NOTES ON THE C^*-ENVELOPE AND THE ŠILOV IDEAL

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1. Definitions

A (concrete) operator space is a (usually) closed linear subspace X of $\mathcal{B}(K,H)$, for Hilbert spaces H, K (indeed the case $H = K$ usually suffices, via the canonical inclusion $\mathcal{B}(K,H) \subset \mathcal{B}(H \oplus K)$). However, sometimes we want to keep track too of the norm $\|\cdot\|_{n,m}$ that $\mathcal{M}_{n,m}(X)$ inherits from $\mathcal{M}_{n,m}(\mathcal{B}(H,K))$, for all $n, m \in \mathbb{N}$. An abstract operator space is a pair $(X, \{\|\cdot\|_n\}_{n\geq 1})$, consisting of a vector space, and a norm on $\mathcal{M}_n(X)$ for all $n \in \mathbb{N}$, such that there exists a complete isometry $u : X \to \mathcal{B}(K,H)$. In this case we call the sequence $\{\|\cdot\|_n\}_{n\geq 1}$ an operator space structure on the vector space X. An operator structure on a normed space $(X, \|\cdot\|)$, will usually mean a sequence of matrix norms as above, but with $\|\cdot\| = \|\cdot\|_1$ ([3, 1.2.2]).

If X is a linear subspace of a C^*-algebra $\mathcal{C}$, then X is an operator space with the matrix norm structure inherited by a faithful representation of $\mathcal{C}$.

Let X be an operator space and $\phi : X \to \mathcal{B}(H)$ a linear map. We define $\phi_n := id_n \otimes \phi : \mathcal{M}_n(X) \to \mathcal{B}(H^n)$ by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. We call ϕ *completely positive, completely contractive or completely isometry* if ϕ_n is positive, contractive or isometry, for every $n \in \mathbb{N}$.

An *operator system* is a selfadjoint linear subspace S of a unital C^*-algebra, that contains the unit. We usually require that the C^*-algebra is generated by S.

We can use the decomposition of an element $x \in S$ in the sum of two positive elements in S, i.e. $x = (1\|x\| + x)/2 + (1\|x\| - x)/2$, to prove that

> a unital linear map $\phi : S \to \mathcal{B}(H)$ is completely positive iff it is completely contractive.

A *concrete operator algebra* $\mathcal{A}$ is a closed subalgebra of some $\mathcal{B}(H)$. Then an operator algebra is both an operator space (with the operator structure inherited by $\mathcal{B}(H)$) and a Banach algebra. Conversely, if $\mathcal{A}$ is both an (abstract) operator space and a Banach algebra, then we call $\mathcal{A}$ an *(abstract) operator algebra* if there exist a Hilbert space H and a complete isometric homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$([3]).

We will consider only unital operator algebras. Note that if $\mathcal{A}$ is an operator algebra then $\mathcal{A}+\mathcal{A}^*$ is an operator system. Also, If $\mathcal{C} = C^*(\mathcal{A})$, then $\mathcal{C} = \mathcal{C}^*(\mathcal{A}+\mathcal{A}^*)$.

Let $\mathcal{A} \subseteq C^*(\mathcal{A})$ be an operator algebra. Given two unital completely contractive maps $\phi_k : \mathcal{A} \to \mathcal{B}(H_k)$, $k = 1, 2$, we write $\phi_1 \leq \phi_2$ if $H_1 \subseteq H_2$ and

$P_{H_1}\phi_2(x)|_{H_1} = \phi_1(x)$, $x \in \mathcal{A}$; ϕ_2 is called a *dilation* of ϕ_1 and ϕ_1 is called a *compression* ϕ_2. The relation $\leq$ is transitive and one has $\phi_1 \leq \phi_2$ and $\phi_2 \leq \phi_1$ iff $(H_1, \phi_1) = (H_2, \phi_2)$. Thus $\leq$ defines a partial ordering of ucc maps of $\mathcal{A}$. Of course there is always a trivial way in dilating a ucc simply by taking the direct sum with any other ucc map.

A dilation ϕ_2 of ϕ_1 need not satisfy $H_2 = [C^*(\phi_2(\mathcal{A}))H_1]$, but it can always be replaced with a smaller dilation of ϕ_1 that has this property; in consequence the dimension of H_2 has an upper bound in terms of the dimension of H_1 and the cardinality of $\mathcal{A}$.

In general, we can have the following scheme. Let $X \subseteq C^*(X)$ be a unital operator space and $\phi : X \to \mathcal{B}(H)$ a completely contractive map. Then ϕ extends uniquely to a ucp map $\tilde{\phi}$ of the operator system $S = X + X^*$ (see [2]). *Arveson's Extension Theorem* implies that there is a completely positive (thus completely contractive) map $\psi : C^*(S) = C^*(X) \to \mathcal{B}(H)$ extending $\tilde{\phi}$. Now, we can apply *Stinespring's Dilation Theorem* on ψ, so that there is a Hilbert space $K \supseteq H$ and a unital representation $\pi : C^*(X) \to \mathcal{B}(K)$ such that $\psi(c) = P_H\pi(c)|_H$, for every $c \in C^*(X)$. When $K = [\pi(C^*(S))H]$, π is called *minimal Stinespring dilation* and it is unique up to unitary equivalence. Hence, $\pi|_X$ is a dilation of ϕ.

Of course, the previous scheme can be applied in case $X = \mathcal{A}$ is an operator algebra and ϕ is a unital completely isometry.

Definition 1.1. A *representation* of an operator algebra $\mathcal{A}$ on a Hilbert space H is an algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ that is also a completely contractive map.

We can see that this definition contains the case when $\mathcal{A}$ is a C^*-algebra, since a homorphism of a C^*-algebra is a *-homomorphism, iff it is completely contractive, iff it is just contractive.

Definitions 1.2. 1. ([4]) A representation $\phi : \mathcal{A} \to \mathcal{B}(H)$ is a *∂-representation* if whenever $\psi : \mathcal{A} \to \mathcal{B}(K)$ dilates ϕ, then H reduces $\psi(\mathcal{A})$.
2. ([4]) A *boundary representation* of a unital operator algebra $\mathcal{A}$ consists of the following three

i. a completely isometric homomorphism $\phi : \mathcal{A} \to \mathcal{C}$,
ii. where $\mathcal{C} = C^*(\phi(\mathcal{A}))$ is a C^*-algebra,
iii. $\pi : \mathcal{C} \to \mathcal{B}(H)$ is a representation of $\mathcal{C}$ such that the only completely positive map on $\mathcal{C}$ agreeing with π on $\phi(\mathcal{A})$ is π itself.

3. ([1]) A ucc map $\phi : \mathcal{A} \to \mathcal{B}(H)$ is said to be *maximal* if it has no nontrivial dilations, i.e. $\phi' \geq \phi \Rightarrow \phi' = \phi \oplus \psi$, for some ucc map ψ.
4. ([1]) A ucc map $\pi : \mathcal{A} \to \mathcal{B}(H)$ is said to have the *unique extension property* if

i. π has a unique completely positive extension $\tilde{\pi} : C^*(\mathcal{A}) \to \mathcal{B}(H)$,
ii. $\tilde{\pi} : C^*(\mathcal{A}) \to \mathcal{B}(H)$ is a representation of $C^*(\mathcal{A})$ on $\mathcal{H}$.

Remarks 1.3. 1. The unique extension property for $\pi : \mathcal{A} \to \mathcal{B}(H)$ is equivalent to the assertion that every extension of π to a ucp map $\phi : C^*(\mathcal{A}) \to \mathcal{B}(H)$ should be multiplicative on $C^*(\mathcal{A})$.

2. Note that the restriction $\pi|_{\mathcal{A}}$ of a boundary representation $\pi : C^*(\mathcal{A}) \to \mathcal{B}(H)$ of the ucis map $id : \mathcal{A} \to C^*(\mathcal{A})$ has the unique extension property.

Theorem 1.4. *([4, theorem 1.1]) Let $\mathcal{A}$ a unital operator algebra. Then $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a ∂-representation if, and only if, given any ucis map $\phi : \mathcal{A} \to \mathcal{C}$, where $\mathcal{C} = C^*(\phi(\mathcal{A}))$, there exists a boundary representation $\pi : C^*(\phi(\mathcal{A})) \to \mathcal{B}(H)$, such that $\pi \circ \phi = \rho$.*

Proof. Suppose first that $\phi : \mathcal{A} \to C^*(\phi(\mathcal{A})) = \mathcal{C}$ is a ucis and $\pi : \mathcal{C} \to \mathcal{B}(H)$ is a boundary representation. Set $\rho = \pi \circ \phi$. Then ρ is a representation and a ucc map. Suppose $\nu : \mathcal{A} \to \mathcal{B}(K)$ be a dilation of ρ. We will show that H reduces $\nu(\mathcal{A})$.
To this end, we define a map $\gamma : \phi(\mathcal{A}) \to \mathcal{B}(K)$ by $\gamma(\phi(a)) = \nu(a)$, $a \in \mathcal{A}$. This map is ucc, since $\|\gamma(\phi(a))\| = \|\nu(a)\| \leq \|a\| = \|\phi(a)\|$. So, by the Arveson's extension theorem, extends to ucp map $\tilde{\gamma} : \mathcal{C} \to \mathcal{B}(K)$, with $\tilde{\gamma} \circ \phi = \gamma \circ \phi = \nu$. Now, the map $c \mapsto P_H \tilde{\gamma}(c)|_H$, $c \in \mathcal{C}$ is ucp and by definition $P_H \gamma(\phi(a))|_H = P_H \nu(a)|_H = \rho(a) = \pi(\phi(a))$, for all $a \in \mathcal{A}$. Since π *is a boundary representation*, in fact we have that $P_H \tilde{\gamma}(c)|_H = \pi(c)$, for all $c \in \mathcal{C}$. Hence, for all $a \in \mathcal{A}$, we get

$$\begin{aligned}
\rho(a)\rho(a)^* &= \pi(\phi(a))\pi(\phi(a))^* = \pi(\phi(a)\phi(a)^*) \\
&= P_H \gamma(\phi(a)\phi(a)^*)|_H \geq P_H \gamma(\phi(a))\gamma(\phi(a)^*)|_H \\
&= P_H \nu(a)\nu(a)^*|_H \geq P_H \gamma(\phi(a)) P_H \gamma(\phi(a)^*)|_H \\
&= P_H \nu(a) P_H \nu(a)^*|_H = \rho(a)\rho(a)^*.
\end{aligned}$$

Hence, $P_H \nu(a)\nu(a)^*|_H = P_H \nu(a) P_H \nu(a)^*|_H$, thus $\nu(a)^* H \subseteq H$. A similar argument gives that $\nu(a) H \subseteq H$. Thus, H reduces $\nu(\mathcal{A})$.
For the converse, let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be a *∂-representation* and $\phi : \mathcal{A} \to C^*(\phi(\mathcal{A})) = \mathcal{C}$ a ucis map. Then we can define the ucis map $\phi^{-1} : \phi(\mathcal{A}) \to \mathcal{A}$ and we get the ucc map $\rho \circ \phi^{-1} : \phi(\mathcal{A}) \to \mathcal{B}(H)$. Then, by the Arveson's extension theorem there is a ucp map $\pi : \mathcal{C} \to \mathcal{B}(H)$, such that $\pi \circ \phi = \rho$. We will show that π is a boundary representation.
To this end let $\tilde{\pi} : \mathcal{C} \to \mathcal{B}(K)$ the (minimal) Stinespring dilation of π. Then $\tilde{\pi} \circ \phi$ is a dilation of ρ and since ρ is a ∂-representation, H reduces $\tilde{\pi} \circ \phi(\mathcal{A})$. Hence,

$$\begin{aligned}
\pi(\phi(a)\phi(a)^*) &= P_H \tilde{\pi}(\phi(a)\phi(a)^*)|_H = P_H \tilde{\pi}(\phi(a))\tilde{\pi}(\phi(a)^*)|_H \\
&= P_H \tilde{\pi}(\phi(a)) P_H \tilde{\pi}(\phi(a)^*)|_H = \pi(\phi(a))\pi(\phi(a)^*).
\end{aligned}$$

A same argument gives also that $\pi(\phi(a)^*\phi(a)) = \pi(\phi(a)^*)\pi(\phi(a))$. Hence, $\phi(\mathcal{A})$ is in the multiplicative domain of π [1]. Thus π is a representation of $\mathcal{C}$, since $\phi(\mathcal{A})$ generates $\mathcal{C}$. Now, let $r : \mathcal{C} \to \mathcal{B}(H)$ a ucp map, such that $r(\phi(a)) = \pi(\phi(a))(= \rho(a))$, for all $a \in \mathcal{A}$. Then, (the same argument shows that) r is also a representation, thus $r = \pi$, since $r|_{\phi(\mathcal{A})} = \pi|_{\phi(\mathcal{A})}$ and $\phi(\mathcal{A})$ generates $\mathcal{C}$. □

Theorem 1.5. *Let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be a ucis representation. Then the following are equivalent*

(1) ρ is a ∂-representation.
(2) ρ has the unique extension property.
(3) ρ is maximal.

[1] The multiplicative domain of π is defined as the set $\{c \in \mathcal{C} : \pi(cc^*) = \pi(c)\pi(c)^* \text{ and } \pi(c^*c) = \pi(c)^*\pi(c)\}$. This is a C*-subalgebra of $\mathcal{C}$ and π is a *-homomorphism when restricted to this set (see [5, Theorem 3.19]).

Proof. (1⇒ 2). Let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be ∂-representation. Let the ucis map $id : \mathcal{A} \to C^*(\mathcal{A})$. Then by the previous theorem, there exists a boundary representation $\pi : C^*(\mathcal{A}) \to \mathcal{B}(H)$, such that $\pi \circ id = \rho$. Hence, π is an extension of ρ. Since π is boundary representation, it is the unique ucp extension of ρ (and also a representation of $\mathcal{C}$). Thus ρ has the unique extension property.
(2 ⇒ 3). Suppose ρ has the unique extension property and let $\phi : \mathcal{A} \to \mathcal{B}(K)$ be a dilation of ρ, such that $K = [C^*(\phi(\mathcal{A}))H]$. It suffices to show that $K = H$ and $\phi = \rho$. By the Arveson's extension theorem, ϕ can be extended to a ucp map $\tilde{\phi} : C^*(\mathcal{A}) \to \mathcal{B}(K)$. Since the compression of $\tilde{\phi}$ to H defines a ucp map of $C^*(\mathcal{A})$ to $\mathcal{B}(H)$ that restricts to ρ on $\mathcal{A}$, the unique extension property implies that $P_H\tilde{\phi}(\cdot)P_H$ is multiplicative on $C^*(\mathcal{A})$. So for $c \in C^*(\mathcal{A})$,

$$P_H\tilde{\phi}(c)^*P_H\tilde{\phi}(c)P_H = P_H\tilde{\phi}(c^*c)P_H \geq P_H\tilde{\phi}(c)^*\tilde{\phi}(c)P_H,$$

since $\tilde{\phi}(c^*c) \geq \tilde{\phi}(c)^*\tilde{\phi}(c)$. Thus $|(1-P_H)\tilde{\phi}(c)P_H|^2 \leq 0$. Hence, H is invariant under the set of operators $\tilde{\phi}(C^*(\mathcal{A})) \supseteq \phi(\mathcal{A})$; so, H is $\phi(\mathcal{A}))$-invariant. Also, since $\tilde{\phi}$ is positive we have that $\phi(a)^*H = \tilde{\phi}(a)^*H = \tilde{\phi}(a^*)H \subseteq H$, for all $a \in \mathcal{A}$, hence H is $\phi(\mathcal{A})^*$-invariant. Therefore H is $C^*(\phi(\mathcal{A}))$-invariant. Thus $K = [C^*(\phi(\mathcal{A}))H] = H$ and it follows that $\phi = \rho$.
(3 ⇒ 1). Let $\rho : \mathcal{A} \to \mathcal{B}(H)$ be a maximal ucis representation. Then for every dilation $\phi : \mathcal{A} \to \mathcal{B}(K)$, we have that $\phi = \rho \oplus \psi$, for a ucc map ψ. Thus $\phi(\mathcal{A})H = \rho(\mathcal{A})H \subseteq H$ and $\phi(\mathcal{A})^*H = \rho(\mathcal{A})^*H \subseteq H$. Hence H reduces $\phi(\mathcal{A})$. So ρ is a ∂-representation. □

References

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