Morita-type equivalences for operator algebras

A. Katavolos

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This talk will describe work of George Eleftherakis (Athens).

1 TRO equivalence

1.1 A TRO $\mathcal{M}$ is a linear subspace of some $B(H_1, H_2)$ such that

$$\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}.$$

TRO's arise as:

• Corners of C*-algebras,
• injective operator spaces,
• normalizing spaces of (possibly nonselfadjoint) algebras (w. Todorov).

• A TRO $\mathcal{M}$ is a Hilbert (bi)-module over the C*-algebras $\mathcal{M}^*\mathcal{M}$ and $\mathcal{M}\mathcal{M}^*$. Conversely, Hilbert modules can be represented as TRO's.

w*-closed TRO's are generated by their partial isometries; they need not contain any unitaries.

1.2 Normalisers: A normaliser of $\mathcal{A} \subseteq B(H_1)$ into $\mathcal{B} \subseteq B(H_2)$ is $T \in B(H_1, H_2)$ such that $T^*\mathcal{B}T \subseteq \mathcal{A}$ and $T^*\mathcal{A}T \subseteq \mathcal{B}$.

TRO equivalence: existence of 'sufficiently many' normalisers.

Definition 1 *Two (w*-closed) algebras $\mathcal{A}, \mathcal{B}$ are called* TRO equivalent *if there exists a TRO $\mathcal{M}$ such that $\mathcal{A} = [\mathcal{M}^*\mathcal{B}\mathcal{M}]^{-w^*}$ and $\mathcal{B} = [\mathcal{M}\mathcal{A}\mathcal{M}^*]^{-w^*}$.*

Examples: unitarily equivalence; Morita equivalence of W^*-algebras.

Example: $\mathcal{A} \sim_{TRO} M_n(\mathcal{A})$ with $\mathcal{M} = C_n(\mathcal{A})$:

$$\begin{pmatrix} * \\ * \end{pmatrix} \cdot (*) \cdot \begin{pmatrix} * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \qquad \begin{pmatrix} * \\ * \end{pmatrix} \cdot \begin{pmatrix} * & * \\ * & * \end{pmatrix} \cdot \begin{pmatrix} * & * \end{pmatrix} = (*)$$

1.3 TRO equivalence is an equivalence relation.

1.4 **Theorem 2** *The reflexive algebras $\mathcal{A}, \mathcal{B}$ are TRO equivalent if and only if there exists a $*-$isomorphism*

$$\theta : \Delta(\mathcal{A})' \to \Delta(\mathcal{B})'$$

such that

$$\theta(\mathrm{Lat}(\mathcal{A})) = \mathrm{Lat}(\mathcal{B}).$$

Reflexive algebras: Given a (unital, w*-closed) algebra $\mathcal{A} \subseteq B(H)$ and a (strongly closed) lattice of projections $\mathcal{L}$, let

$$\begin{aligned} \mathrm{Lat}(\mathcal{A}) =& \{P \in Proj(H) : P^{\perp}\mathcal{A}P = \{0\}\} \\ \mathrm{Alg}(\mathcal{L}) =& \{A \in B(H) : P^{\perp}AP = 0 \ \forall P \in \mathcal{L}\} \\ \mathcal{A} \text{ reflexive } :& \ \mathcal{A} = \mathrm{Alg}\,\mathcal{L}. \end{aligned}$$

Examples: W* algebras,
nest algebras (here $\mathcal{L}$ is totally ordered - a nest)
CSL algebras (elements of $\mathcal{L}$ commute).

2 TRO equivalence vs spacial Morita equivalence.

Example 3 *There exist nests $\mathcal{N}_1, \mathcal{N}_2$ which are isomorphic but the algebras $\mathcal{N}_1'', \mathcal{N}_2''$ are not isomorphic. Thus isomorphism of the lattices does not guarantee TRO equivalence, even for multiplicity free nest algebras. See item 3.1 below.*

On $H_1 = \ell^2(\mathbb{Q} \cap [0,1])$ define, for each $t \in [0,1]$,

$$\begin{aligned} Q_t^+ =& \{f : \text{supp} f \subseteq [0,t]\} \\ Q_t^- =& \{f : \text{supp} f \subseteq [0,t)\}. \end{aligned}$$

On $L^2([0,1], \lambda)$ define, for each $t \in [0,1]$,

$$N_t = \{f : \text{supp} f \subseteq [0,t]\}.$$

Let

$$\begin{aligned} \mathcal{N}_1 =& \{Q_t^{\pm} : t \in [0,1]\} \quad \text{on} \;\; H_1 = \ell^2(\mathbb{Q} \cap [0,1]) \\ \mathcal{N}_2 =& \{Q_t^{\pm} \oplus N_t : t \in [0,1]\} \quad \text{on} \;\; H_2 \equiv H_1 \oplus L^2([0,1], \lambda). \end{aligned}$$

2.2 **Proposition** *Two CSL algebras $\mathcal{A}, \mathcal{B}$ have isomorphic lattices iff they are "spacially Morita equivalent":*

Definition Let $\mathcal{A} \subset B(H_1), \mathcal{B} \subset B(H_2)$ be w^*-closed algebras. If there exist spaces $\mathcal{U} \subset B(H_1, H_2), \mathcal{V} \subset B(H_2, H_1)$ such that $\mathcal{B}\mathcal{U}\mathcal{A} \subset \mathcal{U}, \;\; \mathcal{A}\mathcal{V}\mathcal{B} \subset \mathcal{V}, \;\; [\mathcal{V}\mathcal{U}]^{-w^*} = \mathcal{A}, \;\; [\mathcal{U}\mathcal{V}]^{-w^*} = \mathcal{B}$ then we say that the algebras $\mathcal{A}, \mathcal{B}$ are **spacially Morita equivalent** and the system

$$\left(\begin{smallmatrix} \mathcal{A} & \mathcal{V} \\ \mathcal{U} & \mathcal{B} \end{smallmatrix} \right)$$

is called a **spacial Morita context**.

2.3 In general, TRO equivalence $\Rightarrow$ spacial Morita equivalence.

2.4 Definition: Given a commutative subspace lattice (CSL) $\mathcal{L}$, the class of all w*-closed algebras $\mathcal{A}$ containing a masa and s.t. $\mathrm{Lat}\,\mathcal{A} = \mathcal{L}$ has a *minimal element* $\mathcal{A}_{min}(\mathcal{L})$ [Arveson]. In case $\mathcal{A}_{min}(\mathcal{L}) = \mathrm{Alg}\, L$ we say $\mathcal{L}$ is *synthetic*.

Using these methods, one shows:

Proposition 4 *Synthesis is preserved under lattice isomorphisms.*

Qu: Under epimorphisms?

3 TRO equivalence and CSL algebras.

Definition 5 *Let $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an isomorphism between CSL's. Then ϕ restricts to an isomorphism between the 'atomic parts', but not necessarily between the 'continuous parts' (see example above). If it does, we say ϕ* ***respects continuity****.*

Proposition 6 *Two CSL algebras are TRO equivalent iff there exists an isomorphism between their lattices which respects continuity.*

Consequences of TRO equivalence for CSL algebras:

Let $\mathcal{A}, \mathcal{B}$ be CSL algebras which are TRO equivalent via $\mathcal{M}$. There is a bijective correspondence between the w^*-closed ideals of the algebras $\mathcal{A}$ and $\mathcal{B}$. It follows that

$$\begin{aligned} \mathcal{A} \text{ strongly reflexive } &\Leftrightarrow \mathcal{B} \text{ strongly reflexive} \\ R_1(\mathcal{A}) = 0 &\Leftrightarrow R_1(\mathcal{B}) = 0 \\ \mathcal{A} \text{ semisimple } &\Leftrightarrow \mathcal{B} \text{ semisimple} \\ \mathcal{A} = \mathrm{Rad}(\mathcal{A})^{-w^*} &\Leftrightarrow \mathcal{B} = \mathrm{Rad}(\mathcal{B})^{-w^*} \end{aligned}$$

4 Morita equivalence for W*-algebras (Rieffel)

The category ${}_{\mathcal{A}}\mathfrak{M}$ of Hilbert $\mathcal{A}$-modules for a W* algebra $\mathcal{A}$:
Objects: (H, π) where π is a normal (: w*-continuous) *-representation.
Morphisms:

$$\mathrm{Hom}_{\mathcal{A}}(H_1, H_2) = \{T \in B(H_1, H_2) : T\pi_1(a) = \pi_2(a)T \ \forall a \in \mathcal{A}\}.$$

A functor $\mathcal{F} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$ is called a $*$-functor if
$T \in \mathrm{Hom}_{\mathcal{A}}(H_1, H_2) \Rightarrow \mathcal{F}(T)^* = \mathcal{F}(T^*)$

Example: If $\phi : \mathcal{B} \to \mathcal{A}$ is a *-isomorphism, then $\mathcal{F}(H, \alpha) = (H, \alpha \circ \phi)$ on objects and $\mathcal{F}(T) = T$ on morphisms defines an equivalence *-functor $\mathcal{F} : {}_{\mathcal{A}}\mathfrak{M} \to {}_{\mathcal{B}}\mathfrak{M}$.

Theorem 7 (Rieffel) *${}_{\mathcal{A}}\mathfrak{M}$ and ${}_{\mathcal{B}}\mathfrak{M}$ are equivalent via *-functors (write $A \sim_R B$) iff there is an 'abstract Morita context' $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{X} \\ \mathcal{X}^* & \mathcal{B} \end{smallmatrix}\right)$ for W*-algebras.*

equivalently [Connes]

$$A \sim_R B \quad \Longleftrightarrow \quad \exists \text{ faithful normal reprs. s.t. } \alpha(\mathcal{A})' \simeq \beta(\mathcal{B})'.$$

In view of Theorem 2 this remark has the following equivalent version:

Theorem 8 *The W*-algebras $\mathcal{A}, \mathcal{B}$ are Morita equivalent if and only if they have faithful normal representations α, β on Hilbert spaces such that the algebras $\alpha(\mathcal{A}), \beta(\mathcal{B})$ are TRO equivalent.*

$$A \sim_R B \quad \Longleftrightarrow \exists \textit{ faithful normal * reps: } \quad \alpha(A) \sim_{TRO} \beta(B)$$

5 The category $\mathfrak{M}$ for dual operator algebras.

5.1 Dual operator algebras: (unital) w*-closed subalgebras (A, w^*) of some $B(H)$. There is an abstract characterisation (Le Merdy): they are the operator algebras that have an *operator space* predual.

5.2 The categories ${}_{\mathcal{A}}\mathfrak{M}$ and ${}_{\mathcal{A}}\mathfrak{D}\mathfrak{M}$ of Hilbert $\mathcal{A}$-modules.

Objects $Ob({}_{\mathcal{A}}\mathfrak{M})$:
Completely contractive unital w*-contns. reps (normal reprs.) (H, α).

$\mathrm{Hom}_{\mathcal{A}}(H_1, H_2) = \{T \in B(H_1, H_2) : T\alpha_1(a) = \alpha_2(a)T \ \forall a \in \mathcal{A}\}$.

Objects: $Ob({}_{\mathcal{A}}\mathfrak{D}\mathfrak{M}) = Ob({}_{\mathcal{A}}\mathfrak{M})$.

$\mathrm{Hom}_{\mathcal{A}}^{\mathfrak{D}}(H_1, H_2) = \{T \in B(H_1, H_2) : T\alpha_1(a) = \alpha_2(a)T \ \forall a \in \Delta(\mathcal{A})\}$.

Observe that ${}_{\mathcal{A}}\mathfrak{M} \subseteq {}_{\mathcal{A}}\mathfrak{D}\mathfrak{M}$; If $\mathcal{A}$ is a W^* algebra then ${}_{\mathcal{A}}\mathfrak{M} = {}_{\mathcal{A}}\mathfrak{D}\mathfrak{M}$.

5.3 A Δ-extension of a functor $\mathcal{F}$ is a functor

$$\mathcal{F}^\delta : {}_{\mathcal{A}}\mathfrak{D}\mathfrak{M} \longrightarrow {}_{\mathcal{B}}\mathfrak{D}\mathfrak{M}$$

such that

$$\begin{array}{ccc} {}_{\mathcal{A}}\mathfrak{M} & \hookrightarrow & {}_{\mathcal{A}}\mathfrak{D}\mathfrak{M} \\ \mathcal{F}\downarrow & & \mathcal{F}^\delta\downarrow \\ {}_{\mathcal{B}}\mathfrak{M} & \hookrightarrow & {}_{\mathcal{B}}\mathfrak{D}\mathfrak{M} \end{array}$$

commutes.

6 The main theorem.

6.1 **Definition** We say that the unital dual operator algebras $\mathcal{A}, \mathcal{B}$ are Δ-equivalent if there exists an equivalence functor

$$\mathcal{F} : {}_{\mathcal{A}}\mathfrak{M} \leftrightarrow {}_{\mathcal{B}}\mathfrak{M}$$

with a $\Delta-$extension to an equivalence $*$-functor

$$\mathcal{F}^{\delta} : {}_{\mathcal{A}}\mathfrak{D}\mathfrak{M} \leftrightarrow {}_{\mathcal{B}}\mathfrak{D}\mathfrak{M}.$$

We write $\mathcal{A} \sim_{\Delta} \mathcal{B}$.

Observe that for W*-algebras, $\mathcal{A} \sim_{\Delta} \mathcal{B} \iff \mathcal{A} \sim_{R} \mathcal{B}$.

6.2 **Theorem** Let $\mathcal{A}, \mathcal{B}$ be unital dual operator algebras.

$\mathcal{A} \sim_{\Delta} \mathcal{B} \iff \exists$ compl. isom. normal reps: $\alpha(\mathcal{A}) \sim_{TRO} \beta(\mathcal{B})$.

(cf. theorem 8)

7 Properties of the equivalence functors

Suppose that $\mathcal{A}, \mathcal{B}$ are unital dual operator algebras and $\mathcal{A} \sim_{\Delta} \mathcal{B}$ via $\mathcal{F}$.

7.1 $\mathcal{F}$ is equivalent to a functor $\mathcal{F}_{\mathcal{U}}$ of 'tensoring by' a suitable bimodule $\mathcal{U}$.

7.2 $\mathcal{F}$ maps completely isometric representations to completely isometric representations.

7.3 $\mathcal{F}$ 'preserves reflexivity': If α is a compl. isom. repr. and $\beta = \mathcal{F}(\alpha)$ then $\alpha(\mathcal{A})$ is reflexive iff $\beta(\mathcal{B})$ is.

7.4 $\mathcal{F}$ 'respects' the lattices: If $(H, \alpha) \in {}_{\mathcal{A}}\mathfrak{M}$ with corresponding object $(\mathcal{F}(H), \beta) \in {}_{\mathcal{B}}\mathfrak{M}$ then

$$\mathcal{F}^{\delta}(\mathrm{Lat}(\alpha(\mathcal{A}))) = \mathrm{Lat}(\beta(\mathcal{B})).$$

7.5 $\mathcal{F}$ is a normal functor, i.e. 'w*-continuous'.

8 Examples and applications.

8.1 If $\mathcal{A} \sim_{\Delta} \mathcal{B}$ then $\mathcal{A}$ can be completely isometrically reresented as a CSL algebra iff $\mathcal{B}$ can be so represented.

8.2 If $\mathcal{L}$ is a non-synthetic CSL then $\mathcal{A}_{min}(\mathcal{L})$ cannot be isometrically represented as a CSL algebra, although its diagonal, $\mathcal{L}'$, *is* a CSL algebra (a vN algebra with abelian commutant).But note $\mathcal{A}_{min}(\mathcal{L})$ *can* be represented as a *reflexive* algebra: consider $(\mathcal{A}_{min}(\mathcal{L}))^{(\infty)}$ (infinite ampliation).

8.3 Two CSL algebras are Δ-equivalent iff they are TRO equivalent.

It follows from Theorem 2 that two CSL algebras, both with continuous lattices, or both with totally atomic lattices, are $\Delta-$equivalent iff they have isomorphic lattices.

8.4 If two nest algebras $\mathcal{A}, \mathcal{B}$ are similar then there exists an equivalence functor

$$\mathcal{F} : {}_{\mathcal{A}}\mathfrak{M} \leftrightarrow {}_{\mathcal{B}}\mathfrak{M}.$$

8.5 But they are not always Δ-equivalent: There exists an example of similar nest algebras with unitarily equivalent diagonals which are not $\Delta-$equivalent.

Example 9 Let $\mathcal{N}_1, \mathcal{N}_2$ be the nests of Example 3. Define the 'ordinal sums' of $\mathcal{N}_1$ and $\mathcal{N}_2$, namely the nests

$$\begin{aligned}\mathcal{L}_1 =& \{N \oplus 0 : N \in \mathcal{N}_1\} \cup \{I_{H_1} \oplus M : M \in \mathcal{N}_2\} \subset B(H_1 \oplus H_2)\\ \mathcal{L}_2 =& \{M \oplus 0 : M \in \mathcal{N}_2\} \cup \{I_{H_2} \oplus N : N \in \mathcal{N}_1\} \subset B(H_2 \oplus H_1).\end{aligned}$$

Now $\mathcal{N}_1$ and $\mathcal{N}_2$ are similar (by the similarity theorem [Davidson]), hence so are $\mathcal{L}_1$ and $\mathcal{L}_2$. So by (8.4), if $\mathcal{A} = \mathrm{Alg}(\mathcal{L}_1)$ and $\mathcal{B} = \mathrm{Alg}(\mathcal{L}_2)$ the categories ${}_{\mathcal{A}}\mathfrak{M}$ and ${}_{\mathcal{B}}\mathfrak{M}$ are equivalent. Observe that $\Delta(\mathcal{A}) \simeq_{unit} \Delta(\mathcal{B})$ because $\Delta(\mathcal{A}) = \mathcal{N}_1'' \oplus \mathcal{N}_2''$ and $\Delta(\mathcal{B}) = \mathcal{N}_2'' \oplus \mathcal{N}_1''$.

Suppose $\mathcal{A} \sim_{\Delta} \mathcal{B}$. Then, by (8.3), $\mathcal{A} \sim_{TRO} \mathcal{B}$. So by 2 there exists a $*-$isomorphism

$$\theta : \Delta(\mathcal{A}) \to \Delta(\mathcal{B}) \text{ such that } \theta(\mathcal{L}_1) = \mathcal{L}_2.$$

Since the diagonals are masas the map θ must be unitarily implemented. Now there exist two possibilities:

$$\theta(I_{H_1} \oplus 0) = \begin{cases} M \oplus 0 & \text{for some } \; N \in \mathcal{N}_2 \quad (a) \\ \\ I_{H_2} \oplus N & \text{for some } \; N \in \mathcal{N}_1 \quad (b) \end{cases}$$

$$\begin{aligned}(a) \quad &\Longrightarrow \quad \mathcal{N}_1'' \simeq_{unit} (\mathcal{N}_2''|_{M(H_2)})\\ (b) \quad &\Longrightarrow \quad \mathcal{N}_1'' \simeq_{unit} (\mathcal{N}_2'' \oplus \mathcal{N}_1''|_{N(H_1)})\end{aligned}$$

Both lead to a contradiction because $\mathcal{N}_1''$ is totally atomic while the others are not.