Operator Algebras: An introduction

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$\mathscr{B}(\mathscr{H})$

Let $\mathscr{H}$ be a Hilbert space. The algebra of all bounded linear operators $T : \mathscr{H} \to \mathscr{H}$ is denoted $\mathscr{B}(\mathscr{H})$. It is complete under the norm

$$\|T\| = \sup\{\|Tx\| : x \in \mathrm{b}_1(\mathscr{H})\}$$

Moreover, it has an *involution* $T \to T^*$ defined via

$$\langle T^* x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathscr{H}.$$

This satisfies

$$\|T^* T\| = \|T\|^2 \qquad \text{the } C^* \text{ property.}$$

C*-algebras

Definition

(a) A **Banach algebra** $\mathscr{A}$ is a complex algebra equiped with a complete submultiplicative norm:

$$\|ab\| \leq \|a\| \, \|b\| \, .$$

(b) A **C*-algebra** $\mathscr{A}$ is a Banach algebra equiped with an involution[1] $a \to a^*$ satisfying the **C*-condition**

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } \; a \in \mathscr{A} \, .$$

If $\mathscr{A}$ has a unit $\mathbf{1}$ then necessarily $\mathbf{1}^* = \mathbf{1}$ and $\|\mathbf{1}\| = 1$.
If not, adjoin a unit:
If $\mathscr{A}$ is a C*-algebra let $\qquad \mathscr{A}^\sim =: \mathscr{A} \oplus \mathbb{C}$

with $\quad (a,z)(b,w) =: (ab + wa + zb, zw) \quad (a,z)^* =: (a^*, \bar{z})$

$$\|(a,z)\| =: \sup\{\|ab + zb\| : b \in \mathrm{b}_1 \mathscr{A}\} \quad (\text{i.e. } \; \mathscr{A}^\sim \curvearrowright \mathscr{A})$$

[1] that is, a map on $\mathscr{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$

Basic Examples

A **morphism** $\phi : \mathscr{A} \to \mathscr{B}$ *between C*-algebras is a linear map that preserves products and the involution.*
We will see later that morphisms are automatically contractive, and 1-1 morphisms are isometric (algebra forces topology).

Basic Examples:

- $\mathbb{C}$
- $C(K)$: K compact Hausdorff, $f^*(t) = \overline{f(t)}$: abelian, unital.
- $C_0(X)$: X locally compact Hausdorff, $f^*(t) = \overline{f(t)}$: abelian, nonunital (iff X non-compact).
  We will see later (9) that all abelian C-algebras can be represented as $C_0(X)$ for suitable X.*
- $M_n(\mathbb{C})$: $A^* =$ conjugate transpose,
  $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$: non-abelian, unital.
- $\mathscr{B}(\mathscr{H})$: non-abelian, unital.
  We will see later (26) that all C-algebras can be represented as closed selfadjoint subalgebras of $\mathscr{B}(\mathscr{H})$ for suitable $\mathscr{H}$.*

Nonexamples:

- $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ holomorphic}\}$ [2]
  A closed subalgebra of the C-algebra $C(\overline{\mathbb{D}})$ but not a *-subalgebra, because if $f \in A(\mathbb{D})$ then $\overline{f}$ is not holomorphic unless it is constant: $A(\mathbb{D}) \cap A(\mathbb{D})^* = \mathbb{C}\mathbf{1}$: antisymmetric algebra.*
- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > j\}$ (upper triangular matrices).
  A closed subalgebra of the C-algebra $M_n(\mathbb{C})$ but not a *-subalgebra. Here $T_n \cap T_n^* = D_n$, the diagonal matrices: a maximal abelian selfadjoint algebra (masa) in M_n.*
- $M_{oo}(\mathbb{C})$: infinite matrices with finite support.
  To define norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra, but not complete.
  Its completion is $\mathscr{K}$, the set of compact operators on ℓ^2: a non-unital, non-abelian C-algebra.*

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[2] $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Direct sums, matrix algebras, $C_0(X, \mathscr{A})$ ...

- If X is an index set and $\mathscr{A}$ is a C*-algebra, the Banach space $\ell^\infty(X, \mathscr{A})$ of all bounded functions $a : X \to \mathscr{A}$ (with norm $\|a\|_\infty = \sup\{\|a(x)\|_{\mathscr{A}} : x \in X\}$) becomes a C*-algebra with pointwise product and involution.
Its subspace $c_0(X, \mathscr{A})$ consisting of all $a : X \to \mathscr{A}$ with $\lim_{x\to\infty} \|a(x)\|_{\mathscr{A}} = 0$ is a C*-algebra.
The subset $c_{00}(X, \mathscr{A})$ consisting of all functions of finite support is a dense *-subalgebra, which is proper when X is infinite.

- If X is locally compact Hausdorff then $C_b(X, \mathscr{A})$ is the *-subalgebra of $\ell^\infty(X, \mathscr{A})$ consisting of continuous functions. It is closed, hence a C*-algebra . (This is just $C(X, \mathscr{A})$ when X is compact.)
- The subalgebra $C_0(X, \mathscr{A})$ consists of those $f \in C_b(X, \mathscr{A})$ which 'vanish at infinity', i.e. such that the function $t \to \|f(t)\|_{\mathscr{A}}$ is in $C_0(X)$.

Direct sums, matrix algebras, $C_0(X, \mathscr{A})$ ...

Definition

(i) The direct sum $\mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_n$ of C*-algebras is a C*-algebra under pointwise operations and involution and the norm

$$\|(a_1, \ldots, a_n)\| = \max\{\|a_1\|, \ldots, \|a_n\|\}.$$

(ii) Let $\{\mathscr{A}_i\}$ be a family of C*-algebras . Their **direct product** or ℓ^∞**-direct sum** $\bigoplus_{\ell^\infty} \mathscr{A}_i$ is the subset of the cartesian product $\prod \mathscr{A}_i$ consisting of all $(a_i) \in \prod \mathscr{A}_i$ such that $i \to \|a_i\|_{\mathscr{A}_i}$ is bounded. It is a C*-algebra under pointwise operations and involution and the norm

$$\|(a_i)\| = \sup\{\|a_i\|_{\mathscr{A}_i} : i \in I\}.$$

(iii) The **direct sum** or c_0**-direct sum** $\bigoplus_{c_0} \mathscr{A}_i$ of a family $\{\mathscr{A}_i\}$ of C*-algebras is the closed selfadjoint subalgebra of their direct product consisting of all $(a_i) \in \prod \mathscr{A}_i$ such that $i \to \|a_i\|_{\mathscr{A}_i}$ vanishes at infinity.

In case $\mathscr{A}_i = \mathscr{A}$ for all i, the direct product is just $\ell^\infty(I, \mathscr{A})$.

Direct sums, matrix algebras, $C_0(X, \mathscr{A})$ ...

- If $\mathscr{A}$ is a C*-algebra and $n \in \mathbb{N}$, the space $M_n(\mathscr{A})$ of all matrices $[a_{ij}]$ with entries $a_{ij} \in \mathscr{A}$ becomes a *-algebra with product $[a_{ij}][b_{ij}] = [c_{ij}]$ where $c_{ij} = \sum_k a_{ik} b_{kj}$ and involution $[a_{ij}]^* = [d_{ij}]$ where $d_{ij} = d_{ji}^*$.

How to define a norm?

Special cases:

- Suppose $\mathscr{A}$ is $C_0(X)$; then norm $M_n(C_0(X))$ by identifying it (as a *-algebra) with $C_0(X, M_n)$, i.e. M_n-valued continuous functions on X vanishing at infinity.
- Suppose $\mathscr{A}$ is a C*-subalgebra of some $\mathscr{B}(\mathscr{H})$; then norm $M_n(\mathscr{A}) \subseteq M_n(\mathscr{B}(\mathscr{H}))$ by identifying $M_n(\mathscr{B}(\mathscr{H}))$ with $\mathscr{B}(\mathscr{H}^n)$.

General case: Use Gelfand - Naimark 26.

The spectrum

Definition

If $\mathscr{A}$ is a unital C*-algebra and $GL(\mathscr{A})$ denotes the group of invertible elements of $\mathscr{A}$, the **spectrum** of an element $a \in \mathscr{A}$ is

$$\sigma(a) = \sigma_{\mathscr{A}}(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin GL(\mathscr{A})\}.$$

If $\mathscr{A}$ is non-unital, the spectrum of $a \in \mathscr{A}$ is defined by

$$\sigma(a) = \sigma_{\mathscr{A}^\sim}(a).$$

In this case, necessarily $0 \in \sigma(a)$.

Lemma

The set $GL(\mathscr{A})$ is open in $\mathscr{A}$ and the map $x \to x^{-1}$ is continuous (hence a homeomorphism) on $GL(\mathscr{A})$.

The spectrum

Proposition

The spectrum is a nonempty compact *subset of* $\mathbb{C}$*.*

The **spectral radius**

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

satisfies $\rho(a) \leq \|a\|$. The **Gelfand-Beurling** formula is

$$\rho(a) = \lim_n \|a^n\|^{1/n} \leq \|a\|.$$

The spectrum

Proposition

(i) $a = a^*$ *(we say a is* **selfadjoint***)* $\implies \sigma(a) \subseteq \mathbb{R}$
(ii) $a = b^*b$ *(is it OK to call a* **positive** *??)* $\implies \sigma(a) \subseteq \mathbb{R}^+$
(iii) $u^*u = 1 = uu^*$ *(we say u is* **unitary***)* $\implies \sigma(u) \subseteq \mathbb{T}$

Lemma

If $aa^* = a^*a$ *(we say a is* **normal***) then* $\rho(a) = \|a\|$. [3]

Proposition

*There is at most one norm on a *-algebra making it a C*-algebra.*

[3] This is not true in general: consider any $a \neq 0$ with $a^2 = 0$.

Gelfand Theory

Theorem (Gelfand-Naimark 1)

Every commutative C-algebra $\mathscr{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathscr{A}})$ where $\hat{\mathscr{A}}$ is the set of nonzero morphisms $\phi : \mathscr{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. The map is the Gelfand transform:*

$$\mathscr{A} \to C_0(\hat{\mathscr{A}}): \; a \to \hat{a} \quad \text{where} \quad \hat{a}(\phi) = \phi(a), \; (\phi \in \hat{\mathscr{A}}).$$

The algebra $\mathscr{A}$ is unital iff $\hat{\mathscr{A}}$ is compact.

In more detail:
$\hat{\mathscr{A}}$ is the set of all *nonzero* multiplicative linear forms (*characters*) $\phi : \mathscr{A} \to \mathbb{C}$, (necessarily $\|\phi\| \leq 1$ and, when $\mathscr{A}$ is unital, $\|\phi\| = \phi(\mathbf{1}) = 1$) equipped with the w*-topology: $\phi_i \to \phi$ iff $\phi_i(a) \to \phi(a)$ for all $a \in \mathscr{A}$.

When $\mathscr{A}$ is non-abelian there may be no characters (consider $M_2(\mathbb{C})$ or $\mathscr{B}(\mathscr{H})$, for example).

When $\mathscr{A}$ is abelian there are 'many' characters: for each $a \in \mathscr{A}$ there exists $\phi \in \hat{\mathscr{A}}$ such that $\|a\| = |\phi(a)|$.

When $\mathscr{A}$ is unital $\hat{\mathscr{A}}$ is compact and $\mathscr{A}$ is isometrically *-isomorphic to $C(\hat{\mathscr{A}})$.

The Continuous Functional Calculus

Let A be a selfadjoint element of the unital C*-algebra $\mathscr{B}(\mathscr{H})$.
For any polynomial $p(\lambda) = \sum_n c_k \lambda^k$ we have a (normal) element $p(A) = \sum_n c_k A^k \in \mathscr{B}(\mathscr{H})$.
The map

$$\mathscr{P}(\sigma(A)) \to \mathscr{B}(\mathscr{H}) : \Phi_0 : p \to p(A)$$

is a *-homomorphism.
We wish to extend this map to a map $f \to f(A)$ defined on all continuous functions $f : \sigma(A) \to \mathbb{C}$.

The Continuous Functional Calculus

Theorem

If $A \in \mathscr{B}(\mathscr{H})$ is selfadjoint and p is a polynomial,

$$\|p(A)\| = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\} \equiv \|p\|_{\sigma(A)}.$$

This is a consequence of

Proposition (Spectral mapping Theorem I)

If $A \in \mathscr{B}(\mathscr{H})$ is selfadjoint and p is a polynomial,

$$\sigma(p(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.$$

and the fact that the spectral radius of a normal element ($p(A)$ is normal) equals its norm.

The Continuous Functional Calculus

Definition

Let $A = A^* \in \mathscr{B}(\mathscr{H})$. The **continuous functional calculus** for A is the unique continuous extension

$$\Phi_c : (C(\sigma(A)), \|.\|_{\sigma(A)}) \to (\mathscr{B}(\mathscr{H}), \|.\|) : f \to f(A)$$

of the map $\Phi_o : p \to p(A)$.
Thus if f is continuous on $\sigma(A)$, the operator $f(A) \in \mathscr{B}(\mathscr{H})$ is defined by the limit

$$f(A) = \lim p_n(A) \text{ where } \|p_n - f\|_{\sigma(A)} \to 0.$$

The Spectral Theorem

If $A \in \mathscr{B}(\mathscr{H})$ is selfadjoint and $K = \sigma(A)$, the continuous functional calculus

$$\Phi_c : C(K) \to \mathscr{B}(\mathscr{H}) : f \to f(A)$$

is a *representation* of the (abelian) C*-algebra $C(K)$ on $\mathscr{H}$.
We will construct a 'measure' $E(\cdot)$ with values not numbers, but projections on $\mathscr{H}$, so that

$$\Phi_c(f) = \int_K f(\lambda) dE_\lambda \qquad \text{for each } f \in C(K)$$

and in particular $A = \Phi_c(id) = \int_K \lambda \, dE_\lambda$.

(This generalises $A = \sum \lambda_i E_i$,

where λ_i: eigenvalues, E_i: eigenprojections of $A \in M_n$.)

The Spectral Theorem

Definition

A (regular) ‘spectral measure’ on K is a map
$E : \mathscr{S}(K) \to \mathscr{B}(\mathscr{H})$ such that ($\mathscr{S}(K)$: the Borel σ-algebra)

1. $E(\Omega)^* = E(\Omega)$
2. $E(\Omega_1 \cap \Omega_2) = E(\Omega_1).E(\Omega_2)$
3. $E(\emptyset) = 0$ and $E(K) = I$
4. for $x, y \in H$, the map $\mu_{xy} : \Omega \to \langle E(\Omega)x, y\rangle$ is a σ-additive complex-valued (regular) set function on $\mathscr{S}(K)$.

Theorem

Every representation π of $C(K)$ on a Hilbert space H determines a unique regular Borel spectral measure $E(.)$ on K so that

$$\int_K f dE = \pi(f) \qquad (f \in C(K)).$$

Positivity

Definition

An element $a \in \mathscr{A}$ is **positive** if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$.
We write $\mathscr{A}_+ = \{a \in \mathscr{A} : a \geq 0\}$.
If a, b are selfadjoint, we define $a \leq b$ by $b - a \in \mathscr{A}_+$.

Examples

In $C(X)$: $f \geq 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in X$ because $\sigma(f) = f(X)$.
In $\mathscr{B}(\mathscr{H})$: $T \geq 0$ iff $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.

Remark

Any morphism $\pi : \mathscr{A} \to \mathscr{B}$ between C-algebras preserves order:*

$$a \geq 0 \quad \Rightarrow \quad \pi(a) \geq 0.$$

Remark

If $a = a^*$ then $\qquad -\|a\| \mathbf{1} \leq a \leq \|a\| \mathbf{1}.$

Positivity

Proposition

Every positive element has a unique positive square root.

Theorem

In any C-algebra, any element of the form $a^* a$ is positive.*

For the proof, we need

Proposition

For any C-algebra the set $\mathscr{A}_+$ is a* cone*:*

$$a, b \in \mathscr{A}_+,\ \lambda \geq 0 \quad \Rightarrow \quad \lambda a \in \mathscr{A}_+, a + b \in \mathscr{A}_+.$$

Lemma

In a unital C-algebra if $x = x^*$ and $\|x\| \leq 1$, then*

$$x \geq 0 \quad \Longleftrightarrow \quad \|\mathbf{1} - x\| \leq 1.$$

The GNS construction

Definition

A **state** on a C*-algebra $\mathscr{A}$ is a positive linear map of norm 1, i.e. $\phi : \mathscr{A} \to \mathbb{C}$ linear such that $\phi(a^*a) \geq 0$ for all $a \in \mathscr{A}$ and $\|\phi\| = 1$. A state is called **faithful** if $\phi(a^*a) > 0$ whenever $a \neq 0$.

NB. When $\mathscr{A}$ is unital and ϕ is positive, $\|\phi\| = \phi(\mathbf{1})$.

Examples

- On $\mathscr{B}(\mathscr{H})$, $\phi(T) = \langle T\xi, \xi\rangle$ for a unit vector $\xi \in \mathscr{H}$,
or $\phi(T) = \sum_i \langle T\xi_i, \xi_i\rangle$ where $\sum \|\xi_i\|^2 = 1$ (diagonal 'density matrix').
- On $C(K)$, $\phi(f) = f(t)$ for $t \in K$,
or $\phi(f) = \int f d\mu$ for a probability measure μ.
- For a C*-algebra $\mathscr{A}$, if $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is a representation and $\xi \in \mathscr{H}$ a unit vector, $\phi(a) = \langle \pi(a)\xi, \xi\rangle$.

Conversely,

The GNS construction

Conversely,

Theorem (Gelfand, Naimark, Segal)

For every state f on a C-algebra $\mathscr{A}$ there is a triple $(\pi_f, \mathscr{H}_f, \xi_f)$ where π_f is a representation of $\mathscr{A}$ on $\mathscr{H}_f$ and $\xi_f \in \mathscr{H}_f$ a* cyclic [4] *unit vector such that*

$$f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle \quad \text{for all } a \in \mathscr{A}.$$

The GNS triple $(\pi_f, \mathscr{H}_f, \xi_f)$ is uniquely determined by this relation up to unitary equivalence.

[4] i.e. $\pi_f(\mathscr{A})\xi_f$ is dense in $\mathscr{H}_f$.

The universal representation

Theorem (Gelfand, Naimark)

For every C-algebra $\mathscr{A}$ there exists a representation $(\pi, \mathscr{H})$ which is one to one (called* faithful*).*

Idea of proof Enough to assume $\mathscr{A}$ unital. Let $\mathscr{S}(\mathscr{A})$ be the set of all states. For each $f \in \mathscr{S}(\mathscr{A})$ consider $(\pi_f, \mathscr{H}_f)$ and 'add them up' to obtain $(\pi, \mathscr{H})$. Why is this faithful? Because

Lemma

For each nonzero $a \in \mathscr{A}$ there exists $f \in \mathscr{S}(\mathscr{A})$ such that $f(a^*a) > 0$.

... and then

$$\|\pi(a)\xi_f\|^2 = \langle \pi(a^*a)\xi_f, \xi_f \rangle = \langle \pi_f(a^*a)\xi_f, \xi_f \rangle = f(a^*a) > 0$$

so $\pi(a) \neq 0$.

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