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Bimodules over $\mathrm{VN}(G)$, harmonic operators and the non-commutative Poisson boundary

by

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Abstract. Starting with a left ideal J of $L^1(G)$ we consider its annihilator $J^\perp$ in $L^\infty(G)$ and the generated $\mathrm{VN}(G)$-bimodule in $\mathcal{B}(L^2(G))$, $\mathrm{Bim}(J^\perp)$. We prove that $\mathrm{Bim}(J^\perp) = (\mathrm{Ran}\, J)^\perp$ when G is weakly amenable discrete, compact or abelian, where $\mathrm{Ran}\, J$ is a suitable saturation of J in the trace class. We define jointly harmonic functions and jointly harmonic operators and show that, for these classes of groups, the space of jointly harmonic operators is the $\mathrm{VN}(G)$-bimodule generated by the space of jointly harmonic functions. Using this, we give a proof of the following result of Izumi and Jaworski–Neufang: the non-commutative Poisson boundary is isomorphic to the crossed product of the space of harmonic functions by G.

1. Introduction. Let J be an ideal of the Fourier algebra $A(G)$ of a locally compact group G. There are two 'canonical' ways to construct from J an $L^\infty(G)$-bimodule in $\mathcal{B}(L^2(G))$. One way is to consider the annihilator $J^\perp$ of J within $\mathrm{VN}(G)$ and then take the $L^\infty(G)$-bimodule generated by $J^\perp$, denoted by $\mathrm{Bim}(J^\perp)$. The other way is to take the saturation of J within the trace class on $L^2(G)$, which we call $\mathrm{Sat}\, J$, and then consider its annihilator. This gives a masa bimodule $(\mathrm{Sat}\, J)^\perp$ in $\mathcal{B}(L^2(G))$. In [1], we proved that these two procedures yield the same bimodule:

$$(*)\qquad \mathrm{Bim}(J^\perp) = (\mathrm{Sat}\, J)^\perp.$$

In [22], Neufang and Runde introduced the notion of σ-harmonic operators $\widetilde{\mathcal{H}}_\sigma$ (where σ belongs to the space of completely bounded multipliers $M^{\mathrm{cb}}A(G)$ of $A(G)$) as an extension of the notion of σ-harmonic functionals on $A(G)$ defined and studied by Chu and Lau [4]. One of the main results

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of [22] is that when σ is positive definite and normalised, $\widetilde{\mathcal{H}}_\sigma$ is the von Neumann algebra on $L^2(G)$ generated by the algebra $\mathcal{D}_G$ of multiplication operators together with the space $\mathcal{H}_\sigma$ of harmonic functionals. In [2], for a subset $\Sigma \subseteq M^{\mathrm{cb}}A(G)$ we considered the set of jointly harmonic functionals $\mathcal{H}_\Sigma$ (resp. operators $\widetilde{\mathcal{H}}_\Sigma$). Using the equality $(*)$, we showed that, for any $\Sigma \subseteq M^{\mathrm{cb}}A(G)$, we have $\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}(\mathcal{H}_\Sigma)$, thus obtaining a generalization of the result of Neufang and Runde.

Another concept of harmonicity was introduced and studied by Jaworski and Neufang [19]. Recall that a function $\phi \in L^\infty(G)$ is said to be *harmonic* with respect to a probability measure μ on G [13, 12] if it is a fixed point point of the map P_μ on $L^\infty(G)$ given by

$$(P_\mu \phi)(s) = \int_G \phi(st)\, d\mu(t).$$

The space of μ-harmonic functions is denoted by $\mathcal{H}(\mu)$. If G is abelian, it follows from the Choquet–Deny theorem that if the support of μ generates G as a closed subgroup, then $\mathcal{H}(\mu)$ consists of constants. In particular, it is a subalgebra of $L^\infty(G)$. Consider the natural isometric representation $\mu \mapsto \Theta(\mu)$ of the measure algebra $M(G)$ on $\mathcal{B}(L^2(G))$ introduced by Ghahramani [14]. For $\mu \in M(G)$, the map $\Theta(\mu)$ extends the action $\phi \mapsto P_\mu(\phi)$, $\phi \in L^\infty(G)$. For a probability measure μ, the harmonic operators T are defined in [19] by the relation $\Theta(\mu)T = T$. The collection of all μ-harmonic operators is denoted by $\widetilde{\mathcal{H}}(\mu)$. The *non-commutative Poisson boundary* of μ, denoted by $\widetilde{\mathcal{H}}_\mu$, is defined to be the space $\widetilde{\mathcal{H}}(\mu)$, equipped with a certain von Neumann algebra structure [17]. The space $\mathcal{H}(\mu)$ yields a von Neumann subalgebra of $\widetilde{\mathcal{H}}(\mu)$ denoted by $\mathcal{H}_\mu$. Non-commutative Poisson boundaries were first considered by Izumi for discrete groups in [18] where he showed that $\widetilde{\mathcal{H}}_\mu$ is the crossed product of $\mathcal{H}_\mu$ by G acting by left translations. Jaworski and Neufang [19] extended this to locally compact G, thus answering a question in [18]. This result was further generalised in [20] to locally compact quantum groups.

When G is abelian, the settings described in the previous two paragraphs are connected by the usual Fourier transform. (In particular, $\widetilde{\mathcal{H}}(\mu)$ is a subalgebra of $\mathcal{B}(L^2(G))$ in this case.) We discuss this relation in Section 4.

One may ask: What is a dual version of $(*)$? Can it be used to study the space $\widetilde{\mathcal{H}}(\mu)$ of harmonic operators? The present paper focuses on these questions. Instead of an ideal of $A(G)$, we start with a left ideal J of $L^1(G)$. We then consider its annihilator $J^\perp$ in $L^\infty(G)$ and the $\mathrm{VN}(G)$-bimodule $\mathrm{Bim}(J^\perp)$ generated by the collection $\{M_f : f \in J^\perp\}$ of multiplication operators in $\mathcal{B}(L^2(G))$. We also construct a suitable saturation $\mathrm{Ran}\, J$ of J within the trace class $\mathcal{T}(G)$ on $L^2(G)$. When G is abelian, utilising Fourier transform and using $(*)$ we show (Section 4) that

$$(\operatorname{Ran} J)^{\perp} = \operatorname{Bim}(J^{\perp}).$$

The following question then arises: Is this formula true for any locally compact group G? We show that equality does occur when G is weakly amenable and either discrete (Section 5) or compact (Section 6).

Given a set $\Lambda \subseteq M(G)$ (not necessarily consisting of probability measures), in Section 7 we define the space $\mathcal{H}(\Lambda)$ of *jointly Λ-harmonic* functions to be the set of functions in $L^{\infty}(G)$ which are μ-harmonic for every μ in Λ, and we introduce in an analogous fashion the corresponding space $\widetilde{\mathcal{H}}(\Lambda)$ of *jointly Λ-harmonic operators*. As a consequence of our previous results, we recover $\widetilde{\mathcal{H}}(\Lambda)$ when the group is compact, weakly amenable discrete or abelian: we show that it is the weak-* closed VN(G)-bimodule generated by $\mathcal{H}(\Lambda)$ in $\mathcal{B}(L^2(G))$. When Λ is a singleton consisting of a probability measure μ, using this we give a proof of the above mentioned result of Izumi and Jaworski–Neufang: the non-commutative Poisson boundary $\widetilde{\mathcal{H}}_{\mu}$ is isomorphic to the crossed product of $\mathcal{H}_{\mu}$ by a canonical action of G.

2. Preliminaries. Let G be a second countable locally compact group equipped with a left Haar measure. As usual, the corresponding Lebesgue spaces on G are denoted by $L^p(G)$ for $1 \leq p \leq \infty$. We denote by $\lambda : G \to \mathcal{B}(L^2(G))$, $s \mapsto \lambda_s$, the left regular representation of the group G, given by $(\lambda_s f)(t) = f(s^{-1}t)$; here $\mathcal{B}(L^2(G))$ denotes the algebra of bounded linear operators on $L^2(G)$. We write $(\cdot,\cdot)$ for the inner product and we use $\langle\cdot,\cdot\rangle$ for the various Banach space dualities, in particular for the duality between $L^1(G)$ and $L^{\infty}(G)$. For $\phi \in L^{\infty}(G)$, let M_{ϕ} be the operator on $L^2(G)$ of multiplication by ϕ. We denote by $\mathcal{D}_G$ or $\mathcal{D}$ the algebra $\{M_{\phi} : \phi \in L^{\infty}(G)\}$. This is a maximal abelian selfadjoint algebra (for brevity, masa).

The predual $\mathcal{T}(G)$ of $\mathcal{B}(L^2(G))$ can be identified with the space of all functions the form $h : G \times G \to \mathbb{C}$, defined marginally almost everywhere (see for example [1]) and given by

$$h(x,y) = \sum_{i=1}^{\infty} f_i(x) g_i(y), \tag{1}$$

where $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$. The norm on $\mathcal{T}(G)$ is given by

$$\|h\|_t = \inf\Big\{\sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2\Big\}$$

where the infimum is taken over all representations (1) of h. The pairing between $\mathcal{B}(L^2(G))$ and $\mathcal{T}(G)$ is given by

$$\langle T, h\rangle_t := \sum_{i=1}^{\infty} (T f_i, \bar{g}_i).$$

The *group von Neumann algebra* of G is the algebra

$$\mathrm{VN}(G) = \overline{\mathrm{span}\{\lambda_x : x \in G\}}^{w^*},$$

acting on $L^2(G)$. Its predual can be identified with the *Fourier algebra* $A(G)$ of G [10] which is the (commutative, regular, semisimple) Banach algebra consisting of all complex functions u on G of the form

$$u(x) = (\lambda_x f, g), \quad x \in G, \quad \text{where } f, g \in L^2(G). \tag{2}$$

The pairing between $\mathrm{VN}(G)$ and $A(G)$ is given by $\langle \lambda_x, u\rangle_A = u(x)$. A function $\sigma : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $\sigma u \in A(G)$ for every $u \in A(G)$. If σ is a multiplier of $A(G)$, the map $m_\sigma : A(G) \to A(G)$ given by $m_\sigma(u) = \sigma u$ is automatically bounded. A multiplier σ of $A(G)$ is called *completely bounded* [7] if the dual $m_\sigma^* : \mathrm{VN}(G) \to \mathrm{VN}(G)$ of m_σ is completely bounded. We write $M^{\mathrm{cb}}A(G)$ for the algebra of all completely bounded multipliers of $A(G)$. If σ is in $M^{\mathrm{cb}}A(G)$ and $h \in \mathcal{T}(G)$, it was shown by Gilbert and Bożejko–Fendler [3] that $N(\sigma)h$ is in $\mathcal{T}(G)$, where $N(\sigma)(s,t) = \sigma(ts^{-1})$.

Let J be a closed ideal of $A(G)$. Consider the norm closed masa bimodule

$$\mathrm{Sat}\, J = \overline{\mathrm{span}(N(J)\mathcal{T}(G))}^{\|\cdot\|_t}$$

of $\mathcal{T}(G)$ generated by $N(J)$. Denote by $(\mathrm{Sat}\, J)^\perp$ the annihilator of $\mathrm{Sat}\, J$ in $\mathcal{B}(L^2(G))$. Let $J^\perp$ be the annihilator of J in $\mathrm{VN}(G)$, and $\mathrm{Bim}(J^\perp)$ the weak-* closed masa bimodule generated by $J^\perp$ in $\mathcal{B}(L^2(G))$.

The following result was proved in [1]:

Theorem 2.1. *Let $J \subseteq A(G)$ be a closed ideal. Then* $(\mathrm{Sat}\, J)^\perp = \mathrm{Bim}(J^\perp)$.

3. Ideals of $L^1(G)$ and bimodules over $\mathrm{VN}(G)$. Throughout this section, we fix a locally compact group G. Let $\rho : G \to \mathcal{B}(L^2(G))$, $r \mapsto \rho_r$, be the right regular representation of G on $L^2(G)$, given by

$$(\rho_r f)(s) = \Delta(r)^{1/2} f(sr), \quad f \in L^2(G),\ s, r \in G,$$

where Δ denotes the modular function of G.

Denote by $\mathrm{ad}\,\rho_r$ the map on $\mathcal{B}(L^2(G))$ given by $\mathrm{ad}\,\rho_r(T) = \rho_r T \rho_r^*$, $T \in \mathcal{B}(L^2(G))$. Let $M(G)$ be the measure algebra of G, that is, the (convolution) Banach algebra of all bounded, complex Borel measures on G. We identify $L^1(G)$ with the (closed) ideal of $M(G)$ consisting of all measures absolutely continuous with respect to Haar measure. Define a representation Θ of the algebra $M(G)$ on $\mathcal{B}(L^2(G))$ by

$$\langle \Theta(\mu)(T), h\rangle_t = \int_G \langle \mathrm{ad}\,\rho_r(T), h\rangle_t \, d\mu(r)$$

for every $h \in \mathcal{T}(G)$. This representation was introduced and studied by Størmer for abelian groups [26] and by Ghahramani [14] for locally compact groups. See [21] for more references.

Since $\operatorname{ad}\rho_r$ and $\Theta(\mu)$ are (bounded) weak-* continuous maps, they have (bounded) preduals θ_r and $\theta(\mu) : \mathcal{T}(G) \to \mathcal{T}(G)$. Thus,

$$\theta(\mu)(h) = \int_G \theta_r(h)\, d\mu(r), \qquad h \in \mathcal{T}(G).$$

Note that for $r \in G$ we have [1, Lemma 4.1]

$$\theta_r(h) = \Delta(r^{-1})h_{r^{-1}}, \qquad h \in \mathcal{T}(G). \tag{3}$$

Here $h_r(s,t) = h(sr,tr)$ and $s,t,r \in G$. Therefore, if $f \in L^1(G)$ then

$$\theta(f)(h) = \int_G \Delta(r^{-1})h_{r^{-1}}f(r)\, dr, \qquad h \in \mathcal{T}(G).$$

Let $J \subseteq L^1(G)$ be a closed left ideal; we denote by $J^\perp$ its annihilator in $L^\infty(G)$. Set

$$\operatorname{Ran} J = \overline{\operatorname{span}\{\theta(f)(h) : f \in J, h \in \mathcal{T}(G)\}}^{\|\cdot\|_t} \subseteq \mathcal{T}(G).$$

Given a subspace $\mathcal{U} \subseteq L^\infty(G)$, we let

$$\operatorname{Bim}(\mathcal{U}) = \overline{\operatorname{span}\{AM_aB : A,B \in \operatorname{VN}(G),\ a \in \mathcal{U}\}}^{w^*} \subseteq \mathcal{B}(L^2(G));$$

thus, $\operatorname{Bim}(\mathcal{U})$ is the weak-* closed VN(G)-bimodule generated by the multiplication operators with symbols coming from $\mathcal{U}$.

We denote by $(\operatorname{Ran} J)^\perp$ the annihilator of $\operatorname{Ran} J$ within $\mathcal{B}(L^2(G))$. We are interested in the relation between $(\operatorname{Ran} J)^\perp$ and $\operatorname{Bim}(J^\perp)$.

Lemma 3.1. *The space* $(\operatorname{Ran} J)^\perp$ *is the intersection of the kernels of the maps* $\{\Theta(f) : f \in J\}$. *We write this as*

$$(\operatorname{Ran} J)^\perp = \ker \Theta(J).$$

Consequently, $(\operatorname{Ran} J)^\perp$ *is a* VN(G)*-bimodule.*

Proof. Since $\Theta(f)$ is a VN(G)-bimodule map for every $f \in J$, the space $\ker \Theta(J)$ is a VN(G)-bimodule. The equality $(\operatorname{Ran} J)^\perp = \ker \Theta(J)$ follows directly from the definition. ∎

Remark 3.2. Let $s,t \in G$, $f \in L^1(G)$ and $a \in L^\infty(G)$. Then

$$\Theta(f)(\lambda_s^* M_a \lambda_t) = \lambda_s^*\Theta(f)(M_a)\lambda_t = \lambda_s^*\Big(\int_G \rho_r M_a \rho_r^* f(r)\, dr\Big)\lambda_t = \lambda_s^* M_g \lambda_t,$$

where

$$g(x) = \int_G a(xr)f(r)\, dr = \int_G f(x^{-1}z)a(z)\, dz = \langle a, \lambda_x f\rangle, \qquad x \in G.$$

Lemma 3.3. *Let* $s,t \in G$ *and* $a \in L^\infty(G)$. *Then*

$$\lambda_s^* M_a \lambda_t \in (\operatorname{Ran} J)^\perp \iff a \in J^\perp.$$

Proof. Since $(\operatorname{Ran} J)^{\perp}$ is a $\mathrm{VN}(G)$-bimodule, it suffices to show that $a \in J^{\perp}$ if and only if $M_a \in (\operatorname{Ran} J)^{\perp}$.

Suppose $a \in J^{\perp}$ and $f \in J$. By Remark 3.2,

$$\Theta(f)(M_a) = M_g, \quad \text{where} \quad g(x) = \langle a, \lambda_x f\rangle,\ x \in G.$$

Since $f \in J$ and J is a closed left ideal, $\lambda_x f \in J$ [11, 2.43], and so g vanishes almost everywhere. Thus, $\Theta(f)(M_a) = 0$ for all $f \in J$ and so $M_a \in (\operatorname{Ran} J)^{\perp}$.

Suppose, conversely, that $M_a \in (\operatorname{Ran} J)^{\perp}$. Then for every $f \in J$ we have $\Theta(f)(M_a) = 0$ and so, by Remark 3.2,

$$\langle a, \lambda_x f\rangle = 0 \quad \text{for almost all } x.$$

Thus, for all $g \in L^1(G)$, we have

$$\int_G g(x)\langle a, \lambda_x f\rangle\, dx = 0.$$

Therefore

$$\begin{aligned}\langle a, (g * f)\rangle &= \int_G (g * f)(y) a(y)\, dy = \int_G \Big(\int_G g(x) f(x^{-1}y)\, dx\Big) a(y)\, dy \\ &= \int_G g(x)\Big(\int_G f(x^{-1}y) a(y)\, dy\Big)\, dx = \int_G g(x)\langle a, \lambda_x f\rangle\, dx = 0.\end{aligned}$$

Let (g_i) be an approximate unit for $L^1(G)$. Then

$$\langle a, f\rangle = \lim \langle a, g_i * f\rangle = 0,$$

and hence $a \in J^{\perp}$. ∎

Proposition 3.4. *For every left ideal $J \subseteq L^1(G)$, we have*

$$\operatorname{Bim}(J^{\perp}) \subseteq (\operatorname{Ran} J)^{\perp}. \tag{4}$$

Proof. Since the maps $\Theta(f)$ are weak-* continuous, it suffices, by Lemma 3.1, to show that if $a \in J^{\perp}$ and $s, t \in G$, then $\Theta(f)(\lambda_s^* M_a \lambda_t) = 0$ for all $f \in J$. But this follows from Lemma 3.3. ∎

In the subsequent sections, we will show that equality holds in (4) when G is weakly amenable discrete, compact or abelian. We do not know whether equality holds in (4) for a general locally compact group G; in the next lemma, we establish a useful restricted version, which should be compared to [1, Lemma 4.6]. We identify the annihilator $J^{\perp}$ of an ideal $J \subseteq L^1(G)$ with its image in the masa $\mathcal{D} = \mathcal{D}_G$.

Proposition 3.5. *For every left ideal $J \subseteq L^1(G)$,*

$$\operatorname{Bim}(J^{\perp}) \cap \mathcal{D} = (\operatorname{Ran} J)^{\perp} \cap \mathcal{D} = J^{\perp}.$$

Proof. Trivially, $J^{\perp} \subseteq \mathrm{Bim}(J^{\perp}) \cap \mathcal{D}$, while, by Proposition 3.4, $\mathrm{Bim}(J^{\perp}) \cap \mathcal{D} \subseteq (\mathrm{Ran}\, J)^{\perp} \cap \mathcal{D}$. It remains to show that if $M_a \in (\mathrm{Ran}\, J)^{\perp} \cap \mathcal{D}$, then $a \in J^{\perp}$. But this follows from Lemma 3.3. ∎

4. The abelian case. In [2], we used Theorem 2.1 to investigate the relation between σ-harmonic functionals (where σ is a multiplier of the Fourier algebra) and σ-harmonic operators.

In this section we assume that G is a second countable locally compact abelian group and we obtain the equality

$$\mathrm{Bim}(J^{\perp}) = (\mathrm{Ran}\, J)^{\perp}$$

for an ideal $J \subseteq L^1(G)$.

For this, we use Theorem 2.1 for the dual group $\varGamma$. To see the connection, let μ be a probability measure on G and let σ be the Fourier transform of μ, that is, $\sigma = \hat{\mu}$, where $\hat{\mu}(x) = \int_G \overline{x(r)}\, d\mu(r)$ for $x \in \varGamma$. As $L^1(G)$ is a convolution ideal in $M(G)$ and $A(\varGamma) = \{\hat{f} : f \in L^1(G)\}$, the function σ is a multiplier of $A(\varGamma)$ which, since $\mathrm{VN}(\varGamma)$ is an abelian C^*-algebra, is completely bounded (see, for example, [9, Prop. 2.2.6]). It is not hard to see that in this case the space of μ-harmonic functions on G is identified with the space of $\check{\sigma}$-harmonic functionals on $A(\varGamma)$ (here $\check{\sigma}(t) = \sigma(t^{-1})$) via the dual of the Fourier transform. In [2], we used Theorem 2.1 to investigate the relation between σ-harmonic functionals (where σ is a multiplier of the Fourier algebra) and σ-harmonic operators.

In this section, we consider ideals both of $A(\varGamma)$ and of $L^1(G)$. For clarity, if I is an ideal of $A(\varGamma)$, we will denote by $\mathrm{Bim}_{\mathcal{D}_\varGamma}(I^{\perp})$ the $D_\varGamma$-bimodule of $\mathcal{B}(L^2(\varGamma))$ generated by the annihilator $I^{\perp}$ of I in $\mathrm{VN}(\varGamma)$, while if I is an ideal of $L^1(G)$ we will denote by $\mathrm{Bim}_{\mathrm{VN}(G)}(I^{\perp})$ the $\mathrm{VN}(G)$-bimodule of $\mathcal{B}(L^2(G))$ generated by the multiplication operators with symbols in the annihilator $I^{\perp}$ of I in $L^\infty(G)$.

For a closed ideal $J \subseteq L^1(G)$, we wish to prove the equality

$$(\mathrm{Ran}\, J)^{\perp} = \mathrm{Bim}_{\mathrm{VN}(G)}(J^{\perp}). \tag{5}$$

Let $F : L^2(G) \to L^2(\varGamma)$ be the unitary operator such that $F(f) = \hat{f}$, $f \in L^1(G) \cap L^2(G)$, and

$$\varPhi : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(\varGamma)), \quad \varPhi(T) = FTF^{-1}.$$

It is clear that $\varPhi(\mathcal{D}_G) = \mathrm{VN}(\varGamma)$ and $\varPhi(\mathrm{VN}(G)) = \mathcal{D}_\varGamma$, and it is readily verified that

$$\varPhi(\mathrm{Bim}_{\mathrm{VN}(G)}(J^{\perp})) = \mathrm{Bim}_{\mathcal{D}_\varGamma}(\varPhi(J^{\perp})) \quad \text{and} \quad \varPhi((\mathrm{Ran}\, J)^{\perp}) = \varPsi(\mathrm{Ran}\, J)^{\perp},$$

where $\varPsi : \mathcal{T}(G) \to \mathcal{T}(\varGamma)$ denotes the predual of the map $\varPhi^{-1}$. Hence, (5) is equivalent to

$$(\Psi(\operatorname{Ran} J))^{\perp} = \operatorname{Bim}_{\mathcal{D}_{\Gamma}}(\Phi(J^{\perp})), \tag{6}$$

after identifying $J^{\perp}$ with its image in $\mathcal{D}_G$.

We will need the following lemma.

LEMMA 4.1. *Let $h \in \mathcal{T}(G)$ and $f \in L^1(G)$. Then*

$$\Psi(\theta(f)(h)) = N(\phi(\hat{f}))\Psi(h),$$

where ϕ denotes the map $\phi(u)(x) = u(x^{-1})$ for $x \in \Gamma$ and $N(\sigma)(s,t) = \sigma(ts^{-1})$.

Proof. Since the maps Ψ and $\theta(f)$ are linear and continuous on $\mathcal{T}(G)$, it suffices to prove the lemma when $h(x,y) = \xi(x)\bar{\eta}(y)$, where ξ, η are continuous with compact support. Note that since $F : L^2(G) \to L^2(\Gamma)$ is a unitary operator, the map F_2 given on elementary tensors by $F_2(\xi \otimes \eta) = F(\xi) \otimes F(\eta)$ is a well-defined bounded linear map from $\mathcal{T}(G)$ into $\mathcal{T}(\Gamma)$. Since the function $(s,t,r) \mapsto h(sr^{-1}, tr^{-1})f(r)$ is in $L^1(G \times G \times G)$, for $x, y \in \Gamma$ we have

$$\begin{aligned}
F_2(\theta(f)(h))(x,y) &= \int_G \int_G \overline{x(s)y(t)}(\theta(f)h)(s,t)\, ds\, dt \\
&= \int_G \int_G \int_G \overline{x(s)y(t)} h(sr^{-1}, tr^{-1}) f(r)\, dr\, ds\, dt \\
&= \int_G \int_G \int_G \overline{x(sr)y(tr)} h(s,t) f(r)\, dr\, ds\, dt \\
&= \int_G \int_G \int_G \overline{x(s)x(r)y(t)y(r)} h(s,t) f(r)\, dr\, ds\, dt \\
&= \int_G \int_G \int_G \overline{x(s)y(t)(xy)(r)} h(s,t) f(r)\, dr\, ds\, dt \\
&= \int_G \int_G \overline{x(s)y(t)} \Big(\int_G \overline{(xy)(r)} f(r)\, dr \Big) h(s,t)\, ds\, dt \\
&= \hat{f}(xy) \int_G \int_G \overline{x(s)y(t)} h(s,t)\, ds\, dt = \hat{f}(xy) F_2(h)(x,y).
\end{aligned}$$

But it is not hard to verify that, for all such ξ, η,

$$\Psi(\xi \otimes \bar{\eta})(x,y) = (\hat{\xi} \otimes \overline{\hat{\eta}})(x,y) = F_2(\xi \otimes \bar{\eta})(x, y^{-1})$$

and so

$$\Psi(h)(x,y) = F_2(h)(x, y^{-1}) \quad \text{for all } h \in \mathcal{T}(G). \tag{7}$$

Thus the previous equality gives

$$\begin{aligned}
\Psi(\theta(f)(h))(x,y) &= F_2(\theta(f)(h))(x, y^{-1}) = \hat{f}(xy^{-1}) F_2(h)(x, y^{-1}) \\
&= \phi(\hat{f})(yx^{-1}) \Psi(h)(x,y),
\end{aligned}$$

i.e. $\Psi(\theta(f)(h)) = N(\phi(\hat{f}))\Psi(h)$. ∎

An operator $T \in \mathcal{B}(L^2(\Gamma))$ is in $(\Psi(\operatorname{Ran} J))^\perp$ if and only if $\langle T, \Psi(\theta(f)h)\rangle_t = 0$ for all $f \in J$ and $h \in \mathcal{T}(G)$. It follows from Lemma 4.1 that this is equivalent to the statement that $\langle T, N(\phi(\hat{f}))\Psi(h)\rangle_t = 0$ for all $f \in J$ and $h \in \mathcal{T}(G)$. Noting that Ψ maps $\mathcal{T}(G)$ onto $\mathcal{T}(\Gamma)$, we find that T is in $(\Psi(\operatorname{Ran} J))^\perp$ if and only if it annihilates $N(\phi(\hat{J}))\mathcal{T}(\Gamma)$, i.e. if and only if T is in $(\operatorname{Sat}\phi(\hat{J}))^\perp$. (Here $\hat{J} = \{\hat{f} : f \in J\}$.)

We have thus shown that

$$(\Psi(\operatorname{Ran} J))^\perp = (\operatorname{Sat}\phi(\hat{J}))^\perp.$$

Using Theorem 2.1 for the ideal $\phi(\hat{J}) \subseteq A(\Gamma)$, we see that

$$(\operatorname{Sat}\phi(\hat{J}))^\perp = \operatorname{Bim}_{\mathcal{D}_\Gamma}(\phi(\hat{J})^\perp)$$

and so

$$(\Psi(\operatorname{Ran} J))^\perp = \operatorname{Bim}_{\mathcal{D}_\Gamma}(\phi(\hat{J})^\perp).$$

Thus the required equality (6) becomes

$$\operatorname{Bim}_{\mathcal{D}_\Gamma}(\phi(\hat{J})^\perp) = \operatorname{Bim}_{\mathcal{D}_\Gamma}(\Phi(J^\perp)).$$

It now suffices to prove that

$$(\phi(\hat{J}))^\perp = \Phi(J^\perp).$$

We have

$$\Phi(J^\perp) = \Big\{\Phi(M_g) : M_g \in \mathcal{D}_G,\ \int_G g(s)f(s)\,ds = 0 \text{ for all } f \in J\Big\}.$$

On the other hand, using the equality $\mathrm{VN}(\Gamma) = \Phi(\mathcal{D}_G)$, we have

$$\begin{aligned}(\phi(\hat{J}))^\perp &= \{T \in \mathrm{VN}(\Gamma) : \langle T, \phi(\hat{f})\rangle_A = 0 \text{ for all } f \in J\}\\ &= \{\Phi(M_g) : M_g \in \mathcal{D}_G,\ \langle \Phi(M_g), \phi(\hat{f})\rangle_A = 0 \text{ for all } f \in J\},\end{aligned}$$

where $\langle\cdot,\cdot\rangle_A$ denotes the Banach space duality between $\mathrm{VN}(\Gamma)$ and $A(\Gamma)$.

Thus, it suffices to prove that, for any $f \in L^1(G)$ and $g \in L^\infty(G)$,

$$\langle \Phi(M_g), \phi(\hat{f})\rangle_A = \int_G g(s)f(s)\,ds. \tag{8}$$

Fix $f \in L^1(G)$ and note that both sides of (8) are linear and weak-* continuous functions of g. Since $L^\infty(G)$ is the weak-* closed linear span of the set $\{x : x \in \Gamma\}$ of characters, it suffices to prove (8) when g is a character x. Now $\Phi(M_x) = \lambda_x$. Since $\langle \lambda_x, \hat{f}\rangle_A = \hat{f}(x)$, we have

$$\begin{aligned}\langle \Phi(M_x), \phi(\hat{f})\rangle_A &= \langle \lambda_x, \phi(\hat{f})\rangle_A = \phi(\hat{f})(x) = \hat{f}(x^{-1})\\ &= \int_G f(s)\overline{x^{-1}(s)}\,ds = \int_G f(s)x(s)\,ds,\end{aligned}$$

as required.

This proves

Proposition 4.2. *Let G be a second countable locally compact abelian group. Then, for any closed ideal $J \subseteq L^1(G)$,*

$$(\operatorname{Ran} J)^\perp = \operatorname{Bim}(J^\perp).$$

5. The discrete case. In this section we assume that G is discrete; in this case, the Haar measure coincides with the counting measure. We denote by δ_s the function on G defined by $\delta_s(t) = 1$ if $s = t$ and $\delta_s(t) = 0$ if $s \neq t$; note that $\{\delta_s\}_{s\in G}$ is an orthonormal basis of $L^2(G)$. Let X be an operator in $\mathcal{B}(L^2(G))$. We denote by $[X(s,t)]$ be the matrix of X with respect to the basis $\{\delta_s\}_{s\in G}$. The diagonal $D(X)$ of X is the operator on $L^2(G)$ whose matrix with respect to the basis $\{\delta_s\}_{s\in G}$ is given by $D(X)(s,t) = 0$ if $s \neq t$ and $D(X)(s,t) = X(s,t)$ if $s = t$. For $t \in G$, we denote by $D_t(X)$ the tth diagonal of X, given by $D_t(X) = \lambda_t D(\lambda_{t^{-1}} X)$. Note that the maps $X \mapsto D_t(X)$ are weak-* continuous and linear.

Also note that $D_r(X) = S_{N(\delta_r)}(X)$, Schur multiplication by the matrix $[N(\delta_r)(s,t)]$. Indeed,

$$S_{N(\delta_r)}([X(s,t)]) = [\delta_r(ts^{-1})(X(s,t))] = \left[\begin{cases} X(s,rs), & t = rs \\ 0, & t \neq rs \end{cases}\right].$$

Thus, if $u : G \to \mathbb{C}$ is *finitely supported*, then $S_{N(u)}(X)$ is a linear combination of diagonals of X.

Suppose that the group G is weakly amenable in the sense of [5]. This means that there exists a net $\{u_i\}_{i\in I}$ consisting of finitely supported elements of $A(G)$ and a positive constant L such that $\|u_i\|_{\mathrm{mcb}} \le L$ for all i and $u_i(s) \to 1$ for all $s \in G$ (here $\|u_i\|_{\mathrm{mcb}}$ is the completely bounded norm of u_i as a multiplier of $A(G)$, or equivalently of the Schur multiplier $S_{N(u_i)}$). It follows that for each $h \in \mathcal{T}(G)$ we have

$$\|N(u_i)h\|_t \le L\|h\|_t \quad \text{for all } i.$$

Proposition 5.1. *Let G be a weakly amenable (discrete) group. Then each $A \in \mathcal{B}(L^2(G))$ is in the weak-* closed linear span of its diagonals.*

Proof. Recall that the diagonals of A are $S_{N(\delta_t)}(A)$, $t \in G$. Thus if $h \in \mathcal{T}(G)$ annihilates all diagonals of A, then

$$0 = \langle S_{N(\delta_t)}(A), h\rangle = \langle A, N(\delta_t)h\rangle \quad \text{for all } t \in G.$$

But $N(\delta_t)h(s,r) = \delta_t(rs^{-1})h(s,r) = h(s,ts)$ when $r = ts$, and $N(\delta_t)h(s,r) = 0$ otherwise. Thus A must annihilate all the diagonals of h. If we prove that h is in the trace-norm closed linear span of its diagonals, it will follow that $\langle A, h\rangle = 0$, as required.

It thus remains to prove that h is in the trace-norm closed linear span of its diagonals. For this, observe first that given $\epsilon > 0$ there is an $h_\epsilon \in \mathcal{T}(G)$, supported on finitely many diagonals, such that $\|h - h_\epsilon\|_t < \epsilon$ (it suffices to take h_ϵ of the form php where p is the projection on the span of a suitably large but finite subset $\{\delta_t : t \in F\}$, since such projections tend strongly to the identity).

But note that

$$\lim_i \|N(u_i)h_\epsilon - h_\epsilon\|_t = 0.$$

This is because on each of the finitely many non-zero diagonals $D_t(h_\epsilon)$ we have $N(u_i)D_t(h_\epsilon) = u_i(t)D_t(h_\epsilon)$, hence

$$\|N(u_i)D_t(h_\epsilon) - D_t(h_\epsilon)\|_t = |u_i(t) - 1|\,\|D_t(h_\epsilon)\|_t,$$

and $u_i(t) \to 1$. Therefore we can choose i_0 such that $\|N(u_i)h_\epsilon - h_\epsilon\|_t < \epsilon$ for all $i \geq i_0$.

Thus we finally have, for all $i \geq i_0$,

$$\begin{aligned}\|N(u_i)h - h\|_t &\leq \|N(u_i)(h - h_\epsilon)\|_t + \|N(u_i)h_\epsilon - h_\epsilon\|_t + \|h_\epsilon - h\|_t \\ &\leq L\|h - h_\epsilon\|_t + \|N(u_i)h_\epsilon - h_\epsilon\|_t + \|h_\epsilon - h\|_t < L\epsilon + \epsilon + \epsilon.\end{aligned}$$

This shows that h is in the trace-norm closed linear span of the family $\{N(u_i)h : i \in I\}$; but as observed above, since each u_i is finitely supported, each $N(u_i)h$ is a linear combination of diagonals of h. This proves the claim and concludes the proof of the proposition. ∎

LEMMA 5.2. *Let G be a discrete group and $J \subseteq L^1(G)$ be a closed left ideal. If $X \in (\mathrm{Ran}\, J)^\perp$, then $D_t(X) \in (\mathrm{Ran}\, J)^\perp$ for all $t \in G$.*

Proof. A direct calculation shows that

$$D(\rho_s X \rho_{s^*}) = \rho_s D(X) \rho_{s^*}.$$

It follows by the weak-* continuity of D that

$$\Theta(f)D(X) = D(\Theta(f)(X))$$

for $f \in L^1(G)$. The conclusion follows from Lemma 3.1. ∎

PROPOSITION 5.3. *Let G be a discrete weakly amenable group and let $J \subseteq L^1(G)$ be a closed left ideal. Then*

$$(\mathrm{Ran}\, J)^\perp = \mathrm{Bim}(J^\perp).$$

Proof. Let $X \in (\mathrm{Ran}\, J)^\perp$. Since $(\mathrm{Ran}\, J)^\perp$ is a VN(G)-bimodule, we have $\lambda_{t^{-1}}X \in (\mathrm{Ran}\, J)^\perp$ and it follows from Lemma 5.2 that $D(\lambda_{t^{-1}}X) \in (\mathrm{Ran}\, J)^\perp$. Now, $D(\lambda_{t^{-1}}X) = M_{a_t}$ for some $a_t \in \ell^\infty(G)$. It follows from Lemma 3.3 that $a_t \in J^\perp$, and hence $D_t(X) \in \mathrm{Bim}(J^\perp)$. Since the operator X is in the weak-* closed linear span of its diagonals (Proposition 5.1), we infer that $X \in \mathrm{Bim}(J^\perp)$.

By Proposition 3.4, the proof is complete. ∎

REMARK 5.4. In a previous version of this paper we claimed that Proposition 5.3 holds in any discrete group. We wish to thank J. Crann and M. Neufang who pointed out that our argument was incomplete.

6. The compact case. In this section we assume that G is compact. We denote by $\widehat{G}$ the unitary dual of G, that is, the set of all (equivalence classes of) irreducible representations. If $\pi \in \widehat{G}$, we denote by H_π the space of the representation π, and by d_π its dimension. Suppose that for each irreducible representation (π, H_π) of G we are given a subspace $E_\pi \subseteq H_\pi$ (possibly trivial). If $E_\pi \neq \{0\}$ choose an orthonormal basis $e_1, \ldots, e_{s_\pi}$ of E_π and extend it to an orthonormal basis $e_1, \ldots, e_{d_\pi}$ of H_π. If $E_\pi = \{0\}$ let $e_1, \ldots, e_{d_\pi}$ be an orthonormal basis of H_π. For $\pi \in \widehat{G}$, we denote by π_{ij}, $1 \leq i, j \leq d_\pi$, the matrix coefficients of the representation π with respect to the basis $e_1, \ldots, e_{d_\pi}$ of H_π; thus,

$$\pi_{ij}(s) = (\pi(s)e_j, e_i), \quad s \in G,\ i, j = 1, \ldots, d_\pi. \tag{9}$$

Let $E = \{E_\pi\}_{\pi \in \widehat{G}}$ and consider the set

$$J(E) = \overline{\operatorname{span}\left\{\pi_{ij} : 1 \leq i \leq d_\pi,\ 1 \leq j \leq s_\pi,\ \pi \in \widehat{G},\ E_\pi \neq \{0\}\right\}}^{\|\cdot\|_1},$$

where $\|\cdot\|_1$ is the $L^1(G)$ norm. Clearly, $J(E)$ is a closed left ideal of $L^1(G)$, being invariant under left translations. Conversely, every closed left ideal of $L^1(G)$ is of this form [16, 38.13] for some $E = \{E_\pi\}_{\pi \in \widehat{G}}$.

Denoting by $J(E)^\perp$ the annihilator in $L^\infty(G)$, we have:

PROPOSITION 6.1. *The space $J(E)^\perp$ is the weak-* closure of the linear span of*

$$\begin{aligned}\mathcal{S} := &\left\{\overline{\pi'_{ij}} : 1 \leq i \leq d_{\pi'},\ s_{\pi'} < j \leq d_{\pi'},\ E_{\pi'} \neq \{0\}\right\}\\ &\cup \left\{\overline{\pi'_{ij}} : 1 \leq i, j \leq d_{\pi'},\ E_{\pi'} = \{0\}\right\}.\end{aligned}$$

Proof. Let $\pi' \in \widehat{G}$ be such that $E_{\pi'} \neq \{0\}$ and let $1 \leq k \leq d_{\pi'}$ and $s_{\pi'} < l \leq d_{\pi'}$. Let $\pi \in \widehat{G}$ be such that $E_\pi \neq \{0\}$ and let $1 \leq i \leq d_\pi$ and $1 \leq j \leq s_\pi$. If π' is not equivalent to π, then $\int \overline{\pi'_{kl}(t)}\pi_{ij}(t)\,dt = 0$ for all k, l by the Schur orthogonality relations [11, 5.8]. If π' is equivalent to π, then $\int \overline{\pi'_{kl}(t)}\pi_{ij}(t)\,dt = 0$ for all k since $j \neq l$. Moreover, it is clear that

$$\bigcup \left\{\overline{\pi'_{ij}} : 1 \leq i, j \leq d_{\pi'},\ E_{\pi'} = \{0\}\right\} \subseteq J(E)^\perp.$$

Hence $\mathcal{S} \subseteq J(E)^\perp$.

For the reverse containment, we show that the preannihilator $\mathcal{S}_\perp$ is contained in $J(E)$. Now $\mathcal{S}_\perp$ is a closed left ideal in $L^1(G)$, since the linear span of $\mathcal{S}$ is invariant under left translations. Take $f \in \mathcal{S}_\perp$. Let (g_ν) be an approximate unit for $L^1(G)$ consisting of functions in $L^2(G)$ and set $f_\nu = g_\nu * f$;

so $f_\nu \in L^2(G)$ and $\|f - f_\nu\|_1 \to 0$. Since each f_ν is in $\mathcal{S}_\perp$, it is orthogonal (in the $L^2(G)$ sense) to π'_{ij}'s whose conjugates generate $\mathcal{S}$ and hence, by the Peter–Weyl theorem, f_ν belongs to the $L^2(G)$ closed span of the remaining π'_{ij}'s, that is, to the closure of

$$\operatorname{span}\left\{\pi'_{ij} : 1 \le i \le d_{\pi'},\, 1 \le j \le s_{\pi'},\, E_{\pi'} \ne \{0\}\right\}$$

in $L^2(G)$. But the $L^2(G)$ closure of this set is contained in its $L^1(G)$ closure, which coincides with $J(E)$. Thus $f_\nu \in J(E)$ for each ν, and so $f \in J(E)$. ∎

Remark 6.2. The above proposition may be proved using the theory of strong M-bases in Banach spaces:

Let X be a Banach space. A family (u_i) of vectors is called a *Markushevich basis* or an *M-basis* of X [15, Definition 1.7] if there exists a family (u'_i) in the dual X^* of X such that

(1) $\langle u'_i, u_j\rangle_X = \delta_{ij}$, where $\langle\cdot,\cdot\rangle_X$ is the pairing between X^* and X,
(2) $\overline{\operatorname{span}\{u_i\}}^{\|\cdot\|} = X$,
(3) $\overline{\operatorname{span}\{u'_i\}}^{w^*} = X^*$.

The family (u_i) is called a *strong M-basis* [15, Definition 1.32] if for every $x \in X$ we have

$$x \in \overline{\operatorname{span}\{u_i : \langle u'_i, x\rangle_X \ne 0\}}^{\|\cdot\|}.$$

It follows from [8, 2.9.3] that the family $\{\pi_{ij} : 1 \le i,j \le d_\pi,\, \pi \in \widehat{G}\}$ as defined in (9) is a strong M-basis of the space $L^1(G)$. Proposition 6.1 now follows from [15, Proposition 1.35].

By the Peter–Weyl theorem (see for example [11, Theorem 5.12]), $L^2(G)$ is the orthogonal direct sum

$$L^2(G) = \bigoplus_{\pi\in\widehat{G}} \mathcal{E}_\pi$$

where

$$\mathcal{E}_\pi = \operatorname{span}\left\{\sqrt{d_\pi}\,\pi_{ij} : 1 \le i,j \le d_\pi\right\}.$$

Moreover, $\sqrt{d_\pi}\,\pi_{ij}$, $1 \le i,j \le d_\pi$, is an orthonormal basis of $\mathcal{E}_\pi$. If $\pi \in \widehat{G}$, denote by $P_\pi \in \mathcal{B}(L^2(G))$ the orthogonal projection onto $\mathcal{E}_\pi$.

With respect to this decomposition, each $T \in \mathcal{B}(L^2(G))$ corresponds to an infinite matrix $T = [T_{\pi,\pi'}]$ of operators $T_{\pi,\pi'} \in \mathcal{B}(\mathcal{E}'_\pi, \mathcal{E}_\pi)$ which act on finite-dimensional spaces, where $T_{\pi,\pi'} = P_\pi T P_{\pi'}$.

Remark 6.3. *If* $\pi \in \widehat{G}$ *then* $P_\pi \in \mathrm{VN}(G)$.

Proof. Since $\mathcal{E}_\pi$ is ρ_s-invariant, we have $P_\pi\rho_s = \rho_s P_\pi$ for all $s \in G$. ∎

Remark 6.4. An operator T is in $(\operatorname{Ran} J)^\perp$ (resp. $\operatorname{Bim}(J^\perp)$) if and only if $T_{\pi,\pi'}$ is in $(\operatorname{Ran} J)^\perp$ (resp. $\operatorname{Bim}(J^\perp)$) for all $\pi, \pi' \in \hat{G}$.

Proof. Since $(\operatorname{Ran} J)^{\perp}$ is a $\mathrm{VN}(G)$-bimodule, if $T \in (\operatorname{Ran} J)^{\perp}$ then, by Remark 6.3, $T_{\pi,\pi'} = P_\pi T P_{\pi'}$ is in $(\operatorname{Ran} J)^{\perp}$. Conversely, if $T_{\pi,\pi'} \in (\operatorname{Ran} J)^{\perp}$ for all $\pi, \pi' \in \hat{G}$ then, since T is in the weak-* closed linear span of $\{T_{\pi,\pi'} : \pi, \pi' \in \widehat{G}\}$ and $(\operatorname{Ran} J)^{\perp}$ is a weak-* closed subspace, it follows that T is in $(\operatorname{Ran} J)^{\perp}$.

The proof for $\operatorname{Bim}(J^{\perp})$ is identical. ∎

THEOREM 6.5. *Let G be a compact group and $J \subseteq L^1(G)$ be a closed left ideal. Then*

$$(\operatorname{Ran} J)^{\perp} = \operatorname{Bim}(J^{\perp}).$$

Proof. As noted in the introduction to this section, the ideal J is of the form $J(E)$ for some $E = \{E_\pi\}_{\pi \in \widehat{G}}$. By Proposition 3.4, it is enough to show that if an operator T is in $(\operatorname{Ran} J)^{\perp}$, then $T \in \operatorname{Bim}(J^{\perp})$. By Remark 6.4, it suffices to prove that $T_{\pi,\pi'} \in \operatorname{Bim}(J^{\perp})$ for all $\pi, \pi' \in \widehat{G}$.

Fix $\pi, \pi' \in \widehat{G}$ and write $P := P_\pi$ and $Q := P_{\pi'}$ to simplify notation. We have to prove that $PTQ \in \operatorname{Bim}(J^{\perp})$. Recall that the linear span of the set

$$\{M_{\overline{\pi_{ij}}}\lambda_s : \pi \in \widehat{G},\ 1 \le i, j \le d_\pi,\ s \in G\}$$

is a *-algebra with trivial commutant, and it is weak-* dense in $\mathcal{B}(L^2(G))$. It follows that the linear span of the set

$$(*) \qquad \{PM_{\overline{\pi_{ij}}}\lambda_s Q : \pi \in \widehat{G},\ 1 \le i, j \le d_\pi,\ s \in G\}$$

is weak-* dense in $\mathcal{B}(QL^2(G), PL^2(G))$. Since $\mathcal{B}(QL^2(G), PL^2(G))$ is finite-dimensional, we have

$$\operatorname{span}\{PM_{\overline{\pi_{ij}}}\lambda_s Q : \pi \in \widehat{G},\ 1 \le i, j \le d_\pi,\ s \in G\} = \mathcal{B}(QL^2(G), PL^2(G)).$$

From the generating set $(*)$ we choose an algebraic basis $\{PM_k\lambda_{s_k}Q : 1 \le k \le m\}$ of $\mathcal{B}(QL^2(G), PL^2(G))$, where each M_k is $M_{\overline{\pi_{ij}}}$ for some $\pi \in \widehat{G}$ and some $1 \le i, j \le d_\pi$. There are scalars c_k such that

$$(10) \qquad PTQ = \sum_{k=1}^{m} c_k P M_k \lambda_{s_k} Q.$$

We will show that the only non-zero terms in this sum are those for which $M_k = M_{\overline{\pi_{ij}}}$ for some π, i, j, where either $E_\pi = \{0\}$, or $E_\pi \neq \{0\}$ and $s_\pi < j \le d_\pi$. Since such terms are in $\operatorname{Bim}(J^{\perp})$, it will follow that PTQ is in $\operatorname{Bim}(J^{\perp})$, thus completing the proof.

For a continuous function f we have (recalling that $\Theta(f)$ is a $\mathrm{VN}(G)$-bimodule map)

$$(11) \qquad \Theta(f)(PTQ) = \sum_{k=1}^{m} c_k P\Theta(f)(M_k)\lambda_{s_k} Q.$$

Fix $k \in \{1, \dots, m\}$, and let π_{ij} be such that $M_k = M_{\overline{\pi_{ij}}}$. Then

$$\Theta(f)(M_k) = \Theta(f)(M_{\overline{\pi_{ij}}}) = \int_G f(r)(\rho_r M_{\overline{\pi_{ij}}} \rho_r^*)\, dr = M_g$$

where $g(x) = \int_G f(r)\overline{\pi_{ij}}(xr)\, dr$ (Remark 3.2), that is,

$$g(x) = \Big(f, \sum_k \pi_{ik}(x)\pi_{kj}\Big) = \sum_k \overline{\pi_{ik}(x)}(f, \pi_{kj}).$$

Let $\pi' \in \widehat{G}$ be such that $E_{\pi'} \neq \{0\}$ and choose $f = d_{\pi'}\pi'_{nn}$ where $1 \le n \le s_{\pi'}$. Then, by the orthogonality relations,

$$g(x) = \sum_k \overline{\pi_{ik}(x)}\, \delta_{nk}\delta_{nj}\delta_{\pi\pi'} = \overline{\pi_{in}(x)}\, \delta_{nj}\delta_{\pi\pi'}.$$

It follows that

$$\Theta(f)(M_{\overline{\pi_{ij}}}) = \Theta(d_{\pi'}\pi'_{nn})(M_{\overline{\pi_{ij}}}) = M_{\overline{\pi_{in}}}\delta_{nj}\delta_{\pi\pi'} = M_{\overline{\pi_{ij}}}\delta_{nj}\delta_{\pi\pi'}.$$

Hence all the monomials in the expression (11) for $\Theta(f)(PTQ)$ must vanish, except when $\pi = \pi'$ and $j = n$, in which case they are left unchanged. Thus (11) gives

$$\Theta(f)(PTQ) = \sum_k c_k P M_k \lambda_{s_k} Q, \tag{12}$$

the summation being over those k for which $M_k = M_{\overline{\pi'_{in}}}$.

Now $f \in J$ since $1 \le n \le s_{\pi'}$; thus, by Lemma 3.1, $\Theta(f)(PTQ) = 0$ and therefore the sum (12) must vanish. But the monomials $PM_k\lambda_{s_k}Q$ are linearly independent (they were chosen from an algebraic basis) and so each term must vanish.

Thus, for all π'_{ij} with $E_{\pi'} \neq \{0\}$, $1 \le i \le d_{\pi'}$ and $1 \le j \le s_{\pi'}$, all terms of the form $c_k P M_{\overline{\pi'_{ij}}} \lambda_{s_k} Q$ must vanish in the sum (10). Therefore in this sum the only non-zero terms remaining are of the form $c_k P M_k \lambda_{s_k} Q$ where $M_k = M_{\overline{\pi_{ij}}}$ for some π_{ij} with $E_\pi \neq \{0\}$ and $s_\pi < j \le d_\pi$ or for some π_{ij} with $E_\pi = \{0\}$. By Proposition 6.1, these are in $\mathrm{Bim}(J^\perp)$, hence $PTQ \in \mathrm{Bim}(J^\perp)$ as required. ∎

7. Jointly harmonic operators. In this section, G is a locally compact group. If μ is a probability measure on G, let P_μ be the map on $L^\infty(G)$ given by

$$(P_\mu\phi)(s) = \int_G \phi(st)\, d\mu(t).$$

A function ϕ is called *μ-harmonic* [13, 12] if

$$P_\mu\phi = \phi.$$

More generally, given a set $\Lambda \subseteq M(G)$ (not necessarily consisting of probability measures) we define the set $\mathcal{H}(\Lambda)$ of *jointly Λ-harmonic* functions by letting

$$\mathcal{H}(\Lambda) := \{\phi \in L^\infty(G) : P_\mu\phi = \phi \text{ for all } \mu \in \Lambda\}.$$

Note that $\mathcal{H}(\Lambda)$ is a weak-* closed linear subspace of $L^\infty(G)$. The preannihilator of $\mathcal{H}(\Lambda)$ in $L^1(G)$ is

$$J_\Lambda := \overline{\operatorname{span}\{f * \mu - f : f \in L^1(G),\, \mu \in \Lambda\}}$$

[4, p. 8]. Since $\mathcal{H}(\Lambda)$ is invariant under left translations, the space J_Λ is a left ideal in $L^1(G)$.

The map $\Theta(\mu)$ extends P_μ (under the natural identification of $L^\infty(G)$ with $\mathcal{D}_G$): for every $\phi \in L^\infty(G)$ and any $\mu \in M(G)$, we have

$$\Theta(\mu)(M_\phi) = M_{P_\mu\phi},$$

and so $\phi \in \mathcal{H}(\Lambda)$ if and only if $\Theta(\mu)(M_\phi) = M_\phi$ for all $\mu \in \Lambda$. It is therefore natural to define the set $\widetilde{\mathcal{H}}(\Lambda)$ of all *jointly Λ-harmonic operators* by letting

$$\widetilde{\mathcal{H}}(\Lambda) := \{T \in \mathcal{B}(L^2(G)) : \Theta(\mu)(T) = T \text{ for all } \mu \in \Lambda\}.$$

This weak-* closed subspace of $\mathcal{B}(L^2(G))$ is a $\mathrm{VN}(G)$-bimodule (as $\Theta(\mu)$ is a $\mathrm{VN}(G)$-bimodule map for every μ) and it contains $\{M_a : a \in \mathcal{H}(\Lambda)\}$; hence it contains $\mathrm{Bim}(\mathcal{H}(\Lambda))$.

Theorem 7.1. *If $\Lambda \subseteq M(G)$ then*

$$\widetilde{\mathcal{H}}(\Lambda) = (\operatorname{Ran} J_\Lambda)^\perp.$$

Proof. Recall that $\operatorname{Ran} J_\Lambda$ is the closed linear span of $\theta(u)h$ where $u \in J_\Lambda$ and $h \in \mathcal{T}(G)$. If $u = f * \mu - f$ where $f \in L^1(G)$, $\mu \in \Lambda$ and $T \in \mathcal{B}(L^2(G))$ then

$$\langle T, \theta(u)h\rangle_t = \langle \Theta(f)\Theta(\mu - \delta_e)T, h\rangle_t.$$

By Lemma 3.1, $T \in (\operatorname{Ran} J_\Lambda)^\perp$ if and only if

$$\Theta(f)\Theta(\mu - \delta_e)T = 0, \qquad f \in L^1(G),\ \mu \in \Lambda. \tag{13}$$

Since Θ is the integral of a bounded representation of G, namely $\operatorname{Ad}\rho$, if X is a non-zero operator then $\Theta(f)(X)$ must be non-zero for some $f \in L^1(G)$. Thus, (13) holds true if and only if

$$\Theta(\mu - \delta_e)T = 0 \quad \text{for all } \mu \in \Lambda$$

i.e. if and only if $T \in \widetilde{\mathcal{H}}(\Lambda)$. ∎

Theorems 7.1 and 6.5 and Propositions 5.3 and 4.2 imply the following corollary.

Corollary 7.2. *Let G be a locally compact group such that* $(\operatorname{Ran} J)^{\perp} = \operatorname{Bim}(J^{\perp})$ *for every closed left ideal J of $L^1(G)$. Then*

$$\widetilde{\mathcal{H}}(\Lambda) = \operatorname{Bim}(\mathcal{H}(\Lambda)). \tag{14}$$

In particular, (14) *holds true if G is abelian, or weakly amenable discrete, or compact.*

8. The non-commutative Poisson boundary. In this section, we discuss the case where Λ is a singleton consisting of a probability measure, say μ. There exists a norm one projection $\widetilde{\mathcal{E}}$ on $\widetilde{\mathcal{H}}(\mu) := \widetilde{\mathcal{H}}(\Lambda)$ given by a pointwise-weak-* limit of convex combinations of iterates of $\Theta(\mu)$. The non-commutative Poisson boundary of μ, denoted by $\widetilde{\mathcal{H}}_\mu$, is defined to be the space $\widetilde{\mathcal{H}}(\mu)$, equipped with the unique von Neumann algebra structure defined through the Choi–Effros product $\diamond$ given by $T \diamond S = \widetilde{\mathcal{E}}(TS)$ [17]. The space $\mathcal{H}(\mu) := \mathcal{H}(\Lambda)$ is closed under $\diamond$ and therefore is a von Neumann subalgebra of $\widetilde{\mathcal{H}}_\mu$ denoted by $\mathcal{H}_\mu$.

Thus $\widetilde{\mathcal{H}}(\mu)$ is an injective weak-* closed operator system, and in fact so is its subspace $\mathcal{H}(\mu)$ (it is the range of a contractive projection from $\mathcal{D}$). Moreover, $\mathcal{H}(\mu)$ admits a natural action α of G by weak-* continuous unital completely positive isometries, given by the restriction of the action of G on $L^\infty(G)$ by left translation: $(\alpha_s\phi)(t) = \phi(s^{-1}t)$ (the space $\mathcal{H}(\mu)$ is invariant under translation because P_μ commutes with each α_s).

We wish to show that the operator system $\widetilde{\mathcal{H}}(\mu)$ is isomorphic, as a dual operator system, to the operator system crossed product $G \ltimes_\alpha \mathcal{H}(\mu)$, which we now define:

Let $\mathcal{M}$ be a dual operator system, and let $s \mapsto \alpha_s$ be a point-weak-* continuous action of G on $\mathcal{M}$ by weak-* continuous unital completely positive isometries. The action is encoded by the map

$$\tilde{\alpha} : \mathcal{M} \to L^\infty(G, \mathcal{M}) : v \mapsto (\alpha_s^{-1}(v))_{s\in G},$$

which is a unital completely positive isometry. Let $\mathcal{B} := \mathcal{B}(L^2(G))$ and identify $L^\infty(G,\mathcal{M})$ with $L^\infty(G) \bar{\otimes} \mathcal{M} \subseteq \mathcal{B} \bar{\otimes} \mathcal{M}$. We also have a map

$$G \to \mathcal{B} \bar{\otimes} \mathcal{M} : s \mapsto \tilde{\lambda}_s := \lambda_s \otimes I.$$

Definition 8.1. The *crossed product* $G \ltimes_\alpha \mathcal{M}$ is defined to be the subspace of $\mathcal{B} \bar{\otimes} \mathcal{M}$ generated by $\tilde{\alpha}(\mathcal{M}) \cdot \tilde{\lambda}(G)$; it is the weak-* closed space

$$G \ltimes_\alpha \mathcal{M} := \overline{\operatorname{span}\{\tilde{\alpha}(v)\tilde{\lambda}_s : v \in \mathcal{M},\ s \in G\}}^{w^*} \subseteq \mathcal{B} \bar{\otimes} \mathcal{M}.$$

Remark 8.2. If $\mathcal{M}$ is an *injective* operator system (as in the case $\mathcal{M} = \mathcal{H}(\mu)$ considered here), it follows from the well known corresponding result for von Neumann algebra crossed products [27, Theorem X.1.7] that the crossed product is independent of the representation of $\mathcal{M}$ as a weak-*

closed operator subsystem of some $\mathcal{B}(H)$. This is because $\mathcal{M}$ admits a unique von Neumann algebra structure, $\mathcal{N}$ say, induced by the Choi–Effros product and its original operator space structure. Then $G \ltimes_\alpha \mathcal{M}$ is isomorphic, as a dual operator system, to the von Neumann algebra crossed product $G \ltimes \mathcal{N}$, which does not depend on the representation of $\mathcal{N}$ on Hilbert space.

Let $V \in \mathcal{B} \bar{\otimes} \mathcal{B}$ be the *fundamental unitary*, given by

$$(V\xi)(s,t) = \xi(st,t)\Delta(t)^{1/2}, \qquad \xi \in L^2(G) \otimes L^2(G),$$

and define

$$\tilde{\Gamma} : \mathcal{B} \to \mathcal{B} \bar{\otimes} \mathcal{B} \quad \text{by} \quad \tilde{\Gamma}(T) := V(T \otimes I)V^*.$$

Note that $\tilde{\Gamma}$ is clearly a normal *-homomorphism and an isometry, hence a normal unital completely positive map.

Proposition 8.3. *We have $\tilde{\Gamma}(\mathrm{Bim}(\mathcal{H}(\mu))) = G \ltimes_\alpha \mathcal{H}(\mu)$. In particular,*

$$G \ltimes_\alpha \mathcal{H}(\mu) \subseteq \tilde{\Gamma}(\widetilde{\mathcal{H}}(\mu)). \tag{15}$$

Proof. It is not hard to verify that $V(\lambda_r \otimes I) = (\lambda_r \otimes I)V$ for all $r \in G$ and $(I \otimes M_f)V = V(I \otimes M_f)$ for all $f \in L^\infty(G)$.

Thus, $V \in \mathcal{B} \bar{\otimes} \mathcal{D}$. It follows that

$$\begin{aligned} \tilde{\Gamma}(T) &= V(T \otimes I)V^* \in \mathcal{B} \bar{\otimes} \mathcal{D} && \text{for all } T \in \mathcal{B}, \\ \tilde{\Gamma}(\lambda_r) &= \lambda_r \otimes I = \tilde{\lambda}_r && \text{for all } r \in G. \end{aligned}$$

If $\phi \in \mathcal{H}(\mu)$ and $s \in G$ then the element $(\tilde{\alpha}\phi)(s) = \alpha_s^{-1}(\phi)$ of $\mathcal{H}(\mu)$ acts as a multiplication operator on $L^2(G)$ as follows:

$$((\tilde{\alpha}\phi)(s)\eta)(t) = (\alpha_s^{-1}(\phi))(t)\eta(t) = \phi(st)\eta(t), \qquad \eta \in L^2(G),\ t \in G.$$

We claim that, for every $\phi \in \mathcal{H}(\mu)$ and $r \in G$,

$$\tilde{\Gamma}(M_\phi \lambda_r) = \tilde{\alpha}(\phi)\tilde{\lambda}_r. \tag{16}$$

Now $\tilde{\Gamma}(M_\phi\lambda_r) = \tilde{\Gamma}(M_\phi)\tilde{\lambda}_r$ so it suffices to prove that $\tilde{\Gamma}(M_\phi) = \tilde{\alpha}(\phi)$, or equivalently $(\tilde{\alpha}(\phi))V = V(M_\phi \otimes I)$. Indeed, for all $\xi, \eta \in L^2(G)$,

$$\begin{aligned} (\tilde{\alpha}(\phi)V(\xi \otimes \eta))(s,t) &= (\alpha_{s^{-1}}\phi)(t)V(\xi \otimes \eta))(s,t) = \phi(st)\xi(st)\eta(t)\Delta(t)^{1/2}, \\ (V(M_\phi \otimes I)(\xi \otimes \eta))(s,t) &= (V(\phi\xi \otimes \eta))(s,t)) = (\phi\xi)(st)\eta(t)\Delta(t)^{1/2}, \end{aligned}$$

which proves the claim.

By linearity and weak-* continuity,

$$\tilde{\Gamma}\big(\overline{\mathrm{span}\{M_\phi\lambda_r : \phi \in \mathcal{H}(\mu),\ r \in G\}}^{w^*}\big) = \overline{\mathrm{span}\{\tilde{\alpha}(\phi)\tilde{\lambda}_r : \phi \in \mathcal{H}(\mu),\ r \in G\}}^{w^*},$$

i.e. $\tilde{\Gamma}(\mathrm{Bim}(\mathcal{H}(\mu))) = G \ltimes_\alpha \mathcal{H}(\mu)$.

Since $\mathrm{Bim}(\mathcal{H}(\mu)) \subseteq \widetilde{\mathcal{H}}(\mu)$, we have in particular $G \ltimes_\alpha \mathcal{H}(\mu) \subseteq \tilde{\Gamma}(\widetilde{\mathcal{H}}(\mu))$. ∎

In case G is weakly amenable discrete, compact or abelian, by Corollary 7.2 we know that $\mathrm{Bim}(\mathcal{H}(\mu)) = \widetilde{\mathcal{H}}(\mu)$. Therefore the previous proposition yields:

PROPOSITION 8.4. *Assume that* G *is weakly amenable discrete, compact or abelian. Then* $\tilde{\Gamma}$ *is an isomorphism of dual operator spaces between* $\widetilde{\mathcal{H}}(\mu)$ *and the crossed product* $G \ltimes_\alpha \mathcal{H}(\mu)$.

COROLLARY 8.5. *Assume that* G *is weakly amenable discrete, compact or abelian. Then the crossed product* $G \ltimes_\alpha \mathcal{H}(\mu)$ *is an injective operator system.*

For G weakly amenable discrete, compact or abelian we obtain the following corollary, established by Izumi [18] for discrete groups, by Jaworski and Neufang [19] for locally compact groups and by Kalantar, Neufang and Ruan [20] for locally compact quantum groups. Analogous results were obtained in [25] for complex contractive measures.

Using these results together with Theorem 7.1 we obtain, for any locally compact group G, the equality $(\mathrm{Ran}\, J_\Lambda)^\perp = \mathrm{Bim}(J_\Lambda^\perp)$ when $\Lambda = \{\mu\}$ and μ is a probability measure.

COROLLARY 8.6. *Assume that* G *is weakly amenable discrete, compact or abelian. Let* μ *be a probability measure on* G. *The non-commutative Poisson boundary* $\widetilde{\mathcal{H}}_\mu$ *is* **-isomorphic to the crossed product* $G \ltimes_\alpha \mathcal{H}_\mu$.

Proof. It follows from the definition of the von Neumann algebra structure on $\mathcal{H}_\mu$ that $\alpha_s(\phi \diamond \psi) = \alpha_s(\phi) \diamond \alpha_s(\psi)$ for $\phi, \psi \in \mathcal{H}_\mu$. Thus G acts on the von Neumann algebra $\mathcal{H}_\mu$ by weak-* continuous *-automorphisms. The corollary now follows from Proposition 8.4 and the fact that $\tilde{\Gamma}$ induces a completely positive surjective isometry between von Neumann algebras, which must therefore be a *-homomorphism [9, Corollary 5.2.3]. ∎

Added in proof (June 2019). In the recent preprint [6] it is shown that, for any locally compact group G with the approximation property (AP) of Haagerup and Kraus, the relation $(\mathrm{Ran}\, J)^\perp = \mathrm{Bim}(J^\perp)$ holds for every closed left ideal J of $L^1(G)$. It follows that, in this paper, the conclusions of Corollary 7.2, Proposition 8.4, Corollary 8.5 and Corollary 8.6 hold for any locally compact group G with the AP.

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