Borel equivalence relations, cardinal algebras and structurability

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I would like to start by giving some motivation and general background concerning the theory of Borel equivalence relations and then I will talk about some recent results in this theory.

Classification problems

A classification problem is given by:

- A collection of objects X.
- An equivalence relation E on X.

A complete classification of X up to E consists of:

- A set of invariants I.
- A map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$.

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Equivalence relations and reducibility

The theory of equivalence relations on well-behaved spaces, i.e., standard Borel spaces (like, e.g., Euclidean spaces, separable Banach spaces, Polish groups, etc.), studies the set-theoretic nature of possible (complete) invariants and develops a mathematical framework for measuring the complexity of classification problems.

In this talk I will be interested in Borel equivalence relations E on standard Borel spaces X (i.e., E is Borel as a subset of the product space $X \times X$).

The following simple concept is basic in organizing this study.

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Definition

Let $(X, E), (Y, F)$ be equivalence relations. E is (Borel) reducible to F, in symbols

$$E \leq_B F,$$

if there is Borel map $f : X \to Y$ such that

$$x \; E \; y \Leftrightarrow f(x) \; F \; f(y).$$

Intuitive meaning:

- The classification problem represented by E is at most as complicated as that of F.
- F-classes are complete invariants for E.

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We also put:

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Borel cardinality theory

The preceding concepts can be also interpreted as the basis of a "definable" or Borel cardinality theory for quotient spaces.

- $E \leq_B F$ means that there is a "Borel injection" of X/E into Y/F, i.e., X/E has Borel cardinality less than or equal to that of Y/F, in symbols

$$|X/E|_B \leq |Y/F|_B$$

- $E \sim_B F$ means that X/E and Y/F have the same Borel cardinality, in symbols

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Countable Borel equivalence relations

Definition

E is countable if every E-class is countable.

Example

Any equivalence relation, E_Γ^X, induced by a Borel action of a countable group Γ on X

We actually have:

Theorem (Feldman-Moore)

Every countable E is of the form E_Γ^X.

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Turing equivalence

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Up to bireducibility they also include:

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E_G^X for G second countable locally compact group (e.g., Lie group)

Example (Hjorth-K)

Isomorphism of countable structures that are of "finite type", e.g., finitely generated groups, locally finite trees, finite rank torsion-free abelian groups, finite transcendence degree fields, etc.

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Hyperfinite

We will now consider the structure of $\leq_B$ on the countable Borel equivalence relations.

The simplest countable Borel equivalence relations are the smooth ones, which have a trivial structure. A countable Borel equivalence relation is smooth if it admits a Borel transversal.

The next more complicated ones are the so-called hyperfinite ones.

Definition

E is hyperfinite if $E = \bigcup_n E_n$, with E_n increasing and finite (i.e., having equivalence classes that are finite).

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Theorem (Dougherty-Jackson-K)

Up to Borel bireducibility, there is only one non-smooth, hyperfinite equivalence relation, namely E_0.

Here E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by

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Universal

The hyperfinite equivalence relations are the simplest non-trivial countable equivalence relations. At the other end there are the most complex ones, the so-called universal ones.

Theorem (Dougherty-Jackson-K)

There is a universal countable Borel equivalence relation, E_∞. It satisfies $E \leq_B E_\infty$, for all countable E.

It is of course unique up to bi-reducibility.

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$E_\infty \sim_B$ (the shift equivalence relation on 2^{F_2})

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$$E_0 <_B E_\infty$$

There are countable equivalence relations which are neither hyperfinite nor universal.

Theorem (Adams, Jackson-K-Louveau)

There exist intermediate countable Borel equivalence relations E, i.e.,

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Does non-linearity occur here?

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The key point is that there is a phenomenon of set theoretic rigidity analogous to the measure theoretic rigidity phenomena discovered by Zimmer.

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- **(Measure theoretic rigidity)** Under certain circumstances, when a countable group acts preserving a probability measure, the equivalence relation associated with the action together with the measure "encode" or "remember" a lot about the group (and the action).
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Some set theoretic rigidity results:

Theorem (Adams-K)

$$|\mathbb{T}^m/\mathrm{GL}_m(\mathbb{Z})|_B = |\mathbb{T}^n/\mathrm{GL}_n(\mathbb{Z})|_B \Leftrightarrow m = n$$

Below $\Gamma_p = \mathrm{SO}_7(\mathbb{Z}[1/p])$, p prime. Also E_p is the free part of the shift equivalence relation on 2^{Γ_p}.

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Since that time many other such set theoretic rigidity results have been obtained but they all used sophisticated methods of ergodic theory.

Theorem (Hjorth-K)

Let $\Gamma_p = (\mathbb{Z}/p\mathbb{Z} \star \mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}$. Then if E_p is the free part of the shift equivalence relation on 2^{Γ_p}, then the E_p are incomparable in Borel reducibility. Similarly if for each subset S of odd primes, we consider the group $\Gamma_S = (\star_{p \in S}(\mathbb{Z}/p\mathbb{Z} \star \mathbb{Z}/p\mathbb{Z})) \times \mathbb{Z}$.

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Here are also some striking such rigidity results from algebra:

Let $\cong_n$ be isomorphism of torsion-free abelian groups of rank n, i.e., subgroups of $(\mathbb{Q}^n, +)$. This can be seen to be (up to $\sim_B$) a countable Borel equivalence relation.

Theorem (Thomas)

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Thus

$$E_0 \sim_B (\cong_1) <_B (\cong_2) <_B (\cong_3) <_B \cdots$$

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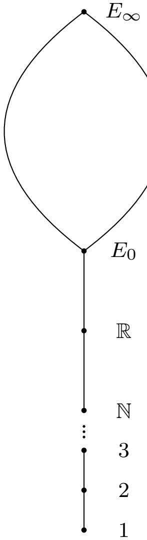
To this day it is unknown how to produce even a single example of a pair of incomparable under Borel reducibility countable Borel equivalence relations without using ergodic theory.

Picture of $\leq_B$ on countable equivalence relations



Algebraic structure

Although it was known for the last 20 years or so that the poset of countable Borel equivalence relations under Borel reducibility was quite complicated, practically nothing was known until very recently about the structure of this poset.

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In the late 1940's Tarski published the book *Cardinal Algebras* in which he developed an algebraic approach to the theory of cardinal addition, devoid of the use of the full Axiom of Choice, which of course trivializes it. A **cardinal algebra** is an algebraic system consisting of an abelian semigroup with identity (viewed additively) augmented with an infinitary addition operation for infinite sequences, satisfying certain axioms.

The theory of cardinal algebras seems to have been largely forgotten but I will show that they appear naturally in the context of the current theory of countable Borel equivalence relations. As a result one can apply Tarski's theory to discover a number of interesting laws governing the structure of countable Borel equivalence relations, which, in retrospect rather surprisingly, have not been realized before.

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Cardinal algebras

In order to give the flavor of the results that one can obtain by applying Tarski's theory of cardinal algebras to Borel equivalence relations, I will mention a few representative examples.

Below if $n > 0$ is a positive integer and E an equivalence relation on X, then nE is the direct sum of n copies of E, i.e., the equivalence relation F on $X \times \{0, 1, \ldots, n-1\}$ (where E lives on X), defined by $(x,i)F(y,j) \iff xEy \ \& \ i = j$.

Cardinal algebras

In order to give the flavor of the results that one can obtain by applying Tarski's theory of cardinal algebras to Borel equivalence relations, I will mention a few representative examples.

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Cardinal algebras

Theorem (K-Macdonald)

(i) (Existence of least upper bounds) *Any increasing sequence $F_0 \leq_B F_1 \leq_B \dots$ of countable Borel equivalence relations has a least upper bound (in the pre-order $\leq_B$).*

(ii) (Interpolation) *If $\mathcal{S}, \mathcal{T}$ are countable sets of countable Borel equivalence relations and $\forall E \in \mathcal{S} \forall F \in \mathcal{T} (E \leq_B F)$, then there is a countable Borel equivalence relation G such that $\forall E \in \mathcal{S} \forall F \in \mathcal{T} (E \leq_B G \leq_B F)$.*

Theorem

(iii) (Cancelation) *If* $n > 0$ *and* E, F *are countable Borel equivalence relations, then*

$$nE \leq_B nF \implies E \leq_B F$$

and therefore

$$nE \sim_B nF \implies E \sim_B F.$$

(iv) (Dichotomy for integer multiples) *For any countable Borel equivalence relation* E*, exactly one of the following holds:*

$$(a)\ E <_B 2E <_B 3E <_B \dots,$$

$$(b)\ E \sim_B 2E \sim_B 3E \sim_B \dots.$$

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What is a cardinal algebra?

A **cardinal algebra** is a system $\langle A, +, \sum\rangle$, where $\langle A, +\rangle$ is an abelian semigroup with identity, which will be denoted by 0, and $\sum\colon A^{\mathbb{N}} \to A$ is an infinitary operation, satisfying the following axioms, where we put $\sum_{n<\infty} a_n = \sum((a_n)_{n\in\mathbb{N}})$:

(A) $\sum_{n<\infty} a_n = a_0 + \sum_{n<\infty} a_{n+1}$.

(B) $\sum_{n<\infty}(a_n + b_n) = \sum_{n<\infty} a_n + \sum_{n<\infty} b_n$.

(C) If $a + b = \sum_{n<\infty} c_n$, then there are $(a_n), (b_n)$ such that

$$a = \sum_{n<\infty} a_n, b = \sum_{n<\infty} b_n, c_n = a_n + b_n.$$

(D) If $(a_n), (b_n)$ are such that $a_n = b_n + a_{n+1}$, then there is c such that for each n, $a_n = c + \sum_{i<\infty} b_{n+i}$.

The poset of a cardinal algebra

In a cardinal algebra A, let

$$a \leq b \iff \exists c(a + c = b).$$

It turns out that this is a partial ordering. Moreover all the expected commutativity, associativity laws for $+, \sum$ and monotonicity with respect to $\leq$ hold.

Definition

Let $\mathcal{E}$ be the class of countable Borel equivalence relations. We denote by $[\mathcal{E}]$ the quotient space of $\mathcal{E}$ by $\sim_B$, i.e., $[\mathcal{E}] = \{[E] \colon E \in \mathcal{E}\}$, where $[E] = \{F \in \mathcal{E} \colon E \sim_B F\}$. We call $[E]$ the **bireducibility type** of E (in $\mathcal{E}$).

If $E_i, i < n$, where $n \leq \infty$, are equivalence relations, with E_i living on X_i, then we let $\bigoplus_{i<n} E_i$ be the equivalence relation on $\bigsqcup_{i<n} X_i = \bigcup_{i<n} X_i \times \{i\}$ given by

$$(x, j) \bigoplus_{i<n} E_i (y, k) \iff j = k \ \& \ x E_j y.$$

We can now define on $[\mathcal{E}]$,

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We now have

Theorem (K-Macdonald)

$\langle [\mathcal{E}], +, \sum \rangle$ is a cardinal algebra. Moreover, for $E, F \in [\mathcal{E}]$, $E \leq_B F \iff [E] \leq [F]$.

Thus all the laws of cardinal algebras that are proved in Tarski's book are valid for countable Borel equivalence relations, in particular those stated in earlier slides.

Remark

It is unknown if the bireducibilty types of arbitrary Borel equivalence relations form a cardinal algebra. It is known however that even it is a cardinal algebra, its partial order will not be that of reducibility.

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Cancelation laws

In particular, recall the cancelation law for sums:

If $n > 0$ and E, F are countable Borel equivalence relations, then

$$nE \sim_B nF \implies E \sim_B F.$$

On can ask whether there an analogous result for products.
If E, F are equivalence relations, on X, Y, resp., their product $E \times F$ is the equivalence relation on $X \times Y$ defined by:

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The proof is inspired by tanother result of Tarski in cardinal arithmetic that states that the Axiom of Choice is equivalent to the statement: For any two infinite cardinals κ, λ, $(\kappa^2 = \lambda^2 \implies \kappa = \lambda)$.

As opposed to the proof of Tarski's Theorem, which makes use of the Hartogs number of an infinite cardinal to produce, assuming the Axiom of Choice fails, two infinite cardinals μ, ν such that if $\kappa = \mu + \nu, \lambda = \mu \cdot \nu$, then $\kappa^2 = \lambda^2$ but $\kappa < \lambda$, the proof of this theorem uses some deep results of ergodic theory and geometric group theory, thus it is far from elementary.

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Idea of the proof: Take a countable group Γ which is the infinite direct sum of a countable group which is torsion free, simple and has property (T). Then take a countable group Δ which is the infinite direct sum of a countable group which is torsion, simple and has property (T) (these exist by results of Ol'shanskiĭ). Let R be the equivalence relation induced by the free part of the shift action of Γ on $[0,1]^\Gamma$ and similarly define S for Δ. Then it turns out that if $E = R \oplus S$ and $F = R \times S$, then $E <_B F$ but $E^2 \sim_B F^2$. Among other things the proof uses Popa's superrigidity theory.

Algebraic structure

Despite these results, in general the algebraic structure of the hierarchical order of Borel reducibility remains quite mysterious. For example consider the following simple question:

Problem

Are there two incomparable, under Borel reducibility, countable Borel equivalence relations that have an infimum in the Borel reducibility order?

Some progress has been achieved very recently by systematically studying the concept of structurability of equivalence relations, that has played an important role in this theory. This is joint work with Ronnie Chen.

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Structurabilty

It will take too long to discuss in detail this theory, so I will concentrate on a couple of results that in particular answer the above question.

Let E be a countable Borel equivalence relation and $\mathcal{K}$ a class of countable structures (like, e.g., linear orders, graphs, etc.). We say that E is $\mathcal{K}$-structurable if there is a Borel way to put a structure in $\mathcal{K}$ on each equivalence class.

Examples

Hyperfinite, treeable, induced by a free Borel action of a fixed countable group Γ, etc.

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Theorem (K-Solecki-Todorcevic, Miller)

There is a $\mathcal{K}$-structurable countable Borel equivalence relation $E_{\infty\mathcal{K}}$, which is invariantly universal for the class of $\mathcal{K}$-structurable countable Borel equivalence relations. This is unique up to Borel isomorphism.

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Structurabilty

The next result shows that the Borel reducibility order contains a very large distributive lattice. Below a countable Borel equivalence relation E is called universally structurable if it is of the form $E_{\infty\mathcal{K}}$.

Theorem (Chen-K)

The order of Borel reducibility among universally structurable relations is a countably complete, distributive lattice, whose (countable) infs and sups coincide with those in the class of all countable Borel equivalence relations. Moreover, every Borel partial order embeds into this lattice.

This in particular shows that there is a large class of incomparable under Borel reducibility countable Borel equivalence relations that admit (even countable) sups and infs and provide a positive answer to the above question.

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