot) P_H$ is a *-homomorphism of $C^*(\mathcal{X})$. So for $c \in C^*(\mathcal{X})$,

$$P_H \tilde{\psi}(c)^* P_H \tilde{\psi}(c) P_H = P_H \tilde{\psi}(c^* c) P_H \geq P_H \tilde{\psi}(c)^* \tilde{\psi}(c) P_H,$$

since $\tilde{\psi}(c^* c) \geq \tilde{\psi}(c)^* \tilde{\psi}(c)$. Thus $|(1 - P_H)\tilde{\psi}(c) P_H|^2 \leq 0$. Hence, H is invariant under the set of operators $\tilde{\psi}(C^*(\mathcal{X})) \supseteq \phi(\mathcal{X})$, and therefore under $C^*(\phi(\mathcal{X}))$. Thus $K = [C^*(\psi(\mathcal{X}))H] = H$ and it follows that $\psi = \phi$.

Proposition 1.5 ([1, theorem 3.1] *Invariance Principle*). *Let $\mathcal{X}_k \subseteq C^*(\mathcal{X}_k)$, $k = 1, 2$, be two operator spaces and let $\theta : \mathcal{X}_1 \to \mathcal{X}_2$ be a ucis and onto map. For every maximal ucis map $\phi_1 : \mathcal{X}_1 \to \mathcal{B}(H)$, the ucis map $\phi_2 : \mathcal{X}_2 = \theta(\mathcal{X}_1) \to \mathcal{B}(H)$, defined by $\phi_2 \circ \theta = \phi_1$ is also maximal.*

Proof. Consider the ucis map $\phi_2 : \mathcal{X}_2 \to \mathcal{B}(H)$ defined by $\phi_2 = \phi_1 \circ \theta^{-1}$. It suffices to show that ϕ_2 is maximal, given that ϕ_1 is maximal. To this end, let $\phi : \mathcal{X}_2 \to \mathcal{B}(K)$ be a dilation of ϕ_2, with $K = [C^*(\phi(\mathcal{X}_2))H]$ (thus ϕ is ucis map). Then $\psi \equiv \phi \circ \theta$ is a ucis map of $\mathcal{X}_1$ to $\mathcal{B}(K)$ that compresses to ϕ_1 and satisfies $K = [C^*(\psi(\mathcal{X}_2))H] = [C^*(\phi \circ \theta(\mathcal{X}_1))H] = [C^*(\phi(\mathcal{X}_1))H]$. Thus, by maximality of ϕ_1 we have that $\phi_1 = \phi = \phi \circ \theta$, hence $\phi = \phi_1 \circ \theta^{-1} = \phi_2$. □

2 Theorems of Existence

The crucial theorem is the following.

Theorem 2.1 ([1, theorem 1.3]) *Let $\mathcal{X}$ be an operator space. Then every ucis map $\phi : \mathcal{X} \to \mathcal{B}(H_0)$ dilates to a maximal ucis map $\rho : \mathcal{X} \to \mathcal{B}(H)$.*

To prove this, we have to make some remarks. First of all, if there is a chain of ucis maps $\phi_1 \leq \phi_2, \leq \cdots$ with $H_1 \subseteq H_2 \subseteq \cdots$, then we can define a ucis map ϕ_∞ on $H_\infty = \overline{\cup_n H_n}$ such that $P_{H_n}\phi_\infty|_{H_n} = \phi_n$. To see this, first observe that if $a_n \in \mathcal{B}(H_n)$ is a sequence of operators, such that $H_n \subseteq H_{n+1}$, $\sup\{\|a_n\| : n \in \mathbb{N}\} < +\infty$ and $a_n = P_{H_n}a_{n+1}|_{H_n}$, then we can define a *unique* operator $a \in \mathcal{B}(H_\infty)$, where $H_\infty = \overline{\cup_n H_n}$, such that $P_{H_n}a|_{H_n}$ for every $n \in \mathbb{N}$. Also, we get that $\|a\| = \sup_n \|a_n\|$. If we wish to do the same thing for a chain of ucc map (H_n, ϕ_n), we set $a_n = \phi_n(x)$ and $\phi_\infty(x) := a$. Uniqueness establishes the existence of (H_∞, ϕ_∞). Also, for every $x_{ij} \in \mathcal{X}$, we get that

$$\|[\phi_n(x_{ij})]\| = \|[x_{ij}]\|,$$

for every $n \in \mathbb{N}$, since ϕ_n are ucis maps. Thus, by taking supremum we get that $\|[\phi_\infty(x_{ij})]\| = \|[x_{ij}]\|$, so ϕ_∞ is ucis.

The same is true if, instead of $\mathbb{N}$ we have a limit ordinal λ, and a chain of ucc maps (H_α, ϕ_α), in the sense that for every $\alpha, \beta < \lambda$ with $\alpha \leq \beta$, then $\phi_\alpha \leq \phi_\beta$.

Also, we have the following definition.

Definition 2.2 Let $\phi : \mathcal{X} \to \mathcal{B}(H)$ be a ucis map and let $\mathcal{F}$ be a (possibly empty) subset of $\mathcal{X} \times H$. We will say that ϕ is maximal on $\mathcal{F}$ if for every dilation ψ of ϕ acting on $K \supseteq H$, we have,

$$\psi(x)\xi = \phi(x)\xi, \quad (x, \xi) \in \mathcal{F}.$$

A ucis map $\phi : \mathcal{X} \to \mathcal{B}(H)$ is maximal if and only if it is maximal on $\mathcal{X} \times H$. If ϕ is maximal on $\mathcal{F} \subseteq \mathcal{X} \times H$ and $\psi \geq \phi$, then ψ is maximal on $\mathcal{F}$.

Lemma 2.3 *For every ucis representation $\phi : \mathcal{X} \to \mathcal{B}(H)$ and every $(x, \xi) \in \mathcal{X} \times H$, there is a dilation of ϕ that is maximal on (x, ξ).*

Proof. Since for every dilation $\psi \geq \phi$ we have $\|\psi(x)\xi\| \leq \|x\| \, \|\xi\| \leq +\infty$, we can find a dilation ϕ_1 of ϕ for which $\|\phi_1(x)\xi\|$ is as close to $\sup\{\|\psi(x)\xi\| : \psi \geq \phi, \psi \text{ is a ucc map}\}$. Note that ϕ_1 will also be a ucis map of $\mathcal{X}$. Continuing inductively, we find a sequence of ucis representations $\phi \leq \phi_1 \leq \phi_2 \leq \cdots$, such that $\phi_n : \mathcal{X} \to \mathcal{B}(H_n)$, $H \subseteq H_1 \subseteq H_2 \subseteq \cdots$, and

$$\|\phi_{n+1}(x)\xi\| \geq \sup_{\psi \geq \phi_n} \|\psi(x)\xi\| - 1/n.$$

Let H_∞ be the closure of the union $\cup_n H_n$ and let $\phi_\infty : \mathcal{X} \to \mathcal{B}(H_\infty)$ be the unique ucis representation that compresses to ϕ_n on H_n, for every n. Note that ϕ_∞ is maximal on (x, ξ). Indeed, if $\psi \geq \phi_\infty$ then $\psi \geq \phi_n$, for every $n \geq 1$, and

$$\|\phi_\infty(x)\xi\| \geq \left\|P_{H_{n+1}}\phi_\infty(x)\xi\right\| = \|\phi_{n+1}(x)\xi\| \geq \|\psi(x)\xi\| - 1/n.$$

Hence, $\|\phi_\infty(x)\xi\| \geq \|\psi(x)\xi\|$. It follows that

$$\begin{aligned} \|\psi(x)\xi - \phi_\infty(x)\xi\|^2 &= \|\psi(x)\xi - P_{H_\infty}\psi(x)\xi\|^2 \\ &= \|\psi(x)\xi\|^2 - \|\phi_\infty(x)\xi\|^2 \leq 0, \end{aligned}$$

so that $\psi(x)\xi = \phi_\infty(x)\xi$, as asserted. □

Proof of Theorem 2.1. We show first that ϕ_0 can be dilated to a ucis map $\phi_1 : \mathcal{X} \to \mathcal{B}(H_1)$ that is maximal on $\mathcal{X} \times H_0$. To that end, let λ be an ordinal sufficiently large that there is a surjection $\alpha \in \lambda \mapsto x_\alpha \in \mathcal{X} \times \mathcal{H}_0$; hence, $\mathcal{X} \times H_0 = \{x_\alpha : \alpha \in \lambda\}$. We claim that there is a family of ucis maps $\phi_\alpha : \mathcal{X} \to \mathcal{B}(H_\alpha)$, indexed by the ordinals $\alpha \leq \lambda$, which satisfy $\phi_\alpha \geq \phi_0$ together with

1. ϕ_α is maximal on $\{x_\beta : \beta < \alpha\}$,
2. $\alpha \leq \beta \Rightarrow \phi_\alpha \leq \phi_\beta$.

Once the existence of this family is established, one can set $\phi_1 = \phi_\lambda$.

Proceeding inductively, for $\alpha = 0$ we set $\phi_\alpha = \phi_0$, noting that (1) is vacuous for $\alpha = 0$. Assuming that $\alpha \leq \lambda$ is an ordinal for which $\{\phi_\beta : \beta < \alpha\}$ has been defined and satisfies (1) and (2) on the initial segment $\{\beta < \alpha\}$, define α as follows:

i. If α has an immediate predecessor $\alpha - 1$, then the previous lemma implies that $\phi_{\alpha-1}$ can be dilated to a ucis map $\phi_\alpha : \mathcal{X} \to \mathcal{B}(H_\alpha)$, that is maximal on x_α.

ii. If α is a limit ordinal, then the Hilbert spaces H_β, $\beta < \alpha$, are linearly ordered by inclusion; we take H_α to be the closure of their union and $\phi_\alpha : \mathcal{X} \to \mathcal{B}(H_\alpha)$ to be the unique ucis map that compresses to ϕ_β on H_β, for every $\beta < \alpha$.

In either case, properties (1) and (2) persist for the augmented family $\{\phi_\beta : \beta \leq \alpha\}$. This defines $\{\phi_\alpha : \alpha \leq \lambda\}$.

Now one can use ordinary induction on the preceding result to find an increasing sequence of Hilbert spaces $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ and ucis maps $\phi_n : \mathcal{X} \to \mathcal{B}(H_n)$ such that ϕ_{n+1} is a dilation on $\mathcal{X} \times H_n$, $n = 0, 1, 2 \ldots$. Let H_∞ be the closure of $\cup_n H_n$ and let $\phi_\infty : \mathcal{X} \to \mathcal{B}(H_\infty)$ be the unique ucis map that compresses to ϕ_n on H_n, for every $n \geq 1$. Note that every dilation $\psi : \mathcal{X} \to \mathcal{B}(K)$ of ϕ_∞ and every $n \geq 1$, both ψ and ϕ_∞ are dilations of ϕ_{n+1}, so by maximality of ϕ_{n+1} on $\mathcal{X} \times H_n$ we have

$$\psi(x)\xi = \phi_{n+1}(x)\xi = \phi_\infty(x)\xi, \quad (x, \xi) \in \mathcal{X} \times H_n.$$

It follows that ϕ_∞ is maximal on $\mathcal{X} \times \cup_n H_n$, hence on its closure $\mathcal{X} \times H_\infty$. $\square$

Now, let $\iota : \mathcal{X} \to \mathcal{B}(H)$ be a ucis map. Then $C^*(\iota(\mathcal{X})) \subseteq \mathcal{B}(H)$ is said to be a *C*-cover of $\mathcal{X}$*. We can define a C*-cover of $\mathcal{X}$ with the following universal property.

Definition 2.4 *Let $\mathcal{X}$ be a unital operator space. The $C_e^*(\mathcal{X}) = C^*(\iota(\mathcal{X}))$ is a C^*-algebra with the following (universal) property:*

> *for every ucis map $\phi : \mathcal{X} \to C^*(\phi(\mathcal{X})) = \mathcal{C}$ there exists a unique representation $\pi : \mathcal{C} \to C_e^*(\mathcal{X})$, such that π is onto and $\pi(\phi(a)) = \iota(a)$, for every $a \in \mathcal{X}$.*

Definition 2.5 *Let $\mathcal{X} \subseteq C^*(\mathcal{X})$ a unital operator space. A boundary ideal for $\mathcal{X}$ is an ideal $J \subseteq C^*(\mathcal{X})$ with the property that the natural projection of $C^*(\mathcal{X})$ onto $C^*(\mathcal{X})/J$ restricts to a ucis map on $\mathcal{X}$. The Šilov ideal is a boundary ideal which contains every other boundary ideal.*

We can see that if the $C_e^*(\mathcal{X})$ exists, then it is unique up to *-isomorphism. Also if the Šilov ideal exist, then it is unique. In the following we prove the existence of the C^*-envelope for an operator space and thus the existence of the Šilov ideal.

Theorem 2.6 *Every operator space has a C^*-envelope. Thus the Šilov ideal exists.*

Proof. Let an operator space $\mathcal{X}$ acting on a Hilbert space K. Then the inclusion map $\iota : \mathcal{X} \to \mathcal{B}(K)$ is a ucis map and thus dilates to a ucis maximal map $\gamma : \mathcal{X} \to \mathcal{B}(H)$. We claim that $C^*(\gamma(\mathcal{X}))$ is the $C^*_e(\mathcal{X})$.
To this end, suppose $\psi : \mathcal{X} \to \mathcal{B}(H_\psi)$ is a ucis map. In this case $\sigma : \psi(\mathcal{X}) \to \mathcal{B}(H) : \psi(a) \mapsto \gamma(a)$ is also ucis map (and thus well-defined). By Invariance Principle we get that σ is maximal for the unital operator space $\psi(\mathcal{X})$, hence it extends uniquely to a *-homomorphism $\widetilde{\sigma} : C^*(\psi(\mathcal{X})) \to \mathcal{B}(H)$. Then $\widetilde{\sigma}(\psi(x)) = \sigma(\psi(x)) = \gamma(x)$, for every $x \in \mathcal{X}$. Also $\widetilde{\sigma}(C^*(\psi(\mathcal{X}))) = C^*(\widetilde{\sigma}(\psi(\mathcal{X}))) = C^*(\gamma(\mathcal{X}))$, hence $\widetilde{\sigma}$ is onto. So, $C^*(\gamma(\mathcal{X}))$ has the (universal) property of the C*-envelope.
Now let $\mathcal{X} \subseteq C^*(\mathcal{X})$. Then there exists an onto representation $\pi : C^*(\mathcal{X}) \to C^*(\gamma(\mathcal{X}))$. We will prove that $\ker \pi$ is the Šilov ideal. First of all, it is boundary since the map $\tilde{\pi} : C^*(\mathcal{X})/\ker \pi \to C^*(\gamma(\mathcal{X}))$ is a *-isomorphism, hence completely isometric, and $\pi(a) = \gamma(a)$. Also note that since $\pi(a) = \gamma(a) = \gamma \circ id(a)$ and γ is a maximal, then by the invariance principle we get that π is also maximal. Now assume that I is another bounary ideal and let q_I the natural projection of $C^*(\mathcal{X})$ onto $C^*(\mathcal{X})/I$. Define the map $\psi : q_I(\mathcal{X}) \to \mathcal{B}(H)$, such that $\psi(q_I(a)) = \pi(a)$. This map is ucis and thus has a ucp extension $\tilde{\psi} : C^*(\mathcal{X})/I \to \mathcal{B}(H)$. Then $\tilde{\psi} \circ q_I$ is a ucp extension of π. But π is maximal, thus $\pi(c) = \tilde{\psi}(q_I(c))$, for every $c \in C^*(\mathcal{X})$. Hence, for $c \in I$ we get that $\pi(c) = \tilde{\psi}(q_I(c)) = 0$, so $c \in \ker \pi$. Hence $I \subseteq \ker \pi$. □

But we can follow the converse direction as well.

Theorem 2.7 *Let $\mathcal{X} \subseteq C^*(\mathcal{X})$ be an operator space. Then the Šilov ideal J exists and thus the C^*-envelope of $\mathcal{X}$ exists.*

Proof. Let an operator space $\mathcal{X}$ acting on a Hilbert space K. Then the inclusion map $\iota : \mathcal{X} \to \mathcal{B}(K)$ is a ucis map and thus dilates to a ucis maximal map $\gamma : \mathcal{X} \to \mathcal{B}(H)$. Thus γ has the unique extension property. Let $\pi : C^*(\mathcal{X}) \to \mathcal{B}(H)$ be the extension representation. We claim that $\ker \pi$ is the Šilov ideal.
First of all, we have that $\|a + \ker \pi\| = \|\pi(a)\| = \|\gamma(a)\| = \|a\|$ (the same argument holds for all the matrix norms as well), thus $\ker \pi$ is a boundary ideal. Now assume that I is another boundary ideal and let q_I the natural projection of $C^*(\mathcal{X})$ onto $C^*(\mathcal{X})/I$. Define the map $\psi : q_I(\mathcal{X}) \to \mathcal{B}(H)$, such that $\psi(q_I(a)) = \pi(a) = \gamma(a)$. This map is ucis and thus has a ucp extension $\tilde{\psi} : C^*(\mathcal{X})/I \to \mathcal{B}(H)$. Then $\tilde{\psi} \circ q_I$ is a ucp extension of $\pi|_{\mathcal{X}} = \gamma$. But γ is maximal, thus $\pi(c) = \tilde{\psi}(q_I(c))$, for every $c \in C^*(\mathcal{X})$. Hence, for $c \in I$ we get that $\pi(c) = \tilde{\psi}(q_I(c)) = 0$, so $c \in \ker \pi$. Hence $I \subseteq \ker \pi$.
To finish the proof we have to prove the universal property for $C^*(\mathcal{X})/\ker \pi$.

We have that $\tilde{\pi} : C^*(\mathcal{X})/\ker\pi \to \mathcal{B}(H)$ is faithful, thus ucis. Let $\phi : \mathcal{X} \to C^*(\phi(\mathcal{X}))$ a ucis map and consider the ucis map $\tilde{\pi} \circ q \circ \phi^{-1} : \phi(\mathcal{X}) \to \mathcal{B}(H)$. Since $\tilde{\pi} \circ q(a) = \pi(a)$, for every $a \in \mathcal{X}$ and π is maximal, then by the invariance principle, $\tilde{\pi} \circ q \circ \phi^{-1}$ is also maximal. Let $\sigma_0 : C^*(\phi(\mathcal{X})) \to \mathcal{B}(H)$ be its unique extension representation. Then $\sigma_0(\phi(a)) = \tilde{\pi}(q(a))$ and therefore $\sigma_0(\phi(a)\phi(a)^*) = \sigma_0(\phi(a))\sigma_0(\phi(a))^* = \tilde{\pi}(q(a))\tilde{\pi}(q(a))^* = \tilde{\pi}(q(aa^*))$. Also $\sigma_0(\phi(a)^*\phi(a)) = \tilde{\pi}(q(a^*a))$. Hence $\sigma_0(C^*(\phi(\mathcal{X}))) = C^*(\sigma_0 \circ \phi(\mathcal{X})) = C^*(\tilde{\pi}(q(\mathcal{X}))) = \tilde{\pi} \circ q(C^*(\mathcal{X})) = \tilde{\pi}(C^*(\mathcal{X})/\ker\pi)$. So the map $\sigma : C^*(\phi(\mathcal{X})) \to C^*(\mathcal{X})/\ker\pi$ defined by $\sigma = \tilde{\pi}^{-1} \circ \sigma_0$ is a representation onto $C^*(\mathcal{X})/\ker\pi$ with $\sigma(\phi(a)) = \tilde{\pi}^{-1} \circ \sigma_0(\phi(a)) = \tilde{\pi}^{-1} \circ \tilde{\pi}(a) = q(a)$. □

3 Operator Algebras

3.1 Unital operator algebras

In the case where $\mathcal{X}$ is a unital operator algebra $\mathcal{A}$ we can have also the following definitions.

Definitions 3.1 1. ([4]) A representation $\phi : \mathcal{A} \to \mathcal{B}(H)$ is a *∂-representation* if whenever $\psi : \mathcal{A} \to \mathcal{B}(K)$ dilates ϕ, then H reduces $\psi(\mathcal{A})$.
2. ([4]) A *boundary representation* of a unital operator algebra $\mathcal{A}$ consists of the following three

i. a completely isometric homomorphism $\phi : \mathcal{A} \to \mathcal{C}$,
ii. where $\mathcal{C} = C^*(\phi(\mathcal{A}))$ is a C^*-algebra,
iii. $\pi : \mathcal{C} \to \mathcal{B}(H)$ is a representation of $\mathcal{C}$ such that the only completely positive map on $\mathcal{C}$ agreeing with π on $\phi(\mathcal{A})$ is π itself.

Theorem 3.2 ([4, theorem 1.1]) *Let $\mathcal{A}$ a unital operator algebra. Then $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a ∂-representation if, and only if, given any ucis map $\phi : \mathcal{A} \to \mathcal{C}$, where $\mathcal{C} = C^*(\phi(\mathcal{A}))$, there exists a boundary representation $\pi : C^*(\phi(\mathcal{A})) \to \mathcal{B}(H)$, such that $\pi \circ \phi = \rho$.*

Proof. Suppose first that $\phi : \mathcal{A} \to C^*(\phi(\mathcal{A})) = \mathcal{C}$ is a ucis and $\pi : \mathcal{C} \to \mathcal{B}(H)$ is a boundary representation. Set $\rho = \pi \circ \phi$. Then ρ is a representation and a ucc map. Suppose $\nu : \mathcal{A} \to \mathcal{B}(K)$ be a dilation of ρ. We will show that H reduces $\nu(\mathcal{A})$.
To this end, we define a map $\gamma : \phi(\mathcal{A}) \to \mathcal{B}(K)$ by $\gamma(\phi(a)) = \nu(a)$, $a \in \mathcal{A}$. This map is ucc, since $\|\gamma(\phi(a))\| = \|\nu(a)\| \leq \|a\| = \|\phi(a)\|$. So, by the Arveson's extension theorem, extends to ucp map $\tilde{\gamma} : \mathcal{C} \to \mathcal{B}(K)$, with $\tilde{\gamma} \circ \phi = \gamma \circ \phi = \nu$. Now, the map $c \mapsto P_H\tilde{\gamma}(c)|_H$, $c \in \mathcal{C}$ is ucp and by

definition $P_H\gamma(\phi(a))|_H = P_H\nu(a)|_H = \rho(a) = \pi(\phi(a))$, for all $a \in \mathcal{A}$. Since π *is a boundary representation*, in fact we have that $P_H\tilde{\gamma}(c)|_H = \pi(c)$, for all $c \in \mathcal{C}$. Hence, for all $a \in \mathcal{A}$, we get

$$\begin{aligned}\rho(a)\rho(a)^* &= \pi(\phi(a))\pi(\phi(a))^* = \pi(\phi(a)\phi(a)^*)\\ &= P_H\gamma(\phi(a)\phi(a)^*)|_H \geq P_H\gamma(\phi(a))\gamma(\phi(a)^*)|_H\\ &= P_H\nu(a)\nu(a)^*|_H \geq P_H\gamma(\phi(a))P_H\gamma(\phi(a)^*)|_H\\ &= P_H\nu(a)P_H\nu(a)^*|_H = \rho(a)\rho(a)^*.\end{aligned}$$

Hence, $P_H\nu(a)\nu(a)^*|_H = P_H\nu(a)P_H\nu(a)^*|_H$, thus $\nu(a)^*H \subseteq H$. A similar argument gives that $\nu(a)H \subseteq H$. Thus, H reduces $\nu(\mathcal{A})$.
For the converse, let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be a *∂-representation* and $\phi : \mathcal{A} \to C^*(\phi(\mathcal{A})) = \mathcal{C}$ a ucis map. Then we can define the ucis map $\phi^{-1} : \phi(\mathcal{A}) \to \mathcal{A}$ and we get the ucc map $\rho \circ \phi^{-1} : \phi(\mathcal{A}) \to \mathcal{B}(H)$. Then, by the Arveson's extension theorem there is a ucp map $\pi : \mathcal{C} \to \mathcal{B}(H)$, such that $\pi \circ \phi = \rho$. We will show that π is a boundary representation.
To this end let $\tilde{\pi} : \mathcal{C} \to \mathcal{B}(K)$ the (minimal) Stinespring dilation of π. Then $\tilde{\pi} \circ \phi$ is a dilation of ρ and since ρ is a ∂-representation, H reduces $\tilde{\pi} \circ \phi(\mathcal{A})$. Hence,

$$\begin{aligned}\pi(\phi(a)\phi(a)^*) &= P_H\tilde{\pi}(\phi(a)\phi(a)^*)|_H = P_H\tilde{\pi}(\phi(a))\tilde{\pi}(\phi(a)^*)|_H\\ &= P_H\tilde{\pi}(\phi(a))P_H\tilde{\pi}(\phi(a)^*)|_H = \pi(\phi(a))\pi(\phi(a)^*).\end{aligned}$$

A same argument gives also that $\pi(\phi(a)^*\phi(a)) = \pi(\phi(a)^*)\pi(\phi(a))$. Hence, $\phi(\mathcal{A})$ is in the multiplicative domain of π. Thus π is a representation of $\mathcal{C}$, since $\phi(\mathcal{A})$ generates $\mathcal{C}$. Now, let $r : \mathcal{C} \to \mathcal{B}(H)$ a ucp map, such that $r(\phi(a)) = \pi(\phi(a))(= \rho(a))$, for all $a \in \mathcal{A}$. Then, (the same argument shows that) r is also a representation, thus $r = \pi$, since $r|_{\phi(\mathcal{A})} = \pi|_{\phi(\mathcal{A})}$ and $\phi(\mathcal{A})$ generates $\mathcal{C}$. $\square$

The notion of ∂-representations is pretty much the same with that of the maximal ucc maps. In fact we have the following.

Proposition 3.3 *A ucc map $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a ∂-representation if and only if ρ is maximal.*

Proof. Let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be ∂-representation. Let the ucis map $id : \mathcal{A} \to C^*(\mathcal{A})$. Then by the previous theorem, there exists a boundary representation $\pi : C^*(\mathcal{A}) \to \mathcal{B}(H)$, such that $\pi \circ id = \rho$. Hence, π is an extension of ρ. Since π is boundary representation, it is the unique ucp extension of ρ (and also a representation of $\mathcal{C}$). Thus ρ has the unique extension property.

Conversely, let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be a ucc maximal map; then ρ has the unique extension property. Let $\pi : C^*(\mathcal{A}) \to B(H)$ be the unique extension of ρ which is *-homomorphism. Then $\rho = \pi|_{\mathcal{A}}$ is a homomorphism. Now, let $\nu : \mathcal{A} \to \mathcal{B}(K)$ be a dilation of ρ. Then, by the maximality of ρ, $\nu = \rho \oplus \psi$ for some $\psi : \mathcal{A} \to B(K \ominus H)$. Hence H is $\nu(\mathcal{A})$-reducing. Thus ρ is a ∂-representation. $\square$

The following is immediate.

Theorem 3.4 ([4, theorem 1.2]) *Every ucis representation $\rho : \mathcal{A} \to \mathcal{B}(H)$ dilates to a ucis ∂-representation $\rho' : \mathcal{A} \to \mathcal{B}(K)$.* $\square$

Remark 3.5 As we have seen in the proof of the existence of the C*-envelope, $C^*_{env}(\mathcal{A})$ is a C*-cover of $\mathcal{A}$, say $C^*(\iota(\mathcal{A}))$, where ι is a ucis maximal map. The previous proposition, induces that ι is a ∂-representation, hence ι is a homomorphism of $\mathcal{A}$.

3.2 Non-unital operator algebras

But what happens when $\mathcal{A}$ is a non-unital operator algebra? Even in that case we can have the existence of a C*-cover with the same universal property, with that of the C*-envelope of a unital operator algebra, which of course we will call *the C*-envelope of the operator algebra.* This can be proven easily if we pass to *the unitization of a non-unital operator algebra.* So, let us have a brief talk on this unitization.

Let $\mathcal{A}$ be a non-unital operator algebra, regarded as a subalgebra of some $\mathcal{B}(H)$, then a unitization of $\mathcal{A}$ may be obtained by taking $\mathcal{A}^1 = \mathrm{span}\{\mathcal{A}, I_H\}$, and is also an operator algebra.

Theorem 3.6 (Meyer) *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an operator algebra and assume that $I_H \not\in \mathcal{A}$. Let $\pi : \mathcal{A} \to \mathcal{B}(K)$ be a contractive (resp. completely contractive, isometric or completely isometric) homomorphism, K being a Hilbert space. We let $\mathcal{A}^1 = \mathrm{span}\{\mathcal{A}, I_H\} \subseteq \mathcal{B}(H)$, and we extend π to $\pi^1 : \mathcal{A}^1 \to \mathcal{B}(K)$ by letting*

$$\pi^1(a + \lambda I_H) = \pi(a) + \lambda I_K, \quad a \in \mathcal{A}, \lambda \in \mathbb{C}.$$

Then π^1 is a contractive (resp. completely contractive, isometric or completely isometric) homomorphism.

Hence, up to completely isometric isomorphism, this unitization does not depend on the embedding $\mathcal{A} \subseteq \mathcal{B}(H)$. Consequently, $\mathcal{A}^1$ will be called *the* unitization of $\mathcal{A}$ and is usually used without any reference to a concrete embedding of $\mathcal{A}$ in $\mathcal{B}(H)$.

It is clear, also, that if $\mathcal{C}$ is a unital operator algebra with unit denoted by $1_{\mathcal{C}}$ and if $\mathcal{A} \subseteq \mathcal{C}$ is a non-unital subalgebra, then $\mathcal{A}^1$ may be taken to be $\text{span}\{\mathcal{A}, 1_{\mathcal{C}}\} \subseteq \mathcal{C}$. If, in particular, $\mathcal{A}, \mathcal{B}$ are non-unital operator algebras with $\mathcal{A} \subseteq \mathcal{B}$, then the units of $\mathcal{A}^1$ and $\mathcal{B}^1$ may be identified and $\mathcal{A}^1$ may be viewed as a unital subalgebra of $\mathcal{B}^1$.

If $\mathcal{A}$ is an already unital operator algebra then Meyer's result shows that there is an essentially unique unital operator algebra containing $\mathcal{A}$ completely isometrically as a codimension 1 ideal. Again we write this strictly larger algebra as $\mathcal{A}^1$.

Remark 3.7 Let $\mathcal{B}$ be a C*-cover of an operator algebra $\mathcal{A}$. Then it is easy to check that every cai of $\mathcal{A}$ is a cai for $\mathcal{B}$ (since, for every $a \in \mathcal{A}$ we have that $e_t a \to a \Rightarrow a^* e_t \to a$). Hence, if $\mathcal{A}$ is approximately unital, then $\mathcal{B}$ is a unital C*-algebra if and only if $\mathcal{A}$ is unital. Indeed, if $\mathcal{A}$ is unital, then as we saw, $\mathcal{B}$ is also unital (with the same unit). Now, if $\mathcal{B}$ is unital and $\mathcal{A}$ approximately unital with (e_t) cai, then (e_t) is cai for $\mathcal{B}$ as well. So, $e_t = e_t 1_{\mathcal{B}} \to 1_{\mathcal{B}}$. Since $\mathcal{A}$ is closed, we get that $1_{\mathcal{B}} \in \mathcal{A}$.

Remark 3.8 But it may happen a C*-cover of a non-unital operator algebra to be unital. For example, let U be the bilateral shift of $\ell^2(\mathbb{Z})$ and $\mathcal{A}$ to be the closed linear span of polynomials $\sum_{n=1}^{k} \lambda_n U^n$. Then $\mathcal{A}$ is not unital, but its C*-cover is unital, since $U^*U = 1$. For this reason we make the following convention.

Let $(\mathcal{B}, j)$ be a C*-cover of a non-unital operator algebra $\mathcal{A}$, and let $\mathcal{B} \curvearrowright H$ and $\mathcal{A} \curvearrowright K$, for some Hilbert spaces H, K. If $\mathcal{B}$ is non-unital, then using Meyer's theorem, we get that $j : \mathcal{A} \to \mathcal{B} \subseteq \mathcal{B}(H)$ extends uniquely to the ucis $j^1 : \mathcal{A}^1 \to \mathcal{B}^1 \subseteq \mathcal{B}(H)$. On the other hand, if $\mathcal{B}$ is unital, then we identify $\mathcal{B}$ to $\mathcal{B}^1$, and $j : \mathcal{A} \to \mathcal{B}(H)$ extends uniquely to $j^1 : \mathcal{A}^1 \to \mathcal{B}(H)$, such that $j^1(1_{\mathcal{A}}) = j^1(I_K) = I_H = 1_{\mathcal{B}}$; hence $j^1(A^1) \subseteq \mathcal{B}$.

Definition 3.9 We define *a C*-envelope of a non-unital operator algebra* $\mathcal{A}$ to be a pair $(\mathcal{B}, \iota)$, where B is the C*-subalgebra generated by the copy $\iota(\mathcal{A})$ of $\mathcal{A}$ inside a C*-envelope $(C^*_{env}(\mathcal{A}^1), \iota)$ of the unitization $\mathcal{A}^1$ of $\mathcal{A}$.

The following theorem provides that a C*-envelope of a non-unital operator algebra is unique up to *-isomorphisms, thus we can refer to it as *the* C*-envelope.

Theorem 3.10 *Let $\mathcal{A}$ be an operator algebra and let $(C^*_{env}(\mathcal{A}), \iota)$ be a C*-envelope of $\mathcal{A}$. Then ι is a homomorphism and $C *_{env} (\mathcal{A})$ has the following universal property:*

> *given a C*-cover $(\mathcal{B}, j)$ of $\mathcal{A}$, there exists a (necessarily unique and surjective) *-homomorphism $\pi : \mathcal{B} \to C^*_{env}(\mathcal{A})$, such that $\pi \circ j = \iota$.*

Proof. If $\mathcal{A}$ is unital, then this is already proven. Now, let $\mathcal{A}$ be non-unital and $(\mathcal{B}, j)$ a C*-cover of $\mathcal{A}$. Then, j extends to a completely isometric unital homomorphism $j^1 : \mathcal{A}^1 \to \mathcal{B}^1$ whose range generates $\mathcal{B}^1$ as a C*-algebra. Thus there is a unique and surjective *-homomorphism $\rho : \mathcal{B}^1 \to C^*_{env}(\mathcal{A}^1)$, such that $\rho \circ j^1 = \iota$, where $\iota : \mathcal{A}^1 \to C^*_{env}(\mathcal{A}^1)$ is the canonical embedding. Let $\pi = \rho|_{\mathcal{B}}$; then π is a *-homomorphism with

$$\pi(j(a)) = \rho(j^1(a)) = \iota(a) \in C^*_{env}(\mathcal{A}),$$

for all $a \in \mathcal{A}$. Since π is a *-homomorphism, $\mathcal{B} = C^*(j(\mathcal{A}))$ and $\pi(j(\mathcal{A})) \subseteq C^*_{env}(\mathcal{A})$, we get that $\pi(\mathcal{B}) \subseteq C^*_{env}(\mathcal{A})$. Also, since $C^*_{env}(\mathcal{A}) = C^*(\iota(\mathcal{A})) \subseteq C^*_{env}(\mathcal{A}^1)$, we get that $C^*_{env}(\mathcal{A}) = C^*(\iota(\mathcal{A})) = C^*(\pi \circ j(\mathcal{A})) = \pi(C^*(j(\mathcal{A}))) = \pi(\mathcal{B})$. Hence, π is onto. □

References

[1] William Arveson. Notes on the unique extension property. *http://math.berkeley.edu/ arveson/Dvi/unExt.pdf*, 2006.

[2] William B. Arveson. On subalgebras of C^*-algebras. *Bull. Amer. Math. Soc.*, 75:790–794, 1969.

[3] D. P. Blecher and C. Le Merdy. *Operator algebras and their modules—an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series.* The Clarendon Press Oxford University Press, Oxford, 2004. Oxford Science Publications.

[4] Michael A. Dritschel and Scott A. McCullough. Boundary representations for families of representations of operator algebras and spaces. *J. Operator Theory*, 53(1):159–167, 2005.