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Idempotents of large norm and homomorphisms of Fourier algebras

by

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Abstract. We provide necessary and sufficient conditions for the existence of idempotents of arbitrarily large norm in the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of a locally compact group G. We prove that the existence of idempotents of arbitrarily large norm in $B(G)$ implies the existence of homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ for every locally compact group H. A partial converse is also obtained: the existence of homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ for some amenable locally compact group H implies the existence of idempotents of arbitrarily large norm in $B(G)$.

1. Introduction. Let G be a locally compact group. The Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of G were introduced by Eymard [Ey]. The Fourier–Stieltjes algebra of G consists of the matrix coefficients $(\pi(\cdot)\xi, \eta)$ of all continuous unitary representations π of G, while the Fourier algebra of G consists of the matrix coefficients of the left regular representation of G. If G is abelian, $A(G)$ and $B(G)$ can be identified, via the Fourier transform, with $L^1(\widehat{G})$ and the measure algebra $M(\widehat{G})$ of the dual group $\widehat{G}$, respectively. In [Co2] Cohen characterized the homomorphisms from $A(H)$ into $B(G)$ in terms of piecewise affine maps when H, G are locally compact abelian groups. To obtain his result he proved a characterization of idempotents in $B(G)$ [Co1]. Host in [Ho] extended the characterization of idempotents in $B(G)$ to general locally compact groups.

Homomorphisms of Fourier algebras for locally compact groups were studied by Ilie [Il] and Ilie and Spronk [IS]. They characterized completely bounded homomorphisms from $A(H)$ into $B(G)$ for locally compact groups H, G with H amenable in terms of piecewise affine maps [IS] (see also [Da]).

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Let G, H be locally compact groups. Let K be a subgroup of G and C a left coset of K in G. A map $\alpha : C \to H$ is called *affine* if there exists a continuous homomorphism $\theta : K \to H$ and elements $s_0 \in H$, $t_0 \in G$ such that $C = t_0^{-1}K$ and

$$\alpha(t) = s_0\theta(t_0t)$$

for all $t \in C$.

A map $\alpha : Y \to H$ is called *piecewise affine* if Y can be written as a disjoint union $Y = \bigcup_{i=1}^m Y_i$, where each Y_i belongs to the open coset ring $\Omega_0(G)$, such that each restriction $\alpha|_{Y_i}$ extends to an affine map $\alpha_i : C_i \to H$ defined on an open coset $C_i \supseteq Y_i$.

Recall that the open coset ring $\Omega_0(G)$ is the ring generated by the open cosets of the group G.

Let Y be an open and closed subset of G and α a piecewise affine map $Y \to H$. Define $\rho : A(H) \to B(G)$ by

$$\rho(u)(t) = \begin{cases} u \circ \alpha(t), & t \in Y, \\ 0, & t \in G \setminus Y. \end{cases} \tag{1}$$

It follows from results of Ilie and Spronk [IS, Proposition 3.1 and Theorem 3.7] and Daws [Da] that ρ is a completely bounded homomorphism and that, if the group H is amenable, every completely bounded homomorphism $\rho : A(H) \to B(G)$ is of this form.

NOTATION. The symbol χ_F denotes the characteristic function of a set F.

To motivate our work, consider the following simple example of completely bounded homomorphisms from $A(\mathbb{Z})$ to $A(\mathbb{Z})$ of arbitrarily large norm: For $F = \{-k, \ldots, k\} \subseteq \mathbb{Z}$ we define the map $\rho_F : A(\mathbb{Z}) \to A(\mathbb{Z})$ by

$$\rho_F(u)(j) = \begin{cases} u(0) & \text{if } j \in F, \\ 0 & \text{if } j \notin F. \end{cases}$$

Then it follows from [Il] that ρ_F is a completely bounded homomorphism. Consider the function $u_0 : \mathbb{Z} \to \mathbb{Z}$ given by $u_0(i) = \delta_{i,0}$. Clearly $u_0 \in A(\mathbb{Z})$ and $\|u_0\|_{A(\mathbb{Z})} \le 1$. Since $\rho_F(u_0) = \chi_F$, its Fourier transform is $\widehat{\rho_F(u_0)}(z) = \widehat{\chi_F}(z) = \sum_{i\in F} z^{-i}$ and so

$$\begin{aligned} \|\rho_F\| \ge \|\rho_F(u_0)\|_{A(\mathbb{Z})} &= \|\chi_F\|_{A(\mathbb{Z})} = \|\widehat{\chi_F}\|_{L^1(\mathbb{T})} \\ &= \int_{\mathbb{T}} \Big|\sum_{i\in F} z^{-i}\Big|\, dz = \int_0^{2\pi} |D_k(x)|\,\frac{dx}{2\pi}, \end{aligned}$$

where D_k is the Dirichlet kernel, and it is known that the L^1 norm of D_k grows like $\log k$.

In the above example we used the existence of idempotents in $A(\mathbb{Z})$ of large norm to construct homomorphisms of $A(\mathbb{Z})$ with large norm.

In this work we study the following questions: (a) for which locally compact groups G there exist idempotents of arbitrarily large norm in the Fourier algebra $A(G)$ (resp. in the Fourier–Stieltjes algebra $B(G)$), and (b) how is the existence of idempotents of arbitrarily large norm related to the existence of homomorphisms of arbitrarily large norm between Fourier algebras.

We provide necessary and sufficient conditions for the existence of idempotents of arbitrarily large norm in the Fourier algebra $A(G)$ (resp. in the Fourier–Stieltjes algebra $B(G)$) of a locally compact group G. To prove our results we reduce the problem to the case where G is totally disconnected. Then we first consider the case where G is a discrete group in Proposition 2.2, and for the general case we use a result of Leiderman, Morris and Tkachenko for totally disconnected groups [LMT]. We also prove that the existence of idempotents of arbitrarily large norm in $B(G)$ implies the existence of homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ for every locally compact group H. Finally, we obtain a partial converse: the existence of homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ for some amenable locally compact group H implies the existence of idempotents of arbitrarily large norm in $B(G)$.

2. Norms of idempotents. Let G be a locally compact group. A function $u : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $uA(G) \subseteq A(G)$. In this case the map $m_u : A(G) \to A(G) : v \mapsto uv$ is bounded. In case it is completely bounded we call u a *completely bounded multiplier*. We denote by $M_{\mathrm{cb}}A(G)$ the algebra of completely bounded multipliers of $A(G)$.

The space $M_{\mathrm{cb}}A(G)$ inherits the operator space structure from the space $\mathrm{CB}(A(G))$ of completely bounded maps $A(G) \to A(G)$. We will write $\|u\|_{\mathrm{CB}(A(G))}$, and simply $\|u\|_{\mathrm{cb}}$ when there is no risk of confusion, for the completely bounded norm $\|m_u\|_{\mathrm{CB}(A(G))}$ of an element $u \in M_{\mathrm{cb}}A(G)$. Note that $B(G)$ consists of completely bounded multipliers on $A(G)$ [DCH, Corollary 1.8]; thus $B(G)$ (and also $A(G)$) inherits the operator space structure from $M_{\mathrm{cb}}A(G)$.

It is shown in [DCH] that the space $M_{\mathrm{cb}}A(G)$ is the dual of the normed space $(L^1(G), \|\cdot\|_{Q(G)})$, where the norm $\|\cdot\|_{Q(G)}$ is given by

$$\|f\|_{Q(G)} = \sup\Big\{\Big|\int_G f(s)\phi(s)\,ds\Big| : \phi \in M_{\mathrm{cb}}A(G),\ \|\phi\|_{\mathrm{cb}} \le 1\Big\}, \qquad f \in L^1(G).$$

We shall use the following theorem, combining [Sp, Corollary 6.3(iv)] and [Ey, 2.26, Corollaire 3, and 3.25, Proposition]:

Theorem 2.1 (Eymard, Spronk). *Let G be a locally compact group and H a closed, normal subgroup of G. Let $\pi : G \to G/H$ be the quotient map.*

The map

$$j_\pi : M_{\mathrm{cb}}(A(G/H)) \to M_{\mathrm{cb}}(A(G)) : u \mapsto u \circ \pi$$

is a complete isometry. Moreover, $j_\pi(B(G/H)) \subseteq B(G)$*; if* H *is compact, then* $j_\pi(A(G/H)) \subseteq A(G)$.

Proposition 2.2. *Let* G *be a discrete infinite group. Then*

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G \textit{ finite}\} = +\infty.$$

Proof. Assuming that

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G \text{ finite}\} = M < +\infty,$$

we shall prove that $\ell^\infty(G) \subseteq M_{\mathrm{cb}}(A(G))$. This means that G is a strong Leinert set, which, by a result of Pisier [Pi, Theorem 3.3], implies that G must be finite.

If v is an extreme point of the positive part Ω of the unit ball of $\ell^\infty(G)$, then $v(t) \in \{0,1\}$ for all $t \in G$. Thus for any finite $F \subseteq G$, the function $\chi_F v$ takes values in $\{0,1\}$ and so $\chi_F v = \chi_{F'}$ for some finite subset F' of G. Thus

$$\|\chi_F v\|_{\mathrm{cb}} \le M$$

by our assumption.

Now fix an arbitrary $u \in \Omega$ and a finite subset $F \subseteq G$. By the Krein–Milman theorem, u is a weak-* limit of a net (u_i) of convex combinations of extreme points of Ω. By the previous paragraph, each u_i will satisfy $\|\chi_F u_i\|_{\mathrm{cb}} \le M$.

Since $M_{\mathrm{cb}}A(G)$ is the dual of $(\ell^1(G), \|\cdot\|_{Q(G)})$, given $\varepsilon > 0$ there exists $f \in \ell^1(G)$ with $\|f\|_{Q(G)} \le 1$ such that

$$\|\chi_F u\|_{\mathrm{cb}} - \varepsilon < \Big|\sum_{t\in G} (\chi_F u f)(t)\Big|.$$

Now

$$\sum_{t\in G} (\chi_F u f)(t) = \lim_i \sum_t (\chi_F u_i f)(t),$$

since u is the weak-* limit of the net (u_i) and so

$$\sum_{t\in G} (\chi_F u f)(t) = \lim_i \sum_t (\chi_F u_i f)(t) = \lim_i \langle f, \chi_F u_i\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $\ell^1(G)$ and $M_{\mathrm{cb}}A(G)$. But

$$|\langle f, \chi_F u_i\rangle| \le \|f\|_{Q(G)} \|\chi_F u_i\|_{\mathrm{cb}} \le M$$

for each i, and therefore

$$\|\chi_F u\|_{\mathrm{cb}} - \varepsilon < \Big|\sum_{t\in G} (\chi_F u f)(t)\Big| = \lim_i |\langle f, \chi_F u_i\rangle| \le M.$$

Thus, for all nonnegative u in the unit ball of $\ell^\infty(G)$ we have

$$\sup_{u\in\Omega}\|\chi_F u\|_{\mathrm{cb}} \le M$$

for every finite subset $F \subseteq G$. In particular, if $u \in c_{00}(G)$ is nonnegative then $u/\|u\|_\infty \in \Omega$ and thus

$$\|u\|_{\mathrm{cb}} \le M\|u\|_\infty.$$

Therefore for all $u \in c_{00}(G)$, we have

$$\begin{aligned}\|u\|_{\mathrm{cb}} &\le \|(\operatorname{Re} u)^+\|_{\mathrm{cb}} + \|(\operatorname{Re} u)^-\|_{\mathrm{cb}} + \|(\operatorname{Im} u)^+\|_{\mathrm{cb}} + \|(\operatorname{Im} u)^-\|_{\mathrm{cb}}\\ &\le M(\|(\operatorname{Re} u)^+\|_\infty + \|(\operatorname{Re} u)^-\|_\infty + \|(\operatorname{Im} u)^+\|_\infty + \|(\operatorname{Im} u)^-\|_\infty)\\ &\le 4M\|u\|_\infty.\end{aligned}$$

We conclude that the norms $\|\cdot\|_{\mathrm{cb}}$ and $\|\cdot\|_\infty$ are equivalent on $c_{00}(G)$. Thus the identity map $\mathrm{id} : (c_{00}(G), \|\cdot\|_\infty) \to (M_{\mathrm{cb}}(A(G)), \|\cdot\|_{\mathrm{cb}})$ is continuous.

Since $M_{\mathrm{cb}}(A(G))$ is a dual Banach space, we can consider the unique weak-* continuous extension of id to the double dual $\ell^\infty(G)$ of $c_{00}(G)$ (see e.g. [BlM, Lemma A.2.2]), which we denote by T:

$$T : (\ell^\infty(G), \|\cdot\|_\infty) = (c_{00}(G), \|\cdot\|_\infty)^{**} \to (M_{\mathrm{cb}}(A(G)), \|\cdot\|_{\mathrm{cb}}).$$

We claim that T is the identity. If $u \in \ell^\infty(G)$, we will show that $u = Tu$. Indeed, if (u_i) is a net in $c_{00}(G)$ such that $u = \lim_i u_i$ in the weak-* topology $\sigma(\ell^\infty(G), \ell^1(G))$, then (Tu_i) converges to Tu in the weak-* topology of $M_{\mathrm{cb}}(A(G))$, and hence pointwise (since $(Tu)(t) = \langle Tu, \delta_t\rangle$ for $t \in G$). Thus, for all $t \in G$,

$$(Tu)(t) = \lim_i (Tu_i)(t) = \lim_i u_i(t) = u(t)$$

since $u = \lim_i u_i$ in the weak-* topology $\sigma(\ell^\infty(G), \ell^1(G))$ and hence pointwise. This proves our claim.

We have shown that $\ell^\infty(G) \subseteq M_{\mathrm{cb}}(A(G))$ and thus G must be finite, as observed above. ∎

Since $\|u\|_{\mathrm{cb}} \le \|u\|_{B(G)}$ when $u \in B(G)$ [DCH, Corollary 1.8], we obtain

COROLLARY 2.3. *If G is a discrete infinite group then*

$$\sup_F \{\|\chi_F\|_{A(G)} : F \subseteq G \textit{ finite}\} = +\infty.$$

NOTE. We thank the referee for providing the following alternative argument for Corollary 2.3: Using analogous arguments to those in the proof of Proposition 2.2, we can show that if $\sup_F \{\|\chi_F\|_{A(G)} : F \subseteq G \text{ finite}\} < +\infty$ then $\ell^\infty(G) = B(G)$. It follows from this equality that G must be finite. Indeed if $\ell^\infty(G) = B(G)$, then $\ell^1(G) = C^*(G)$ with equivalent norms. However, this would imply that $\ell^1(G)$ is Arens regular, which by [Y] shows that G is finite.

Theorem 2.4. *Let G be an infinite totally disconnected group. Then*

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G,\ \chi_F \in A(G)\} = +\infty.$$

Proof. From the theorem of van Dantzig [vD], [HR, Theorem II.7.7] there exists a compact open subgroup $H \subseteq G$.

If H is finite then $\{e\}$ is an open subgroup of G and thus G is discrete. In this case the conclusion follows from Proposition 2.2.

If H is infinite, by [LMT, Theorem 2.6] there exists a closed normal subgroup N of H such that the quotient H/N is homeomorphic to a countably infinite product of finite groups. We write $K = H/N$. Clearly K is compact and separable. We also denote by K_d the group K with the discrete topology. The inclusion

$$\iota : K_d \to K$$

is a continuous homomorphism; thus it induces a contractive homomorphism

$$\rho : A(K) \to B(K_d) : u \mapsto u \circ \iota.$$

Let $\epsilon > 0$. By Proposition 2.2 there exists a finite $F \subseteq K$ such that $\chi_F \in A(K_d)$ and

$$\|\chi_F\|_{A(K_d)} > \epsilon.$$

Since K is a totally disconnected and separable group, there exists a decreasing *sequence* of compact open subgroups such that

$$\bigcap_{n=1}^{\infty} K_n = \{e\} \quad \text{and hence} \quad \bigcap_{n=1}^{\infty} FK_n = F.$$

Now FK_n is a finite disjoint union of sets of the form x_iK_n where $x_i \in F$, and since each K_n is a compact open subgroup, $\chi_{x_iK_n}$ is in $A(K)$ and has norm 1. Indeed, the constant function 1 on the compact group K_n belongs to $A(K_n)$ and has norm 1. It follows from [Ey, Proposition 3.21(1)] that χ_{K_n} belongs to $A(K)$ and has norm 1 and hence the same holds for its translate $\chi_{x_iK_n}$. Thus χ_{FK_n} is in $A(K)$ and the sequence $(\|\chi_{FK_n}\|_{A(K)})_n$ is bounded by the cardinality of F. Since ρ is bounded, the sequence $(\|\chi_{FK_n}\|_{B(K_d)})_n$ is also bounded. For all $f \in \ell^1(K_d)$, since $(\chi_{FK_n})_n$ converges pointwise to χ_F, by dominated convergence we have

$$\lim_n \sum_{t\in K_d} f(t)\chi_{FK_n}(t) = \sum_{t\in K_d} f(t)\chi_F(t).$$

Since $\ell^1(K_d)$ is dense in the predual $C^*(K_d)$ of $B(K_d)$, we obtain

$$\text{w}^*\text{-}\lim_n \chi_{FK_n} = \chi_F$$

in the weak-* topology of $B(K_d)$. Therefore $\sup_n \|\chi_{FK_n}\|_{B(K_d)} > \epsilon$ and hence

$$\sup_n \|\chi_{FK_n}\|_{A(K)} > \epsilon,$$

which implies that there exists $\chi_{FK_n} \in A(K)$ such that

$$\|\chi_{FK_n}\|_{A(K)} > \epsilon.$$

This shows that

$$\sup\{\|\chi_V\|_{A(H/N)} : V \subseteq H/N,\ \chi_V \in A(H/N)\} = +\infty$$

and since H/N is compact, it follows that

$$\sup\{\|\chi_V\|_{\mathrm{cb}} : V \subseteq H/N,\ \chi_V \in A(H/N)\} = +\infty.$$

Let $\pi : H \to H/N$ be the quotient map. It follows from Theorem 2.1 that if $F \subseteq H/N$ satisfies $\chi_F \in A(H/N)$, then $\chi_F \circ \pi = \chi_{\pi^{-1}(F)} \in A(H)$ and

$$\|\chi_F\|_{\mathrm{CB}(A(H/N))} = \|\chi_F \circ \pi\|_{\mathrm{CB}(A(H))} = \|\chi_{\pi^{-1}(F)}\|_{\mathrm{CB}(A(H))}.$$

We conclude that

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq H,\ \chi_F \in A(H)\} = +\infty.$$

Let $F \subseteq H$ be such that $\chi_F \in A(H)$. Since H is an open subgroup, by [Ey, 3.21(1)], we have $\chi_F \in A(G)$.

Since by [Sp, Corollary 6.3(iii)] the map

$$M_{\mathrm{cb}}A(G) \to M_{\mathrm{cb}}A(H) : u \mapsto u|_H$$

is completely contractive, we obtain

$$\|\chi_F\|_{\mathrm{CB}(A(G))} \geq \|\chi_F|_H\|_{\mathrm{CB}(A(H))} = \|\chi_F\|_{\mathrm{CB}(A(H))}.$$

We conclude that

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G,\ \chi_F \in A(G)\} = +\infty. \blacksquare$$

Note the crucial use of [LMT] in obtaining a *countable* family (K_n) of compact open subgroups with $\bigcap_{n=1}^{\infty} K_n = \{e\}$.

The proof of the above theorem is not constructive. Below we provide a different proof for the case where G is an infinite direct product of finite groups. Ilie and Spronk [IS, Theorem 2.1] proved that if χ_F is an idempotent in $B(G)$, then $\|\chi_F\|_{B(G)} = 1$ if and only if F is a coset of an open subgroup of G. Forrest and Runde [FR] and Stan [St] proved that if the cb norm of an idempotent $\chi_F \in B(G)$ satisfies $\|\chi_F\|_{\mathrm{cb}} < \frac{2}{\sqrt{3}}$ then F is a coset of an open subgroup of G. The 'gap' $\big[1, \frac{2}{\sqrt{3}}\big)$ was improved by Mudge and Pham [MP] to $\big[1, \frac{1+\sqrt{2}}{2}\big)$.

Proposition 2.5. *Let G be an infinite direct product of finite groups. Then*

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G,\ \chi_F \in A(G)\} = +\infty.$$

Proof. Since G is compact, we have $A(G) = B(G)$ and $\|u\|_{\mathrm{cb}} = \|u\|_{A(G)}$ for all $u \in A(G)$ [KL, Corollary 5.4.11] and hence it is sufficient to prove the proposition for $\|\cdot\|_{A(G)}$.

Let G_0 be a finite group with $|G_0| \geq 3$. Then there exists $A \subseteq G_0$ such that A is not a coset of a subgroup of G_0 (for example, take A such that $|A|$ does not divide $|G_0|$). Then, since $A \in \Omega_0(G_0)$, it follows from [MP] that $\|\chi_A\|_{A(G_0)} \geq \frac{1+\sqrt{2}}{2}$.

Now consider finite groups $G_1, \dots, G_n$ with $|G_i| \geq 3$ for all $i = 1, \dots, n$ and set $G = \prod_{i=1}^n G_i$. Choose $A_i \in G_i$ with $\|\chi_{A_i}\|_{A(G_i)} \geq \frac{1+\sqrt{2}}{2}$ and set $A = A_1 \times \dots \times A_n$. Since $A(G)$ is isometrically isomorphic to the operator space projective tensor product $A(G_1) \hat{\otimes} \cdots \hat{\otimes} A(G_n)$ [KL, Lemma 4.1.2], we obtain

$$\|\chi_A\|_{A(G)} = \|\chi_{A_1} \otimes \cdots \otimes \chi_{A_n}\|_{A(G_1)\hat{\otimes}\cdots\hat{\otimes}A(G_n)} \geq \left(\frac{1+\sqrt{2}}{2}\right)^n.$$

Now let $G = \prod_{i=1}^\infty G_i$ be an infinite product of finite groups G_i. Without loss of generality we may assume that $|G_i| \geq 3$ for all $i \in \mathbb{N}$ (lumping together some of the G_i's if necessary). Set $H_n = \prod_{i=n+1}^\infty G_i$ and let π_n be the quotient map $G \to G/H_n$. It follows from Theorem 2.1 that the map $A(G/H_n) \to A(G) : u \mapsto u \circ \pi_n$ is isometric. Since $G/H_n \simeq \prod_{i=1}^n G_i$, we can choose $A \subseteq G/H_n$ such that $\|\chi_A\|_{A(G/H_n)} \geq \big(\frac{1+\sqrt{2}}{2}\big)^n$. Setting $F = \pi_n^{-1}(A)$, we obtain $\|\chi_F\|_{\mathrm{cb}} = \|\chi_F\|_{A(G)} \geq \big(\frac{1+\sqrt{2}}{2}\big)^n$ and the conclusion follows. ∎

Corollary 2.6. *Let G be a locally compact group and G_0 be the connected component of $e \in G$. If the quotient G/G_0 is infinite then*

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G,\ \chi_F \in B(G)\} = +\infty.$$

Proof. Since G/G_0 is infinite and totally disconnected, it follows from Theorem 2.4 that

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G/G_0,\ \chi_F \in A(G/G_0)\} = +\infty.$$

Let $\pi : H \to G/G_0$ be the quotient map. Since $\chi_{\pi^{-1}(F)} = \chi_F \circ \pi$, it follows from Theorem 2.1 that

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G,\ \chi_F \in B(G)\} = +\infty. \ ∎$$

Remark. Let G be a locally compact group and N a closed normal subgroup of G. It follows from Theorem 2.1 that if

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G/N,\ \chi_F \in B(G/N)\} = +\infty$$

then

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \subseteq G,\ \chi_F \in B(G)\} = +\infty.$$

3. Groups with idempotents with large norms. Let G be a locally compact group and G_0 the connected component of the identity of G. In this section we show that $B(G)$ contains idempotents of arbitrarily large norm if and only if G/G_0 is infinite. We also prove a related result for $A(G)$. It follows from [Ho] that an idempotent is in $B(G)$ if and only if it is of the form χ_F with F in the open coset ring of G.

Let H be an open subgroup of G. Then $H\cap G_0$ is open and closed in G_0, and hence equal to G_0; thus $H\supseteq G_0$. Since G_0 is contained in every open subgroup of G, we see that if E is an left coset of an open subgroup in G, then $E=EG_0$. It is easy to check that if E is a left coset of an open subgroup G, we also have $E^c=E^cG_0$ (here E^c is the complement of E).

Lemma 3.1. *Let X be in the open coset ring $\Omega_0(G)$. Then $X=XG_0$.*

Proof. We show that if $X=XG_0$ and $Y=YG_0$, then $X\cap Y=(X\cap Y)G_0$. Indeed, let $z\in X\cap Y$ and $g\in G_0$. Then $z=xg'=yg''$ for some $x\in X$, $y\in Y$ and $g',g''\in G_0$. Therefore $zg=xg'g=yg''g$, and since $xg'g\in X$ and $yg''g\in Y$, we obtain $zg\in X\cap Y$. In view of the above remark on complements, the assertion follows. ∎

Corollary 3.2. *Let ϕ be the map defined on $\Omega_0(G)$ by $X\mapsto q(X)$ (where $q:G\to G/G_0$ is the quotient map). Then ϕ is a ring isomorphism from $\Omega_0(G)$ onto $\Omega_0(G/G_0)$.*

Proof. Clearly $\phi(X\cap Y)=\phi(X)\cap\phi(Y)$ and $\phi(X^c)=\phi(X)^c$. Let aK be a coset of an open subgroup in G/G_0. Then $\phi^{-1}(aK)$ is a coset of an open subgroup in G and $\phi(\phi^{-1}(aK))=aK$. Finally, $XG_0=YG_0$ for $X,Y\in\Omega_0(G)$ implies $X=Y$, hence ϕ is injective. ∎

Theorem 3.3. *Let G be a locally compact group and G_0 be the connected component of $e\in G$. The following are equivalent:*

(1) *The quotient G/G_0 is infinite and G_0 is compact.*
(2) $\sup\{\|\chi_F\|_{\mathrm{cb}}:\chi_F\in A(G)\}=+\infty$.
(3) $\sup\{\|\chi_F\|_{A(G)}:\chi_F\in A(G)\}=+\infty$.

Proof. That (1) implies (2) follows from Theorems 2.1 and 2.4.

That (2) implies (3) follows since $\|\chi_F\|_{\mathrm{cb}}\le\|\chi_F\|_{A(G)}$.

We show that (3) implies (1): If G_0 is not compact, it follows from Lemma 3.1 that there are no idempotents in $A(G)$. If G/G_0 is finite, the open coset ring is finite by Corollary 3.2, and hence the set of idempotents is finite. ∎

The proof of the following theorem is similar.

Theorem 3.4. *Let G be a locally compact group and G_0 be the connected component of $e\in G$. The following are equivalent:*

(1) *The quotient G/G_0 is infinite.*
(2) $\sup\{\|\chi_F\|_{\mathrm{cb}}:\chi_F\in B(G)\}=+\infty$.
(3) $\sup\{\|\chi_F\|_{B(G)}:\chi_F\in B(G)\}=+\infty$.

4. Norms of homomorphisms. In this section we show that if G is a locally compact group with connected component G_0 of the identity such that G/G_0 is infinite, and H is a locally compact group, then there exist homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$. We also prove that if there exists an amenable group H such that homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ exist, then G/G_0 is infinite.

Proposition 4.1. *Let G, H be locally compact groups and $F \in \Omega_0(G)$. For $u \in A(H)$ define*

$$\rho_F(u)(t) = \begin{cases} u(e), & t \in F, \\ 0, & t \notin F. \end{cases}$$

Then ρ_F is a completely bounded homomorphism $A(H) \to B(G)$ and

$$\|\rho_F\|_{\mathrm{cb}} = \|\rho_F\| = \|\chi_F\|_{B(G)}.$$

Proof. It follows from [IS, Proposition 3.1] that ρ_F is a completely bounded homomorphism. Choose $u \in A(H)$ such that $u(e) = 1$ and $\|u\|_{A(H)} \le 1$. Then

$$\|\rho_F\| \ge \|\rho_F(u)\|_{B(G)} = \|u(e)\chi_F\|_{B(G)} = \|\chi_F\|_{B(G)}.$$

We also have, for $u \in A(H)$,

$$\|\rho_F(u)\|_{B(G)} = \|u(e)\chi_F\|_{B(G)} = |u(e)|\,\|\chi_F\|_{B(G)} \le \|u\|_{A(H)}\|\chi_F\|_{B(G)},$$

and hence $\|\rho_F\| = \|\chi_F\|_{B(G)}$.

Since the image of ρ_F is one-dimensional, it follows that

$$\|\rho_F\|_{\mathrm{cb}} = \|\rho_F\| = \|\chi_F\|_{B(G)}. \ \blacksquare$$

Applying Theorem 3.4 to Proposition 4.1, we obtain the following

Corollary 4.2. *Let G, H be locally compact groups and assume that*

$$\sup\{\|\chi_F\|_{\mathrm{cb}} : F \in \Omega_0(G)\} = +\infty.$$

Then

$$\sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty.$$

Theorem 4.3. *Let G be a locally compact group and G_0 be the connected component of $e \in G$. The following are equivalent:*

(i) *For every locally compact group H,*

$$\sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty.$$

(ii) *There exists an amenable locally compact group H such that*

$$\sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty.$$

(iii) *The group G/G_0 is infinite.*

(iv) $\sup\{\|\chi_F\|_{B(G)} : F \in \Omega_0(G)\} = +\infty$.

Proof. Clearly (i) implies (ii). The equivalence (iii)⇔(iv) follows from Theorem 3.4. Also the implication (iv)⇒(i) follows from Corollary 4.2.

It remains to show that (ii) implies (iii). Suppose that $|G/G_0| < +\infty$. By Corollary 3.2, there exists $m \in \mathbb{N}$ such that $|\Omega_0(G)| \leq m$. Let $\rho : A(H) \to B(G)$ be a completely bounded homomorphism. By [IS, Theorem 3.7] and [Da], ρ is of the form (1) for some $Y \in \Omega_0(G)$ and a piecewise affine map $\alpha : Y \to H$. By [IS, Proposition 3.1] we have

$$\|\rho\|_{\mathrm{cb}} \leq m \cdot \sum_{F \in \Omega_0(G)} \|\chi_F\|_{B(G)} \leq m^2 \max\{\|\chi_F\|_{B(G)} : F \in \Omega_0(G)\},$$

which is a contradiction. ■

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